

Bayesian inference of peculiar motions of low-redshift galaxies from Type Ia supernova observations

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Accepted 2026 September 9. Received 2026 August 31; in original form 2026 April 7

ABSTRACT

The peculiar motions of galaxies are powerful cosmological probes that trace the growth of structures and the distribution of matter in the universe, providing a means to investigate the nature of dark energy and test gravity on cosmological scales. However, their direct observation is extremely challenging, as it requires independent and precise distance measurements to galaxies. We present a Bayesian approach to estimate the radial component of peculiar velocities of galaxies hosting Type Ia supernovae (SNe Ia), relying solely on the background cosmological model and the precision of the SN Ia data. Unlike other peculiar velocity estimators based on Hubble residuals, our method does not assume local linearity of the magnitude–redshift relation or a fixed cosmology, reducing potential bias even for large peculiar velocities and self-consistently avoiding bias due to a wrong cosmology. We validate our method using simulated supernova data with the precision of current and upcoming surveys, and further compare it with the linearized estimator to test its efficacy. We show that our estimator has lower bias than the standard estimator and remains consistent even for peculiar velocities comparable to the Hubble flow. We also present a Bayesian derivation for the linearized estimator generalized to include the supernova magnitude covariance.

Key words: methods: statistical – cosmological parameters – large-scale structure of Universe.

1 INTRODUCTION

Observations of Type Ia supernovae (SNe Ia) have been pivotal in shaping our understanding of the cosmos, particularly the dynamics of the late-time universe. Their discovery provided the first direct evidence for the accelerated expansion of the universe (A. G. Riess & others 1998; S. Perlmutter & others 1999), a phenomenon attributed to dark energy or a cosmological constant. As standardizable candles, SNe Ia serve as precise distance indicators, enabling measurements of cosmic expansion across a wide redshift range and placing tight constraints on cosmological parameters. Analyses of magnitude–redshift data, for example, have yielded precise estimates of the Hubble constant, such as $H_0 = 73.04 \pm 1.04 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (A. G. Riess & others 2022). Despite the growing number of observed supernovae, the precision of cosmological inference from SNe Ia remains limited by systematic uncertainties in their standardization and photometric calibration (M. Vincenzi & others 2024; S. Dhawan, B. Popovic & A. Goobar 2025; G. Efstathiou 2025; M. Ginolin & others 2025). Nevertheless, current analyses have achieved percent-level precision on key cosmological parameters using SN Ia data alone. A significant source of uncertainty arises from the peculiar motions of supernova host galaxies (R. A. Vanderveld 2008; T. M. Davis et al. 2011; B. Carreres & others 2025) that are deviations from the

pure Hubble flow caused by gravitational interactions with large-scale structure (P. J. E. Peebles 1980). These motions perturb the observed redshifts and introduce additional scatter in distance measurements. Understanding the redshift dependence of these effects is crucial not only for refining cosmological constraints from SNe Ia, but also for probing growth of structures and testing gravity on cosmological scales (D. Huterer & others 2015).

The peculiar motions arise from the gravitational interaction of overdense regions and consist of a large-scale coherent component and small-scale random motions. From linear perturbation theory, the coherent velocities can be predicted from the density contrast on very large scales (P. J. E. Peebles 1980; M. A. Strauss & J. A. Willick 1995; F. Bernardeau et al. 2002); however, this approximation fails on small, nonlinear scales where the random, virialized motions of individual galaxies dominate (M. Davis & P. J. E. Peebles 1983). Since gravitational instabilities give rise to both the density contrast and peculiar motions, the latter contain roughly half of the information in the phase space of the matter distribution, especially on non-linear scales (M. A. Strauss & J. A. Willick 1995; C. Howlett, L. Staveley-Smith & C. Blake 2017). Moreover, while the observed galaxy distribution represents only baryonic matter and must be converted to the total matter density using bias modelling, peculiar velocities directly trace the gravitational potential and are therefore free from such bias (N. Kaiser 1984; V. Desjacques, D. Jeong & F. Schmidt 2018). As a result, independent measurements of peculiar velocities can provide complementary information to galaxy clustering surveys.

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At quasi-linear and nonlinear scales, peculiar velocities and their statistical measures, such as the mean pairwise velocity of matter, can be computed from state-of-the-art cosmological N -body (V. Springel et al. 2005; E. Jennings 2012; T. Okumura et al. 2014). When combined with higher-order galaxy clustering statistics, these peculiar velocities become powerful probes for constraining modified gravity and dark energy models (N. Kaiser 1987; E. V. Linder 2005; T. Clifton et al. 2012). In particular, while many of these models predict similar background cosmological evolutions, they exhibit distinct signatures in perturbations, especially on small scales (K. Koyama 2016). Since dark energy began to dominate only recently, such features cannot be effectively constrained using the CMB, which primarily probes the early universe and large scales. Therefore, peculiar-velocity measurements provide a unique and essential probe of small-scale physics in the late universe (C. Howlett et al. 2017).

Given their unique advantages as cosmological probes, numerous efforts have been devoted to measuring peculiar velocities with increasing precision. The most common approach for coherent motions is velocity field reconstruction, which infers the peculiar velocity field from the observed galaxy distribution by converting galaxy densities into total matter overdensities and applying a linear integral operator, assuming a fiducial cosmology, to relate density to velocity (A. Nusser & M. Davis 1994; J. Carrick et al. 2015). However, since the reconstructed velocity field is derived from the galaxy density field under an assumed cosmology, it does not introduce additional independent degrees of freedom; rather, it reflects the same information encoded in the density field. Another approach uses empirical relations such as the Fundamental Plane for elliptical galaxies (S. Djorgovski & M. Davis 1987; A. Dressler et al. 1987) and the Tully–Fisher relation for spirals (R. B. Tully & J. R. Fisher 1977) to infer redshift-independent distances and subtract the Hubble flow; while individual measurements are noisy, their correlations ensembled over many galaxies yield cosmological insights. SNe Ia offer a more precise probe, as residuals from fitting the magnitude–redshift relation in a given cosmological model can be translated into peculiar velocities via standard error propagation (L. Hui & P. B. Greene 2006; C. Gordon, K. Land & A. Slosar 2007; D. Huterer et al. 2017). The standard Hubble residual-based peculiar velocity estimators commonly used in the literature are derived under the assumption of local linearity of the magnitude–redshift relation (or distance–redshift relation), which is valid only when the peculiar velocities are very small compared to the Hubble flow ($v_p \ll cz_{\text{cos}}$; T. M. Davis et al. 2011; B. Carreres et al. 2023). Moreover, the estimator has to be evaluated at the unknown cosmological redshift, z_{cos} , and with the true cosmological parameters. In practice, the estimator is evaluated at the observed redshift and with the fiducial values of the cosmological parameters. For small peculiar velocities, the first approximation is valid; however, the choice of a wrong cosmology can significantly bias the velocity estimates. For $v_p \sim cz_{\text{cos}}$, the linear approximation breaks, and the standard estimator becomes biased even for the correct cosmology (B. Carreres et al. 2023) and therefore cannot be used reliably. Although peculiar velocities can, in principle, be inferred using the exact nonlinear relation, most practical implementations adopt this linear approximation, which is the focus of our comparison (A. Carr et al. 2022; E. R. Peterson & others 2022).

In this work, we present a Bayesian approach to estimating the line-of-sight peculiar velocities of host galaxies using SN Ia observations. We employ a method developed in a previous work (U. Upadhyay, T. D. Saini & S. K. Sethi 2026) for fitting general

non-linear models with errors in both dependent and independent variables. Using the magnitude–redshift relation in the standard Λ CDM model without fixing its parameters, we estimate the peculiar velocities at the redshifts of the observed SNe Ia. The method is validated with simulated data having the precision of current and upcoming supernova surveys. We also present a generalized Bayesian version of the standard linear-approximation-based peculiar velocity estimator that accounts for the covariance of SN Ia magnitudes. Using simulated data, we show that our approach yields consistent estimates even for larger peculiar velocities, where the linear approximation breaks down. Further, it self-consistently avoids bias due to incorrect cosmology and is free of the assumption of Gaussianity of the peculiar velocity distribution and therefore works for both random and coherent peculiar motion.

This paper is organized as follows. In Section 2, we briefly discuss the linear theory of peculiar velocities. In Section 3, we discuss the methodology. First, formulation of the Bayesian method for fitting general errors-in-variables models and its specialization for the magnitude–redshift relation in Section 3.1, and then the linear approximation-based estimator generalized to account for SN Ia magnitude covariance in Section 3.2. In Section 4, we present the results of testing and validating our methods using simulated supernova data and their application to the Pantheon+ sample of SNe Ia. In Section 5, we conclude with a summary of the work discussing the significance and limitations of our approach compared to the existing literature on this problem of peculiar velocity estimation. In Appendix A, we present a Bayesian derivation of the linear approximation-based estimator generalized to account for the supernova magnitude covariance. In Appendix B, we present trace plots of the velocity estimates at different redshifts and discuss the convergence issues arising from the sensitivity of the likelihood to small redshift perturbations.

