

CHAPTER I

CONTINUUM THEORY OF THE HYDROMECHANICS OF NEMATIC LIQUID CRYSTALS

Introduction

The first continuum theory of liquid crystals was due to Oseen (1929). He described the preferred direction of molecular alignment by a unit vector field $\vec{n}$, now called the director field. In terms of this, in a perfectly aligned nematic, molecular orientation in every part of the fluid is given by the vector $\vec{n}$ with $\vec{\nabla} \cdot \vec{n} = \vec{n} \cdot \vec{\nabla} \times \vec{n} = (\vec{n} \cdot \vec{\nabla}) \vec{n} = 0$. A perfectly aligned cholesteric exhibits a pure twist [$\vec{\nabla} \cdot \vec{n} = 0 = \vec{n} \times (\vec{\nabla} \times \vec{n})$; $\vec{n} \cdot (\vec{\nabla} \times \vec{n}) = q = 2\pi/P$ where P is the pitch]. Any deviation from this configuration involves a free energy of deformation whose density can be expressed in terms of the director and its gradients with three independent constants of deformation referred to as the elastic constants. Frank (1968) reformulated Oseen's hydrostatic theory (Oseen 1933) on the basis of curvature elasticity. He showed that any deformation in a nematic or a cholesteric can be associated with one or all of the three principal deformations, viz., splay, twist and bend which can be mathematically characterised by $\vec{\nabla} \cdot \vec{n}$, $\vec{n} \cdot \vec{\nabla} \times \vec{n}$ and $(\vec{n} \cdot \vec{\nabla}) \vec{n}$ respectively. A more complete picture of the hydrostatic behaviour of

liquid crystals was given by Ericksen (1962). Static deformations due to magnetic fields have been analyzed by Leslie (1970) in terms of the Ericksen theory.

The theory of the hydrodynamical behaviour of liquid crystals was developed by various people in stages. The first attempts at formulating the hydrodynamical properties of anisotropic liquids was by Anselmus (1931). He considered an anisotropic liquid (which is made up of long rod like molecules) which has no elastic properties. He expressed the stress tensor in terms of the velocity gradients and the director field. Ericksen (1960) was perhaps the first to give a simple but a complete theory of the hydrodynamics of anisotropic liquids. He used the laws of conservation of mass, linear momentum, angular momentum and energy along with the invariance principle, in setting up the constitutive equations. He also obtained an expression for the stress tensor of an incompressible anisotropic liquid having a director of constant magnitude, in terms of the velocity gradients and the director. This envisaged the anisotropic liquid as having seven viscosity coefficients. Later Leslie (1966) reduced this number to six by showing that one of the viscosity coefficients vanishes identically as a consequence of the law of increase of entropy. One consequence of this model is that though velocity has gradients the director orientation is the same

throughout the fluid. This is a consequence of the elasticity of the medium being ignored. Thus this theory does not reduce in the hydrostatic limit to the hydrostatic theory of Oseen, Frank and Ericksen.

A proper hydromechanical theory of liquid crystals should give the hydrostatic theory of Oseen, Frank and Ericksen in the limit of zero velocity. Leslie (1968) has developed one such theory which will be briefly reviewed in this Chapter. [Martin et al. (1972) and Lee & Eringen (1974) have also proposed alternative approaches to this problem. However, these theories will not be dealt with in this thesis.] Leslie incorporated elastic properties into the basic structure of Ericksen's theory of anisotropic liquids. The resulting theory called the Ericksen-Leslie theory is quite general and can be applied to nematics and to cholesterics. This theory has been widely used in ^{the} literature to account for the behaviour of liquid crystals. Leslie has himself demonstrated the usefulness of this theory in accounting for the results of the viscometric studies of Porter and Johnson (1962). Since the problems presented in Chapters II to VII are entirely based on the Ericksen-Leslie formulation a survey of it should be useful. The theory as worked out for an incompressible nematic having constant director magnitude will be given below. Its

extension to cholesterics (due to Leslie 1968a) will be considered in Chapter V.

