

CHAPTER III

POISEUILLE FLOW OF NEMATICS

Introduction

The viscosity of a nematic in Poiseuille flow (i.e., flow through a capillary tube) was studied experimentally by Peter and Peters (1955). They measured the apparent viscosity η (the viscosity calculated by using the well known Poiseuille formula) as a function of shear rate for para-azoxyanisole (PAA) at different temperatures. At any given temperature, in the nematic phase η decreased by as much as 50% as the shear increased approaching a constant value at high shear rates. In the isotropic phase η did not change with shear rate. Similar measurements were carried out by Porter and Johnson (1962). However it should be stressed that in all these experiments special care was not taken to align the molecules at the tube wall. This is an important factor because ^a surface can influence the orientation of molecules in its vicinity depending upon the type of chemical treatment or rubbing given to it (Zocher 1933, Chatelain 1943). To study the effect of wall

orientation On η Fisher and Fredrickson (1969) used a capillary viscometer and took special steps to treat the walls to effect a perpendicular or parallel director orientation. For perpendicular alignment at the wall (achieved by treating the surface with a mixture of sulphuric acid and dichromate) a plot of η as a function of $4Q/\pi R$ (where Q is the rate of flow and R the tube radius) resulted in a universal curve in that it was true for all values of tube radius. For parallel alignment at the walls (obtained by passing a hair through the tube several times) they were unable to obtain a uniform boundary alignment and they were unable to get a universal curve.

The first analytical theory for Poiseuille flow based on the Ericksen-Leslie theory was given by Atkin (1970) for the case of zero magnetic field. By using scaling laws he was able to show that η is a function of $4Q/\pi R$ alone. Tseng et al. (1972) solved Atkin's equations numerically and calculated η as a function of $4Q/\pi R$ using measured values of the viscosity and elastic constants of PAA. The theoretical results agree well with the experimental data of Fisher and Fredrickson. However Tseng et al. did not investigate

the effect of an axial magnetic field or of changing boundary condition of director orientation on η . In this chapter these problems have been considered. Since nematics have a positive diamagnetic susceptibility anisotropy ($\Delta\chi$) a magnetic field tends to align the director parallel to the field direction. It follows from the theoretical calculations that an axial magnetic field always brings down the value of η at practically any shear rate. Also any alignment, other than the homeotropic one at the walls brings down the value of η . This shows that η is strongly influenced by weak anchoring of the director at the walls.

Theory

The capillary tube is assumed to be horizontal, with its axis parallel to the z direction. The fluid is assumed to flow in the $+z$ direction. We seek steady state solutions for the director and velocity fields in the form

$$\begin{aligned} n_x &= \sin \theta(r), & n_y &= 0, & n_z &= \cos \theta(r) \\ v_x &= 0, & v_y &= 0, & v_z &= v(r) \end{aligned} \tag{1}$$

The applied magnetic field is assumed to be

$$H_r = 0 = H_\psi ; \quad H_z = H$$

Ignoring body forces due to gravity equations (25) of Chapter I can now be written as

$$\frac{d\pi_{rr}}{dr} + \frac{\pi_{rr} - \pi_{\psi\psi}}{r} + g_r = 0 \quad (a)$$

$$\frac{d\pi_{rz}}{dr} + \frac{\pi_{rz}}{r} + g_z = 0 \quad (b)$$

$$\frac{d\hat{t}_{rr}}{dr} + \frac{1}{r}(t_{rr} - t_{\psi\psi}) = \frac{\partial p}{\partial r} \quad (c) \quad (2)$$

$$\frac{dt_{rz}}{dr} + \frac{t_{rz}}{r} = \frac{\partial p}{\partial z} \quad (d)$$

$$\frac{\partial p}{\partial \psi} = 0 \quad (e)$$

The components of π_{ji} , g_k , G_i and t_{pq} which are relevant to this problem are

$$\pi_{rr} = c_\theta \theta_r [K_{11} + K_{33} s_\theta^2] + \frac{K_{11} s_\theta}{r} ,$$

$$\pi_{\psi\psi} = K_{11} [c_\theta \theta_r + s_\theta / r] ,$$

$$\pi_{rz} = -K_{33} s_\theta^3 \theta_r ,$$

$$g_r = -K_{33} s_\theta \theta_r^2 + \frac{(\lambda_1 + \lambda_2)}{2} \frac{dv}{dr} c_\theta + \gamma s_\theta ,$$

