

CHAPTER 7

DYNAMICS OF KINKS IN SMECTIC-C

7.1 Introduction

On the basis of the theory of hydrodynamics of smectic- C formulated in chapter four, some results on the dynamics of the uniform state and hydrodynamic instabilities have been described in chapters five and six respectively. In this chapter, the same theory is used to study an interesting problem associated with the dynamics of kinks, *viz.* front propagation. Propagation of fronts or kinks (also called domain walls or solitons) has been studied in the analyses of reaction-diffusion equations and in physical models of different phenomena in fluid dynamics, dendritic growth, population growth, chemical reactions and other biological models [1-12]. The non-linear dynamics of liquid crystals under the action of external fields has invoked great theoretical and experimental interest in this subject [13-19]. In this context, it may be noted that "solitons" describing kinks in liquid crystals are not solitons in the true sense. They are more in the nature of solitary waves. They do not have the essential characteristics of true solitons namely preservation of shape and momenta on a pairwise-collision. This is a consequence of the fact that they exist

in overdamped systems.

It was pointed out in the first chapter that a kink is a permitted topological defect in an external electric or magnetic field. It is in the nature of a nonsingular wall connecting two uniform states. It was pointed out that these start moving uniformly with a constant velocity when either the base states or the core becomes asymmetric. In this chapter, the dynamical effects of oscillating and rotating magnetic fields on the propagation of kinks in smectic-C will be considered. Many of the results of this chapter have been published [20].

7.2 Theory

In smectic-C, in an external electric or magnetic field, kinks or domain walls in the $\mathbf{c}$ vector field can exist without introducing layer deformations. They can be twist kinks, or splay-rich or bend-rich kinks. To simplify the problem further, we ignore coupling constants C_1 and C_2 . We consider the dynamics of only twist kinks since they need not be associated with backflow effects since the confining walls which are perpendicular to the smectic layers can exert a force due to pressure gradients which prevent backflow. This is a permitted solution of the hydrodynamic equations of smectic-C. It should be noted that there could exist other solutions which permit backflow but we do not consider them in this chapter. Even in such a complex case, if the macroscopic velocity is assumed to be small, the results which we will present in this chapter will not be changed drastically. We point out this especially because most studies to date [16, 17] on the propagation of kinks in chiral or achiral smectic-C do not seem to have appreciated the fact that generally backflow exists. In this chapter, the dynamics of kinks without macroscopic flow will be considered, and the relevant dynamical equation is the Oseen equation (4.30) presented in chapter four. The free energy density includes the elastic deformations of the $\mathbf{c}$ vector and the energy density due to an applied external electric or magnetic

field. Here, we shall describe the effects of a magnetic field $\mathbf{H}$. Incidentally, electric fields would produce similar results only if the material is free from ionic impurities.

In the cartesian coordinate system, the layer normal is taken along the z -axis, that is, $\mathbf{N} = (0, 0, 1)$, and the director $\mathbf{n}$ is

$$\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

where $\phi = \phi(z, t)$ is the angle that the $\mathbf{c}$ vector makes with the x -axis and θ is the constant angle that $\mathbf{n}$ makes with the layer normal. Though smectic-C is biaxial, often we get a deep insight even in the limit of uniaxial approximation where the system is assumed to be uniaxial about $\mathbf{n}$. Introducing additional terms due to biaxiality only leads to complicated algebra but no new effects. The free energy density is

$$F = \frac{1}{2} \left[B_3 \left(\frac{\partial \phi}{\partial z} \right)^2 - \chi_a (\mathbf{n} \cdot \mathbf{H})^2 \right] \quad (7.1)$$

where B_3 is the twist elastic constant and χ_a is the diamagnetic anisotropy of the Sm-C in the local uniaxial approximation. The dynamical equation for $\mathbf{c}$ is then given by

$$\lambda \frac{\partial \phi}{\partial t} = - \frac{\delta F}{\delta \phi} \quad (7.2)$$

where $\lambda = -\gamma_1$ is the twist viscosity coefficient. We consider a spatially homogeneous magnetic field described by

$$\mathbf{H} = H_0 (a + b \cos \gamma t) (\cos \alpha \cos \omega t, \cos \alpha \sin \omega t, \sin \alpha) \quad (7.3)$$

where γ is the angular frequency of the oscillating field, α is the angle that the field makes with the layers, and ω is the angular frequency of rotation about the layer normal, $\mathbf{N}$. The dynamics of twist kinks will be discussed in the three geometries depicted in Figure 7.1.

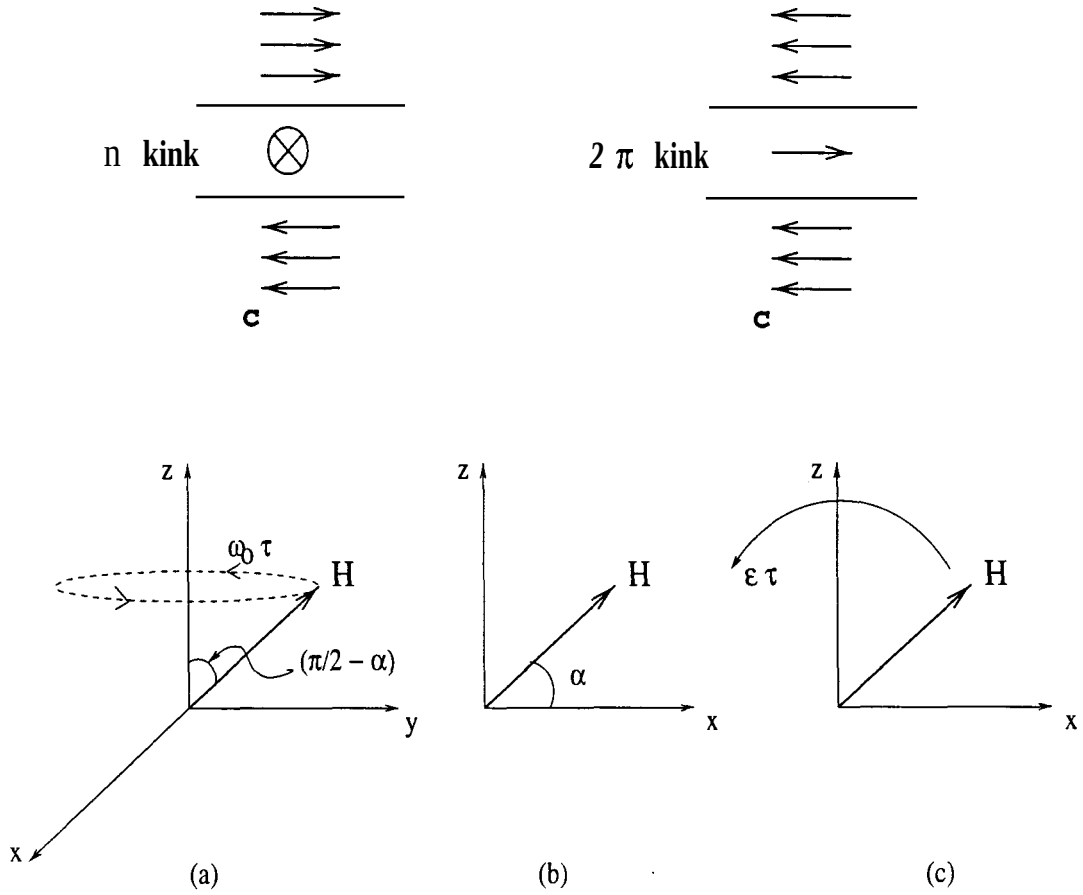


Figure 7.1: Schematic representation of a π twist kink, a 2π twist kink and the three geometries in which the dynamics of such kinks in an external magnetic field, H , are considered. (a) A field tilted at a constant angle α with respect to the layers rotating uniformly about the z axis, the layer normal. (b) A non-rotating field at a constant tilt angle, α with respect to the layers. (c) A constant field uniformly rotating in a plane xz , normal to the layers.

In terms of the dimensionless quantities, $\tau = (\chi_a H_0^2 / \lambda) t$ and $\eta = \sqrt{\chi_a H_0^2 / B_3} z$, the dynamical equation (7.2) can be written in the form

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2} - (a + b \cos \gamma_0 \tau)^2 [\sin \theta \cos \theta \sin \alpha \cos \alpha \sin(\phi - \omega_0 \tau) + \sin^2 \theta \cos^2 \alpha \sin(\phi - \omega_0 \tau) \cos(\phi - \omega_0 \tau)] \quad (7.4)$$

where $\gamma_0 = \gamma t / \tau$ and $\omega_0 = \omega t / \tau$. Time, distance, velocity and frequencies (of rota-

tion and oscillation) have been expressed in terms of these dimensionless quantities, *viz.* $\tau = t/t_0$, $\eta = z/z_0$, $\omega_0 = \omega t_0$, $\gamma_0 = \gamma t_0$ and, velocity (dimensionless) $= (z_0/t_0) \times$ velocity. The characteristic relaxation time and coherence length, t_0 and z_0 , are $\lambda/\chi_a H_0^2$ and $\sqrt{B_3/\chi_a H_0^2}$ respectively. Since the results are independent of the phase of the oscillating field, it has been taken to be zero.

