

CHAPTER II

THE SHEAR FLOW OF NEMATICS

Introduction

In this chapter we discuss the effects of magnetic fields and boundary conditions on the plane shear flow of nematics. This was the first flow problem to have been solved analytically (Leslie 1968, 1971). The nematic is assumed to fill the gap between two horizontal, plane parallel plates one of which moves with a constant velocity $\vec{V}$ relative to the other. Leslie developed the general theory for this flow in the presence of a magnetic field applied in the plane of shear and predicted a change in apparent viscosity (η) when either the shear rate or the magnetic field is varied. However for analysing the flow in detail one needs to apply numerical techniques for solving the differential equations. This has been done by Finlayson (1974) who studied the effect of a magnetic field applied perpendicular to the plates with homeotropic wall alignment. He did not however investigate the effect of a field applied along the flow. Moreover neither Leslie nor Finlayson considered other equally important geometries in shear flow. In this chapter studies of four out of the eight

possible geometries are presented. Only those cases where the director orientation is confined to the plane of shear have been considered. Effects of magnetic fields applied along the flow ^{or} perpendicular to the plates have been worked out for symmetric (homeotropic) and asymmetric director alignments at the plates.

Theory

The nematic is assumed to occupy the space between planes $y = 0$ and $y = 2h$, $2h$ being the sample thickness. The plate $y = 2h$ is assumed to move along x axis with a constant velocity V . A constant magnetic field $\vec{H}$ is applied in the xy plane (which is the plane of shear) making an angle ψ with x axis. External body forces are ignored. Steady state solutions for the director and velocity fields are assumed to be

$$\begin{aligned}\vec{n} &= [\cos \theta(y), \sin \theta(y), 0] \\ \vec{v} &= [v(y), 0, 0]\end{aligned}\quad (1)$$

The equations 23 of Chapter I simplify to the following differential equations [first obtained by Leslie (1971)] for θ and v :

$$\begin{aligned}2f(\theta) \frac{d^2\theta}{dy^2} + \frac{df}{d\theta} \left(\frac{d\theta}{dy}\right)^2 + \frac{e(\lambda_1 + \lambda_2 \cos 2\theta)}{g(\theta)} \\ + (\Delta\chi)H^2 \sin(2\psi - 2\theta) = 0\end{aligned}\quad (2)$$

$$\frac{dv}{dy} = \frac{c}{g(\theta)} \quad (3)$$

Hen, $f(\theta) = K_{11}\cos^2 \theta + K_{33}\sin^2 \theta$ and

$$2g(\theta) = 2\mu_1\sin^2 \theta \cos^2 \theta + (\mu_5 - \mu_2)\sin^2 \theta \\ + (\mu_3 + \mu_6)\cos^2 \theta + \mu_4 \quad \text{and}$$

$$t_{xy} = c = \text{a constant.}$$

For a given value c of the applied shear we integrate (2), obtaining θ as a function of y subject to the boundary condition $\theta(0) = \theta^{(1)}$ and $\theta(2h) = \theta^{(2)}$. We then integrate (3) to obtain the velocity V of the upper plate as

$$V = c \int_0^{2h} \frac{dy}{g[\theta(y)]} \quad (4)$$

η which is defined as usual as the ratio of the applied shear stress (c) to the average velocity gradient ($V/2h$) is calculated

$$\eta = \frac{2hc}{V} \quad (5)$$

Results

Computations have been carried out for para-

azoxyanisole (PAA). The elastic and viscosity coefficients have been assumed to be the same as those used by Tseng et al. (see table 1). These have been found to fit the experimental results of Fisher and Fredrickson (1969) very well. The diamagnetic susceptibility anisotropy $\Delta\chi$ has been taken to be 0.136×10^{-6} cgs (Gasparoux and Prost 1971). The differential equations have been solved by the orthogonal collocation method used by Finlayson (1972; Tseng et al. 1972; Villadsen and Stewart 1967) a brief summary of which can be found in the appendix to this Chapter. The sixteen zeroes of the sixteenth order Legendre polynomial P_{16} have been chosen as the collocation points. For cases involving large deformations calculations have been repeated with 24 collocation points (the zeroes of P_{24}) to check the accuracy of results. η from the two calculations agrees to within 0.01%. The technique has also been found to work well upto above 1000 μ sample thickness. But results for a 50 μ gap width have been presented. Double precision arithmetic has been used in all computations.

Effect of Boundary Conditions

Two types of boundary conditions have been considered for θ , which are easily realizable in practice.