2 PECULIAR VELOCITIES IN LINEAR THEORY

In cosmological theory, peculiar velocities of galaxies arise from density perturbations. In linear theory, the curl-free velocity field can be related to the density contrast, in Fourier space, as (P. J. E. Peebles 1980):

$$\mathbf{v}(\mathbf{k}) = \frac{ia(t)\mathbf{k}}{k^2} \frac{\partial \delta(\mathbf{k}, t)}{\partial t} \quad (1)$$

In linear theory, the spatial and time dependence of the density contrast are separable: $\delta(\mathbf{k}, t) = D^+(t)\delta(\mathbf{k})$, where $D^+(t)$ is the growing mode of density contrast. This gives us:

$$\mathbf{v}(\mathbf{k}) = \frac{ia(t)\mathbf{k}}{k^2} \frac{dD^+(t)}{dt} \delta(\mathbf{k}) \quad (2)$$

The peculiar velocities of galaxies cannot be computed using only the linear theory, as non-linear effects become important at the scale of galaxies. The peculiar velocity field of galaxies is modelled as the sum of a large-scale coherent velocity field and a random, uncorrelated component owing to small-scale effects. The large-scale, coherent velocity field can be simulated using equation (2) for Gaussian density perturbations, an observed feature of linear perturbations. In this work, we use linear theory only to simulate the coherent large-scale component of the peculiar velocity field, which serves as the fiducial velocity field for validating the proposed Bayesian inference method. We simulate the coherent, large-scale velocity field. These simulations yield

the linear component of the line-of-sight velocity field. The proposed inference framework itself does not rely on the assumption of linear theory and can, in principle, be applied to peculiar velocity fields obtained from more realistic N -body simulations or observational reconstructions. The details of the simulation are discussed in Section 4, where the mock SN Ia catalogue is constructed.

3 METHODOLOGY

The standard likelihood for estimating cosmological parameters from magnitude–redshift data is sensitive to the contribution of the peculiar velocity of supernova host galaxies to the observed redshifts, especially at low redshifts. Therefore, if not properly accounted for, these peculiar velocities can introduce bias into estimation of cosmological parameters such as matter density (Ω_M), Hubble constant (H_0), and the dark energy equation of state (w). This effect is most pronounced at low redshifts ($z \lesssim 0.1$), where typical peculiar velocities of a few hundred km s^{-1} are comparable to the Hubble flow, although coherent peculiar motions can also induce correlated uncertainties at higher redshifts (L. Hui & P. B. Greene 2006; T. M. Davis et al. 2011; A. Carr et al. 2022; E. R. Peterson et al. 2022). Due to this dependence, however, it also presents an opportunity to estimate these peculiar velocities jointly with the cosmological parameters. In order to estimate the peculiar motion of supernova host galaxies, we treat the fitting of the magnitude–redshift relation to the SN Ia data as an *errors-in-variables* model with the peculiar motions contributing to the errors in the independent variable, z . For this, we use a Bayesian estimator developed in our previous work (U. Upadhyay et al. 2026) to study and correct for the effect of peculiar motion of galaxies on the inference of cosmological parameters from SN Ia data. We also discuss a Bayesian derivation of the standard peculiar velocity estimator based on a local linear approximation of the magnitude–redshift relation, assuming Gaussian redshift uncertainties induced by peculiar velocities, and generalize it to incorporate supernova magnitude covariance (see Appendix A for the details). Later in the manuscript, we validate and compare these estimators using simulated SN Ia data with magnitude uncertainties corresponding to the current and upcoming supernova surveys.

3.1 General errors-in-variables method

Here, we briefly outline the general method for fitting *errors-in-variables* models and its specific cosmological application to fitting the magnitude–redshift relation to supernova observations as described in detail in U. Upadhyay et al. (2026), and then discuss its application in peculiar velocity inference, which is the primary focus of the present study. Consider a general model expressed in functional form,

$$y = f(\theta, x). \quad (3)$$

where $\theta = \{\theta_1, \theta_2, \dots, \theta_m\}$ are the model parameters, and x and y are the independent and dependent variables of the model, respectively. In the usual scenario where only the dependent variable, y , has noisy measurements or the uncertainty in the independent variable is negligible, the likelihood is derived from the distribution of errors in the dependent variable. However, in reality, all measurements are subject to uncertainties, and we often encounter model-fitting problems in which the errors in both variables must be taken into account. Such model-fitting problems

are called *errors-in-variables* models (C. Dougherty 2007). For the simple linear model and Gaussian distribution of the errors, the problem can be easily solved by analytically marginalizing over the uncertainty in the independent variable to obtain a likelihood based on errors in both dependent and independent variables (W. H. Press et al. 1992). For an arbitrary non-linear model with *errors-in-variables*, our approach is to treat the unknown true value of the independent variable as parameters while treating the errors in the dependent variables as usual. In the usual case with errors only in the dependent variable, we proceed to write the likelihood function assuming that the model predicts the true values, y_i^* and the measurements acquire some random noise, ϵ_{Y_i} , yielding $y_i = y_i^* + \epsilon_{Y_i}$. If the random noise in the measurement has a probability distribution $P_Y(\epsilon_Y)$, the likelihood function is given by

$$\mathcal{L}_Y(\mathcal{D} | \theta, y^*) = \prod_i P_Y(y_i - y_i^*) \delta(y_i^* - f(\theta, x_i)), \quad (4)$$

where $\mathcal{D}$ represents the data, and the Dirac delta function ensures that the true values of the dependent variable y depend on θ and the measured independent variable x through the model function $f(\theta, x)$. Using Bayes' theorem, the posterior distribution is given by

$$P(\theta, y^* | \mathcal{D}; I) \propto \mathcal{L}_Y(\mathcal{D} | \theta, y^*) P(\theta | I) \quad (5)$$

where, $P(\theta | I)$ is the prior distribution for the model parameters θ , and I denotes any other background information. When we have errors in both dependent and independent variables, the likelihood in equation (4) can be generalized to

$$\mathcal{L}_{XY}(\mathcal{D} | \theta, x^*, y^*) = \prod_i P_Y(y_i - y_i^*) P_X(x_i - x_i^*) \delta(y_i^* - f(\theta, x_i^*)), \quad (6)$$

where x_i^* are the true values of the independent variables, which we treat as parameters, and P_X is the probability distribution of errors in the independent variable, $x_i = x_i^* + \epsilon_{X_i}$. Again, using Bayes' theorem, the posterior distribution for the model parameters θ and the independent variable x^* can be written as

$$P(\theta, x^* | \mathcal{D}; I) = \prod_i P_Y(y_i - f(\theta, x_i^*)) P_X(x_i - x_i^*) P(\theta | I) P_{\chi^2}(\chi_Y^2), \quad (7)$$

where P_X contains our prior information about the independent variable, and P_{χ^2} is a χ^2 -distribution in χ_Y^2 given by

$$P_{\chi^2}(\chi_Y^2) = \frac{\chi_Y^{k-2} \exp(-\chi_Y^2/2)}{2^{k/2} \Gamma(k/2)}, \quad (8)$$

with k as the desired degrees of freedom of the fit. Here, P_{χ^2} is a regularization factor, often introduced to mitigate overfitting (D. S. Sivia 1996) – a common issue in high-dimensional problems such as this, where the number of parameters exceeds the number of data points. Equation (7) allows us to estimate the parameters θ along with the correction in the independent variable x . In this work, we are interested in fitting the magnitude–redshift relation in the standard Λ CDM model, where θ corresponds to the cosmological parameters and the independent variable is the redshift. Equation (7) thus enables joint estimation of cosmological parameters and redshift uncertainties, which we assume to originate from peculiar velocities. This approximation is justified for the idealized mock data considered here and for the low-redshift regime, where peculiar velocities dominate over other sources of redshift and magnitude perturbations, while the effects of gravitational lensing and spectroscopic redshift uncertainties

are negligible (C. Bonvin, R. Durrer & M. A. Gasparini 2006; T. M. Davis et al. 2011; A. Carr et al. 2022).

The relation between the apparent magnitude (m) and redshift (z) of a source is given by

$$m = M + 5 \log \left(\frac{D_L}{\text{Mpc}} \right) + 25, \quad (9)$$

where M is the absolute magnitude, and D_L is the luminosity distance of the source. For a flat Friedmann–Lemaître–Robertson–Walker universe, the luminosity distance of a source at a cosmological redshift z can be expressed as

$$D_L = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{h(z')}, \quad (10)$$

where the dimensionless Hubble parameter is given by

$$h(z) = [\Omega_M(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}]^{1/2}. \quad (11)$$

Here Ω_M and Ω_{DE} are the present-day matter and dark energy density parameters, respectively, and w is the dark energy equation of state parameter. The radiation component has been neglected, as it makes a negligible contribution to the late-time expansion history over the redshift range probed by SNe Ia.