The equation (7.4) is an overdamped double sine-Gordon equation. This equation is also applicable to other systems. When $a = 0$, equation (7.4) reduces to the normal sine-Gordon equation describing nematics in an external field; while, when $a \neq 0$, it describes field effects in ferronematics which are usual nematics with doped ferromagnetic grains. A variant of equation (7.4) can describe front propagation in chiral smectic liquid crystals [16].

This partial differential equation has been numerically solved by the method of lines using Numerical Algorithms Group (NAG) Fortran Library subroutines. We look for kink solutions which necessitate the boundary condition, $[\partial\phi/\partial\eta]_{\eta=\pm\infty} = 0$. Since numerical solutions can be got only for finite systems, we replace this by $[\partial\phi/\partial\eta]_{\eta=\pm L} = 0$, where L is the extent of the system in dimensionless units. We have taken $L = 1000$ with a mesh size, $\delta L = 0.1$. It is large enough to treat the solutions as a representative of an infinite sample, *i.e.*, L is very much larger than the coherence length, ξ . First, a front is excited in the form of a discontinuous twist in the homogeneous state. This is equivalent to, numerically, a step function in distortion. Then, its time evolution to a stable form is studied. For example, to create a propagating 2π kink, it is sufficient to have a step of 2π at the origin as the initial profile. The shape of the initial profile does not alter the asymptotic or late-time results as long as the net amplitude is large enough.

7.3 Instability of a 2π kink

We consider the case when a static field $\mathbf{H}$ is applied along the layers, and here, a simple situation in which kinks could become unstable is considered. A π kink is a natural topological defect in this case. As a result of this, two twist π kinks would repel each other. In the literature, there does not seem to exist a complete investigation of the dynamics of this situation. This problem necessarily involves numerical investigation. In this case, it has to be emphasized that the two kinks will not be in uniform motion. As an example, a pair of π twist kinks is studied.

We consider an initial state of $\mathbf{c}$ vector orientation described by a total winding of 2π connecting two stable states, i.e., a 2π twist kink in the presence of a magnetic field applied along the z -axis. Naturally, this is not a stable solution of equation (7.4) which permits only a π kink. We look at the time evolution of this state which can be called a front in director distortion.

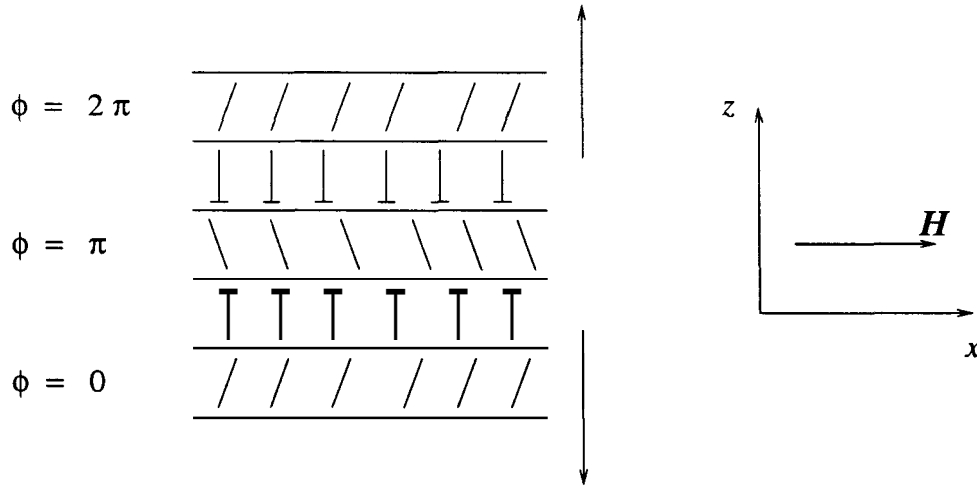


Figure 7.2: A schematic representation of the splitting of a 2π kink. The field $\mathbf{H}$ is applied along the layers.

In Figure 7.2, a schematic representation of the splitting of a 2π kink is shown. The evolution of the 2π step is shown in Figure 7.3. It splits into two π kinks which start

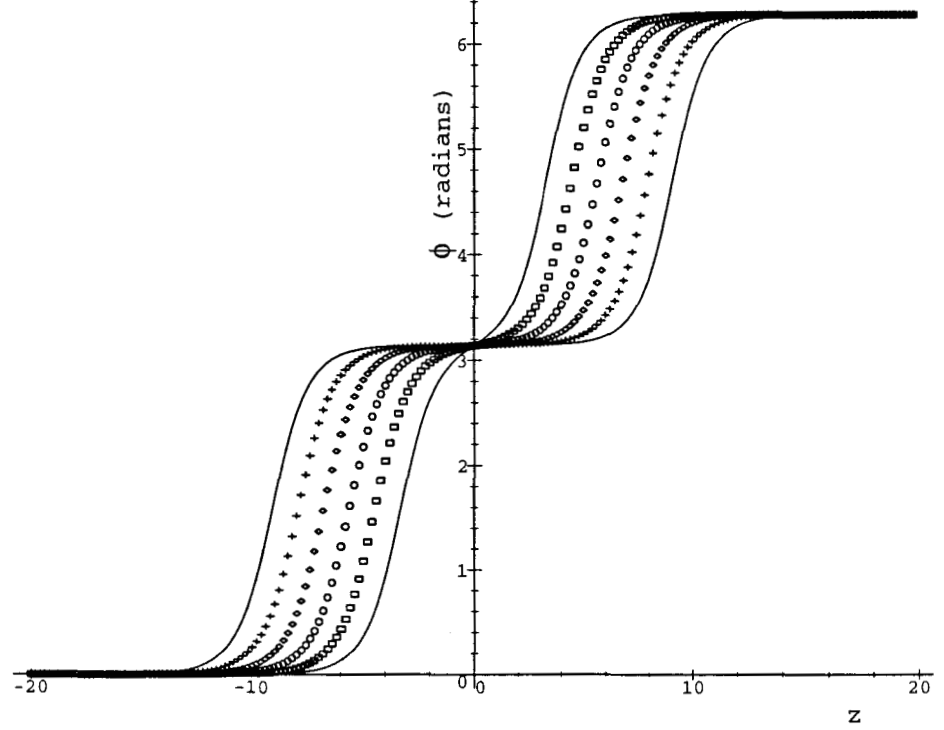


Figure 7.3: The splitting of a 2π kink is shown for $r = 10^2$ (—), $\tau = 10^3$ ($\square$), $\tau = 10^4$ ($\circ$), $\tau = 10^5$ ($\diamond$), $\tau = 10^6$ (+), $\tau = 10^7$ (—). At $\tau = 0$, it is a step function of height 2π .

moving apart. The time has been chosen such that any two adjacent kinks have the same separation. The "snapshots" of the kinks have been taken at $\tau = 10^{i+1}$ for $i = 1, \dots, 6$. It appears from the calculation that the separation between the two π kinks varies with time as:

$$d_{i+1} - d_i \propto \ln \left(\frac{\tau_{i+1}}{\tau_i} \right) \quad (7.5)$$

where d_i corresponds to the spatial separation between the centres of the two π kinks at time, τ_i . Due to this logarithmic dependence, the 2π kink appears to be

slowly separating into two individual π kinks.

7.4 Velocity selection

It has been already pointed out in the first chapter that a π kink which has a topological winding of 180° moves in a static field applied at an angle to the smectic layers. It is easy to see that this motion is due to the potential difference between the uniform base states since they are oriented at different angles with respect to the field. One of the states will be stable and the other unstable or metastable. Schiller, Pelzl, and Demus [17] obtained an analytical solution which describes a uniformly moving π kink. In this case depicted in Figure 7.4, $\gamma_0 = \omega_0 = 0$, and equation (7.4) permits a uniformly moving π kink, described by

$$\phi = 2 \tan^{-1} \exp[(\eta - \tau \cos \theta \sin \alpha) \sin \theta \cos \alpha] \quad (7.6)$$

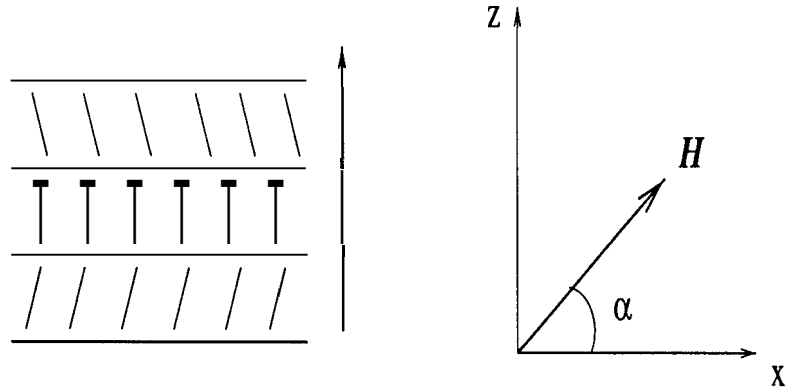


Figure 7.4: Schematic representation of a uniformly moving π kink when the field is applied at an angle, α , with respect to the layers.

