

CHAPTER V

FLOW OF CHOLESTERICS PARALLEL TO THE HELICAL AXIS

Introduction

In previous chapters we discussed the hydrodynamics of nematics. We shall now consider the hydrodynamical behaviour of cholesteric liquid crystals which is markedly different from that of nematics. Cholesterics are highly non-Newtonian in their flow properties. For instance Porter ^{et al.} (1966) and Sakamoto ^{et al.} (1969) found experimentally that the apparent viscosity η of a cholesteric increases by nearly a million-fold as the shear rate drops from a very high to a low value. This anomalously high value of η has been accounted for by Helfrich (1969) on the basis of the permeation model a brief account of which can be found below. Helfrich assumed that the cholesteric flows along its helical axis with a flat velocity profile and with the molecules spiralling as they move down the axis. He also assumed that the helical structure does not move owing to the anchoring effects of the wall. In describing Helfrich's model, we show in this chapter that the anomalous hydrodynamical behaviour of

cholesterics can be understood on the basis of the continuum theory. With no assumptions made regarding the anchoring of the structure by the wall, it emerges that at low shear rates the main features of the permeation model follow as a natural consequence of Leslie's continuum theory of incompressible cholesterics with constant director magnitude (Leslie 1968) which is itself an extension of the Ericksen-Leslie theory discussed earlier (Chapter I). A brief summary of Leslie's theory is also given.

Helfrich's permeation model

The cholesteric is assumed to flow under a constant pressure gradient in a capillary tube of radius R such that the tube axis and the helical axis of the cholesteric coincide with the z direction. The helical structure is assumed to be held immobile by the anchoring effects at the wall. The velocity v is assumed to be a constant. Helfrich calculated the rate of energy dissipated (at constant temperature) in such a flow and equated this to the work done by the pressure gradient.

Assume that the director orientation at a point z is given by

$$n_x = \cos(qz + c), \quad n_y = \sin(qz + c), \quad n_z = 0$$

where $q = 2\pi/P$, P being the pitch and c is an arbitrary constant. The energy dissipated per second is given by (see eqn. (12), Chapter I)

$$\tau \frac{dS}{dt} = t_{ji} d_{ij} + \pi_{ji} n_{ij} - g_i n_i + \frac{dW}{dt} \equiv \epsilon \text{ say.}$$

Since $\mathbf{v} = (0, 0, v)$ and since v is a constant

$$\epsilon = \pi_{ji} \frac{d}{dt} n_{ij} - g_i \frac{dn_i}{dt} + \frac{\partial W}{\partial z} v.$$

The Frank's energy density W turns out to be zero.

Hence $\epsilon = -g_i \frac{dn_i}{dt}$. We find that

$$\frac{dn_x}{dt} = -qv \sin(qz + c), \quad \frac{dn_y}{dt} = qv \cos(qz + c),$$

$$g_x = \lambda_1 \frac{dn_x}{dt} + \gamma n_x, \quad g_y = \lambda_1 \frac{dn_y}{dt} + \gamma n_y$$

$$\text{so that } \epsilon = -q^2 v^2 \lambda_1.$$

Now the amount of work done by the pressure gradient $\partial p / \partial z$ in one second is clearly $\epsilon' = \frac{\partial p}{\partial z} v$. Equating ϵ and ϵ' one gets

$$\frac{\partial p}{\partial z} = \lambda_1 q^2 v .$$

The volume rate of flow of the fluid is $A = \pi v R^2$.

The apparent viscosity η can be calculated from the Poiseuille formula

$$\eta = - \frac{\lambda_1 q^2 R^2}{8} = - \frac{\lambda_1 \pi^2}{2} \frac{R^2}{p^2}$$

Thus η is independent of the velocity of the fluid, directly proportional to the square of the tube radius and inversely proportional to the square of the pitch.

At this stage one may make the following observation. The existence of velocity gradient is a natural and necessary consequence of the existence of viscosity in the medium, and the assumption of a flat velocity profile violates this basic principle. A more appropriate procedure even when discussing flow at low shear rates would be to assume the existence of at least an infinitesimal velocity gradient along with corresponding infinitesimal distortions in the director field. We shall develop such a theory in the next section.