$$g_\theta = \frac{(\lambda_2 - \lambda_1)}{2} \frac{dv}{dr} s_\theta + \gamma c_\theta ,$$

$$g_z = (\Delta\chi) H^2 c_\theta ,$$

$$t_{rz} = g(\theta) \frac{dv}{dr} ,$$

where

$$\theta_r = \frac{d\theta}{dr} , \quad s_\theta = \sin \theta , \quad c_\theta = \cos \theta , \text{ and}$$

$$2g(\theta) = 2\mu_1 s_\theta^2 c_\theta^2 + \mu_4 + (\mu_5 - \mu_2) s_\theta^2 + (\mu_3 + \mu_6) c_\theta^2 .$$

Equations 2(a) and 2(b) can be written as

$$\theta_{rr} [K_{11} c_\theta + K_{33} s_\theta^2 c_\theta] + \theta_r^2 [2K_{33} s_\theta c_\theta^2 - K_{33} s_\theta - K_{33} s_\theta^3 - K_{11} s_\theta]$$

$$- \frac{\theta_r}{r} [K_{11} c_\theta + K_{33} c_\theta s_\theta^2] - \frac{K_{11} s_\theta}{r^2} + \frac{(\lambda_1 + \lambda_2)}{2} c_\theta \frac{dv}{dr} + \gamma s_\theta = 0$$

$$K_{33} s_\theta^3 \theta_{rr} + 3K_{33} s_\theta^2 c_\theta \theta_r^2 + \frac{K_{33} s_\theta^3 \theta_r}{r} + \frac{(\lambda_1 - \lambda_2)}{2} s_\theta \frac{dv}{dr} \dots (3)$$

$$- \gamma c_\theta = 0$$

An elimination of γ from the equations (3) yields

$$2f(\theta)\theta_{rr} + \frac{df}{d\theta}\theta_r^2 + \frac{2f(\theta)}{r}\theta_r - \frac{K_{11}\sin 2\theta}{r^2} + (\lambda_1 + \lambda_2 \cos 2\theta)\frac{dv}{dr} - (\Delta\chi)H^2 \sin 2\theta = 0 \quad (4)$$

where

$$f(\theta) = K_{11}\theta^2 + K_{33}S_\theta^2.$$

Also (4) integrate to yield

$$g(\theta)\frac{dv}{dr} = \frac{b}{r} - \frac{ar}{2}$$

*en $-a = \frac{\partial p}{\partial z}$ is the pressure gradient which A^* assumed to be a (negative) constant and b is an arbitrary constant which has to be equated to zero so that the velocity field might not be singular at any point in a section of the flow. Hence

$$\frac{dv}{dr} = -\frac{ar}{2g(\theta)} \quad (5)$$

Substituting in (4) one gets

$$2f(\theta)\theta_{rr} + \theta_r^2 \frac{df}{d\theta} + \frac{2f(\theta)\theta_r}{r} - \frac{K_{11}\sin 2\theta}{r^2} - \frac{a(\lambda_1 + \lambda_2 \cos 2\theta)r}{2g(\theta)} - (\Delta\chi)H^2 \sin 2\theta = 0 \quad (6)$$

On solving (6) with suitable boundary conditions for θ and obtaining θ or a function of r (5) can be solved with the boundary condition $v(R) = 0$ yielding v as a function of r . This enables a determination of Q the volume of the nematic flowing out of the tube

per second $Q = 2\pi \int_0^R vr \, dr$ and subsequently the apparent viscosity η [$\eta = \pi a R^4 / 8Q$].