Table 1

Viscosity and elastic constants of PAA

μ 's (poise)	K's (dyne)
$\mu_1 = -0.038$	
$\mu_2 = -0.068$	
$\mu_3 = 0.000$	$K_{11} = 4.9 \times 10^{-7}$
$\mu_4 = 0.068$	$K_{33} = 10.5 \times 10^{-7}$
$\mu_5 = 0.048$	
$\mu_6 = -0.020$	
$\lambda_1 = -\lambda_2$	

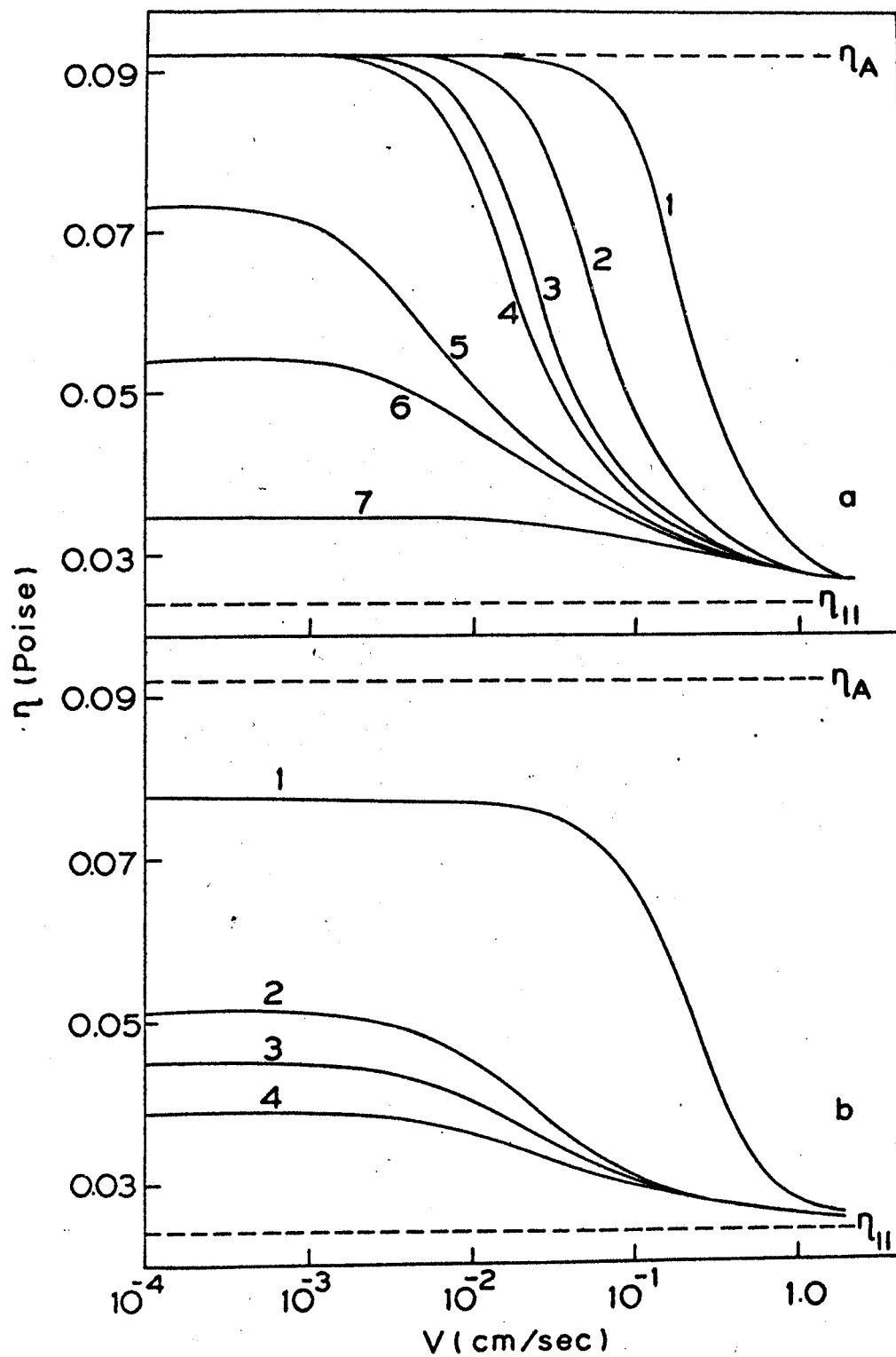
(1) $\underline{\theta^{(1)} = 2 = \theta^{(2)}}$: In this case in the absence of flow the director is homeotropically aligned in the gap ~~plates~~ in the absence of magnetic fields. At low shear rates (or equivalently at low velocities of the upper plate), in the absence of a magnetic field η remains nearly a constant at the Miesowicz value $\eta_A = (\mu_4 + \mu_5 - \mu_2)/2$ ($= 0.092$ poise for PAA). With increasing shear rate η decreases approaching the Miesowicz value , $(\mu_3 + \mu_4 + \mu_6)/2$ ($= 0.024$ poise for PAA) corresponding to the director being oriented along the flow direction over a major section of the flow. The parameters that have been used are such that for PAA $\cos 2\theta_0 = -\lambda_1/\lambda_2$. This means that the director orientation at high shear rates is parallel to the flow. It has to be admitted that there is as yet no conclusive evidence that this is indeed the case for PAA. Figure 1a illustrates the variation of η with the velocity of the upper plate.