Ericksen-Leslie Continuum Theory

A nematic liquid crystal is made up of long molecules which tend, at any point in the fluid, to align approximately parallel to one another. A unit vector field $\vec{a}(\vec{r}, t)$ called the director field describes the preferred direction of orientation while $\vec{v}(\vec{r}, t)$ describes the velocity field. The nematic is assumed to be enclosed in a volume V bounded by a surface σ . Let $\rho(\vec{r}, t)$ be the density, $p_1 = \rho v_1$ the momentum density, t_{ji} the stress tensor, f_1 the external body force per unit volume, ρ_1 the moment of inertia of the director and U the internal energy per unit volume. The stress gradient $t_{ji,j}$ and the external body force f_1 cause a change in the velocity field v_1 of the director. Oseen (1929) proposed that even in the absence of external fields the surface as well as the bulk of the nematic contribute forces - the former in the form of the gradient $\pi_{ji,j}$ of the surface director stress π_{ji} and the latter in the form of the intrinsic director body force per unit volume g_1 - towards changing the rotational velocity $\dot{n}_1$ of the director. If G_1 is the

external director body force per unit volume Oseen's proposition can be written as

$$\rho_1 \frac{d}{dt} \int_V \dot{n}_1 dV = \int_V (\overset{\text{body}}{G_1} + \overset{\text{body}}{S_1}) dV + \int_{\sigma} \overset{\text{surface}}{\pi_{j1}} d\sigma_j \quad (1)$$

The law of conservation of mass becomes

$$\frac{d}{dt} \int_V \rho dV = 0 \quad (2)$$

The rate of change of linear momentum of the director equals the sum of the surface and body forces per unit volume. This is the law of conservation of momentum which is symbolically

$$\frac{d}{dt} \int_V p_1 dV = \int_{\sigma} t_{j1} d\sigma_j + \int_V f_1 dV \quad (3)$$

The total energy density is $\rho \frac{v_1 v_1}{2} + \frac{\rho_1}{2} \dot{n}_1 \dot{n}_1 + U$.

Any change in total energy is brought about by surface and body forces. If $t_1 = t_{j1} d\sigma_j$ and $\sum_k = \pi_{jk} d\sigma_j$ are the surface forces due to t_{j1} and π_{jk} respectively the equation expressing the law of conservation of energy becomes

$$\begin{aligned}
\frac{d}{dt} \int_V \left[\rho \frac{\mathbf{v}_1 \mathbf{v}_1}{2} + \rho_1 \frac{\dot{\mathbf{n}}_1 \dot{\mathbf{n}}_1}{2} + U \right] dV \\
= \int_V \left(\mathbf{f}_1 \mathbf{v}_1 + \overset{g_1 \dot{n}_1 +}{\wedge} \mathbf{G}_1 \dot{\mathbf{n}}_1 \right) dV + \int_{\sigma} \left(\mathbf{t}_1 \mathbf{v}_1 + \sum_1 \mathbf{n}_1 \right) d\sigma \quad (4)
\end{aligned}$$

The angular momentum of the fluid is the sum of the angular momentum corresponding to linear motion ($\vec{r} \times \vec{v}$) and that corresponding to the rotation of the director ($\vec{n} \times \dot{\vec{n}}$). The law of balance of angular momentum states that the rate of change of angular momentum equals the applied torque and can be written as

$$\begin{aligned}
e_{ijk} \frac{d}{dt} \int_V \left[\rho x_j v_k + \rho_1 n_j \dot{n}_k \right] dV \\
= e_{ijk} \int_V \left[x_j \mathbf{f}_k + \overset{q_j g_k +}{\wedge} n_j \mathbf{G}_k \right] dV + e_{ijk} \int_{\sigma} \left[x_j \mathbf{t}_k + n_j \sum_k \right] d\sigma \quad (5)
\end{aligned}$$

On integrating equations 1 to 3 by using Gauss' theorem where necessary one finds that for an incompressible nematic

$$\rho_1 \ddot{\mathbf{n}}_1 = \mathbf{G}_1 + \pi_{j1,j} + \mathbf{G}_1 \quad (6)$$

$$\dot{\rho} = 0 \quad (7)$$

$$\rho \dot{\mathbf{v}}_1 = \mathbf{t}_{j1,j} + \mathbf{f}_1 \quad (8)$$

On integrating equation (4) and on using equations (6) and (8) one gets

$$\dot{U} = t_{ji} d_{ij} + \pi_{ji} N_{ij} - \varepsilon_i N_i + \widetilde{t}_{ji} w_{ij} \quad (9)$$

$$\text{when } d_{ij} = (v_{i,j} + v_{j,i})/2 \quad w_{ij} = (v_{i,j} - v_{j,i})/2,$$

$$N_i = \dot{n}_i - w_{ik} n_k, \quad N_{ij} = \dot{n}_{i,j} - w_{ik} n_{k,j},$$

$$\text{and } \widetilde{t}_{ji} = t_{ji} - \pi_{kj} n_{i,k} + \varepsilon_j n_i.$$

Also since the nematic is incompressible $d_{kk} = v_{k,k} = 0$ and since the director is of constant magnitude $n_i N_i = 0$.