For purposes of computation it is convenient to transform to a new variable ξ defined as

$$\xi = a^{1/3} r$$

Equations (5) and (6) now become

$$\begin{aligned} \frac{d^2\theta}{d\xi^2} + \frac{1}{2f} \frac{df}{d\theta} \left(\frac{d\theta}{d\xi} \right)^2 + \frac{1}{\xi} \frac{d\theta}{d\xi} - \frac{K_{11} \sin 2\theta}{2\xi^2 f(\theta)} \\ - \frac{(\Delta\chi)H^2 \sin 2\theta}{2f(\theta) a^{2/3}} - \frac{(\lambda_1 + \lambda_2 \cos 2\theta)}{4f(\theta)g(\theta)} = 0 \end{aligned} \quad (7)$$

$$\frac{dv}{d\xi} = - \frac{a^{1/3} \xi}{2g(\theta)} \quad (8)$$

θ is singular at $\xi = 0$. But it appears to be reasonable to assume that $\theta(0) = \theta_0$. The other boundary condition for θ is that at the walls $\theta(R) = \theta_b = -\pi/2$. The boundary value problem can be solved quite simply as an initial value problem. For any value of $\frac{d\theta}{d\xi}(\xi=0)$ (7) is integrated by the fourth order Runge-Kutta method (see Appendix 1 of this Chapter) till θ attains the value θ_b at $\xi = \xi_m$ say. Then $R = \xi_m a^{-1/3}$. Equation (8) is simultaneously integrated to yield Q :

$$Q = -2\pi a^{-1/3} \left[\int_0^{\xi_m} w(\xi) d\xi - \frac{1}{2} w(\xi_m) \xi_m^2 \right] \quad (9)$$

where

$$w(\xi) = - \int_0^{\xi} \frac{d\xi}{2g(\theta)} \quad \text{and} \quad w(0) = 0.$$

The velocity profile is $v(\xi) = a^{1/3} [w(\xi) - w(\xi_m)]$.

η is calculated from the relation

$$\eta = \frac{\pi a R^4}{8Q} = \frac{\xi_m^3}{8(Q/\pi R)}$$

Results

In discussing the theoretical results it is often convenient to invoke certain scaling laws (Eriksen 1969) which enable one to present computations in the form of a universal curve that is independent of the dimensions of the viscometer, etc. For the particular case of Poiseuille flow (In the absence of a field) the scaling laws were first derived by Atkin (1970) (see Appendix 2 of this Chapter); the plot of η against $4Q/\pi R$ should be a universal curve. For the field free case Fisher and Fredrickson confirmed that the experimental data did indeed fall on such a universal curve for different tube radii. It should however be emphasized that these scaling laws do not hold good in the presence of externally applied perturbations like electric or magnetic fields. As the calculations deal with various values of the magnetic field it will be more meaningful to present the results for one given tube radius.

To work out the consequences of the theory calculations on a typical nematic like PAA have been presented. η has been calculated for two values of $(\Delta\chi)H^2$, 24.2 and 1.23 cgs. (respectively corresponding

to field strengths of 13340 and 3000 gauss approximately) with $\theta_0 = -\pi/2$ corresponding to homeotropic director alignment. The tube radius was taken to be 55.5μ . The values of μ_1 , K_{11} and $\Delta\phi$ are the same as those used in Chapter II.

Figure 1 illustrates the dependence of η on $4Q/\pi R$. An axial magnetic field brings down the value of η at any shear rate; larger the field greater the decrease in η . If there is no anchoring at the walls and there is non-homeotropic alignment ($\theta_0 \neq -\pi/2$) η gets considerably decreased. The dashed curve in figure 1 corresponds to $\theta_0 = -\pi/4$, for the field free case and shows that at every shear rate η is less than its value for the homeotropic case. This is in qualitative agreement with the observations of Fisher and Fredrickson who found not only a decrease in η but also a non-universality of the curve of η re. $4Q/\pi R$ when they attempted to effect parallel director alignment at the tube wall.

At very high fields with $\theta_0 = -\pi/2$ the director is oriented along the field over a major section of flow excepting for a boundary layer near the tube wall where θ changes from its zero value to $-\pi/2$; the thickness of the boundary layer decreases with increasing magnitude