(11) $\underline{\theta^{(1)} = 0, \theta^{(2)} = \pi/2}$: The variation of η with the shear rate (figure 1b) in this case is also similar. But at low shear rates the constant value of η is about $0.5 \eta_A$ in the absence of magnetic fields, i.e., it is about half the value for case (1). This is because of the fact that the director field is deformed even

Figure 1

Variation of apparent viscosity with shear rate and magnetic field.

- (a) $\theta_1 = \pi/2 = \theta_2$; Magnetic field normal to the plates $H_y =$ (1) 5000 (2) 2000 (3) 0 gauss; magnetic field along the flow $H_x =$ (4) 1000 (5) 1800 (6) 2000 (7) 3000 gauss;
- (b) $\theta_1 = 0$; $\theta_2 = \pi/2$; $H_y =$ (1) 5000 (2) 1000 (3) 0 gauss, (4) $H_x = 1000$ gauss.



in the absence of flow or external magnetic field due to the asymmetric value of θ at the plates. Thus there is already a finite projection of the director along the flow direction.

Effect of magnetic fields

In general a magnetic field (H_x) which is applied parallel to the flow brings down η . This is because the field, in this case tries to orient the director field along the flow and thus aids the viscous torques in dominating over the elastic torques. Similarly a field (H_y) applied perpendicular to the plates acts against the deforming action of viscous torques with the result that such a field in general tends to enhance the value of q .

(1) $\theta^{(1)} = \pi/2 = \theta^{(2)}$: At low shear rates a field H_y does not; make its presence felt since the director is everywhere aligned normal to the flow. But on increasing the shear rate η decreases but more slowly than in the field free case, larger the magnitude of H_y slower the decrease in η (figure 1a). But at high shear rates η attains the value $\eta_{||}$. If the shear rate is kept constant at a high value (such that $\eta \approx \eta_{||}$) and the magnitude of H_y is increased (figure 2b) η remains nearly constant at about $\eta_{||}$ and then increases till it

approaches the value η_A at very high field strengths.

The effect of a field H_x is more interesting. If at low shear rates the value of H_x is increased η remains constant at η_A (figure 2a) till H_x attains a critical value $H_c \approx \frac{\pi}{2h} \left(\frac{K_{33}}{\Delta\chi} \right)^{\frac{1}{2}}$ (≈ 1750 gauss for PAA) which in the static case corresponds to the Freedericksz threshold below which there is no deformation. For $H_x > H_c$ η decreases fairly rapidly attaining the value $\eta_{||}$ at high values of H_x . At low shear rates the value of η is η_A or less than η_A depending upon whether H_x is less than or greater than H_c . But on increasing the shear rate η decreases (figure 1a) attaining the value $\eta_{||}$ at high shear rates regardless of the magnitude of H_x .

(11) $\theta^{(1)} = 0, \theta^{(2)} = \pi/2$: At low shear rates and with increasing applied field H_y , the value of η increases from $0.5 \eta_A$ and ~~figure 2b~~ tends to the value η_A at high values of H_y . ^{(Figure 2b).} This is due to the director field getting aligned perpendicular to the flow over a larger section of the flow as H_y is increased. At any given value of H_y and at low shear rates, η is almost a constant at a value η'_A (higher than $0.5 \eta_A$) depending upon the magnitude of H_y . However on increasing the shear rate (figure 1b) η decreases and finally attains the value $\eta_{||}$ at high shear rates.