On integrating equation (5) and on using equation (8) it is found that the tensor $\widetilde{t}_{ji}$ is symmetric so that equation (9) reduces to

$$\dot{U} = t_{ji} d_{ij} + \pi_{ji} N_{ij} - \varepsilon_i N_i \quad (10)$$

When a nematic flows there is a dissipation of energy and a consequent change in entropy owing to the viscous torques. The Clausius-Duhem inequality states that the rate of change of entropy with time in any thermodynamic process is always non-negative. If S is the entropy density of the nematic the Clausius-Duhem inequality requires that

$$\frac{d}{dt} \int S \, dV \geq 0.$$

If W is the free energy density (in the case of a nematic W is the elastic energy density of deformation)

$$W = U - TS \quad (11)$$

where T is the absolute temperature.

Since the fluid is incompressible $\dot{\rho} = 0$ so that the entropy inequality yields

$$T \frac{dS}{dt} \geq 0$$

On using equations (10) and (11) one obtains, at constant temperature T

$$T \frac{dS}{dt} = t_{ji} d_{ij} + \pi_{ji} N_{ij} - g_i N_i - \frac{dW}{dt} \geq 0 \quad (12)$$

Let W , t_{ji} , π_{ji} and g_i be single valued functions of the variables

$$n_i, n_{i,j}, \dot{n}_i \text{ and } v_{i,j}.$$

in order that a mechanical phenomenon might be described by a law whose form does not change under transformations $\mathcal{T}$ to any other arbitrarily moving right-handed coordinate system it is necessary to use variables which transform as tensors under such transformations. Though ϵ_{ij} and $n_{i,j}$

transform as tensors under the proper orthogonal transformation $\mathcal{T}$, $\dot{n}_i$ and $v_{i,j}$ do not. But the vector $N_i = n_i - w_{ik}n_k$ and the tensors d_{ij} and w_{ij} retain their tensor character under $\mathcal{T}$. Also, since w_{ij} is a skew symmetric tensor it changes sign under reflections. Hence W , $t_{ji} = \pi_{ji}$ and g_i are assumed to be single valued functions of n_i , $n_{i,j}$, N_i and d_{ij} . It is also assumed that the director is non-polar, i.e., that n_i and $-n_i$ are physically indistinguishable.

The entropy inequality (equation 12) is now evaluated as follows:

$$\begin{aligned} \frac{dW}{dt} &= \frac{\partial W}{\partial n_i} \frac{dn_i}{dt} + \frac{\partial W}{\partial n_{i,j}} \frac{d}{dt}(n_{i,j}) \\ &+ \frac{\partial W}{\partial N_i} \frac{dN_i}{dt} + \frac{\partial W}{\partial d_{ij}} \frac{d(d_{ij})}{dt} \end{aligned}$$

Using the fact that $N_i = \dot{n}_i - w_{ik}n_k$, $N_{ij} = \dot{n}_{i,j} - w_{ik}n_{k,j}$ and $\frac{dn_{i,j}}{dt} = N_{ij} + w_{ik}n_{k,j} - w_{kj}n_{i,k} - d_{kj}n_{i,k}$ equation (7) can be written as

$$\begin{aligned} &(t_{ji} + \frac{\partial W}{\partial n_{k,j}} n_{k,i})d_{ij} + (\pi_{ji} - \frac{\partial W}{\partial n_{i,j}})N_{ij} - (g_i + \frac{\partial W}{\partial n_i})N_i \\ &- w_{ij}[n_j \frac{\partial W}{\partial n_i} + n_{j,k} \frac{\partial W}{\partial n_{i,k}} + n_{k,j} \frac{\partial W}{\partial n_{k,i}}] - \frac{\partial W}{\partial N_i} \frac{dN_i}{dt} \\ &= \frac{\partial W}{\partial d_{ij}} \frac{dd_{ij}}{dt} \geq 0 \end{aligned} \quad (15)$$