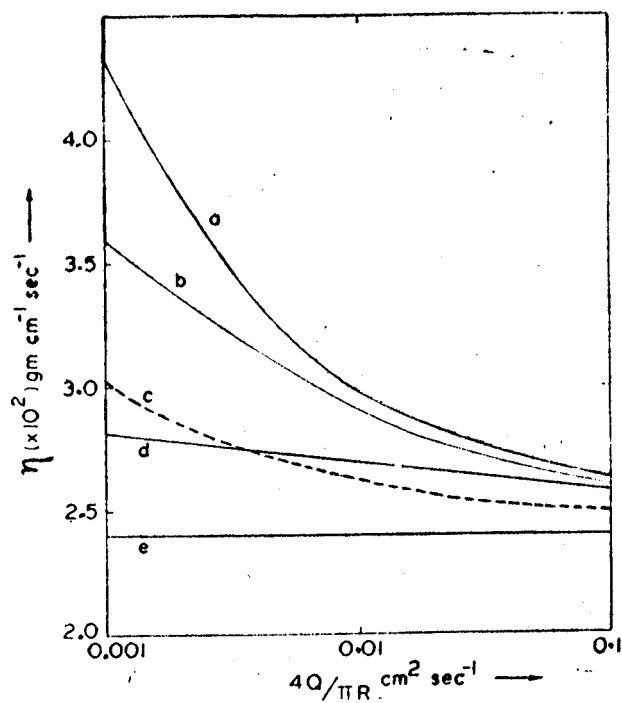


Figure 1. Apparent viscosity η of PAA vs. $4Q/\pi R$ for a tube radius of $R = 55.5 \mu$. The values of $(\Delta\chi)H^2$ and θ_0 are (a) 0, $-\frac{1}{2}\pi$; (b) 1.23, $-\frac{1}{2}\pi$; (c) 0, $-\frac{1}{4}\pi$; (d) 24.2, $-\frac{1}{2}\pi$; (e) ∞ , $-\frac{1}{2}\pi$ or 0, 0.

of the field. In the absence of a field if $\Theta_0 = 0$ the director is oriented along the flow throughout the section of the flow. In both these cases the nematic behaves as if it were an ordinary fluid with an effective viscosity given by $\eta = (\mu_3 + \mu_4 + \mu_6)/2$ ($= 0.024$ poise for PAA).

In figures 2(A) and 2(B) orientation profiles have been given for different shear rates for $(\Delta\chi)H^2 = 24.2$ and 1.23 cgs. For these cases at a given shear rate the transition of the director orientation from $\Theta = 0$ to $\Theta = -\pi/2$ becomes more abrupt with an increase in the field. The same takes place on increasing shear rate at a given field strength. This is the decrease in the thickness of the boundary layer referred to earlier. The velocity profile (figure 3) which is non-parabolic at low fields (or shear rate) tends to become truly parabolic as the field or shear rate is increased.

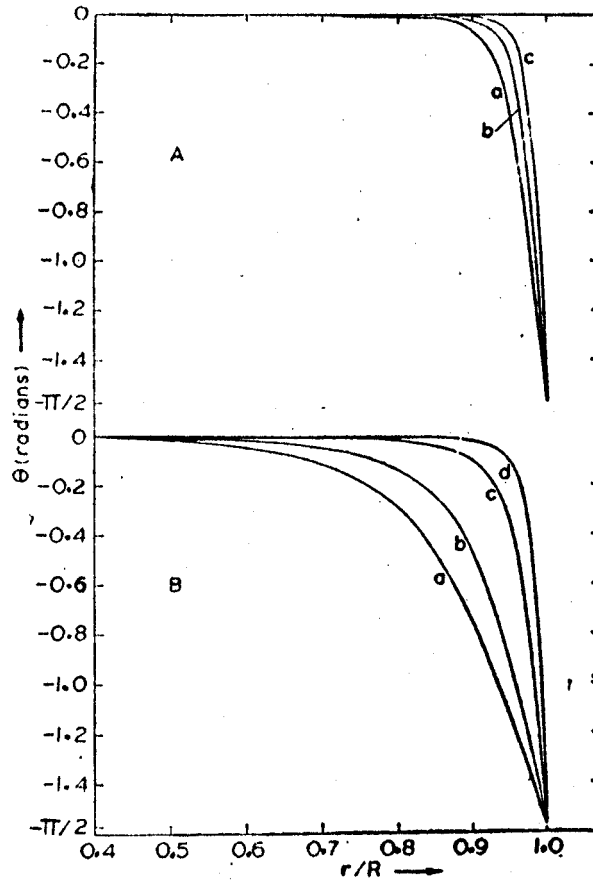


Figure 2. Orientation profiles at different shear rates for $(\Delta\chi)H^2 = 24.2$ (A) and for $(\Delta\chi)H^2 = 1.23$ (B), with homeotropic wall alignment. The values of $4Q/\pi R$ in (A) are (a) 0.003045; (b) 0.03354; (c) 0.1345 and in (B) (a) 0.001245; (b) 0.004167, (c) 0.032438; and (d) 0.1372.