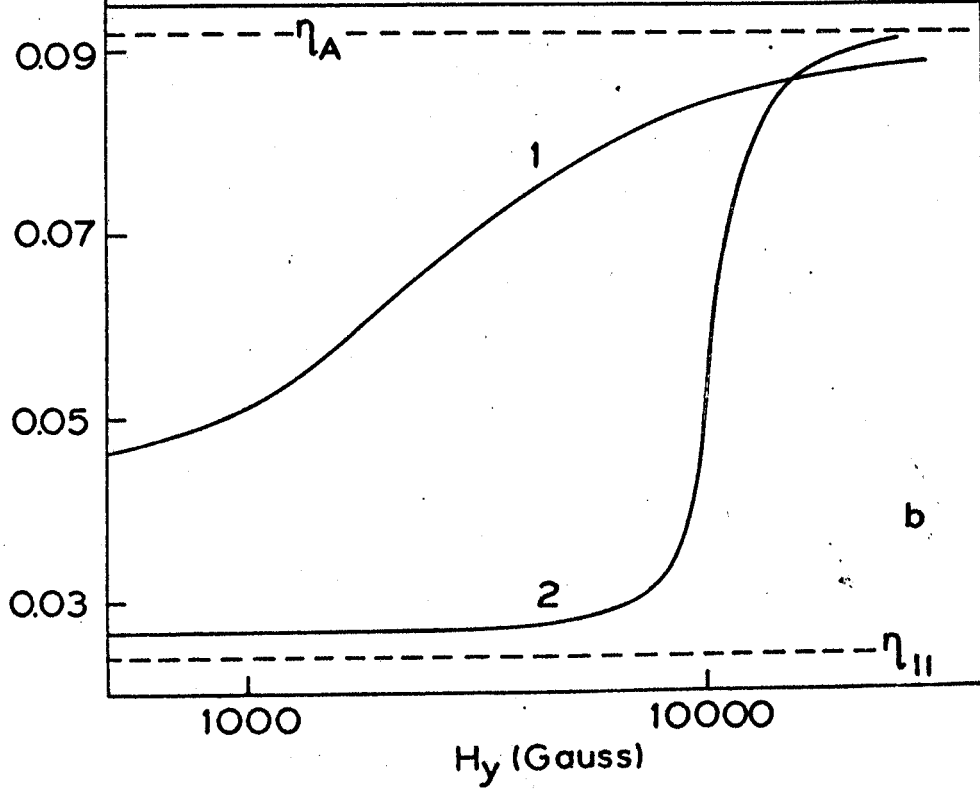
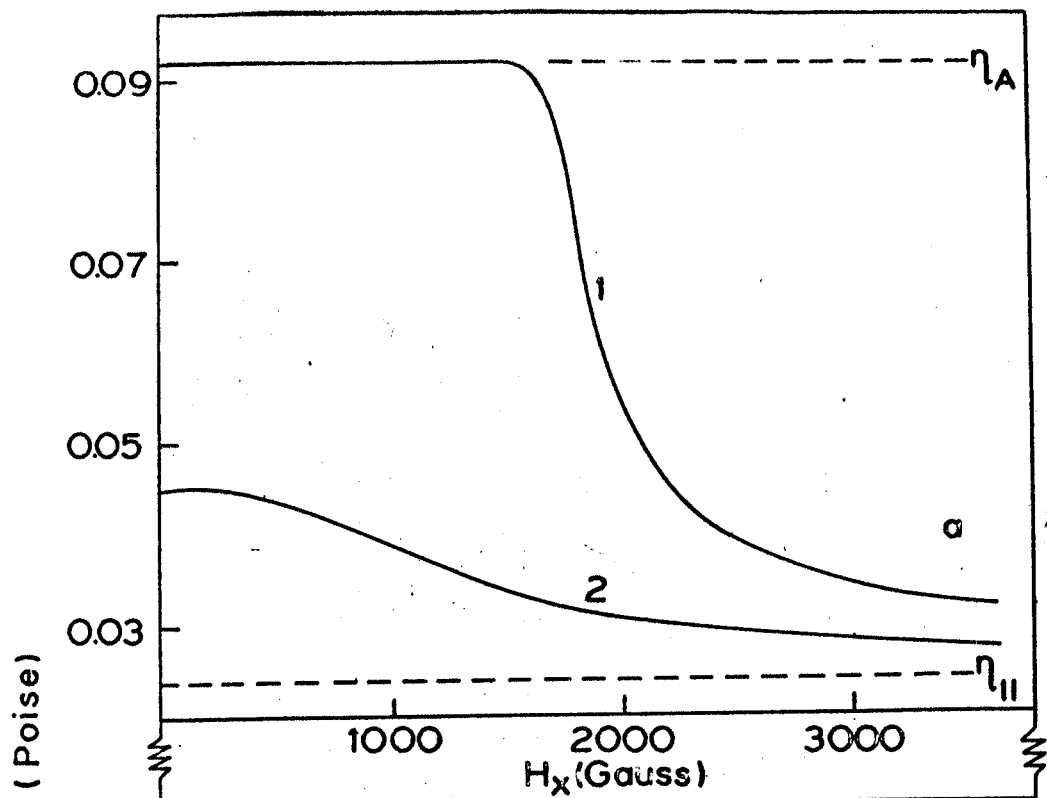
Figure 2

Variation of apparent viscosity with magnetic field.