Since the velocity and hence its gradients are under external control N_1 and d_{1j} can be chosen arbitrarily. If the inequality should hold one gets

$$\frac{\partial W}{\partial N_1} = 0, \quad \frac{\partial W}{\partial d_{1j}} = 0$$

or

$$W \equiv W(n_1, n_{1,j}) \quad (14)$$

Since w_{1j} can be chosen arbitrarily and $W \equiv W(n_1, n_{1,j})$ the coefficient of w_{1j} should be a symmetric tensor so that the inequality is satisfied. Hence

$$\begin{aligned} n_1 \frac{\partial W}{\partial n_j} + n_{1,k} \frac{\partial W}{\partial n_{j,k}} + n_{k,1} \frac{\partial W}{\partial n_{k,j}} \\ = n_j \frac{\partial W}{\partial n_1} + n_{j,k} \frac{\partial W}{\partial n_{1,k}} + n_{k,j} \frac{\partial W}{\partial n_{k,1}} \end{aligned} \quad (15)$$

Since π_{ji} arises entirely from the surface of the nematic, $\pi_{ji} - \frac{\partial W}{\partial n_{1,j}}$ can be taken to be independent of N_{1j} . This yields

$$\pi_{ji} = \frac{\partial W}{\partial n_{1,j}} \quad (16)$$

The inequality (13) now becomes

$$(t_{ji} + \frac{\partial W}{\partial n_{k,j}} n_{k,1}) d_{1j} - (g_1 + \frac{\partial W}{\partial n_1}) N_1 \geq 0 \quad (17)$$

Let t_{ij}^0 and g_i^0 be the static isothermal parts of t_{ij} and g_i so that

$$\begin{aligned} t_{ji} &= t_{ji}^0 + t'_{ji} \\ g_i &= g_i^0 + g'_i \end{aligned} \quad (18)$$

Equations (17), (18) and (13) give

$$\begin{aligned} (t_{ji}^0 + \frac{\partial W}{\partial n_{k,j}} n_{k,i}) d_{ij} - (g_i^0 + \frac{\partial W}{\partial n_i}) N_i \\ + t'_{ji} d_{ij} - g'_i N_i \geq 0 \end{aligned}$$

Since $t_{ji}^0 + \frac{\partial W}{\partial n_{k,j}} n_{k,i}$ and $g_i^0 + \frac{\partial W}{\partial n_i}$ are independent of d_{ij} and N_i the following expressions result

$$\begin{aligned} t_{ji}^0 &= - \frac{\partial W}{\partial n_{k,j}} n_{k,i} - p^0 \delta_{ij} \\ g_i^0 &= - \frac{\partial W}{\partial n_i} + \gamma^0 n_i \end{aligned} \quad (19)$$

where p_0 and γ_0 are arbitrary constants. They are introduced as Lagrangian multipliers corresponding to the constraints of incompressibility and constancy of director magnitude.

The entropy inequality can now be written in terms of non-equilibrium parts t'_{ji} and g'_i alone

$$t'_{ji}d_{ij} - g'_iN_i \geq 0 \quad (20)$$

Using the symmetry of $\bar{t}_{ji}$ as also equations (19), (18) and (16) one obtains

$$t'_{ji} + g'_jn_i = t'_{ij} + g'_in_j \quad (21)$$

To find the forms of t'_{ji} and g'_i we use a first order theory, expanding them in terms of N_i and d_{ij} using tensor coefficients A and B whose elements are functions of the director alone

$$\begin{aligned} t'_{ji} &= A_{ji}^{(0)} + A_{jik}^{(1)} N_k + A_{jikm}^{(2)} d_{km} \\ g'_i &= B_i^{(0)} + B_{ij}^{(1)} N_j + B_{ijk}^{(2)} d_{jk} \end{aligned} \quad (22)$$