The dimensionless velocity, v , of the propagating kink is

$$v = \cos \theta \sin \alpha \quad (7.7)$$

We now consider the time evolution of a π twist front or a step. It was recently shown by Sarloos, Hecke, and Holyst [16] that there can exist two different regimes, which are referred to as linear and nonlinear marginal stability regimes. In the linear marginal-stability regime, the front speed can be calculated explicitly from perturbation analysis at the leading edge of the front. The deviations from the unstable state are small, and one can obtain the dispersion equation from the linearized equations. The velocity, v_l , obtained from such a linear marginal-stability analysis is given by

$$v_l = 2 \sqrt{\sin \theta \cos \theta \sin \alpha \cos \alpha - \sin^2 \theta \cos^2 \alpha} \quad (7.8)$$

In the nonlinear marginal stability regime the front speed depends on the fully nonlinear behaviour of the equation. Here, it has been shown [16] that equation (7.7) corresponds to the nonlinear marginal-stability regime. Equations (7.7, 7.8) are valid for $0 \leq \alpha \leq \frac{\pi}{2}$. Similar solutions exist for other values of α . These authors conjecture that there exists multiple stable travelling-wave solutions satisfying the dynamical equation but the system chooses that particular solution which has the fastest spatial decay at the leading edge of the front. This leads to a crossover in velocity selection from the velocity in the nonlinear marginal stability regime to the velocity obtained from the linear marginal stability analysis, viz. from v to v_l .

This crossover phenomenon is a topic of general interest [4, 6-8, 10, 16]. Under a sudden destabilization, systems in a wide class of problems respond by forming a front which propagates from a stable to an unstable or a metastable state. And the problems of interest are whether the front evolves in time into a kink of a constant, unique speed, and if so, how the system selects this speed from many possible speeds.

The interest has been in equations of the type:

$$\frac{\partial \beta}{\partial t} = F_1(\beta_{zz}, \dots) + F_2(\beta) \quad (7.9)$$

where F_1 is a linear function of the spatial derivatives of β , and F_2 is a nonlinear function of β . We shall not discuss the various conjectures put forward to explain the phenomenon of velocity-selection. In Figure 7.5, we show the dependence of the

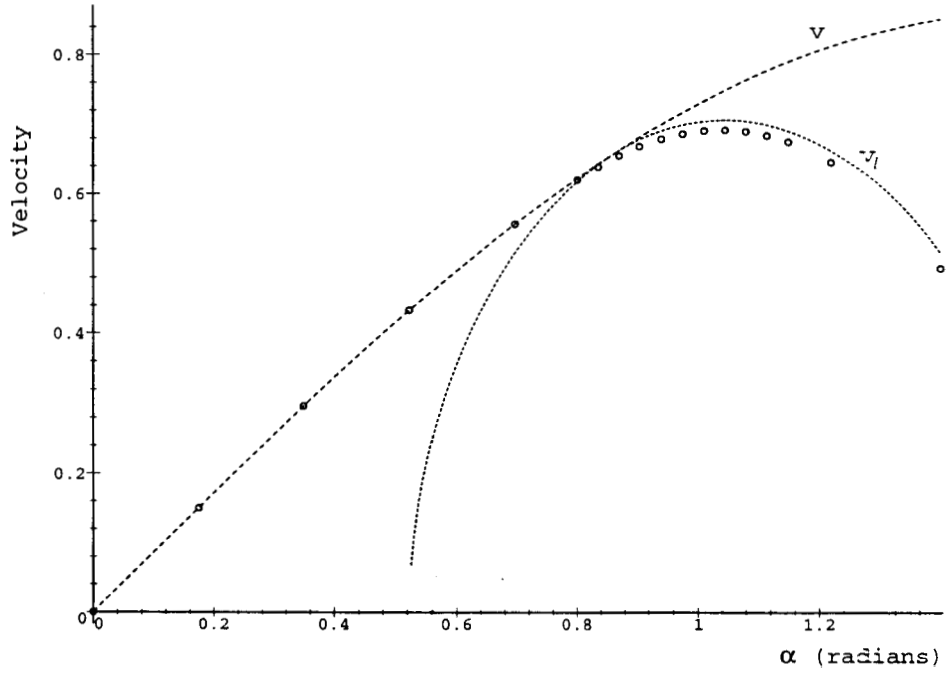


Figure 7.5: Velocity of a π kink as a function of the tilt, α , of the field. Long dashed curve corresponds to the velocity, v , given by equation (7.7) in the nonlinear marginal stability regime and the short dashed curve corresponds to the velocity, v_l , given by equation (7.8) in the linear marginal stability regime. The circles are obtained from the numerical solution of equation (7.4). Here, $\theta = \frac{\pi}{6}$ radians.

velocity of a π kink on a , for $\theta = 30^\circ$ as given by equations (7.7, 7.8). We have also shown in the same figure, our numerical results (indicated by open circles) obtained by studying the evolution of a π front as dictated by the equation of motion. It should be noted that the curves for velocity, v_l , in the linear marginal-stability regime, and the velocity, v , in the nonlinear marginal-stability regime, are mutually tangential to one another at the crossover point, which is obtained from the relation

$$\cot \theta \tan \alpha = 2 \quad (7.10)$$

It should be noted that such a crossover in velocity-selection has been predicted only in cases where an analytical solution could be obtained.

7.4.1 Tilted non-rotating oscillating field

It is important to know as to what would happen when the field also has an oscillating component. We have studied the effects of a tilted non-rotating but oscillating field on a 2π kink. The kink is stationary, as is to be expected from pure symmetry considerations since the base states are at the same potential energy and the potential energy is symmetric about the centre of the kink. The structure changes periodically with time. The variations are symmetric about the centre. The state $\phi = \pi$ is stationary and the average velocity of any other state is also zero.

In this context, we must recall the work of Stewart, Carlsson. and Leslie [18, 19]. They predicted that in an oscillating tilted electric field, under certain assumptions, Melnikov analysis leads to chaotic instabilities in π kinks in the c vector field. In this case of $\omega_0 = 0$ and $\gamma_0 \neq 0$. These authors concluded that for a weak slowly oscillating electric field at a small tilt, chaos occurs for certain ranges of the parameters. Their analysis assumes the existence of a uniform moving frame in which the dynamical equations reduce to a system of ordinary differential equations.

We have reanalyzed this problem. The range of different parameters are :

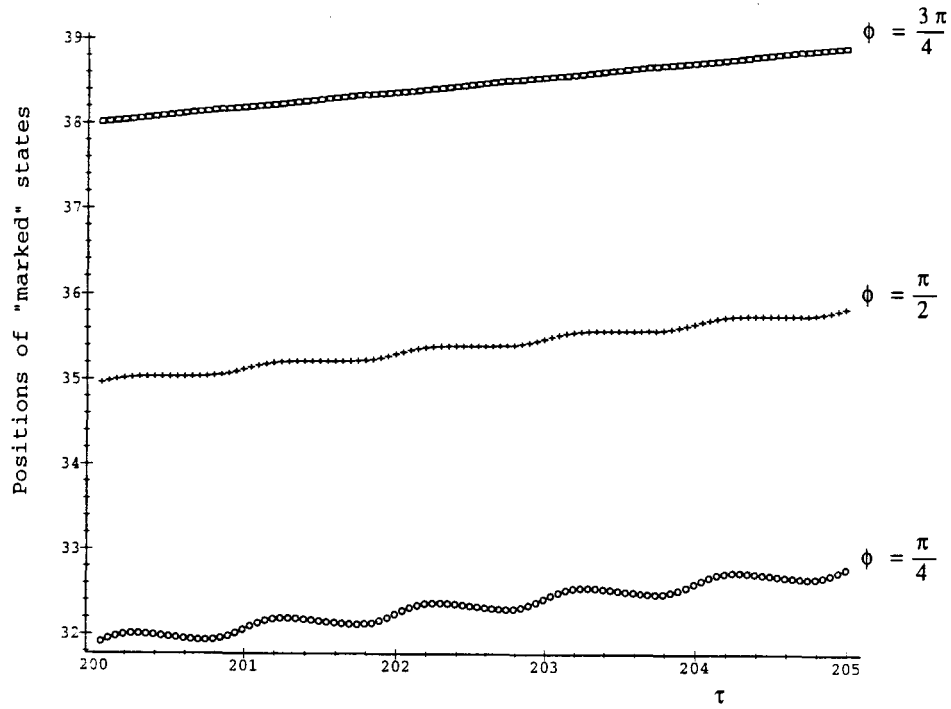


Figure 7.6: Time evolution of the "marked" states, $\phi = \frac{\pi}{4}$ ($\circ$), $\frac{\pi}{2}$ ($+$) and, $\frac{3\pi}{4}$ ($\square$), of a π kink in a tilted oscillating field [Figure 7.41. Here, $a = b = 0.5$, $\alpha = 20^\circ$, $\gamma_0 = 2\pi$, $\theta = \frac{\pi}{6}$ radians.