(a) Field H_x along the flow; $c = 10^{-3}$ dynes cm^{-2}

(1) $\theta_1 = \pi/2 = \theta_2$ (2) $\theta_1 = 0, \theta_2 = \pi/2$.

(b) Field H_y normal to the plates; (1) $\theta_1 = 0, \theta_2 = \pi/2, c = 10^{-3}$ dynes cm^{-2} (2) $\theta_1 = \pi/2 = \theta_2, a = 10.0$ dynes cm^{-2} .



In the presence of a field H_x η continuously decreases with increasing field even at low shear rates, dropping from $0.5 \eta_A$ to $\eta_{||}$ at high values of H_x [figure 2a). The critical behaviour which is exhibited in case (1) cannot be expected here because a deformation in the director field exists even prior to the application of shear or field. At any given value of H_x , η remains constant at low shear rates at a value η_A'' (lower than $0.5 \eta_A$) depending upon the magnitude of H_x . But on increasing the shear rate (figure 1b) η decreases and attains the value $\eta_{||}$ at high shear rates.

Orientation profiles

The director orientation profiles for different shear rates and magnetic fields for both boundary conditions have been given in figures 3 and 4. For $\theta^{(1)} = \pi/2 = \theta^{(2)}$ the θ profile is found to be symmetric between the plates. On increasing either the shear rate or the field H_x the director orientation approaches zero at the centre of the gap but increases to $\pi/2$ near the plates. The diagrams (figure 3) indicate that for very high shear rates or H_x one can expect to find a boundary layer phenomenon for the θ profile. If at a high shear rate a field H_y is applied the director tends

to align normal to the flow in the gap. At a high field (figure 3) the orientation does not vary much from $\pi/2$ away from the plates.

For the case $\theta^{(1)} = 0$, $\theta^{(2)} = \pi/2$ the θ profile is nearly a straight line at low shear rates. θ increases from the zero value at one plate till it attains the value $\pi/2$ at the other. But on increasing the shear rate or H_x , θ remains small over a larger section of the flow (figure 3) increasing to $\pi/2$ in a narrow region near the moving plate. The θ profile becomes concave upward. In this geometry the θ profile is distorted even by a field H_y . Such a field tends to align the director normal to the flow over a greater distance in the gap. The θ profile becomes concave downward. The larger the field H_y , more rapidly does θ increase from zero at the stationary plate to a value close of $\pi/2$ (figure 3). The θ profiles in both the boundary conditions give a clear pictorial explanation for the variations of η with shear rate and magnetic fields.