Hydrodynamics of cholesterics

Cholesterics have a spontaneous twist imposed on a nematic structure. This twist results in a basic difference between the properties of the two. Though in both cases the director $\mathbf{a}$ is assumed to be non-polar the nematic structure has a centre of inversion which is lacking in cholesterics. Thus the director of a cholesteric is not invariant under reflexions in planes containing the director. The other difference between nematics and cholesterics lies in the mechanism of entropy production. In a nematic entropy can change ~~But~~ to the action of viscous forces and due to disruption of molecular alignment (or equivalently change in director magnitude). But in an isothermal change the latter can be ignored. In cholesterics entropy generation arises not only due to the action of viscous forces and disruption of molecular alignment but also due to changes in the inherent twist. If h_i is the heat flux and e_i the entropy flux one can write $h_i = Te_i + \varphi_i$ where φ_i is the part of the heat flux which is utilized for changing the structure. Conservation laws of Chapter I hold in toto except that eqns.(9) and (12) are written as

$$\frac{dU}{dt} = t_{ji}d_{ij} + \pi_{ji}N_{ij} - g_i N_i - h_{i,i} \quad (1)$$

$$t_{ji}d_{ij} + \pi_{ji}N_{ij} - \varepsilon_1 N_1 - e_1 T_{,1} - \frac{dW}{dt} - \varphi_{1,1} \geq 0 \quad (2)$$

On assuming that U , S , H_1 , e_1 , t_{ji} , π_{ji} and ε_1 are single valued functions of T , n_1 , $n_{1,j}$, N_1 , d_{ij} and $T_{,k}$ equation (2) becomes

$$\begin{aligned} & [t_{ji} + \frac{\partial W}{\partial n_{k,j}} n_{k,1}] d_{ij} + [\pi_{ji} - \frac{\partial W}{\partial n_{1,j}} - \frac{\partial \varphi_1}{\partial N_1}] N_{ij} \\ & - [\varepsilon_1 + \frac{\partial W}{\partial N_1}] N_1 - [e_1 + \frac{\partial \varphi_1}{\partial T}] T_{,1} - \frac{\partial \varphi_1}{\partial n_j} n_{j,1} \\ & + [n_1 \frac{\partial W}{\partial n_j} + n_{1,k} \frac{\partial W}{\partial n_{j,k}} + n_{k,1} \frac{\partial W}{\partial n_{k,j}}] w_{1j} \\ & - \frac{\partial \varphi_1}{\partial n_{k,j}} n_{k,1j} - \frac{\partial \varphi_1}{\partial T_{,j}} T_{,j1} - [\frac{\partial \varphi_1}{\partial d_{jk}} + n_k \frac{\partial \varphi_1}{\partial N_1} - n_1 \frac{\partial \varphi_k}{\partial N_j}] d_{jk,1} \\ & - (\frac{\partial W}{\partial T} + S) \frac{dT}{dt} - \frac{\partial W}{\partial N_1} \frac{dN_1}{dt} - \frac{\partial W}{\partial d_{ij}} \frac{dd_{ij}}{dt} - \frac{\partial W}{\partial T_{,1}} \frac{d}{dt}(T_{,1}) = 0 \end{aligned}$$

..(3)

One can now conclude that W satisfies the relations

$$W \equiv W(T, n_1, n_{1,j}) ,$$

$$S = - \frac{\partial W}{\partial T} ,$$

$$\begin{aligned}
 n_1 \frac{\partial W}{\partial n_j} + n_{1,k} \frac{\partial W}{\partial n_{j,k}} + n_{k,i} \frac{\partial W}{\partial n_{k,j}} \\
 = n_j \frac{\partial W}{\partial n_1} + n_{j,k} \frac{\partial W}{\partial n_{j,k}} + n_{k,j} \frac{\partial W}{\partial n_{k,i}}
 \end{aligned}$$