$\frac{2\pi}{2048} \leq \gamma_0 \leq 2\pi$, $0^\circ \leq \alpha \leq 85^\circ$, $10'' \leq \theta \leq 80^\circ$, $0 \leq a \leq 1$, $0 \leq b \leq 1$. We have also considered the domain of parameters, viz. $a = 1, b = 0.1, \theta = 30^\circ, \alpha = 5''$ and $\gamma_0 = \frac{\pi}{150}, \frac{\pi}{125}, \frac{\pi}{100}$ in which these authors predict chaos. The numerical analysis of this problem leads to an entirely different result. To describe the long-time or asymptotic features in the dynamics of a π kink, we follow the time evolution of three "marked" states, of the $\mathbf{c}$ vector, in the π kink. They are at the orientations $\phi = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ with respect to the field. The distance between $\phi = \frac{\pi}{4}$ and $\phi = \frac{3\pi}{4}$ is a measure of the width of the kink. The results are shown in Figure 7.6 where we give the time evolution of these three "marked" states.

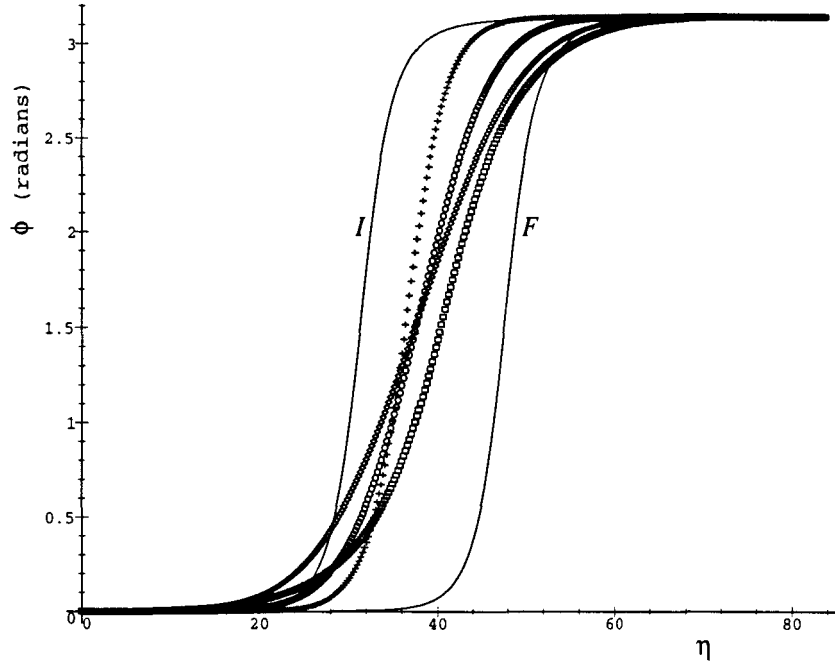


Figure 7.7: “Snapshots” of a π kink at different times $\tau = 200$ (I), 220 (+), 240 (o), 260 (o), 280 ($\square$), 300 (F). Here, $a = b = 0.5$, $\alpha = 20^\circ$, $\gamma_0 = \frac{2}{3}$, $\theta = \frac{\pi}{6}$ radians.

It is found that the instantaneous velocity at any point of the kink and the kink width are periodic in time in all the cases we have considered. The motion is asymmetric in space since the three "marked" states behave differently. The frequency of the kink's structural pulsations and that of the velocity modulation of the "marked" states are equal to the frequency of the oscillating field when $a \neq 0$, and twice of it when $a = 0$. Figure 7.7 shows six "snapshots" of the motion of the entire kink. It depicts all the above mentioned features. Obviously, there is no chaos in the dynamics of this structure. Its dynamics is periodic.

In this case, it is also important to know whether there is a crossover in

velocity selection due to the system choosing from families of solutions. There is no simple way of getting an analytical solution to this problem and this is probably why it has not been addressed to so far by others. Further, methods employed in linear marginal stability analysis do not work in this case since perturbations at the leading edge of the front have a time-dependent frequency and spatial decay. As a first attempt in understanding this problem, we work out the dependence of a time-averaged kink velocity with the tilt angle, α , of the field.

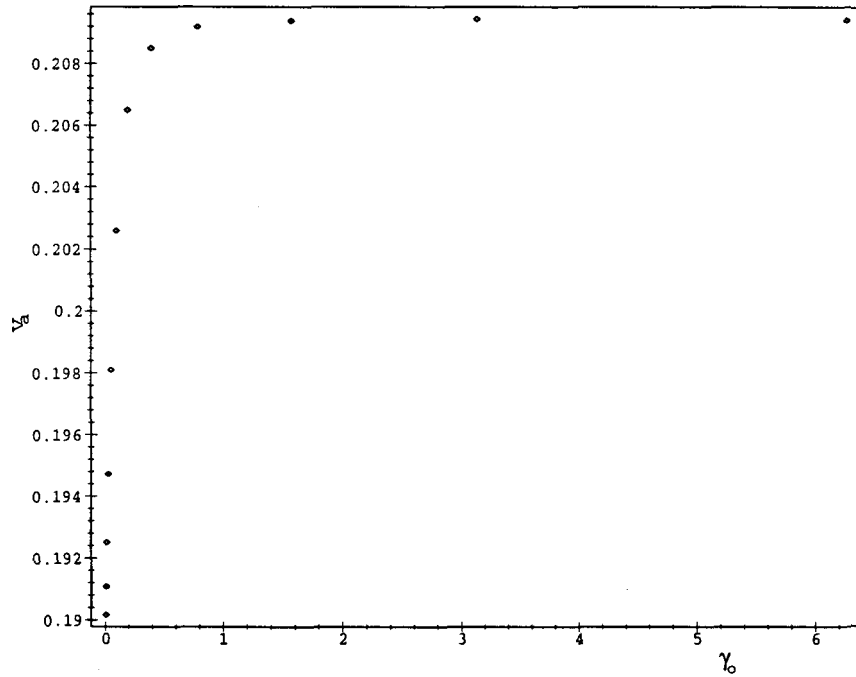


Figure 7.8: Average velocity of a π kink as a function of frequency, γ_0 of oscillating field [Figure 7.4]. Here, $\alpha = 0, b = 1, \alpha = 20^\circ, \theta = \frac{\pi}{6}$ radians.

In view of the periodicity in the dynamics of a π kink, an average velocity, v_a , can

be defined for every state, ϕ_i at the site η_i , on the kink:

$$v_a = \frac{\eta_i(\tau + T) - \eta_i(\tau)}{T} \quad (7.11)$$

Since the kink pulsates with the frequency, $\gamma_0 = \frac{2\pi}{T}$, v_a is unique for the entire kink. In Figure 7.8, we have shown the dependence of v_a on γ_0 at fixed values of a, b, θ and α . A question of relevance at this point is whether v_a approaches the "DC limit" as one goes to the limit $\gamma_0 \rightarrow 0$. At $\gamma_0 = 0$, the time-averaged velocity is certainly an ill-defined quantity and hence, such a limit is not meaningful. This is analogous to the behaviour of a charged particle in an oscillating field where the average velocity is zero for any finite frequency. Such an average does not go to the limit of non-zero velocity when the frequency goes to zero. The breakdown of the defined average velocity can be ascribed to the fact that the "DC limit" is a bifurcation point separating periodic behaviour from non-periodic behaviour. It is found that as γ_0 increases, v_a quickly approaches a fixed asymptotic value, v_r , which we call the *root mean square* or *rms velocity*. It can be seen that v_a is nearly v_r at $\gamma_0 \sim 2\pi$. In all the cases considered by us, we have found that this velocity is given by either of the two following equations:

$$v'_r = (a^2 + \frac{b^2}{2})^{\frac{1}{2}} \cos \theta \sin \alpha \quad (7.12)$$

$$v''_r = 2 (a^2 + \frac{b^2}{2})^{\frac{1}{2}} \sqrt{\sin \theta \cos \theta \sin \alpha \cos \alpha - \sin^2 \theta \cos^2 \alpha} \quad (7.13)$$

These are akin to equations (7.7, 7.8). In Figure 7.9, v_a versus α is shown for $\gamma_0 = 2\pi$ for fixed values of a, b and θ .