Due to the peculiar velocity of the host galaxy with respect to the comoving frame, the observed redshift differs from the cosmological redshift. Unless explicitly stated otherwise, we use z to denote the observed redshift and z^* to denote the corresponding cosmological redshift, which we treat as a free parameter in the Bayesian analysis presented below. Thus, z_i and z_i^* represent the observed and cosmological redshifts of the i -th supernova, respectively. To first order in the line-of-sight peculiar velocity ($v_p \ll c$), the two are related by (T. M. Davis et al. 2011)

$$1+z = (1+z^*) \left(1 + \frac{v_p}{c} \right), \quad (12)$$

In this context, the general expression in equation (7) takes the following form:

$$P(\theta, z^* | \mathcal{D}; I) \propto \exp \left[-\frac{\chi_m^2(\theta, z^*)}{2} \right] P_Z(z_i - z_i^*) P(\theta | I) P_{\chi^2}(\chi_m^2), \quad (13)$$

where, χ_m^2 given by

$$\chi_m^2(\theta, z^*) = \mathbf{m}^T \mathbf{C}_m^{-1} \mathbf{m}, \quad (14)$$

with the magnitude residual vector and covariance defined by

$$\mathbf{m} = \begin{pmatrix} m_1 - m(z_1^*) \\ \vdots \\ m_N - m(z_N^*) \end{pmatrix}, \text{ and } \mathbf{C}_m = \begin{pmatrix} \sigma_{m_1}^2 & \cdots & \sigma_{m_1 m_N} \\ \vdots & \ddots & \vdots \\ \sigma_{m_N m_1} & \cdots & \sigma_{m_N}^2 \end{pmatrix}. \quad (15)$$

For estimating the peculiar velocities, we sample from the above posterior (equation 13) by varying both the Λ CDM parameters, $\theta = \{H_0, \Omega_M\}$, as well as the true redshifts z_i^* . Thus, we obtain the posterior distribution of redshift errors, $\Delta z_i \equiv z_i - z_i^*$, at each redshift in the data, which can be converted into peculiar velocity using the relation, $\Delta z = (v_p/c)(1+z)$. We observe that the regularization term is not required for this problem; therefore, we remove it later. Since our method does not require fixing cosmological parameters, it self-consistently estimates peculiar velocities and cosmological parameters within the assumed cosmological model, and the supernova data, reducing bias due to fixed cosmological parameters. Moreover, by exploring the full posterior of each redshift parameter, it naturally accounts for

correlations among the peculiar velocities. It is important to note that this method can capture only the line-of-sight component of the peculiar velocities. Although this approach provides a consistent estimate of the peculiar velocities of supernovae as discussed in the next section, it involves a variation in the number of parameters that exceeds the number of data points, which can be very large, making it computationally very expensive and statistically too ambitious for any realistic data. We assess the efficacy and robustness of this approach using simulated data sets that mimic both current observations and those anticipated from future surveys. For comparison, we discuss the standard peculiar velocity estimator based on locally linear approximation of the magnitude–redshift relation and the Gaussian assumption of the redshift uncertainty in the next section.

3.2 Linear approximation

In this section, we present a Bayesian derivation of the standard Hubble residual-based peculiar velocity estimator from equation (13), by assuming local linearity of the magnitude–redshift relation and Gaussian distribution for the redshift errors. Apart from being a complementary Bayesian derivation, it also generalizes the standard estimator by including supernova magnitude covariance.

Starting with the posterior distribution in equation (13) without the regularization factor, substituting χ_m^2 from equation (14), and assuming that the redshift errors, ϵ_{z_i} , follow a Gaussian distribution, we can write

$$P(\theta, z^* | \mathcal{D}; I) \propto \exp \left[-\frac{1}{2} \mathbf{m}^T \mathbf{C}_m^{-1} \mathbf{m} \right] \cdot \exp \left[-\frac{1}{2} \mathbf{z}^T \mathbf{C}_z^{-1} \mathbf{z} \right], \quad (16)$$

where the redshift residual vector and associated covariance matrix are given by

$$\mathbf{z} = \begin{pmatrix} z_1 - z_1^* \\ \vdots \\ z_N - z_N^* \end{pmatrix}, \text{ and } \mathbf{C}_z = \begin{pmatrix} \sigma_{z_1}^2 & \cdots & \sigma_{z_1 z_N} \\ \vdots & \ddots & \vdots \\ \sigma_{z_N z_1} & \cdots & \sigma_{z_N}^2 \end{pmatrix}. \quad (17)$$

Both the redshift vector $\mathbf{z}$ and its covariance matrix $\mathbf{C}_z$ are unknowns of the problem, since neither the true cosmological redshifts z_i^* nor their correlations are known *a priori*. Therefore, these quantities define the prior probability distribution represented by the second factor in equation (16). In the present work, we adopt a diagonal Gaussian covariance matrix, $\mathbf{C}_z = \text{diag}[\sigma_{z_1}^2, \dots, \sigma_{z_N}^2]$, where σ_z is chosen to correspond to a prior peculiar velocity dispersion of $\sigma_v = 1000 \text{ km s}^{-1}$. This value is substantially larger than the characteristic peculiar velocity dispersion of nearby galaxies ($\sim 300 \text{ km s}^{-1}$) commonly adopted in supernova cosmology, thereby providing a weakly informative prior that assumes the cosmological redshifts to be independent *a priori*. If prior information on the correlations between peculiar velocities is available, for example from a model of the peculiar velocity field, it can be incorporated straightforwardly through a non-diagonal covariance matrix. The diagonal Gaussian prior assumed here is introduced solely to derive an analytical expression for the generalized linearized estimator, which extends the commonly used linearized estimator to incorporate both magnitude and redshift covariance. This estimator serves only as a benchmark for comparison with the proposed Bayesian estimator. The proposed Bayesian framework itself is not restricted to this assumption and can readily accommodate more general prior distributions.

Here, regularization is not required since we are deriving an analytic expression for the redshift errors. In the MCMC analysis, by contrast, the goal was to sample from the distribution, where regularization could help stabilize parameter exploration and prevent overfitting. Further, since we are interested in estimating the redshift parameters (or peculiar velocities), we fix the cosmological parameters at their best-fitting values $\theta = \theta_0$ to obtain

$$P(z^* | \mathcal{D}; \theta_0; I) \propto \exp \left[-\frac{1}{2} \mathbf{m}^T \mathbf{C}_m^{-1} \mathbf{m} \right] \cdot \exp \left[-\frac{1}{2} \mathbf{z}^T \mathbf{C}_z^{-1} \mathbf{z} \right], \quad (18)$$

Taylor expanding the apparent magnitude about the observed redshift, z , and keeping terms up to linear order in $(z^* - z)$, we obtain

$$m(z^*) = m(z) + m'(z)(z^* - z), \quad (19)$$

where $m'(z)$ is the derivative of $m(z^*)$ w.r.t. z^* evaluated at $z^* = z$.

Substituting (19) into (18) and rearranging the terms, we obtain a multivariate Gaussian in z^* with mean and covariance given by

$$\mathbf{z}^* = - \left[\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}' \right]^{-1} \left[\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{m}^0 + \mathbf{C}_z^{-1} \mathbf{z}^0 \right], \quad (20)$$

and

$$\Sigma_{z^*} = \left[\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}' \right]^{-1}, \quad (21)$$

respectively. Here, $\mathbf{M}' = \text{diag}[m'_1, m'_2, \dots, m'_N]$, while $\mathbf{m}^0$ and $\mathbf{z}^0$ are column vectors given by

$$\mathbf{m}^0 = \begin{pmatrix} m_1 - m(z_1) + m'(z_1)z_1 \\ \vdots \\ m_N - m(z_N) + m'(z_N)z_N \end{pmatrix}_{N \times 1}, \quad \text{and} \quad \mathbf{z}^0 = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}. \quad (22)$$

This provides estimates of the true cosmological redshifts and associated uncertainties in their estimation. Subtracting the observed redshift from equation (20) and converting the difference Δz to velocities using the relation $\Delta z = (v_p/c)(1 + z^*)$, we get an estimate of the peculiar velocities, under the assumption of locally linear approximation of the magnitude–redshift relation and the assumption of Gaussian distribution for the redshift errors. This is a Bayesian derivation of the standard peculiar velocity estimator used in the literature (see B. Carreres et al. 2023) expressed in matrix form. The main difference here is the generalization to include supernova magnitude covariance and the explicit appearance of the prior through the redshift covariance $\mathbf{C}_z$.

The linear approximation holds for small peculiar velocities ($v_p \ll cz^*$), which occur at high redshifts where the Hubble flow dominates. At low redshifts, $v_p \sim cz^*$, and the estimator becomes biased. Another source of bias in the standard estimator comes from fixed cosmology. The quantities in equations (20) and (21) are supposed to be computed at the cosmological redshifts (z_{cos}) and with the correct values of cosmological parameters. However, z_{cos} is an unknown of the problem, and the inferred cosmological parameters themselves could be biased due to the effect of unknown peculiar velocities. In practice, the linearized estimator is evaluated with the fiducial values of the cosmological parameters and at the observed redshifts. The errors-in-variables approach discussed in the previous section is free of both sources of bias, as we show in the results section.