The tensors A and B can be expanded in terms of the director and the kronecker delta in the form

$$\begin{aligned} A_{ji}^{(0)} &= \alpha_0 \delta_{ij} + \alpha_1 n_i n_j \\ A_{jik}^{(1)} &= \alpha_2 \delta_{ji} n_k + \alpha_3 \delta_{jk} n_i + \alpha_4 \delta_{ik} n_j + \alpha_5 n_i n_j n_k \\ A_{jikm}^{(2)} &= \alpha_6 \delta_{ji} n_k n_m + \alpha_7 n_j n_m \delta_{ik} + \alpha_8 n_i n_j \delta_{km} \\ &\quad + \alpha_9 \delta_{jk} n_i n_m + \alpha_{10} \delta_{jm} n_i n_k + \alpha_{11} \delta_{im} n_j n_k \\ &\quad + \alpha_{12} \delta_{ji} \delta_{km} + \alpha_{13} \delta_{jk} \delta_{im} + \alpha_{14} n_i n_j n_k n_m \end{aligned}$$

$$B_i^{(0)} = \gamma_0 n_i$$

$$B_{ij}^{(1)} = \gamma_1 \delta_{ij} + \gamma_2 n_i n_j$$

$$B_{ijk}^{(2)} = \gamma_3 \delta_{ij} n_k + \gamma_4 \delta_{ik} n_j + \gamma_5 \delta_{jk} n_i + \gamma_6 n_i n_j n_k$$

where the α s and γ s are material constants.

Then (22) reduces to

$$t'_{ji} = -p' \delta_{ij} + (\mu_7 + \mu_1 d_{kp} n_k n_p) n_i n_j + \mu_2 n_j n_i + \mu_3 n_i n_j \\ + \mu_4 d_{ij} + \mu_5 n_j d_{ik} n_k + \mu_6 n_i d_{jk} n_k$$

$$g'_i = \gamma_0 n_i + \lambda_1 n_i + \lambda_2 d_{ik} n_k$$

$$\text{where } -p' = \alpha_0 + \alpha_6 d_{km} n_k n_m, \quad \mu_7 = \alpha_1, \quad \mu_1 = \alpha_{14},$$

$$\mu_2 = \alpha_4, \quad \mu_3 = \alpha_3, \quad \mu_4 = \alpha_{13}, \quad \mu_5 = \alpha_7 + \alpha_{11},$$

$$\mu_6 = \alpha_9 + \alpha_{10}, \quad \lambda_1 = \gamma_1, \quad \lambda_2 = \gamma_3 + \gamma_4.$$

Use of equation (21) shows that λ 's and μ 's are related

$$\lambda_1 = \mu_2 - \mu_3$$

$$\lambda_2 = \mu_5 - \mu_6$$

t'_{ji} and g'_i should also satisfy the entropy inequality

(20) which yields

$$\mu_1(d_{kp}n_kn_p)^2 + \mu_4(d_{ij})^2 + (\mu_5 + \mu_6)(d_{ik}n_k)^2 \\ - \lambda_1 N_1^2 + \mu_7 d_{ij}n_in_j + (\mu_2 + \mu_3 - \lambda_2)d_{ik}n_kN_i \geq 0.$$

If μ_7 is equated to zero the rest of the expression can be treated as a quadratic form and the μ 's satisfy the following relations:

$$\mu_7 = 0; \mu_4 \geq 0; 2\mu_1 + 3\mu_4 + 2\mu_5 + 2\mu_6 \geq 0; \\ 2\mu_4 + \mu_5 + \mu_6 \geq 0; -4\lambda_1(2\mu_4 + \mu_5 + \mu_6) \geq (\mu_2 + \mu_3 - \lambda_2)^2$$

For the elastic energy density W , the expression given by Frank, viz.,

$$2W = K_{11}(n_{i,i}n_{j,j}) + K_{22}e_{ijk}n_{k,j}e_{ipq}n_{q,p} - K_{33}n_in_jn_{k,i}n_{k,j}$$

is used. In vector notation this can be written as

$$2W = K_{11}(\vec{\nabla} \cdot \vec{n})^2 + K_{22}(\vec{n} \cdot \vec{\nabla} \times \vec{n})^2 \\ + K_{33}[(\vec{n} \cdot \vec{\nabla})\vec{n}]^2$$

The method of derivation of W is briefly indicated below.