It can be seen that there is a close agreement between the numerically calculated points (shown in boxes) and the analytical curves. This can be taken as an

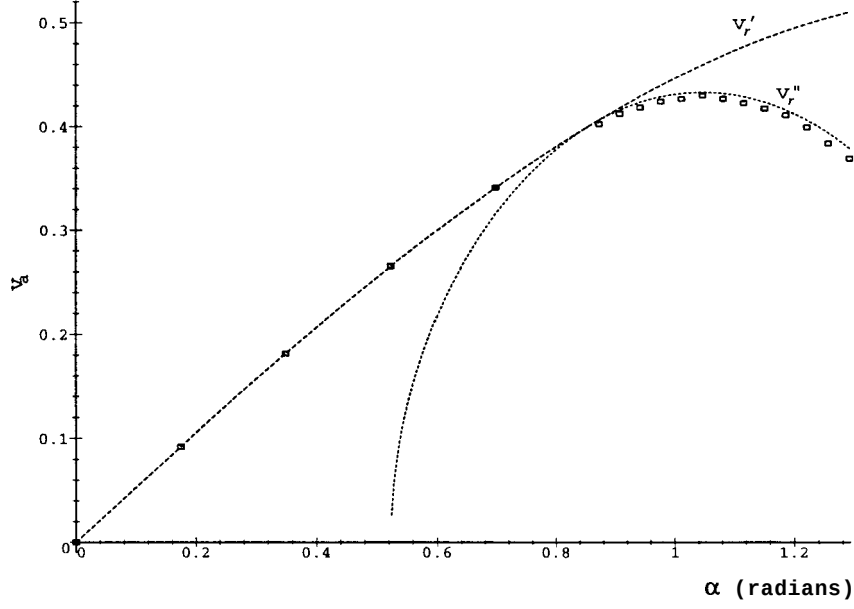


Figure 7.9: Average velocity of a π kink as a function of the tilt, α , in an oscillating field [Figure 7.41. Long dashed curve corresponds to v'_r , and the short dashed curve corresponds to v''_r . The boxes are from the numerical solution of equation (7.4). The numerical results agree with v'_r below the threshold tilt angle given by equation (7.12) and above that, it agrees with v''_r given by equation (7.13). Here, $a = b = 0.5, \gamma_0 = 2\pi, \theta = \frac{\pi}{6}$ radians.

indication that even in the case of oscillating fields, there is a crossover in velocity selection. In other words, the structure and velocity of a π kink are not described by a single function of the tilt angle. In this case, too, the system seems to be choosing a unique solution from a family of solutions. As in the analysis of Sarloos, Hecke, and Holyst [16], here also, the numerically computed velocities are slightly below the ones predicted by equation (7.13) in the linear marginal stability regime and this is possibly due to slow convergence to the asymptotic value in this regime. For other values of γ_0 also, v_a has a similar dependence on α , but we could not fit an

analytical function.

Our analysis raises an important question about the velocity selection in systems more general than that described by equation (7.9). The results presented here indicate that this phenomenon might occur in a broader class of physical systems obeying

$$\frac{\partial \beta}{\partial t} = G_1(\beta_{zz}, \dots) + G_2(t, \beta) \quad (7.14)$$

where G_1 is a linear function of the spatial derivatives of β , and G_2 is a nonlinear function of β and **time** also. To our knowledge, this is the first indication of a system choosing between families of fronts which are not uniformly travelling, that is, $\beta(z, t) = U(z - ct)$ but varying periodically in time.³

7.4.2 Non-oscillating field rotating in a plane normal to the layers

Another interesting case of velocity selection occurs in the dynamics of π kinks when a non-oscillating field is rotating in a plane normal to the layers.

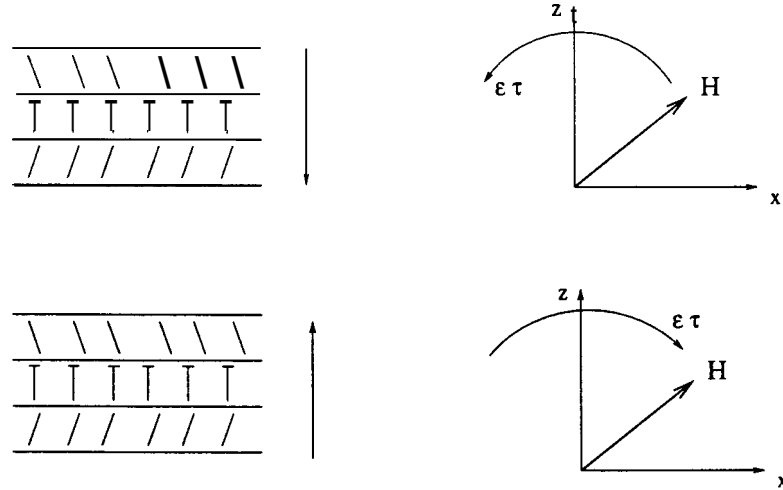


Figure 7.10: A constant field uniformly rotating in a plane xz , normal to the layers. In the case of a field rotating in the anticlockwise direction, the state $\phi = \pi$ invades into the state $\phi = 0$ whereas, in the case of a field rotating in the clockwise direction, the state $\phi = 0$ invades into the state $\phi = \pi$.

³The recent work of Armero *et al.* [21] where they consider the effects of external noise in a one-dimensional model of front propagation is an example of a system described by (7.14).

In a non-oscillating field rotating uniformly in a plane normal to the smectic layers, the potential energy varies periodically in time. Strangely, in this case, a kink instead of oscillating back and forth develops a drift velocity over and above an oscillatory motion and this drift is dependent on the sense of rotation.

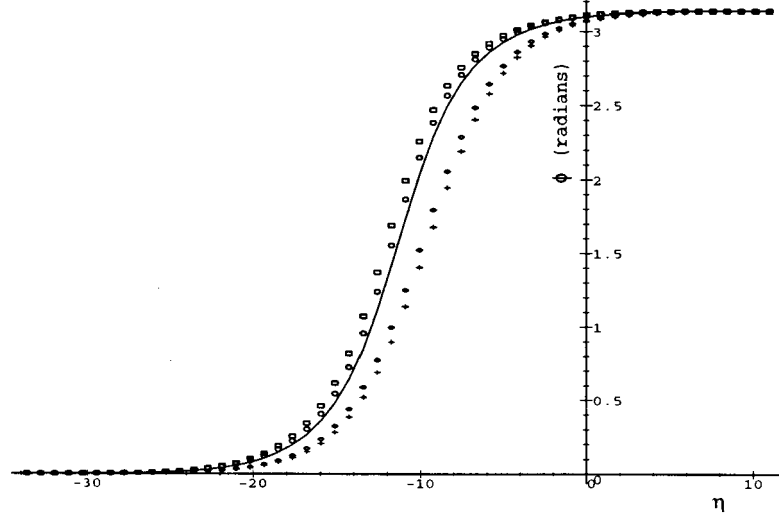


Figure 7.11: "Snapshots" of a π kink in a field uniformly rotating in the anticlockwise direction. [Figure 7.101. The snapshots are at the times $\tau = 300$ (-), 305 (+), 310 (o), 315 (o), 320 ($\square$). The π kink fluctuates periodically with a frequency of $\frac{2\pi}{10}$. The kink drifts such that the state $\phi = \pi$ expands at the expense of the state $\phi = 0$. For clockwise rotation, the drift is such that the state $\phi = 0$ expands at the expense of the state $\phi = \pi$. Here, $a = 1, b = 0, a = \epsilon\tau, e = \frac{2\pi}{20}, \theta = \frac{\pi}{6}$ radians.

Here [see Figure 7.10], we have $b = 0, \omega_0 = 0$, and $a = \epsilon\tau$ where ϵ is the rotational frequency of the field. It is found that the structure and velocity of a π kink oscillates with a frequency equal to 2ϵ . But the surprising result is that there is an average drift velocity for the kink. In Figure 7.11, snapshots of the motion of a π kink are shown.

Since equation (7.4) is invariant under the transformation. $\phi \rightarrow (\pi - \phi), \epsilon \rightarrow -\epsilon$, the drift would be in the opposite direction if the direction of rotation of the field is reversed. It is found that for an anticlockwise rotation, *i.e.* $\epsilon > 0$, the kink propagates such that the state $\phi = \pi$ expands at the expense of the state $\phi = 0$. For a clockwise rotation, *i.e.* $\epsilon < 0$, it is the state $\phi = 0$ that expands at the expense of the state $\phi = \pi$. Also, it is found that the drift velocity of the kink tends to zero as ϵ increases.