Apart from the bias discussed, there is another limitation in its application – the estimates depend on the values of σ_{z_i} , which is what we want to measure. At this point, it is important to

note that the Gaussian distribution in redshift errors can be interpreted as a prior on the true redshift parameters. A very small value of σ_{z_i} corresponds to a narrow prior which will not allow the redshift parameters, z^* , to take any values far from the observed redshifts, z . Similarly, a large value of σ_{z_i} corresponding to a wide prior will yield a different estimate of $\sigma_{z_i}^*$ depending on the likelihood based on the precision of the apparent magnitude of the SN Ia. This effect can be understood more thoroughly by taking various limits of equations (20) and (21), which we discuss in Appendix A. It is safer to adopt a wide prior so that the posterior is dominated by the likelihood rather than the prior; therefore, we take $\sigma_v \sim 1000 \text{ km s}^{-1}$, which is substantially larger than the characteristic peculiar velocity dispersion of nearby galaxies ($\sim 300 \text{ km s}^{-1}$), thereby providing a deliberately weakly informative prior. As shown by the limiting behaviour of equations (20) and (21) in Appendix A, the influence of the prior is governed by the relative widths of the prior and the likelihood. In particular, when $\sigma_v \gg \sigma_m/m'(z)$, where σ_m is the uncertainty in the apparent magnitude and $m'(z)$ is the slope of the magnitude–redshift relation, the posterior is primarily determined by the likelihood, whereas for $\sigma_v \ll \sigma_m/m'(z)$, it becomes increasingly prior dominated, which can introduce bias in the inferred peculiar velocities.

4 RESULTS

Peculiar velocities constitute both an important source of systematic uncertainty in supernova cosmology and a valuable probe of the growth of cosmic structure. In this section, we apply the Bayesian errors-in-variables framework developed in Section 3 to jointly infer peculiar velocities and cosmological parameters from SN Ia data.

We first validate the proposed Bayesian estimator using simulated magnitude–redshift data for which the input peculiar velocities and cosmological parameters are known, and recover their posterior probability distributions. We then compare these results with those obtained using the linearized estimator applied to the same data set, thereby assessing the performance of the general method relative to the linearized approach. Finally, we apply both methods to the Pantheon+ sample of SNe Ia to investigate the extent to which current data constrain peculiar motions.

4.1 Validation with simulated data

In order to validate our method, we simulate magnitude–redshift data for 1701 SNe Ia at the redshifts corresponding to the Pantheon+ sample. For the simulations, we assume the standard Λ CDM model with fiducial values of the relevant cosmological parameters: $\Omega_M = 0.30$, $\Omega_\Lambda = 0.70$, and $H_0 = 73 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The magnitude errors are drawn from a Gaussian distribution, $\Delta m \sim \mathcal{N}(0, \sigma_m)$ for three different cases with $\sigma_m = 0.2, 0.05$, and 0.02 that correspond to the current and future precision in the SN Ia observations. In this work, we assume Gaussian magnitude uncertainties, as is commonly adopted in supernova cosmology. This assumption is appropriate for the idealized simulated data considered here, while extending the analysis to more realistic non-Gaussian observational uncertainties is left for future work. It is worth noting that computing the apparent magnitudes using the above cosmological parameters, or equivalently treating H_0 as a free parameter in the cosmological inference, requires specifying the absolute magnitude of SNe Ia because of the well-known M_B – H_0 degeneracy. Accordingly,

Table 1. Priors on the cosmological parameters and the redshift parameters used in the analysis.

Parameter	Prior
Ω_M	$\mathcal{U}(0.1, 0.5)$
H_0 (km s ⁻¹ Mpc ⁻¹)	$\mathcal{U}(50, 80)$
z^*	$\mathcal{U}(z - 0.01, z + 0.01)$

throughout our simulations, we fix the absolute magnitude to $M_B = -19.25$, corresponding to the central value inferred from the SHOES calibration. The effect of random peculiar motions on supernova cosmology tends to average out as the number of SNe Ia increases, contributing primarily to the statistical uncertainty. In contrast, coherent peculiar motions, such as large-scale bulk flows, induce correlated shifts in the observed redshifts and can therefore introduce systematic biases in cosmological parameter estimation. The impact of both random and coherent peculiar motions on cosmological inference has been extensively studied in the literature (C. Gordon et al. 2007; T. M. Davis et al. 2011), including in our previous work (U. Upadhyay et al. 2026).

Therefore, to capture this effect, first we simulate the peculiar velocities using the velocity power spectrum in the fiducial Λ CDM cosmology. Essentially, it involves generating a random Gaussian field in real space to represent the matter density contrast, shaping it in Fourier space using the matter power spectrum, computing the peculiar velocity field using equation (2), and taking an inverse Fourier transform to obtain the real-space velocity field. The real-space velocity field simulated on the grid is then mapped to (RA, Dec., z) coordinates so that velocities can be assigned to each SN Ia in the sample. After this, we compute the line-of-sight peculiar velocity of the SN Ia, convert it to the redshift error, and add it to the observed redshift. In addition to this coherent component, we also add a random motion drawn from a Gaussian distribution $\Delta z \sim \mathcal{N}(0, \sigma_z)$ with σ_z corresponding to peculiar velocity, $\sigma_{v_p} = 300$ km s⁻¹. For the matter power spectrum, we use the Boltzmann code CLASS (D. Blas, J. Lesgourgues & T. Tram 2011) with the fiducial Λ CDM parameters.

4.1.1 General MCMC method

First, we apply the general MCMC method to the simulated data to obtain the joint posterior distribution (equation 13) for cosmological parameters $\theta = \{\Omega_M, H_0\}$ and the true redshifts, z^* . For this, we need to provide prior information $P(\theta | I)$ and $P_Z(z_i - z_i^*)$. We take uniform priors for all the parameters as listed in Table 1.

We employ the MCMC sampling technique implemented through the Metropolis–Hastings algorithm using a custom Python implementation developed for the present work to efficiently handle the large parameter space ($\sim \mathcal{O}(10^3)$ free parameters). We run 12 independent chains starting randomly from the prior ranges for the cosmological parameters. The natural starting point for the redshift parameters is $z^* = z$, i.e. assuming that there is no contribution to the observed redshift from the peculiar motion. For the convergence, we use the Gelman–Rubin criterion, $R - 1 < 0.05$, for the cosmological parameters. It is important to mention here that all redshift parameters z_i^* do not converge, marking the limitation of the method and the precision of the data. The convergence depends on the sensitivity of the likelihood to the changes in the parameters, which in turn depends on the curvature of the likelihood and the precision of the data. For

the magnitude–redshift relation, the likelihood is sensitive to the redshift errors (or peculiar velocities) at low redshift. A detailed discussion of this convergence issue and its dependence on the redshift is provided in Appendix B.

Fig. 1 shows the posterior distributions of the cosmological parameters together with the peculiar velocities for the three lowest-redshift SNe Ia in the simulated sample. The black dashed lines indicate the true values used in the simulation. All five parameters are found to be statistically consistent with their true values, indicating the potential of the proposed method for jointly inferring the cosmological parameters and peculiar velocities from the simulated supernova data. With improved SN Ia magnitude precision from future surveys, this approach can enable reliable estimation of peculiar velocities directly from SN Ia magnitude–redshift measurements. The contribution of peculiar velocities to the observed redshift decreases with increasing redshift relative to the Hubble expansion, reducing their impact on cosmological parameter inference (T. M. Davis et al. 2011; A. Carr et al. 2022). However, there is another effect that is more relevant to our discussion – the sensitivity of the apparent magnitude to changes in redshift, which determines the information content of the supernova data. Due to the logarithmic dependence of the magnitude–redshift relation, the apparent magnitude at high redshift becomes increasingly insensitive to small changes in the redshift, reducing the constraining power of the data and leaving the redshift parameters effectively prior dominated. This can be observed in the posterior distribution of peculiar velocities corresponding to high-redshift SNe Ia. However, given the large number of these parameters, it is more convenient to plot the estimated peculiar velocities with their uncertainties at different redshifts rather than the full posterior distributions.

Fig. 2 shows the estimated peculiar velocities as a function of redshift, along with their 1σ uncertainties. The true velocities are also shown (in red) for comparison. We find that the estimates are statistically consistent with the true values at all redshifts. However, the uncertainties increase towards higher redshifts, and the estimates become effectively prior dominated, as the information content of the data decreases. Consequently, the corresponding MCMC chains do not converge to a well-defined posterior in this regime, reflecting the limited information content of the data rather than a limitation of the Bayesian inference method. This behaviour is quantified through the Hessian of the likelihood discussed in Appendix B. In contrast, at low redshifts, the peculiar velocities are well constrained and consistent with the true values. The estimates also depend on the precision of the supernova data, as illustrated by the root mean square error (RMSE) in Fig. 3. Different marker shapes correspond to different levels of magnitude precision, while the colour scale indicates the contribution of variance to the RMSE. As expected, the uncertainty in the estimates increases as the precision of the supernova magnitudes decreases. For a fixed precision and redshift, larger values of the RMSE are primarily driven by increased variance rather than bias. This indicates that the overall error budget is dominated by statistical fluctuations, supporting the statistical consistency of the estimator.