Elastic Free Energy Density

Whenever there is a deformation in the director

field $\vec{a}$ the director gradient $\vec{\nabla} \vec{n}$ (or $n_{i,j}$) comes into existence. If any of the three deformations (splay, twist or bend) is imposed on a uniformly aligned nematic the medium loses its centro symmetry. Consequently the curvature strain tensor can be written in terms of $n_{i,j}$ and is an axial tensor a_{ij} which, in one index notation can be written as a_i . The free energy density W can be expanded in a Taylor series in a function of a_i and terms upto $a_i a_j$ are retained:

$$W = K_1 a_i + \frac{1}{2} K_{ijkl} a_i a_j$$

where K_1 is an axial tensor (of rank two) which corresponds to spontaneous deformations (if any) present in the medium and K_{ijkl} is a symmetric tensor (of rank four) which corresponds to deformations imposed by external forces. Using the Neumann principle the symmetry of the medium (which is determined by the director, which in turn is assumed to have cylindrical symmetry) is imposed on the tensors K_1 and K_{ijkl} . The fact that the director is also non polar is used. K_{ijkl} can be written as a 6 x 6 matrix

$$K_{ijkl} = \begin{bmatrix} K_{11} & 0 & 0 & 0 & (K_{11}-K_{22}-K_{24}) & 0 \\ & K_{22} & 0 & K_{24} & 0 & 0 \\ & & K_{33} & 0 & 0 & 0 \\ & & & K_{22} & 0 & 0 \\ & & & & K_{11} & 0 \\ & & & & & K_{33} \end{bmatrix}$$

If the structure is non-enantiomorphic and non-polar $K_1 = 0$. But *if* there is enantiomorphy it is possible to write

$$K_1 = \begin{bmatrix} 0 & k_2 & 0 \\ -k_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This would of course correspond to the case of a cholesteric. One can now write the free energy density for cholesterics as

$$\begin{aligned} w = & K_2 (\vec{n} \cdot \vec{\nabla} \times \vec{n}) + \frac{1}{2} K_{11} (\vec{\nabla} \cdot \vec{n})^2 + \frac{1}{2} K_{22} [\vec{n} \cdot \vec{\nabla} \times \vec{n}]^2 \\ & + \frac{1}{2} K_{33} [(\vec{n} \cdot \vec{\nabla}) \vec{n}]^2 - (K_{22} + K_{24}) \left[\frac{\partial n_x}{\partial x} \frac{\partial n_y}{\partial y} \right. \\ & + \frac{\partial n_y}{\partial y} \frac{\partial n_z}{\partial z} + \frac{\partial n_z}{\partial z} \frac{\partial n_x}{\partial x} - \frac{\partial n_y}{\partial x} \frac{\partial n_x}{\partial y} - \frac{\partial n_z}{\partial y} \frac{\partial n_y}{\partial z} \\ & \left. - \frac{\partial n_x}{\partial z} \frac{\partial n_z}{\partial x} \right] \end{aligned}$$

For nematics the same expression holds, but with $K_2 = 0$. Ericksen (1962a) showed that the coefficient of $(K_{22} + K_{24})$ satisfies the equation of equilibrium identically. Thus K_{24} plays NO part in the mechanics of nematics and cholesterics. For cholesterics K_2 is related to the pitch P by the relation

$$\frac{K_2}{K_{22}} = \frac{2\pi}{P}$$

Thus it is obvious that for nematics which have infinite pitch ($P = \infty$) K_2 can be equated to zero.

The Missowicz Viscosity Coefficients

Certain simple types of flow may be considered to illustrate the dependence of the apparent viscosity on the director orientation. Consider the three flows:

- (i) The director is aligned along the flow (x axis) but perpendicular to the velocity gradient (y);
- (ii) The director is aligned parallel to the velocity gradient (y) but the flow is along x axis;
- (iii) The director, which is aligned along z axis is perpendicular to both the velocity and the velocity gradient. It is a very simple matter to work out the apparent viscosity in these cases. Since director deformations may be ignored one need not utilize (23a), but only (23b) with assumptions of steady flow and zero external body force. A simple calculation shows that η in the three cases has the values respectively

$$\eta_{//} = \frac{(\mu_3 + \mu_4 + \mu_6)}{2} ,$$

$$\eta_A = \frac{(\mu_4 + \mu_5 - \mu_2)}{2}$$

and

$$\eta_3 = \frac{\mu_4}{2}.$$

These flows were investigated in detail, nearly two decades before the birth of the Ericksen-Leslie theory by Miesowicz (1936, 1946).