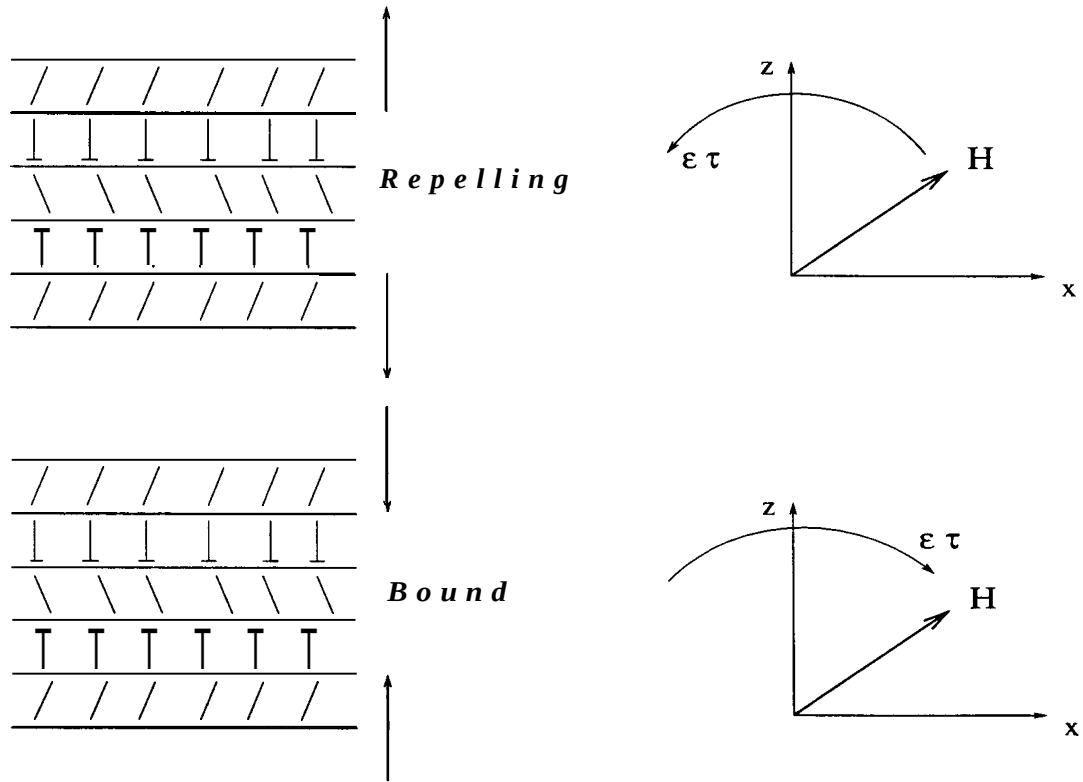


Figure 7.12: A constant field uniformly rotating in a plane xz , normal to the layers. In the case of a field rotating in the anticlockwise direction, the 2π kink splits into two π kinks. And, in the case of a field rotating in the clockwise direction, the 2π kink stays bound.

Since at any given α , a 2π kink is a permitted static solution when the field is non-rotating and non-oscillating, the stability of a 2π kink in the same geometry [see Figure 7.12] has been considered. A 2π kink always splits when $\phi = \pi$ becomes a

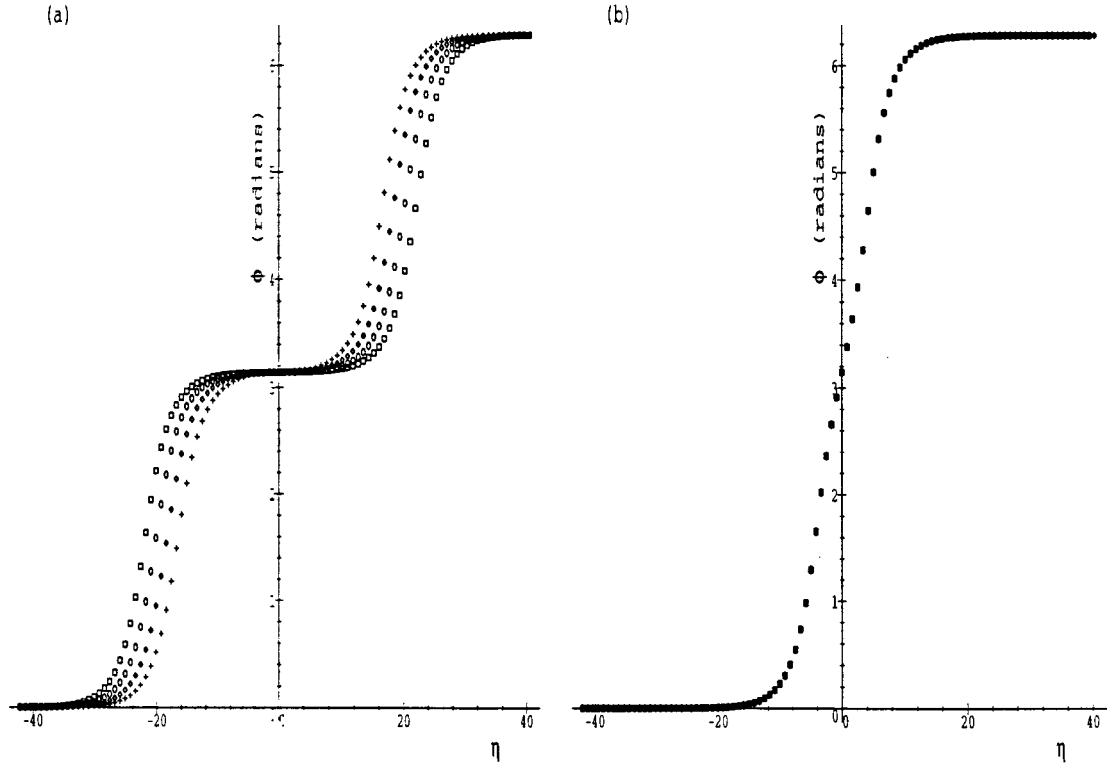


Figure 7.13: “Snapshots” of the motion of a 2π kink in a rotating field [Figure 7.12], at $\tau = 300$ (+), 350 (o), 400 (◊) and 450 (◻). (a) Anticlockwise rotation with $a = \frac{2\pi}{20}$ leads to a splitting of a 2π kink. (b) Clockwise rotation with $a = -\frac{2\pi}{20}$ leads to a bound 2π kink state. Here, $a = 1$, $b = 0$, $\alpha = \epsilon\tau$, $\theta = \frac{\pi}{6}$ radians.

metastable state. In view of what has been said in the previous paragraph, for clockwise rotation, the individual pulsating π kinks drift towards one another. Hence, we end up with a bound oscillating 2π kink state. On the other hand, for anticlockwise rotation, the individual pulsating π kinks drift apart leading to a complete splitting of the 2π kink state. These structural changes are depicted in Figures 7.13(a) and 7.13(b).

7.5 Kinks in a field rotating about the layer normal

As stated earlier, in an external field, a kink starts moving when the core of the kink that connects the uniform states becomes asymmetric. The uniform state will rotate in a rotating field. But, as described earlier it can be in the synchronous regime where it co-rotates with the field or in the asynchronous regime where it experiences phase-slippage. Most of the theoretical studies to date on the motion of kinks are in the synchronous state, where a stable π or 2π kink moves with a uniform velocity [14, 15, 22]. As the asynchronous state is approached, this uniform velocity gradually increases and at the frequency of rotation at which asynchronous motion of the uniform state sets in, the plot of velocity versus rotational frequency tends to a finite velocity with infinite slope. Not much attention has been paid to the dynamics of such kinks in the asynchronous regime. And, there has not been any work regarding the behaviour of kinks in an oscillating rotating field. In chapter five, we considered the dynamical behaviour of the uniform state in this geometry. The results obtained there helps us considerably in predicting the essential dynamical features of kinks in this situation.

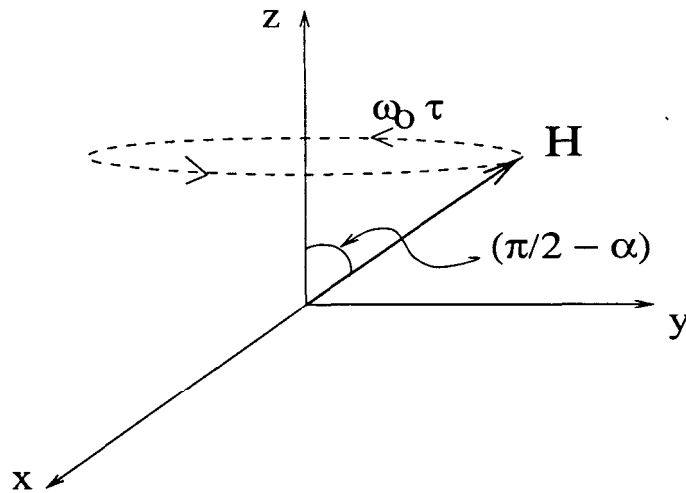


Figure 7.14: A field tilted at a constant angle α with respect to the layers (in the xy -plane) and rotating uniformly about the layer normal (z -axis).

7.5.1 Dynamics of kinks in a non-oscillating field rotating about the layer normal

The geometry under discussion is shown in Figure 7.14. In this case, the following results are found.

Synchronous regime: In the synchronous regime, a uniformly moving kink exists. Biittiker and Landauer [22] have shown that the kink velocity monotonically increases as the rotational frequency, ω_0 , tends to the critical value, ω_c , at which asynchronous motion sets in.