4.1.2 Linear versus exact method

Having discussed the results of the general MCMC method above, and tested its effectiveness in estimating peculiar velocities at low

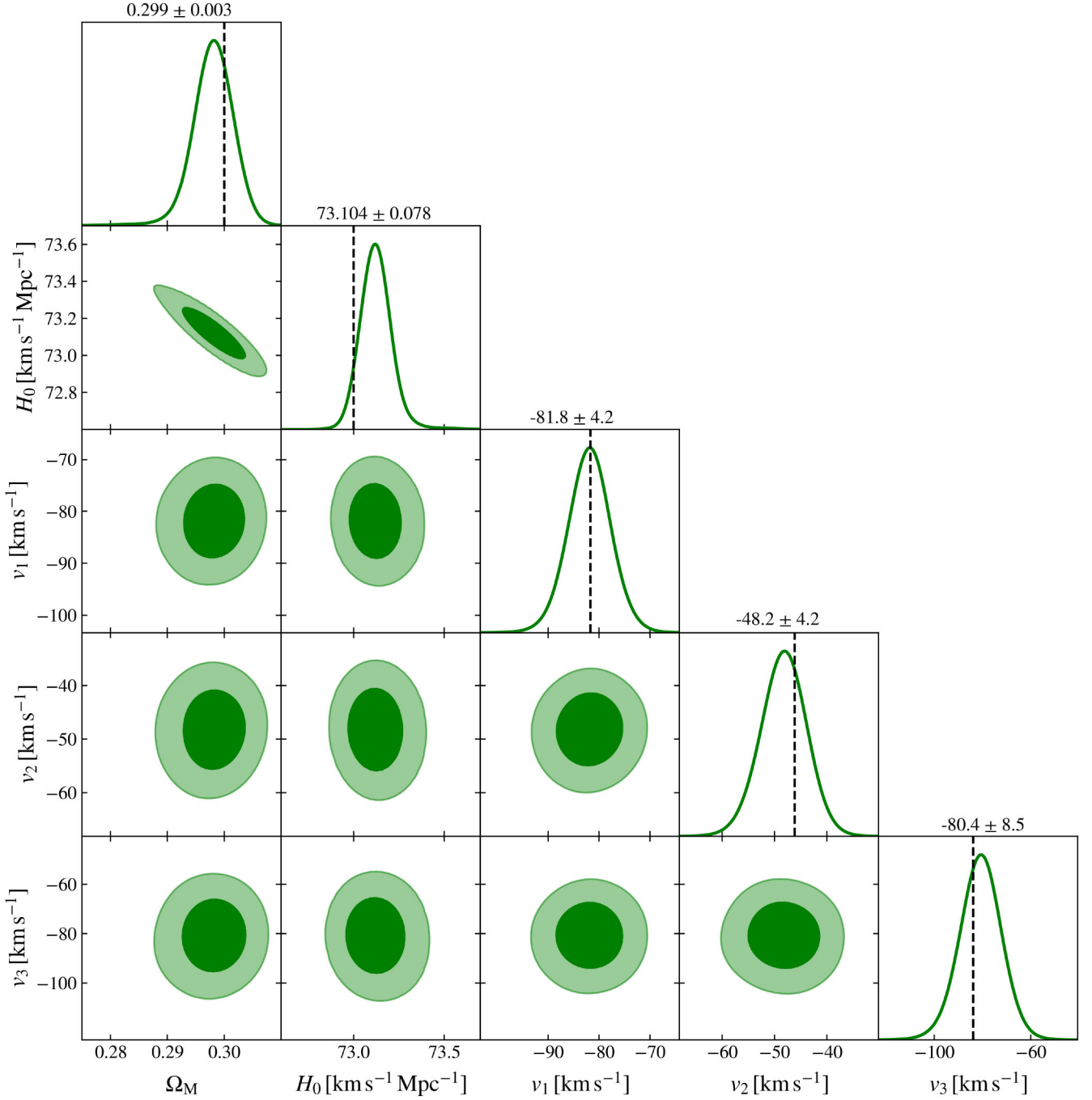


Figure 1. 1D and 2D marginalized posterior distribution for the cosmological parameters and peculiar velocities for the three lowest-redshift supernovae in the sample obtained from simulated data with $\sigma_m = 0.02$. The black-dashed lines show the true values of the parameters.

redshifts, we now proceed in this section to compare the general method with the standard estimator based on linear approximation.

Fig. 4 shows the estimated peculiar velocities, with 1σ uncertainties, plotted against the true velocities used in the simulation for $\sigma_m = 0.02$, comparing the exact (blue circles) and linear (red squares) methods. The black solid line corresponds to the ideal relation, $v_{\text{est}} = v_{\text{true}}$. For small peculiar velocities, the estimates from both the linear and exact methods lie along the black solid line, indicating accurate recovery, as expected since the linear approximation is valid in this regime. However,

as v_{true} increases, the linear method begins to overestimate the velocities. In contrast, the general method based on the errors-in-variables approach remains consistent with the true peculiar velocities, although with slightly larger uncertainties than those obtained with the linear method. Despite the larger uncertainty, the general method provides estimates that are closer to the true values, and therefore performs better overall. This is more clearly understood in terms of the bias-variance trade-off: while the linear estimator exhibits smaller variance, it suffers from increasing bias at larger velocities, whereas the general method shows no significant bias within the uncertainties.

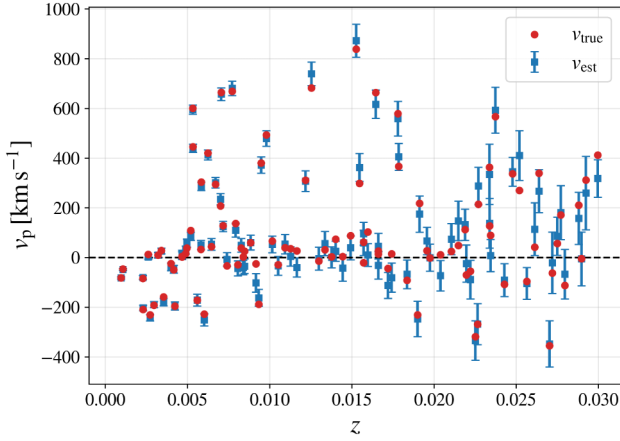


Figure 2. Estimated peculiar velocities along with 1σ error bars at different redshifts obtained using the general method from simulated data with $\sigma_m = 0.02$.

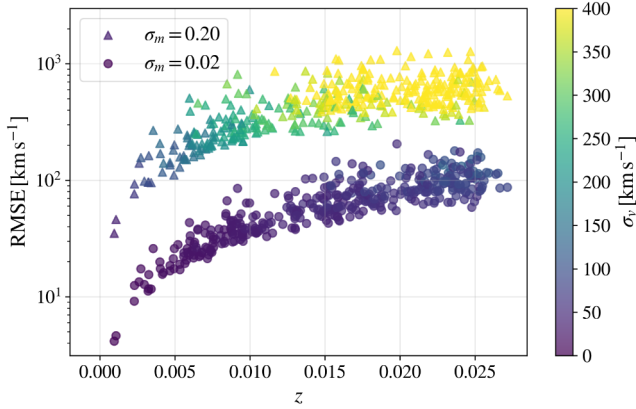


Figure 3. The root mean square errors (RMSE) of estimation as a function of redshift for two different supernova magnitude precision shown by different marker shapes. The colour bar on the right shows the contribution of variance.

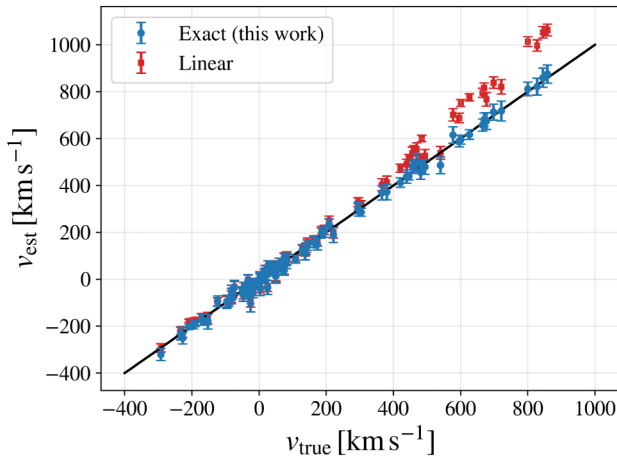


Figure 4. Estimated peculiar velocities along with 1σ error bars versus the true values used in the simulation for simulated data with $\sigma_m = 0.02$. Different colours represent the methods used.

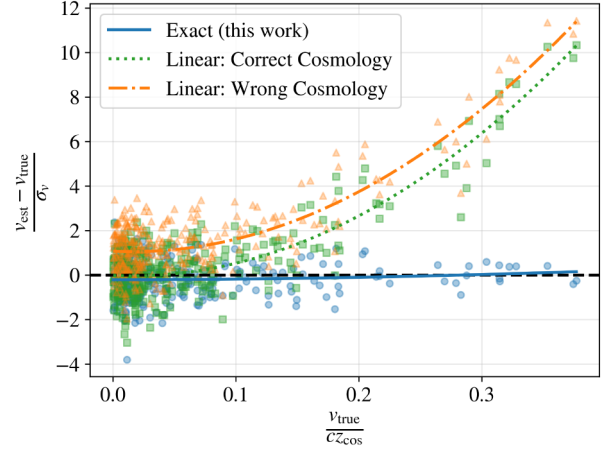


Figure 5. Comparison of the bias arising from the breakdown of the linear approximation and the use of an incorrect cosmology. The three methods are distinguished by different marker shapes and line styles, while the corresponding lines are polynomial fits to the scatter used to illustrate the overall trends. The general method remains statistically consistent within a 68 per cent credible region, even for $v_{\text{true}} \approx cz$, where the linear approximation is no longer valid.