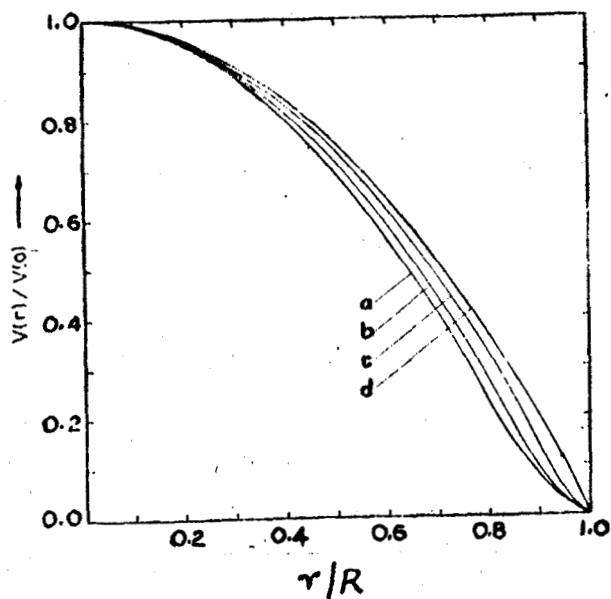


Figure 3. Velocity profiles at different values of $4Q/\pi R$ and same field (curves a and b) and nearly same value of $4Q/\pi R$ and different fields (curves b and c). Values of $(\Delta x)H^2$ and $4Q/\pi R$ are (a) 1.23, 0.001245; (b) 1.23, 0.004167; (c) 24.2, 0.003045; and (d) $(\Delta x)H^2 = \infty$ or $4Q/\pi R = \infty$ (truly parabolic).

Appendix 1

Here a brief description of the fourth order Runge-Kutta method for solving second order ordinary differential equations is given (for greater details see Collatz 1960, Scheid 1968). This method is very handy when an initial value problem is to be solved. That is, if the values of the function θ and its derivative $d\theta/d\xi$ are known at a point $\xi = a$ it is possible to work out the values of θ and $d\theta/d\xi$ at a neighbouring point $\xi = a + h$ such that the value of θ is accurate upto the term h^4 in the Taylor series (hence the name 'fourth order method') h is called the step; in general smaller the value of h greater the accuracy.

To apply this method the differential equation (6) is written as

$$\begin{aligned} \frac{d^2\theta}{d\xi^2} = & -\frac{1}{2f} \frac{df}{d\theta} \left(\frac{d\theta}{d\xi}\right)^2 - \frac{1}{\xi} \frac{d\theta}{d\xi} + \frac{K_{11} \sin 2\theta}{2\xi^2 f(\theta)} \\ & + \frac{(\Delta\chi)H^2 \sin \theta \cos \theta}{f(\theta) a^{2/3}} + \frac{(\lambda_1 + \lambda_2 \cos 2\theta)}{4f(\theta) g(\theta)} \end{aligned}$$

Thus $d^2\theta/d\xi^2$ is known as a function of $d\theta/d\xi$, θ and ξ . We can write

$$\frac{d^2\theta}{d\xi^2} = F\left(\frac{d\theta}{d\xi}, \theta, \xi\right)$$

Suppose we know θ and $d\theta/d\xi$ at a point $\xi = \xi_0$ a

$\theta(\xi_0) = \theta_0$; $(d\theta/d\xi)_{\xi=\xi_0} = \theta'_0$ Then it is possible to write the value of θ at $\xi = \xi_0 + h$ (where h is small enough) such that this value of θ is accurate upto the term h^4 in the Taylor series. The value of $d\theta/d\xi$ at $\xi_0 + h$ can also be written:

$$\text{Let } k_1 = \frac{h^2}{2} F(\theta'_0, \theta_0, \xi_0) ,$$

$$k_2 = \frac{h^2}{2} F\left(\theta'_0 + \frac{k_1}{h}, \theta_0 + \frac{h}{2}\theta'_0 + \frac{k_1}{4}, \xi_0 + \frac{h}{2}\right) ,$$