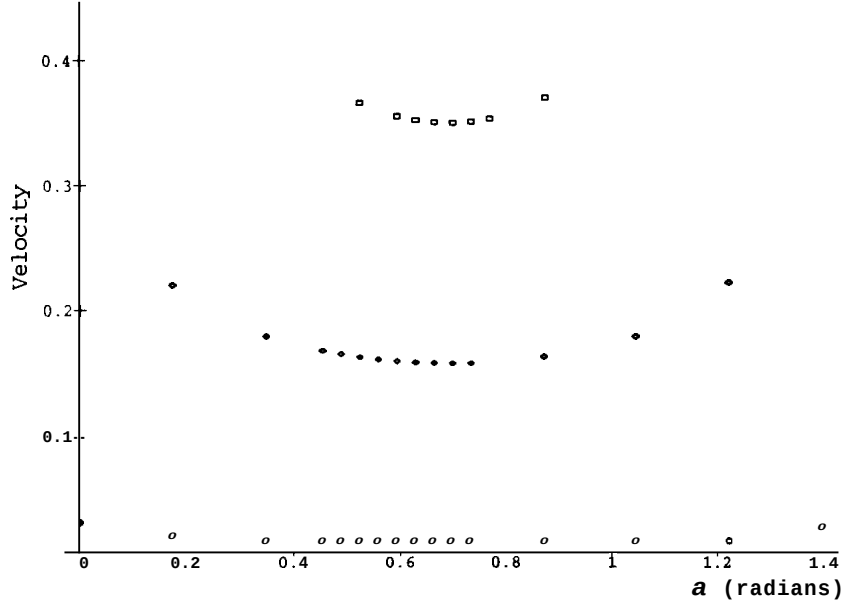


Figure 7.15: Velocity of a 2π kink as a function of the tilt, a , in geometry of Figure 7.14. Here, $a = 1, b = 0$ and, $\omega_0 = 0.01$ ($\circ$), 0.1 ($\diamond$) and 0.2 ($\square$).

In Figure 7.15, variation of the kink velocity with a is shown for a given θ and for three different values of ω_0 . The kink velocity goes through a minimum as a is increased. The data has been given only upto the onset of asynchronous regime.

It is known that the direction of kink propagation gets reversed when the sense of rotation of the external field is reversed.

Asynchronous regime: In chapter five, the dynamical behaviour of the uniform state in the asynchronous regime was described. It was shown that there is a timescale over which the angle between $\mathbf{H}$ and the $\mathbf{c}$ vector is nearly a constant and then the $\mathbf{c}$ vector experiences a sudden slip over a short interval of time. This is the "stick-slip" process. Now, we come to a 2π kink in this regime. The "stick-slip" process makes the structure of the kink time-dependent.

The kink propagation velocity is sensitively dependent on the angle between the base state $\mathbf{c}$ vector and $\mathbf{H}$, the field. Hence, we can expect a nearly uniformly moving kink during the time in which the angle made by the uniform state with the field is hardly perturbed. It is not difficult to see that in the time interval in which the uniform state experiences the "slip" by 2π , the structural features of the kink like its width and velocity changes rapidly. Thus, the structure of the 2π kink would have an oscillatory motion superimposed on a translatory motion. These features have been depicted in Figure 7.16 which shows the time evolution of the three "marked" positions, at $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ with respect to the base state of the kink. Within the range of our numerical investigation, we can conclude, qualitatively, that a 2π kink in the asynchronous regime has the following features :

(i) On long timescales, the centre of the kink as represented by $u_m = \pi$ state where $u_m = u(\eta) - u(-\infty)$ oscillates back and forth.

(ii) Even on a timescale of $\tau \simeq 5000$, *i.e.* the asymptotic or long-time limit, the kink does not seem to have a uniform velocity or width. Due to the effects of a natural finite relaxation time of the director and because of the rapid variation due to the "stick-slip", the width and velocity of the kink are aperiodic in time.

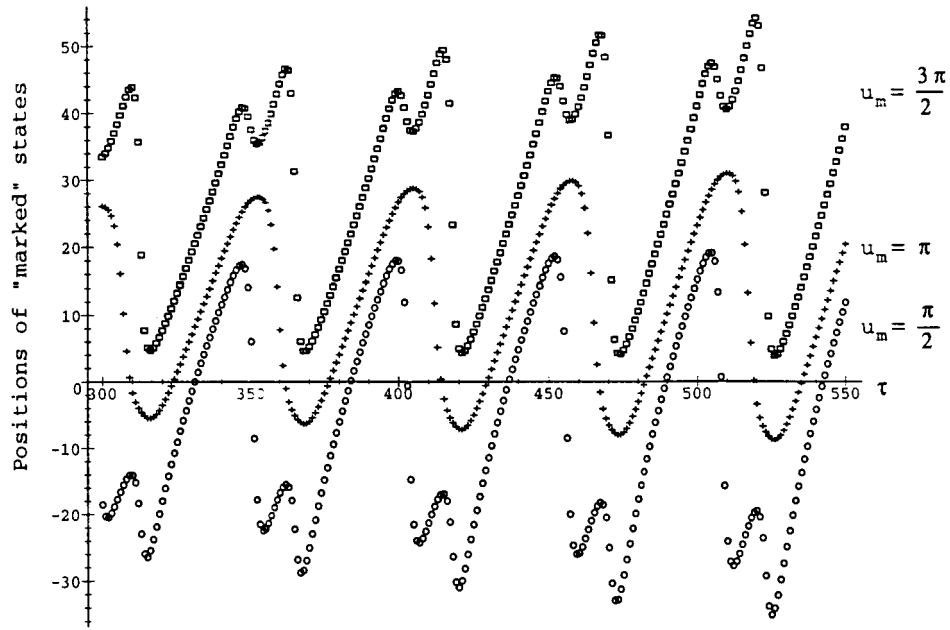


Figure 7.16: A 2π kink in a field rotating about the layer normal and in the asynchronous regime. Time evolution of the "marked" states, with the $\mathbf{c}$ vector orientation relative to the base state, $u_m = \frac{\pi}{2}$ ($\circ$), π ($+$), and $\frac{3\pi}{2}$ ($\circ$). In the asynchronous regime, a 2π kink with a unique structure is not permitted. It is seen that the kink has a translational motion superimposed on oscillatory motion. Here, $a = 1, b = 0, \omega_0 = 0.2, \alpha = 10^\circ, \theta = \frac{\pi}{6}$ radians.

7.5.2 Dynamics of kinks in an oscillating field rotating about the layer normal

In chapter five, it was seen that the dynamics of the uniform state, in an oscillating field rotating about the layer normal, is characterized by the pair of numbers (m, n) describing the phase slip frequency, w . In the light of these observations on the uniform state, we naturally expect the structure and dynamics of kinks to be

time-dependent. We have not been able to determine quantitatively the general characteristics of this state. Attention has been paid to the behaviour of a 2π kink in the phase-oscillating regime for which $w = 0$, and the phase-slipping regime for which $w \neq 0$. It shall be shown that the dynamics of a 2π kink sensitively depends upon the parameters of the geometry. And, we shall quantitatively arrive at the direction of kink propagation.

In the phase-oscillating regime, the structure and width of the 2π kink fluctuates with the period, T , of the oscillating field. This can be thought of as a consequence of the fact that in this case the uniform state faithfully follows the field oscillations. It oscillates with a time period, T , about a direction fixed relative to the field.

We now consider the phase-slipping regime. For certain values of w , there again exists a fluctuating kink structure but with a periodicity equal to nT . To visualize the dynamics of a 2π kink, the scheme of following the “marked” states at $u_m = (u(\eta) - u(-\infty)) = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ has been employed. The width of the soliton is taken to be the distance between $u_m = \frac{\pi}{2}$ and $u_m = \frac{3\pi}{2}$ states.

In Figures 7.17 and 7.18, the features of a kink which correspond to values of $(m, n) = (1, 2)$ and $(m, n) = (1, 1)$ are shown respectively. The different periodicities are clearly evident. It may be mentioned in passing that in the numerical technique of the method of lines, the accuracy with which the system of ordinary differential equations are integrated forward in time should be much greater in the case where $w \neq 0$ than the case $w = 0$.