Fig. 5 shows the normalized bias of the estimator, defined as $(v_{\text{est}} - v_{\text{true}})/\sigma_v$, plotted as a function of the ratio of the true peculiar velocity to the Hubble recessional velocity, v_{true}/cz . The results are shown for the exact likelihood based on the general method (blue circles with solid lines), the linear method with the correct cosmology (green squares with dotted lines), and the linear method assuming an incorrect cosmology (orange triangle with dash-dotted lines). The lines are the polynomial fit to the scatter to show the trend. The black dashed line represents no bias. We find that for the linear method, the normalized bias remains small when the peculiar velocities are much smaller than the Hubble flow. However, as the true velocity becomes comparable to the Hubble flow, the bias increases significantly, reaching several standard deviations from zero. The use of an incorrect cosmology, primarily through an incorrect value of H_0 , introduces an additional systematic bias. In contrast, for the exact likelihood-based estimator, the normalized bias remains within ± 1 , indicating that the estimates are consistent with the true values within the 68 per cent credible interval.

As discussed earlier, the estimates degrade towards higher redshifts because the sensitivity of the likelihood to the small redshift perturbations induced by peculiar velocities decreases with increasing redshift. To illustrate this behaviour, Fig. 6 shows the absolute bias, the standard deviation, and the normalized bias (bias divided by standard deviation) as functions of redshift in three vertical panels, for both the linear and the errors-in-variables-based exact likelihood. For both estimators, the absolute bias and the standard deviation increase with redshift. However, the normalized bias remains approximately constant, indicating that the bias stays within the quoted uncertainties and that the estimators are statistically consistent. Nevertheless, the growing variance at high redshift renders the estimates effectively uninformative, as the posteriors become prior-dominated. For the linear method, we further observe statistically inconsistent estimates even at some low redshifts. These correspond to cases with large values of v_{true}/cz , where the linear approximation breaks down. In contrast, the exact likelihood estimator based on the general method remains consistent in this regime and therefore performs

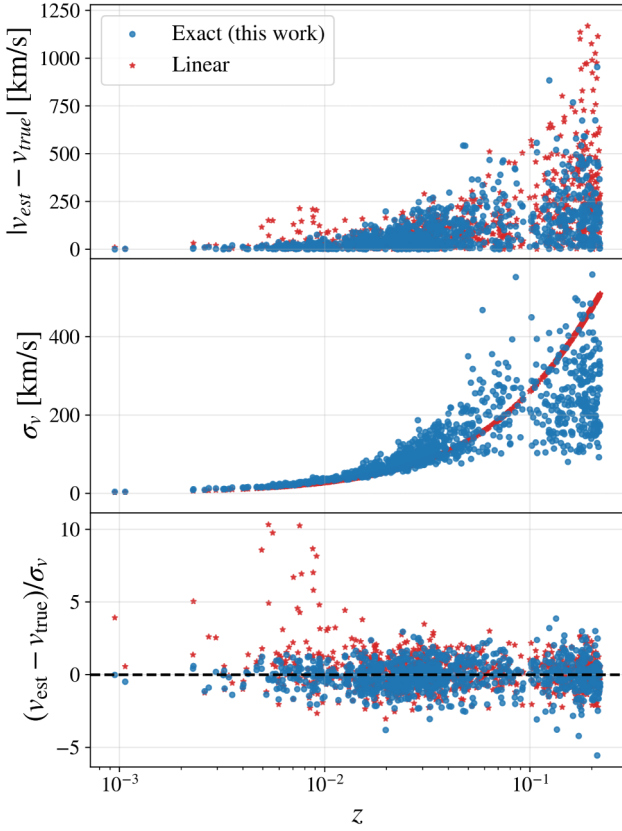


Figure 6. Comparison of the redshift dependence of bias and variance for the linearized estimator (stars) and the exact likelihood based on the errors-in-variables approach (circles). Both the bias and the variance increase with redshift; however, the estimators remain statistically consistent in most cases. An exception occurs when $v_{\text{true}} \approx cz$, where the linear approximation breaks down and the corresponding estimator becomes significantly biased.

better, particularly at low redshifts where the peculiar velocity is comparable to the Hubble flow.

4.1.3 Estimating peculiar velocity correlation

The peculiar velocity correlation function encodes information about the underlying cosmological perturbations. Once measured, it provides a powerful probe of the growth of structure, tests of gravity on cosmological scales, and the nature of dark energy. This approach is particularly promising in light of the large supernova samples expected from upcoming surveys. However, any bias introduced by the linear approximation propagates into the inferred correlation function and can consequently affect the cosmological constraints derived from it. In Fig. 7, we compare the velocity correlation functions obtained using the linear estimator and the exact likelihood-based errors-in-variables approach. However, it is important to note that, given the relatively small number of supernovae considered here, we have not employed a state-of-the-art estimator for the correlation function, and the results should therefore be regarded as illustrative.

A more realistic assessment requires simulations of the peculiar velocity field based on N -body calculations, together with supernova samples constructed to reflect the characteristics of specific surveys. Such an analysis is necessary to quantify the impact on the inferred growth of structure and to assess the po-

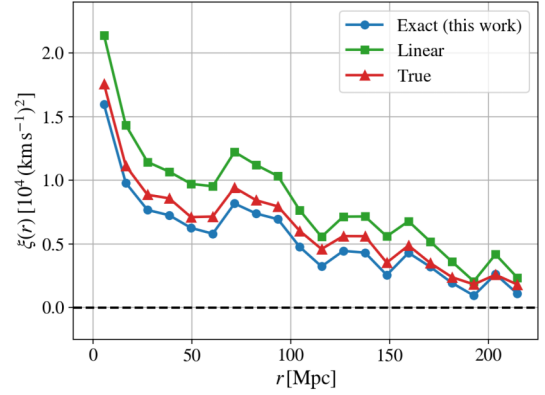


Figure 7. Estimated and true binned two-point correlation function of line-of-sight peculiar velocity for simulated data with $\sigma_m = 0.02$.

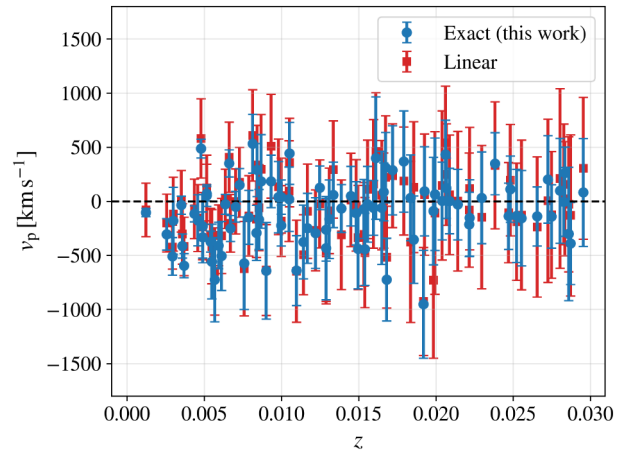


Figure 8. Estimated peculiar velocities along with 1σ error bars obtained from Pantheon+ SNe Ia using both linearized and the exact likelihood based on the errors-in-variables method.

tential for distinguishing between dark energy, modified gravity, and interacting dark-sector scenarios. We leave this detailed investigation for future work.

4.2 Application to Pantheon+ data

In this section, we present the estimates of peculiar velocities derived from the Pantheon+ sample of SNe Ia using both the linear and exact methods. The Pantheon+ compilation includes 1701 SNe Ia with spectroscopic redshifts extending up to $z = 2.26$. Supernova surveys measure the apparent magnitudes and redshifts of SNe Ia; converting luminosity distances into apparent magnitudes requires knowledge of the absolute magnitude, for which we adopt the SH0ES calibration. In the Pantheon+ data set, redshifts are already corrected for the motion of the solar system relative to the CMB rest frame, as well as for the peculiar motions of host galaxies based on measurements using the Tully–Fisher and Fundamental Plane relations. In our analysis, however, we make use of the CMB-frame redshifts, z_{cmb} , since our goal is to estimate these peculiar velocities directly from the supernova data and compare them with existing measurements.

Fig. 8 presents the peculiar velocity estimates from the Pantheon+ sample, along with their associated 1σ uncertainties. At low redshifts ($z < 0.02$), the estimates are consistent with

zero, with a standard deviation of $\sigma_{v_p} \sim 300 \text{ km s}^{-1}$. At higher redshifts, however, the method loses constraining power due to the limited precision of current data and the reduced sensitivity of the likelihood to peculiar velocities. Thus, with present SN Ia data, peculiar velocities can only be reliably estimated at very low redshifts. Looking ahead, the much larger samples expected from LSST and ZTF will provide the statistical precision needed to extend these estimates to higher redshifts, as demonstrated with our simulated data. The key strength of our approach lies in its robustness, as it relies on fewer assumptions and remains straightforward to implement compared to galaxy survey-based methods.