One also finds for φ_1 that

$$\begin{aligned}
 \frac{\partial \varphi_1}{\partial n_{k,j}} + \frac{\partial \varphi_1}{\partial n_{k,i}} = 0, \quad \frac{\partial \varphi_1}{\partial T_{,j}} + \frac{\partial \varphi_1}{\partial T_{,i}} = 0 \quad \text{and} \\
 2 \frac{\partial \varphi_1}{\partial d_{jk}} + n_k \frac{\partial \varphi_1}{\partial N_1} + n_j \frac{\partial \varphi_k}{\partial N_1} - n_1 \frac{\partial \varphi_1}{\partial N_k} - n_1 \frac{\partial \varphi_k}{\partial N_j} = 0 \quad (4)
 \end{aligned}$$

The inequality (3) reduces to

$$\begin{aligned}
 [t_{j1} + \frac{\partial W}{\partial n_{k,j}} n_{k,i}] d_{1j} + [\pi_{j1} - \frac{\partial W}{\partial n_{1,j}} - \frac{\partial \varphi_1}{\partial N_1}] N_{1j} \\
 - [\epsilon_1 + \frac{\partial W}{\partial n_1}] N_1 - [\theta_1 + \frac{\partial \varphi_1}{\partial T}] T_{,1} - \frac{\partial \varphi_1}{\partial n_j} n_{j,1} \geq 0 \quad (5)
 \end{aligned}$$

For an incompressible cholesteric with a director of constant magnitude equations (4) dictate the following form for φ_1

$$\varphi_1 = \alpha e_{1jk} n_k (N_k + d_{kp} n_p)$$

where α is a function of the temperature alone.

From (4) one can work out expressions for π_{j1} and for

the hydrostatic parts of t_{ji} and g_i a

$$\pi_{ji} = -\frac{\partial W}{\partial n_{i,j}} + \alpha e_{ijk} n_k$$

$$g_i^0 = -\frac{\partial W}{\partial n_i} - \alpha e_{ijk} n_{k,j}$$

$$t_{ji}^0 = -p^0 \delta_{ij} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + \alpha e_{jkp} (n_p n_i)_{,k}$$

The hydrodynamic parts t'_{ji} and g'_i of t_{ji} and g_i satisfy the inequality

$$t'_{ji} d_{ij} - g'_i n_j - (e_i + \frac{\partial \varphi_i}{\partial T}) T_{,i} \geq 0$$

In working out the expressions for h_i , t'_{ji} and g'_i one has to include their dependence not only on N_i and d_{jm} but also on the temperature gradient $T_{,k}$. One finds that

$$\begin{aligned} t'_{ji} = & \mu_1 d_{kp} n_k n_p n_i n_j + \mu_2 n_j N_i + \mu_3 n_i N_j + \mu_4 d_{ij} \\ & + \mu_5 n_j d_{ik} n_k + \mu_6 n_i d_{jk} n_k + \mu_7 n_j e_{ipq} n_p T_{,q} \\ & + \mu_8 n_i e_{jpq} n_p T_{,q} \end{aligned}$$

$$g'_i = \lambda_1 N_i + \lambda_2 d_{ij} n_j + \lambda_3 e_{ik} n_j T_{,k}$$

$$h_i = \kappa_1 T_{,i} + \kappa_2 n_j T_{,j} n_i + \kappa_3 \epsilon_{ijk} n_j n_k + \kappa_4 \epsilon_{ijk} n_j d_{kp} n_p$$

where $\lambda_1 = \mu_2 - \mu_3$, $\lambda_2 = \mu_5 - \mu_6$, $\lambda_3 = \mu_7 - \mu_8$ and κ_j are thermal conductivities.

Thus in any problem of cholesteric flow one has to solve for the director $\vec{n}$ the velocity $\vec{v}$ and the temperature T utilizing the equations of motion

$$\pi_{ji,j} + g_i + G_i = \rho_i \dot{n}_i \quad (6a)$$

$$t_{ji,j} + f_i = \rho \dot{v}_i \quad (6b)$$

along with equation (1) for internal energy with ~~and~~ suitable boundary conditions for $\vec{n}$, $\vec{v}$ and T .

One also has to use the expression for W which contains a term corresponding to the intrinsic twist, viz.,

$$2W = 2K_2(\vec{n} \cdot \vec{\nabla} \times \vec{n}) + K_{11}(\vec{\nabla} \cdot \vec{n})^2 + K_{22}(\vec{n} \cdot \vec{\nabla} \times \vec{n})^2 \\ + K_{33}[(\vec{n} \cdot \vec{\nabla}) \vec{n}]^2$$

where $K_2 = \frac{2\pi}{P} K_{22}$ with P as the pitch of the structure.