For those values of w at which there does not exist a well-defined periodicity, it is found that the behaviour of the kink does not depend on the fraction $\frac{m}{n}$ but rather on n . For example, we get oscillatory behaviour for $(m, n) = (5, 1)$ but aperiodic behaviour for $(m, n) = (13, 10)$. In the second case, the dynamics is similar to its

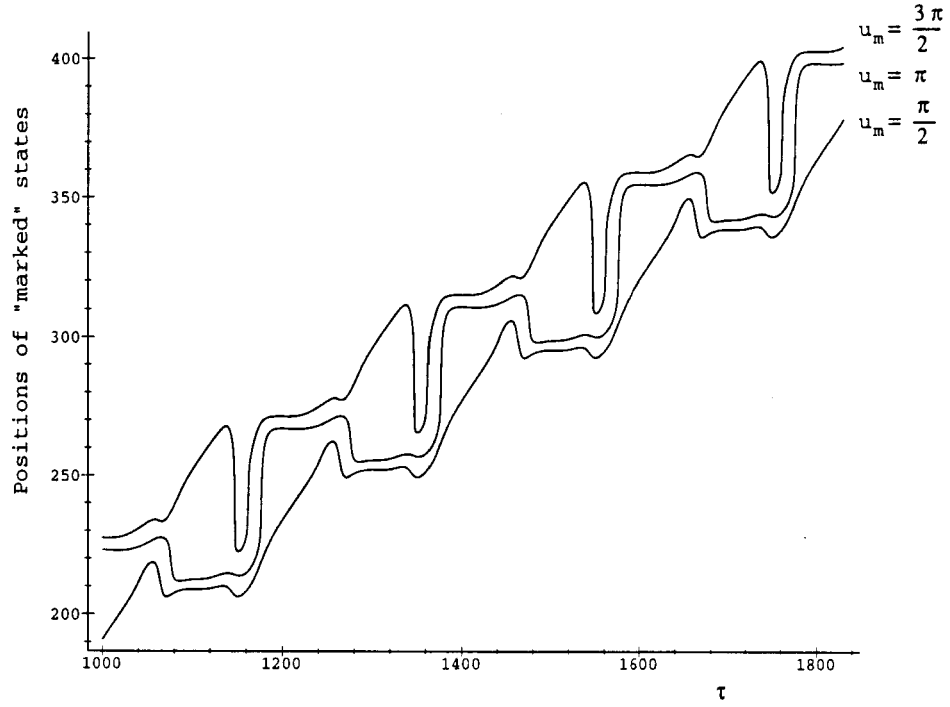


Figure 7.17: A 2π kink in geometry of Figure 7.14. The time evolution of the "marked" states, $u_m = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, in an oscillating field. Here, $a = 1, b = 0.5, \gamma_0 = \frac{2\pi}{100}, \omega_0 = 0.1, \alpha = 10^\circ, (m, n) = (1, 2)$. The state $\phi = 0$ expands at the expense of the state $\phi = \pi$. The time period of the oscillating kink is $\tau = 200$, *i.e.* $2T$.

dynamics in the asynchronous regime. This means that after $\tau = nT$, the structure does not return to its original state. Further, when observed over long timescales, the kink width increases gradually.

The last question which will be addressed to is whether the direction of propagation of a 2π kink can be determined in cases where there exists periodic behaviour. From our calculations, we find that the answer is not as simple as in the synchronous regime in a non-oscillating field. Compare Figures 7.17 and 7.18 for which $\alpha = 10^\circ$

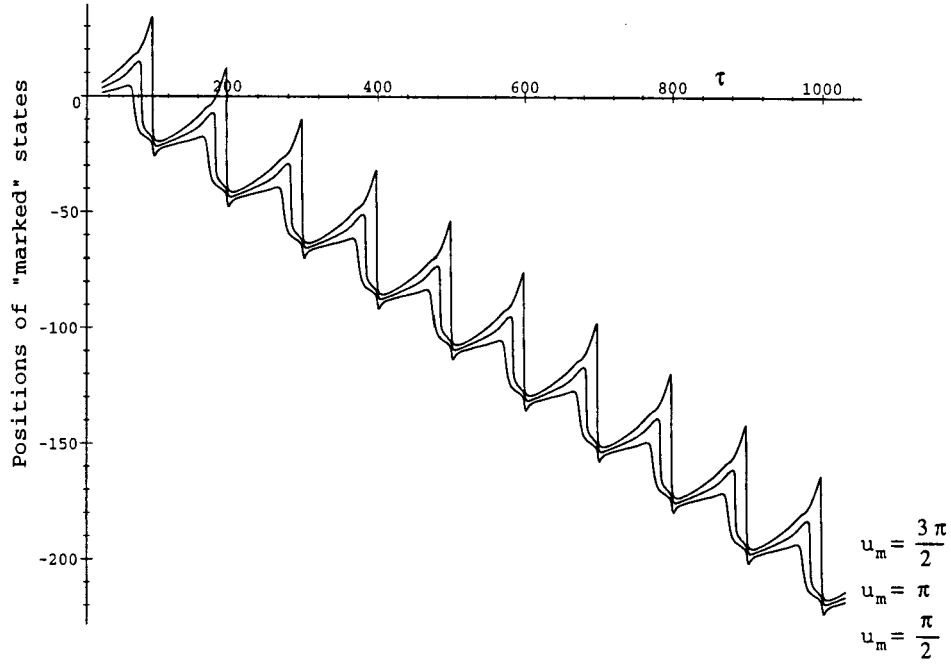


Figure 7.18: A 2π kink in geometry of Figure 7.14. The time evolution of the "marked" states, $u_m = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, in an oscillating field. Here, $a = 1, b = 0.5, \gamma_0 = \frac{2\pi}{100}, \omega_0 = 0.1, a = 20^\circ, (m, n) = (1, 1)$. The state $\phi = \pi$ expands at the expense of the state $\phi = 0$. The time period of the oscillating kink is $\tau = 100$, *i.e.* T.

and $a = 20^\circ$ respectively and for which the only parameter varied is the tilt. We do see that the direction of kink drift in Figure 7.17 is opposite of what is indicated in Figure 7.18. The direction of drift can be determined in a simpler way for these cases where the kink fluctuates periodically.

A kink moves because its internal structure breaks the spatial symmetry of the potential energy. In the frame of reference of the rotating field, the azimuthal angle, u , that the uniform director makes with the field decides the direction of

propagation. When $a = 0$ (or $a \neq 0$), the direction of propagation reverses as u changes by $\frac{\pi}{2}$ (or n). This can be easily understood. The orientation of the base states determine the region within the kink that is at a higher potential energy. Once this is known, we know that the states of lower potential energy would then propagate into the higher potential states. When there is phase-slipping, there will be periods of nearly constant motion and periods of "slip" during which the structure, direction of propagation and velocity changes rapidly. An average azimuthal angle, u_{av} , of the uniform state can be defined by

$$u_{av} = \frac{1}{nT} \int_{\tau_0}^{\tau_0+nT} u(\tau) d\tau \quad (7.15)$$

Here, n still defines the number of cycles of the oscillating field after which the uniform state reaches an equivalent state - as defined by w . Even if there is phase slipping, if the kink fluctuates periodically, it is found that when $a = 0$ (or $a \neq 0$), the direction of kink propagation reverses as u_{av} changes by $\frac{\pi}{2}$ (or n). For the cases shown in Figures 7.17 and 7.18, $\frac{u_{av}}{2\pi} = -0.5135$ and $\frac{u_{av}}{2\pi} = -0.3707$ respectively. Hence, by solving the ordinary differential equation (5.12), u_{av} can be calculated from which the direction of drift can be obtained.

7.5.3 Metastable kinks in a field rotating about the layer normal

In the dynamics of the uniform state, it was shown that for certain angles of tilt a of the field, one of the uniform states becomes metastable. This state drifts towards the stable state as frequency of rotation increases, and at a critical value of the rotational frequency, ω'_c , it collapses to the stable state. This result helps us to understand the dynamics of a kink whose base states are respectively stable and metastable. These structures are simulated by the evolution of an initial front of height n . Referring to Figure 5.7 we can say that these structures can exist with nearly the same topological winding of π up to a rotational frequency, ω'_c . At ω'_c ,

the metastable state suddenly becomes unstable and it can either unwind into a uniform state or wind further through π depending upon the sense of rotation of the field. Interestingly, in an oscillating field, the π kink does not get ironed out to the uniform state even in the phase-slipping regime. But, the non-integral topological winding number becomes time-dependent.

7.5.4 Lattice of kinks

Though in a static field, only a single kink can exist, yet in a rotating field, we can bind together many kinks of the same topological strength and in particular, we can have a lattice of kinks. In an oscillating field, a knowledge of the dynamics of a single kink is sufficient to predict the behaviour of a lattice of kinks. Equivalent states behave identically. In the cases where the single kink is a uniformly travelling front, or its structure pulsates, it is found that the lattice also behaves the same way. In the asymptotic limit, the distance between equivalent states is a constant, while the distance between non-equivalent states vary periodically with time. In the asynchronous regime or in the equivalent phase-slipping regime, the behaviour of the lattice is nothing but that of a single kink with equivalent states behaving identically.

7.6 Remarks

In this chapter, the dynamics of kinks in periodically varying magnetic fields have been investigated.

- In a non-rotating oscillating field tilted with respect to the smectic layer normal, it is found that the structure, the width and the instantaneous velocity of a π kink fluctuates with the period of the oscillating field. Dependence of average velocity of the kink on the tilt angle of the field indicates that there is

velocity-selection, and that the system is possibly choosing between families of solutions.

- When a non-oscillating field is rotated in a plane normal to the smectic layers, a π kink has a non-zero drift velocity which depends upon the sense of rotation. In this geometry, a 2π kink could be in a bound oscillating state or split into oppositely drifting π kinks depending upon the direction of rotation of the field.
- In a non-oscillating field rotating about the layer normal, in the asynchronous regime, the kink fluctuates aperiodically with a slow migration.
- In an oscillating rotating field rotating about the layer normal, kink motion is basically decided by a phase-slip frequency which varies discontinuously with the frequency of rotation of the field. Its dynamics has both periodic and aperiodic features, being very sensitive to the parameters of the geometry.

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