5 CONCLUSION AND DISCUSSION

The peculiar motion of galaxies is an unbiased tracer of matter distribution. This makes it a very promising probe of the growth of structures in the universe. Since peculiar velocities contain information complementary to galaxy clustering, they can be used to investigate the nature of dark energy, test deviations from general relativity on cosmological scales, and distinguish between these two effects. Furthermore, cosmological observations that rely on galaxy redshifts, such as those using standard candles or standard sirens, can be significantly biased if the contribution of galaxy peculiar velocities is not properly accounted for (S. Mukherjee et al. 2021; H. Nimmonkar & S. Mukherjee 2023). For instance, neglecting peculiar motion of host galaxies can bias our inference of cosmological parameters from the SN Ia data by a few percent (L. Hui & P. B. Greene 2006; J. D. Neill, M. J. Hudson & A. J. Conley 2007; T. M. Davis et al. 2011; U. Upadhyay et al. 2026). This has led to numerous approaches for reconstructing peculiar velocities from observations, encompassing both traditional distance-indicator methods and modern Bayesian or machine-learning-based techniques (E. Branchini, A. Eldar & A. Nusser 2002; R. Mohayaee, R. B. Tully & U. Frisch 2004; Z. Wu et al. 2023; B. R. Guachalla et al. 2024; K. Said & others 2025). However, these methods rely on empirical relations, such as the Tully–Fisher and Fundamental Plane relations, which introduce additional assumptions. Combined with the assumptions inherent to galaxy surveys, this makes the inference of peculiar velocities more methodology-dependent and potentially biased. SNe Ia provide another way to estimate peculiar velocities by propagating the Hubble residuals to redshift using standard error propagation. However, the standard Hubble residual-based estimators (B. Carreres et al. 2023) rely on a locally linear approximation to the magnitude–redshift relation, which is valid only when the peculiar velocity is small compared to the Hubble flow ($v_p \ll cz$). In this work, we present an alternative Bayesian method for estimating the line-of-sight peculiar velocities of SN Ia host galaxies from magnitude–redshift data without assuming local linearity. The method relies solely on the cosmological model and the precision of supernova observations, making it applicable even at low redshifts where $v_p \approx cz$ and the linear approximation breaks down.

Our approach to estimating peculiar velocities from supernova data is based on a method developed for fitting non-linear models with errors in both dependent and independent variables (U. Upadhyay et al. 2026). In this framework, we fit the magnitude–redshift relation to SN Ia data, treating the apparent magnitudes and redshifts as the dependent and independent variables, respectively. Following U. Upadhyay et al. (2026), and as described in Section 3, we treat the redshifts as free param-

eters and infer their posterior distributions. The difference between the observed and inferred redshifts is then primarily attributed to peculiar motion, assuming that other contributions, such as gravitational lensing and spectroscopic redshift uncertainties, are subdominant in the low-redshift regime where the method is most effective (C. Bonvin et al. 2006; T. M. Davis et al. 2011; A. Carr et al. 2022), which we convert into estimates of peculiar velocities. To obtain the posterior distribution, we perform MCMC sampling over all redshift parameters. By exploring the full parameter space, the method naturally enables us to capture correlations in the velocity field. We validate the approach using simulated SNe Ia with a peculiar velocity field generated from a fiducial Λ CDM cosmology. For comparison, we also derive an analytic expression for the standard peculiar velocity estimator based on the locally linear approximation of the magnitude–redshift relation and the assumption of Gaussian redshift errors, and extend it to include the supernova magnitude covariance.

The main results of this work are shown in Figs 1 and 4. We find that treating redshifts as free parameters allows peculiar velocities to be estimated using the exact likelihood, with estimates that remain statistically consistent even when the peculiar velocity is comparable to the Hubble flow, as illustrated in Fig. 4. This is particularly relevant at low redshifts. In addition, the method provides the full posterior distribution, which contains additional information about velocity correlations. However, the standard method based on the linear approximation remains computationally more efficient and therefore performs better within the regime where the approximation is valid.

Overall, our method complements existing approaches to estimating peculiar velocities and can be naturally extended to other distance indicators, such as standard sirens. Important directions for future work include studying the applicability of this approach to the larger samples of SNe Ia expected from upcoming surveys and incorporating more realistic peculiar-velocity fields from N -body simulations, particularly on non-linear scales, which could help test various models of dark energy and gravity on cosmological scales, which we leave for future work.

ACKNOWLEDGEMENTS

UU acknowledges Yashi Tiwari for her valuable comments and suggestions throughout this project. We also acknowledge the use of the HPC cluster at the Raman Research Institute, Bangalore, India.

DATA AVAILABILITY

The Pantheon+ sample of Type Ia supernova data analysed in this work is publicly available (D. Scolnic et al. (2022)).

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APPENDIX A: DERIVATION OF THE LINEARIZED ESTIMATOR

Here, we provide a detailed derivation of the linearized Bayesian estimator outlined earlier in Section 3.2. Again, starting with the posterior distribution in equation (13) without the regularization factor, substituting χ_m^2 from equation (14), and assuming that the

redshift errors, ϵ_{z_i} , follow a Gaussian distribution, we can write

$$P(\theta, z^* | \mathcal{D}; I) \propto \exp \left[-\frac{1}{2} \mathbf{m}^T \mathbf{C}_m^{-1} \mathbf{m} \right] \cdot \exp \left[-\frac{1}{2} \mathbf{z}^T \mathbf{C}_z^{-1} \mathbf{z} \right], \quad (\text{A1})$$

Taking the logarithm, we can write

$$\ln P(\theta, z^* | \mathcal{D}; I) = -\frac{1}{2} \left[\mathbf{m}^T \mathbf{C}_m^{-1} \mathbf{m} + \mathbf{z}^T \mathbf{C}_z^{-1} \mathbf{z} \right], \quad (\text{A2})$$

where the matrices $\mathbf{m}$ and $\mathbf{z}$ represent the supernova magnitude and redshift residual vectors, respectively, and $\mathbf{C}_m$ and $\mathbf{C}_z$ are the associated covariance matrices, defined as:

$$\mathbf{m} = \begin{pmatrix} m_1 - m(z_1^*) \\ \vdots \\ m_N - m(z_N^*) \end{pmatrix}, \text{ and } \mathbf{C}_m = \begin{pmatrix} \sigma_{m_1}^2 & \cdots & \sigma_{m_1 m_N} \\ \vdots & \ddots & \vdots \\ \sigma_{m_N m_1} & \cdots & \sigma_{m_N}^2 \end{pmatrix}. \quad (\text{A3})$$

$$\mathbf{z} = \begin{pmatrix} z_1 - z_1^* \\ \vdots \\ z_N - z_N^* \end{pmatrix}, \text{ and } \mathbf{C}_z = \begin{pmatrix} \sigma_{z_1}^2 & \cdots & \sigma_{z_1 z_N} \\ \vdots & \ddots & \vdots \\ \sigma_{z_N z_1} & \cdots & \sigma_{z_N}^2 \end{pmatrix}. \quad (\text{A4})$$

Taylor expanding the magnitude–redshift relation about the observed redshift, and keeping terms up to linear order in z , we get

$$m(z^*) = m(z) + m'(z)(z^* - z). \quad (\text{A5})$$

Since we attribute the difference between true redshift and observed redshift to the peculiar velocity, the linear approximation breaks when $v_p \approx cz^*$ and higher order terms begin to contribute significantly. Under this approximation, the magnitude residual vector takes the form,

$$\mathbf{m} = \begin{pmatrix} m_1 - m(z_1) - m'(z_1)(z_1^* - z_1) \\ \vdots \\ m_N - m(z_N) - m'(z_N)(z_N^* - z_N) \end{pmatrix} \quad (\text{A6})$$

For subsequent calculations, it is convenient to decompose the residual vectors into the sum of two components: one that depends solely on the observed quantities, and another that contains the true redshift parameters, z^* , as follows. Let $\mathbf{m} = \mathbf{m}^0 + \mathbf{m}^*$, and $\mathbf{z} = \mathbf{z}^0 + \mathbf{z}^*$, with

$$\mathbf{m}^0 = \begin{pmatrix} m_1 - m(z_1) + m'(z_1)z_1 \\ \vdots \\ m_N - m(z_N) + m'(z_N)z_N \end{pmatrix}_{N \times 1}, \mathbf{m}^* = -\begin{pmatrix} m'(z_1)z_1^* \\ \vdots \\ m'(z_N)z_N^* \end{pmatrix}_{N \times 1} \quad (\text{A7})$$

and

$$\mathbf{z}^0 = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \mathbf{z}^* = -\begin{pmatrix} z_1^* \\ \vdots \\ z_N^* \end{pmatrix} \quad (\text{A8})$$