At this stage one observes that a does not enter (6a) at all; it gets totally eliminated. Even (6b) does not contain a when one uses flat coordinate systems (such as cartesian and cylindrical polar where

$A_{i,kj} = A_{i,jk}$ for a tensor A_i) to describe isothermal flows. Because

$$\begin{aligned} [\alpha e_{jkp}(n_p n_i), k]_{,j} &= \alpha e_{jkp}(n_p n_i), kj \\ &= \alpha e_{jkp}(n_p n_i), jk = \alpha e_{pkj}(n_p n_i), kj \\ &= -\alpha e_{jkp}(n_k n_i), kj \end{aligned}$$

so that the term containing a drops out.

In the problems treated in this chapter and in the next, the temperature has been assumed to be constant throughout the fluid. Thus our expressions for t_{ji} , g_i and π_{ji} are formally identical to those of Chapter I. The thermal conductivities κ_j have also been ignored.

Flow between two fixed plane parallel plates

The plates are assumed to occupy the planes $x = \pm h/2$ in a right-handed cartesian coordinate system. The cholesteric is assumed to flow along the z axis, under the notion of a constant pressure gradient $p_{,z}$; the helical axis of the cholesteric is assumed to coincide with the z axis.

Solutions are sought for the director and

velocity fields in the form

$$\begin{aligned} n_x &= \cos(qz + \varphi) \cos \theta & v_x &= 0 \\ n_y &= \sin(qz + \varphi) \cos \theta & v_y &= 0 \\ n_z &= \sin \theta & v_z &= v \end{aligned} \quad (7)$$

where $\theta = \theta(x, z)$; $\varphi = \varphi(x, z)$; $v = v(x)$.

For $v = \theta = \varphi = 0$ this gives a cholesteric of pitch $P = 2\pi/q$. Considering low pressure gradients, only the first powers of v , θ , φ and their derivatives are retained. Then (7) reduce to

$$\begin{aligned} n_x &= C - \varphi S \\ n_y &= S + \varphi C \\ n_z &= \theta \end{aligned}$$

where $C = \cos qz$; $S = \sin qz$.

Then equations (6a) become

$$\begin{aligned} \theta_{,xx}(K_{11} - K_{22}S^2) - \varphi_{,xx}(K_{11}S + K_{33}SC^2) - \varphi_{,zz}(K_{22}S) \\ - \theta_{,x}[(K_{33} + 3K_{22})qSC] - \varphi_{,z}(2K_{22}qC) - \lambda_1 v q S + \gamma(C - S\varphi) = 0 \end{aligned} \quad (8)$$

$$\begin{aligned} \theta_{,xx}(K_{22}SC) + \varphi_{,xx}(K_{33}C^3) + \varphi_{,zz}(K_{22}C) + \theta_{,x}[K_{33}qC^2 + K_{22}(C^2 - 2S^2)q] \\ - \varphi_{,z}(2K_{22}qS) + \lambda_1 v q C + \gamma(S + C\varphi) = 0 \end{aligned} \quad (9)$$

$$\begin{aligned} & \theta_{,xx}(K_{22}s^2 + K_{33}c^2) + \varphi_{,xx}[(K_{22} - K_{11})s] \\ & + \theta_{,xx}(K_{11}) - \varphi_{,x}[(K_{11} + K_{33})qc] - \theta(K_{33}q^2) \\ & + \frac{(\lambda_2 - \lambda_1)}{2} v_{,x} c + \gamma \theta = 0 \end{aligned}$$

where $\theta_{,x} = \frac{\partial \theta}{\partial x}$; $\theta_{,xx} = \frac{\partial^2 \theta}{\partial x^2}$ etc.