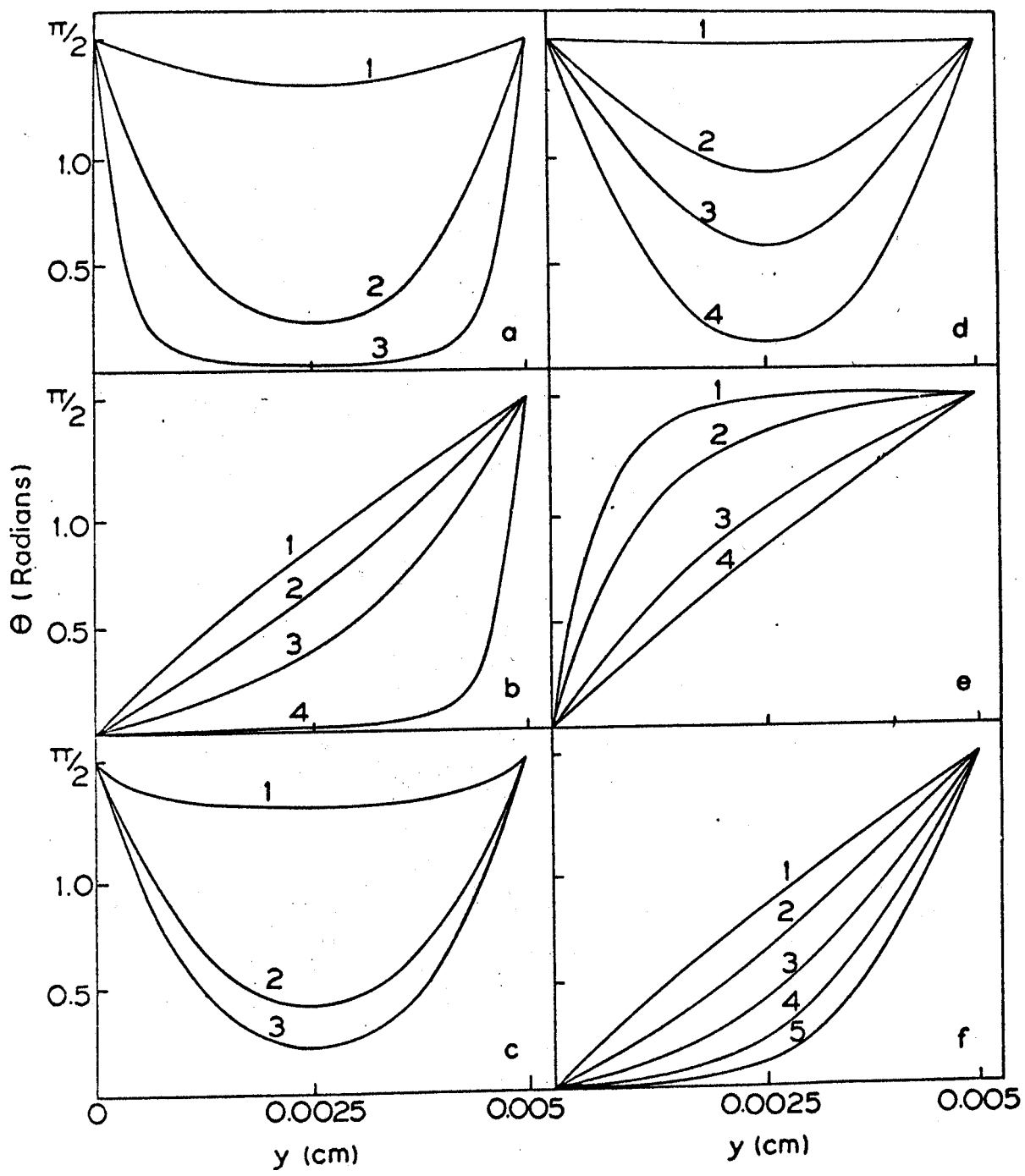
Velocity Profiles

Velocity profiles are presented in figure 4. (The velocity is plotted logarithmically to cover the large range involved. In general the velocity V of the upper plate is greater for greater applied shear and at

Figure 3

Orientation profiles for different shear rates and magnetic fields.

- (a) $H = 0$; $\theta_1 = \pi/2 = \theta_2$; $c =$ (1) 0.1 (2) 1.0 (3) $10.0 \text{ dynes cm}^{-2}$.
- (b) $H = 0$; $\theta_1 = 0$; $\theta_2 = \pi/2$; $c =$ [1] 10^{-3} (2) 0.1 (3) $10^{\frac{1}{2}}$ (4) $10.0 \text{ dynes cm}^{-2}$.
- (c) $c = 10.0 \text{ dynes cm}^{-2}$; $\theta_1 = \pi/2 = \theta_2$;
 $H_y =$ (1) 5000 (2) 2000 (3) 0 gauss.
- (d) $c = 10^{-3} \text{ dynes cm}^{-2}$; $\theta_1 = \pi/2 = \theta_2$;
 $H_x =$ (1) 0 (2) 1800 (3) 2000 (4) 3000 gauss.
- (e) $c = 10^{-3} \text{ dynes cm}^{-2}$; $\theta_1 = 0$; $\theta_2 = \pi/2$;
 $=$ (1) 5000 (2) 25000 (3) 1000 (4) 0 gauss.
- (f) $c = 10^{-3} \text{ dynes cm}^{-2}$; $\theta_1 = 0$; $\theta_2 = \pi/2$;
 $H_x =$ (1) 0 (2) 1000 (3) 1500 (4) 2000 (5) 2500 gauss.