$$k_3 = \frac{h^2}{2} F\left(\theta'_0 + \frac{k_2}{h}, \theta_0 + \frac{h}{2}\theta'_0 + \frac{k_1}{4}, \xi_0 + \frac{h}{2}\right) \text{ and}$$

$$k_4 = \frac{h^2}{2} F\left(\theta'_0 + \frac{2k_3}{h}, \theta_0 + h\theta'_0 + k_3, \xi_0 + h\right)$$

be calculated in the order given. Then the values of θ and $d\theta/d\xi$ at the point $\xi_0 + h$ can be written as

$$\theta(\xi_0 + h) = \theta_0 + h\theta'_0 + \frac{1}{3}(k_1 + k_2 + k_3)$$

$$\frac{d\theta}{d\xi}(\xi_0 + h) = \theta'_0 + \frac{1}{3h}[k_1 + k_4 + 2(k_2 + k_3)] .$$

In a similar way using these values θ and $d\theta/d\xi$ can

be calculated at the points $\xi_0 + 2h, \xi_0 + 3h \dots$
 $\xi_0 + Nh$ where N is called the number of steps. In
 our calculations N varied between 5000 and 20000.
 Along with eqn. (6), eqn. (8) was also integrated but
 by using the Simpson's Rule, viz.,

$$\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{3}[f_0 + 4f_1 + 2f_2 + \dots + 2f_{n-2} + 4f_{n-1} + f_n]$$

where

$$h = \frac{x_n - x_0}{n}, \quad f_i = f(x_0 + ih)$$

and n is an even integer. Thus at the end of the step
 N , Q was obtained directly to enable a calculation of η .
 The program was written in Fortran IV (double precision)
 and run on the IBM 360/44. The total computer time
 used was about 13 hours.

Appendix 2

The method of applying scaling laws to the field free Poiseuille flow of a nematic is outlined below.

New variables r' , z' , t' are defined such that $r = Rr'$; $z = Rz'$; $t = kt'$ (k is a positive constant). Quantities after transformation bear a zero superscript.

$$v = \frac{R}{k} v^0 \quad \dot{n} = \frac{1}{k} \dot{n}^0; \quad \text{grad } v = \frac{1}{k} \text{grad } v^0;$$

$$\text{grad } n = \frac{1}{R} \text{grad } n^0; \quad N = \frac{1}{k} N^0$$

If $k = R^2$, $p = \frac{1}{R^2} p^0$ and $\gamma = \frac{1}{R^2} \gamma^0$ one gets

$$t_{ij} = \frac{1}{R^2} t_{ij}^0; \quad \pi_{ji} = \frac{1}{R} \pi_{ji}^0; \quad \varepsilon_1 = \frac{1}{R^2} \varepsilon_1^0; \quad \frac{dp}{dz} = \frac{1}{R^3} \frac{dp^0}{dz'}$$

The volumetric flow rate is given by

$$Q = 2\pi \int_0^R v_z(r) r dr$$

$$Q^0 = 2\pi \int_0^1 v_z^0(r') r' dr' = \frac{2\pi}{R} \int_0^R v_z(r) r dr = Q/R$$

The equations 2b, 24 and 2% are three differential equations for the two director components and for the

velocity v_z . Using the above transformation the corresponding system of equations for v^0 and n^0 are obtained which has the same form as the original system. If a unique solution exists for this new system of equations Q^0 and dp^0/dz' are uniquely related so that

$$Q^0 = \mathcal{F}\left(\frac{dp^0}{dz'}\right)$$

$$\text{i.e., } Q = R \mathcal{F}\left(R^3 \frac{dp}{dz}\right)$$

or Q/R is a function of $R^3 \frac{dp}{dz}$. Conversely $R^3 \frac{dp}{dz}$ is a function of Q/R , say

$$R^3 \frac{dp}{dz} = L\left(\frac{Q}{R}\right).$$

$$\begin{aligned} \eta &= \frac{\pi \frac{dp}{dz} R^4}{8Q} = \frac{\pi \left(\frac{dp}{dz} R^3\right)}{8(Q/R)} \\ &= \frac{\pi L(Q/R)}{8(Q/R)}. \end{aligned}$$

Thus η should be a function of Q/R .

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