γ elimination from (8) and (9) yields

$$\begin{aligned} & \theta_{,xx}[(K_{11} - K_{22})s] - \varphi_{,xx}(K_{11}s^2 + K_{33}c^2) \\ & - \varphi_{,xx}(K_{22}) - \theta_{,x}[(K_{22} + K_{33})qc] - \lambda_1 vq = 0 \end{aligned} \quad (10)$$

The equations (6b) yield

$$\begin{aligned} p_{,x} &= -v_{,x}(\mu_2 + \mu_6)qs \\ p_{,y} &= v_{,x}[q(\mu_2 c^2 - \mu_3 s^2) + \frac{q(\mu_6 - \mu_3)}{2}(c^2 - s^2)] \\ p_{,z} &= \theta_{,xx}[(K_{11} - K_{22})qs] - \varphi_{,xx}[(K_{11}s^2 + K_{33}c^2)q] \\ & - \varphi_{,xx}(K_{22}q) - \theta_{,x}[(K_{22} + K_{33})q^2c] \\ & + v_{,xx}[\frac{\mu_4 + (\mu_5 - \mu_2)c^2}{2}] \end{aligned} \quad (11)$$

Using (10) and (11) a differential equation results for the velocity field $v(x)$

$$v_{,xx} \left[\frac{\mu_4 + (\mu_5 - \mu_2)C^2}{2} \right] + v(\lambda_1 q^2) - p_{,z} = 0$$

The term $\frac{1}{2}[\mu_4 + (\mu_5 - \mu_2)C^2]$ is replaced by an average value $\bar{\eta}$ so that

$$\bar{\eta} v_{,xx} + v \lambda_1 q^2 - p_{,z} = 0$$

A solution of this equation with boundary conditions $v(\pm h/2) = 0$ is

$$v(x) = \frac{p_{,z}}{\lambda_1 q^2} \left[1 - \frac{\cosh kx}{\cosh kh/2} \right] \quad (12)$$

where

$$k = \left[-\frac{\lambda_1 q^2}{\bar{\eta}} \right]^{\frac{1}{2}}$$

The velocity is symmetric about $x = 0$. The volume rate of flow is

$$Q = \int_{-h/2}^{h/2} v(x) dx = 2 \int_0^{h/2} v(x) dx = \frac{p_{,z} h}{\lambda_1 q^2} \left[1 - \frac{\tanh kh/2}{kh/2} \right]$$

Thus the apparent viscosity for this geometry is given by

$$\eta = -\frac{p_{,z} h^3}{12Q} = \frac{-\lambda_1 q^2 h^2}{12 \left[1 - \frac{\tanh kh/2}{kh/2} \right]}$$

When $h = 100 \mu$, $P = 1 \mu$ (i.e., for large values of kh) (12) shows that the velocity attains 0.99 of its maximum value within a thickness of about 0.5μ of the boundary. Thus in all practical situations the velocity profile is flat over most of the region between the plates. Thus in this case $\eta \approx -\lambda_1 q^2 h^2 / 12$. This is analogous to Helfrich's equation which will be derived below.

3. Flow in a capillary tube: The capillary tube is assumed to have a radius R and its axis parallel to z . Solutions for the director and velocity fields are sought in cylindrical polars in the form

$$\begin{aligned} n_r &= \cos(qz + \varphi - \psi) \cos \theta & v_r &= 0 \\ n_\psi &= \sin(qz + \varphi - \psi) \cos \theta & v_\psi &= 0 \\ n_z &= \sin \theta & v_z &= v(r) \end{aligned}$$

where $\theta = \theta(r, \psi, z)$, $\varphi = \varphi(r, \psi, z)$.

At low pressure gradients, assuming that φ and θ are small we obtain to first order in θ and φ