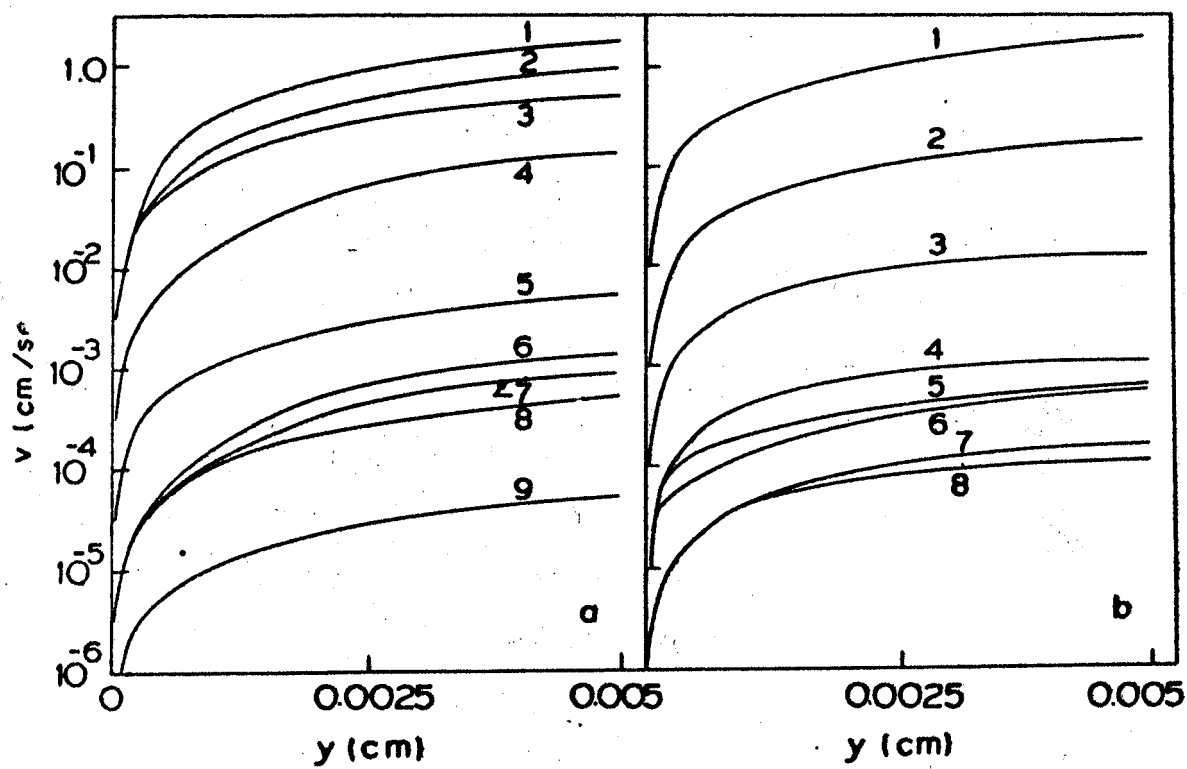


any shear rate V has to be greater with increasing H_x . A field H_y has the opposite effect. This accounts for the dependence of the apparent viscosity on magnetic field in either boundary condition. OM can also observe that at any shear rate, V for the symmetric case is less than that for the asymmetric case, showing clearly that η in the former case is more than that in the latter case.

Figure 4

Velocity profiles for different shear rates and magnetic fields.

- (a) $\theta_1 = \pi/2 = \theta_2$; $c = 10.0 \text{ dynes cm}^{-2}$,
 $H_y = (1) 0 (2) 10000 (3) 25000 \text{ gauss}$,
 $H = 0$; $c = (4) 1.0 (5) 0.1 \text{ dynes cm}^{-2}$,
 $c = 10^{-2} \text{ dynes cm}^{-2}$; $H_x = (6) 3000 (7) 2000$
 $(8) 0 \text{ gauss}$, $(9) c = 10^{-3} \text{ dynes cm}^{-2}$; $H = 0$.
- (b) $\theta_1 = 0$; $\theta_2 = \pi/2$; $H = 0$; $c = (1) 10.0 (2) 1.0$
 $(3) 0.1 (4) 10^{-2} \text{ dynes cm}^{-2}$; $c = 10^{-2} \text{ dynes cm}^{-2}$,
 $H_y = (5) 5000 (6) 20000 \text{ gauss}$; $c = 10^{-3} \text{ dynes}$
 cm^{-2} , $H_x = (7) 2000 (8) 0 \text{ gauss}$.



Appendix

Here the application of the orthogonal collocation method to the problem of the shear flow of nematics is described. As will become clear the orthogonal collocation method is simply a modification and extension of the Gaussian quadrature to the solution of boundary value problems involving differential equations.

Let us first transform the independent variable y to $\bar{y}$ such that

$$\bar{y} = \frac{y - h}{h}$$

Clearly the boundaries $y = 0$ and $y = 2h$ now coincide with the values $\bar{y} = -1$ and $\bar{y} = +1$ respectively. Equations (2) and (3) can be written as

$$2f(\theta) \frac{d^2\theta}{d\bar{y}^2} + \frac{df}{d\theta} \left(\frac{d\theta}{d\bar{y}} \right)^2 + \frac{c(\lambda_1 + \lambda_2 \cos 2\theta)h^2}{g(\theta)} + (\Delta\chi)h^2H^2 \sin(2\psi - 2\theta) = 0 \quad (A1)$$

$$\frac{d\psi}{d\bar{y}} = \frac{ch}{g(\theta)} \quad (A2)$$

The reason for this transformation is to make the range of

$\bar{y}$ coincide with the range in which Legendre polynomials $P_n(\bar{y})$ are orthonormal:

$$\int_{-1}^{+1} P_n(\bar{y}) P_m(\bar{y}) d\bar{y} = \delta_{mn}$$

Also every $P_n(\bar{y})$ vanishes exactly n times in this interval.