An angle (popularly referred to as θ_0) also makes its appearance frequently in the literature. In any flow of nematics, whatever the boundary conditions for the director it is found that at high shears where elastic forces can be ignored the director in almost the entire bulk of the nematic makes an angle θ_0 given by $\cos 2\theta_0 = -\lambda_1/\lambda_2$. Interestingly enough for all known nematics (at temperatures far removed from critical temperatures) $|\lambda_1| \leq |\lambda_2|$. Also λ_1 is found to be negative whereas λ_2 is positive.

The basic equations of motion of the hydrodynamics of liquid crystal can be written as

$$\pi_{ji,j} + \epsilon_i + G_i = \rho_1 \ddot{n}_i \quad (23a)$$

$$t_{ji,j} + f_i = \rho \dot{v}_i \quad (23b)$$

where

$$\pi_{ji} = \frac{\partial W}{\partial n_{i,j}} ;$$

$$g_i = -\frac{\partial W}{\partial n_i} + \lambda_1 n_i + \lambda_2 d_{ik} n_k + \gamma n_i$$

$$\begin{aligned} t_{ji} = & -p\delta_{ij} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + \mu_1 d_{kp} n_k n_p n_i n_j \\ & + \mu_2 n_j n_i + \mu_3 n_j n_i + \mu_4 d_{ij} + \mu_5 n_j d_{ik} n_k \\ & + \mu_6 n_i d_{jk} n_k \equiv -p\delta_{ij} + \hat{t}_{ji} \end{aligned}$$

$$\lambda_1 = \mu_2 - \mu_3 ; \quad \lambda_2 = \mu_5 - \mu_6 ;$$

$$p = p^0 + p' ; \quad \gamma = \gamma^0 + \gamma_0$$

$$2W = K_{11}(n_{i,i})^2 + K_{22}(e_{ijk} n_{k,j})^2 + K_{33}(n_i n_j n_{k,i} n_{k,j}).$$

The equations (23) have to be solved together to obtain the director and velocity fields. This also illustrates the phenomenon of the intimate coupling between the director and velocity fields which, as we shall see in subsequent chapters leads to the various interesting flow properties of nematics.

We have not so far said anything about the form of the force G_i . Since magnetic fields are commonly used as external fields we shall derive G_i in the case

of a magnetic field $\vec{H}$, which need not be homogeneous.

Let $\vec{M}$ be the magnetization. Then the body force per unit volume is $M_k H_{k,i}$ and the torque per unit volume $\epsilon_{ijk} M_j H_k$.

For nematics, which are uniaxial the diamagnetic susceptibility tensor can be written as

$$C_{ij} = (\chi_1 - \chi_2) n_i n_j + \chi_2 \delta_{ij}$$

where χ_1 and χ_2 are the principal diamagnetic susceptibilities.

The magnetization can be written as $M_i = C_{ij} H_j$.

The body force per unit volume is

$$f_i = M_k H_{k,i} = [(\Delta\chi)(H_1 n_1) n_k + \chi_2 H_k] H_{k,i}$$

where $\Delta\chi = \chi_1 - \chi_2$.

The torque per unit volume is

$$\epsilon_{ijk} n_j G_k = \epsilon_{ijk} M_j H_k = \epsilon_{ijk} n_j (H_p n_p) H_k.$$

Hence

$$G_k = (\Delta\chi)(H_p n_p) H_k$$

Physically one can understand the dynamical behaviour of nematics as follows: when a nematic flows, viscous torques which come into existence try to deform the structure setting up gradients in the director field. These deformations bring into play elastic torques which try to combat the deformations so introduced. On the whole the flow of a nematic is a complex competition between elastic and viscous torques. It is this feature which is responsible for the anomalous hydrodynamic behaviour of nematics.

Scaling Laws

Scaling analysis (Eriksen 1969) is an important concept connected with the flow of nematics. In any type of flow it is possible to transform spatial and temporal coordinates such that the experimental results can be expressed in a form which is independent of the geometry of the equipment. Thus one is able to choose proper parameters so that results can be presented as a 'universal curve'. It is then possible to test whether the universal curve which is predicted by the theory indeed exists. This forms a check on the theory. The application of scaling laws to Poiseuille flow has been illustrated in Chapter III. It should be emphasized that scaling is possible only when (i) there is no external field, (ii) the director inertia is negligible, and (iii) director and velocity fields have fixed values at the boundary (fixed boundary conditions).

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