With this decomposition of the vectors, the log-posterior can be written as

$$\begin{aligned} \ln P &= -\frac{1}{2} \left[(\mathbf{m}^0 + \mathbf{m}^*)^T \mathbf{C}_m^{-1} (\mathbf{m}^0 + \mathbf{m}^*) + (\mathbf{z}^0 + \mathbf{z}^*)^T \mathbf{C}_z^{-1} (\mathbf{z}^0 + \mathbf{z}^*) \right] \quad (\text{A9}) \\ &= -\frac{1}{2} \left[\mathbf{m}^{0T} \mathbf{C}_m^{-1} \mathbf{m}^0 + \mathbf{m}^{0T} \mathbf{C}_m^{-1} \mathbf{m}^* + \mathbf{m}^{*T} \mathbf{C}_m^{-1} \mathbf{m}^0 + \mathbf{m}^{*T} \mathbf{C}_m^{-1} \mathbf{m}^* \right. \\ &\quad \left. + \mathbf{z}^{0T} \mathbf{C}_z^{-1} \mathbf{z}^0 + \mathbf{z}^{0T} \mathbf{C}_z^{-1} \mathbf{z}^* + \mathbf{z}^{*T} \mathbf{C}_z^{-1} \mathbf{z}^0 + \mathbf{z}^{*T} \mathbf{C}_z^{-1} \mathbf{z}^* \right] \quad (\text{A10}) \end{aligned}$$

Further, $\mathbf{m}^*$ can be written as $\mathbf{m}^* = \mathbf{M}' \mathbf{z}^*$, where $\mathbf{M}'$ is an $N \times N$ square matrix given by $\mathbf{M}' = \text{diag}[m'_1, \dots, m'_N]$. Substituting this and completing the square in $\mathbf{z}^*$ (see C. M. Bishop 2006 for

quadratic completion of matrix expressions), we get

$$\begin{aligned} \ln P = & -\frac{1}{2} \left[\mathbf{z}^* + (\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}')^{-1} (\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{m}^0 + \mathbf{C}_z^{-1} \mathbf{z}^0) \right]^T \\ & \times (\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}') \\ & \times \left[\mathbf{z}^* + (\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}')^{-1} (\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{m}^0 + \mathbf{C}_z^{-1} \mathbf{z}^0) \right] \\ & + \text{constant}, \end{aligned} \quad (\text{A11})$$

where the constant denotes the remaining terms that are independent of $\mathbf{z}^*$. Thus, we get a posterior distribution which is a Gaussian in $\mathbf{z}^*$ centred at $\bar{\mathbf{z}}^*$ with the associated covariance Σ_{z^*} , given by

$$\bar{\mathbf{z}}^* = - \left[\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}' \right]^{-1} \left[\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{m}^0 + \mathbf{C}_z^{-1} \mathbf{z}^0 \right], \quad (\text{A12})$$

and

$$\Sigma_{z^*} = \left[\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}' \right]^{-1}. \quad (\text{A13})$$

It is instructive to examine various limiting cases of the above expressions in order to better understand the behaviour of the general estimator.

Case 1: High z

Due to the logarithmic form of the magnitude–redshift relation, we have $m'(z) \rightarrow 0$ as $z \rightarrow \infty$. In this limit, $\bar{\mathbf{z}}^* = \mathbf{z}^0$ and $\Sigma_{z^*} = \mathbf{C}_z$. This implies that the predicted apparent magnitude becomes insensitive to redshift uncertainties, and the estimator returns zero redshift correction as the best-fitting value, with an uncertainty entirely determined by the observational redshift covariance. Therefore, in the limit $z \rightarrow \infty$, the data provide no additional information beyond the redshift prior, although the estimate remains statistically consistent with the true value.

Case 2: Intermediate z

At intermediate redshifts, the behaviour of the estimator reflects an interplay between the precision of the supernova magnitude measurements, the sensitivity of the likelihood to redshift perturbations, the magnitude of the peculiar velocity, and the redshift prior. In the regime where $\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}' \gg \mathbf{C}_z^{-1}$, the magnitude data dominate over the prior, and the posterior uncertainty is determined primarily by the supernova magnitude covariance, i.e. $\Sigma_{z^*} \simeq [\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}']^{-1}$. In this limit, the inferred true redshift is driven mainly by consistency with the magnitude–redshift relation, and the estimator effectively extracts peculiar velocity information from the data.

Case 3: Low z

At very low redshifts, the data term dominates the prior, since it includes the derivative of the magnitude. As $z \rightarrow 0$, $m'(z) \rightarrow \infty$, and we obtain $\bar{\mathbf{z}}^* = \mathbf{z}^0$, and $\Sigma_{z^*} = \mathbf{O}$. This implies that, in the low-redshift limit, we obtain very tight constraints, primarily determined by the precision of supernova magnitude measurements. However, in this regime, the linear approximation begins to break down, as higher-order terms in the Taylor expansion of the magnitude–redshift relation become significant. As a result, the linear estimator can yield biased estimates.

While the general method exhibits the same qualitative behaviour of constraints at different redshifts, it remains free from this bias, since it does not rely on a linear expansion of the magnitude–redshift relation.

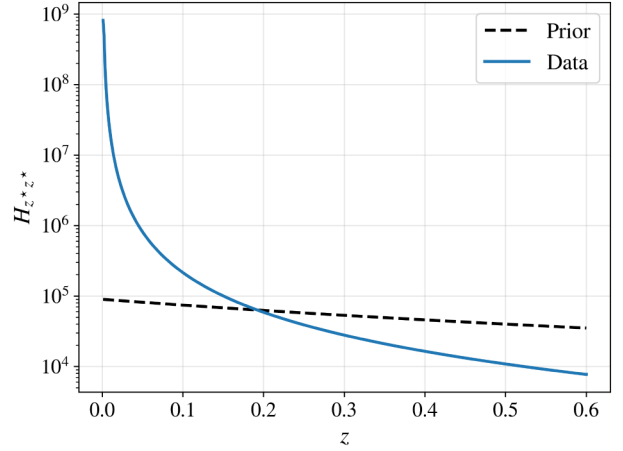


Figure B1. Comparison of the prior and data contributions to the diagonal elements of the Hessian.

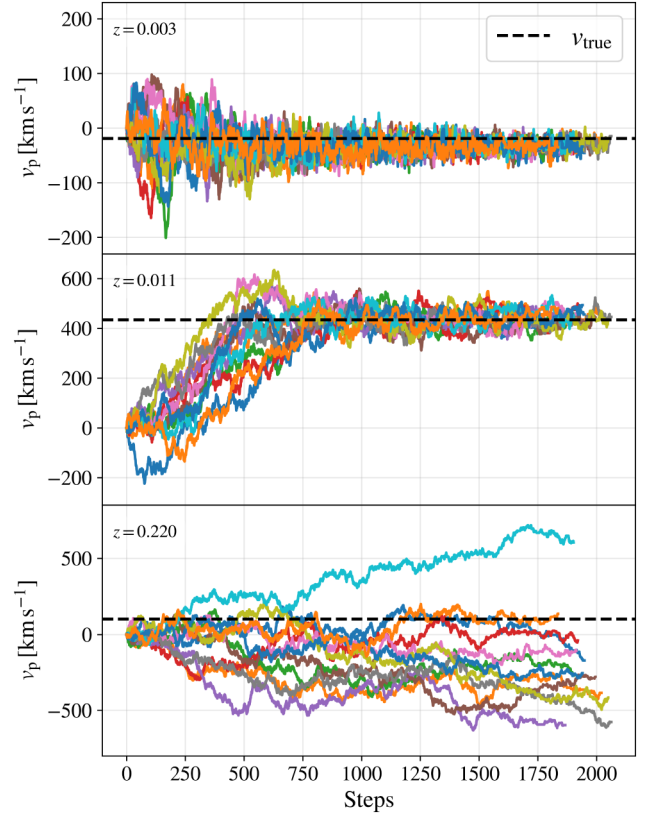


Figure B2. Trace plots illustrating the redshift dependence of MCMC convergence for the simulated data with $\sigma_m = 0.02$.

APPENDIX B: TRACE PLOTS AND CONVERGENCE ISSUES

The issue of convergence of redshift parameters (or equivalently, peculiar velocities) discussed earlier can be easily understood with the curvature of the likelihood or the Hessian. From equation (A11), the Hessian can be easily read off as the negative of the precision matrix:

$$\mathbf{H} \equiv -\Sigma_{z^*}^{-1} = - \left[\mathbf{C}_z^{-1} + \mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}' \right]. \quad (\text{B1})$$

It describes the total precision of the estimate, which is the sum of the prior contribution and the magnitude precision. It is more informative to compare these individual contributions to the total precision. For the simple case where the matrices $\mathbf{C}_z$ and $\mathbf{C}_m$ are diagonal, this comparison can be made by examining the diagonal elements of the Hessian,

$$H_{z^*z^*} = - \left[C_{z^*z^*} + (\mathbf{M}'^T \mathbf{C}_m^{-1} \mathbf{M}')_{z^*z^*} \right]. \quad (\text{B2})$$

The first term represents the contribution from the prior, while the second term corresponds to the contribution from the data. The redshift dependence of these two terms is shown in Fig. B1, where it is evident that the data dominate at low redshifts, while

the prior becomes increasingly dominant at higher redshifts. The same qualitative behaviour is reflected in the trace plots of the

MCMC chains for the general method shown in Fig. B2. At low redshifts, where the data dominate, the peculiar velocities are well constrained, and the chains rapidly reach a stationary behaviour. As the redshift increases, convergence deteriorates, and at high redshifts the chains wander across the full range permitted by the prior, effectively yielding no additional constraints from the data.

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