We now approximate θ by a series of the form

$$\theta = \sum_{i=1}^{N+2} d_i \bar{y}^{i-1}$$

where d_i are constants. Let $\bar{y}_j$ be any one of the N roots of $P_N(\bar{y})$. The value θ_j of θ at $\bar{y}_j$ can be written as

$$\theta_j = \sum_{i=1}^{N+2} d_i \bar{y}_j^{i-1} \quad (A3)$$

In matrix form (A3) can be written as $\theta = \underline{Q} \underline{d}$ or $\underline{d} = \underline{Q}^{-1} \theta$ or equivalently

$$d_i = \sum_{j=1}^{N+2} [Q^{-1}]_{ij} \theta_j$$

where

$$Q_{j1} = \bar{y}_j^{1-1}.$$

The N roots $\bar{y}_j$ together with the boundaries $\bar{y} = \pm 1$ are called the collocation pointer.

The derivatives of θ at the roots $\bar{y}_j$ can be expressed in terms of the values $\theta_j \equiv \theta(\bar{y}_j)$ in the form

$$\left[\frac{d\theta}{d\bar{y}} \right]_{\bar{y}=\bar{y}_j} = \sum_{k=1}^{N+2} A_{jk} \theta_k, \quad \left[\frac{d^2\theta}{d\bar{y}^2} \right]_{\bar{y}=\bar{y}_j} = \sum_{k=1}^{N+2} B_{jk} \theta_k \quad (A4)$$

where

$$A_{jk} = \sum_{i=1}^{N+2} \left[\frac{d}{d\bar{y}} \bar{y}^{i-1} \right]_{\bar{y}=\bar{y}_j} [Q^{-1}]_{ik}$$

and

$$B_{jk} = \sum_{i=1}^{N+2} \left[\frac{d^2}{d\bar{y}^2} \bar{y}^{i-1} \right]_{\bar{y}=\bar{y}_j} [Q^{-1}]_{ik}$$

Substituting in (A1) and (A2) it is now possible to evaluate the two equations at the collocation points

$$2f(\theta_j) \sum_{k=1}^{N+2} B_{jk} \theta_k + \frac{df(\theta_j)}{d\theta_j} \left[\sum_{k=1}^{N+2} A_{jk} \theta_k \right]^2 + \frac{ch^2(\lambda_1 + \lambda_2 \cos 2\theta_j)}{g(\theta_j)} + (\Delta\chi)h^2R^2 \sin(2\psi - 2\theta_j) = 0 \quad (A5)$$

[j = 2, 3 N + 1]

$$\sum_{k=1}^{N+2} A_{jk} v_k = \frac{ch}{g(\theta_j)} \quad (A6)$$

$$[j = 2, 3, \dots, N+2].$$

where $\bar{y}_1 = -\bar{y}_{N+2} = -1$, $\theta_1 = \theta^{(1)}$ the boundary value of θ at $y = 0$ (or $\bar{y} = -1$), $\theta_{N+2} = \theta^{(2)}$ the bound — value of θ at $y = 2h$ (or $\bar{y} = +1$), $v_1 = v(y = 0) \neq 0$ and $v_{N+2} = v(y = 2h) = V$ remains to be determined. Now (A5) provides N equations for the N unknowns θ_j ($j = 2 \dots N+1$). These equations can be solved by the Newton-Raphson method at any given value of c and the θ profile obtained. Now this set of $N+1$ linear equations (A6) in the $N+1$ unknowns v_j ($j = 2 \dots N+2$) can be solved and the velocity profile obtained.

At this stage one might ask why other methods like the finite difference method were NOT used. The present writer did indeed use the finite difference method (3000 steps, double precision) and found that the values of η obtained agreed to 1 part in 10^4 with those obtained from the collocation method. But the finite difference method was found to be slower and could not yield results for $c > 1.0$ in all cases. Hence the collocation method was chosen for our calculations.

The programs were all written in Fortran IV. Standard subroutines (provided by the IBM in their Scientific Subroutine Package) were used for inverting matrices and for solving linear equations. The programs were run on the IBM 360/44 at the Computer Centre, Indian Institute of Science, Bangalore 560012, with double precision arithmetic. The total time for computations of this chapter was about 20 hours.

References

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