$$n_r = C - \varphi S; \quad n_\psi = S + \varphi C; \quad n_z = 0$$

where

$$C = \cos(qz - \psi); \quad S = \sin(qz - \psi).$$

Equations (6a) can be written as

$$\begin{aligned}
 & \Theta_{,sr}(K_{11} - K_{22}S^2) + K_{22}S\Theta \frac{\Theta_{,r}\psi}{r} - K_{22}S\varphi_{,zz} - 2K_{22}q\Theta_{,z} \\
 & + \frac{\varphi_{,r}\psi}{r}(K_{11}C - 2K_{33}S^2C) - \varphi_{,rr}(K_{11}S + K_{33}S^2C) \\
 & - \frac{K_{33}}{r^2}\varphi_{,rr}S^3 + \frac{\varphi_{,r}\psi}{r^2}(2K_{33}S^2C - K_{11}C) - \Theta_{,r}(K_{33} + 3K_{22})qSC \\
 & - \frac{\varphi_{,r}}{r}K_{33}S^3 + \frac{\Theta_{,r}\psi}{r}(2K_{22}qC^2 - K_{22}qS^2 - K_{33}qS^2) \\
 & - \lambda_1 \nabla qS + \gamma(C - S\varphi) = 0
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 & \Theta_{,sr}K_{22}S\Theta + \frac{\Theta_{,r}\psi}{r}(K_{11} - K_{22}C^2) + K_{22}C\varphi_{,zz} \\
 & - 2K_{22}qS\varphi_{,z} + \frac{\varphi_{,r}\psi}{r}[2K_{33}S\Theta^2 - K_{11}S] \\
 & + K_{33}C^3\varphi_{,rr} + \frac{\varphi_{,r}\psi}{r^2}(K_{11}C + K_{33}S^2C) \\
 & + \frac{\varphi_{,r}\psi}{r^2}(K_{11}S - 2K_{33}S\Theta^2) + \Theta_{,r}[K_{33}qC^2 + K_{22}qC^2 - 2K_{22}qS^2] \\
 & + \frac{\varphi_{,r}}{r}[K_{11}C + K_{33}S^2C] + \frac{\Theta_{,r}\psi}{r}(K_{33} + 3K_{22})qSC + \lambda_1 \nabla qC \\
 & + \gamma(S + C\varphi) = 0
 \end{aligned} \tag{14}$$

$$\begin{aligned}
& \frac{2\theta_{,\gamma r} (K_{33} - K_{22})s\phi}{r} + \varphi_{,rr} s(K_{22} - K_{11}) \\
& + \varphi_{,z\gamma} (K_{11} - K_{22})\phi + \frac{\theta_{,\gamma}}{r^2} 2(K_{22} - K_{33})s\phi \\
& + K_{11} \theta_{,zz} + \theta_{,rr} (K_{22}s^2 + K_{33}\phi^2) + \frac{\theta_{,\gamma\gamma}}{r^2} (K_{22}\phi^2 + K_{33}s^2) \\
& + \frac{\theta_{,r}}{r} (K_{22}s^2 + K_{33}\phi^2) - (K_{11} + K_{33})\phi\phi_{,r} \\
& - \frac{qs(K_{11} + K_{33})}{r} \varphi_{,\gamma} - K_{33}q^2\theta + \gamma\theta = 0
\end{aligned}$$

Equations (6b) reduce to

$$\begin{aligned}
p_{,r} &= -qs\phi(\mu_2 + \mu_3)v_{,r} \\
\frac{1}{r}p_{,\gamma} &= q(\mu_2\phi^2 - \mu_3s^2)v_{,r} + \mu_2q(\phi\theta_{,z} - s\phi\theta) \\
&+ \frac{(\mu_6 - \mu_3)}{2} [v_{,r}(\phi^2 - s^2)q] \\
p_{,z} &= \theta_{,rr}qs(K_{11} - K_{22}) + \frac{\theta_{,z\gamma}}{r}q\phi(K_{22} - K_{11}) \\
&- K_{22}q\varphi_{,zz} + \frac{2\varphi_{,\gamma r}}{r}qs\phi(K_{11} - K_{33}) \\
&- \varphi_{,rr}(K_{11}s^2 + K_{33}\phi^2)q - \frac{\varphi_{,\gamma\gamma}}{r^2}q(K_{11}\phi^2 + K_{33}s^2) \\
&+ \frac{2\varphi_{,\gamma}}{r^2}(K_{33} - K_{11})qs\phi - q\phi\theta_{,r}(K_{33} + K_{22})
\end{aligned}$$

$$\begin{aligned}
& - q(K_{11}c^2 + K_{33}s^2)\frac{\varphi_{,r}}{r} - \frac{\theta_{,r}}{r}q^2s(K_{22} + K_{33}) \\
& + \left[\frac{\mu_4}{2} + \frac{(\mu_5 - \mu_2)}{2}c^2 \right] v_{,rr} \\
& + \frac{v_{,r}}{2r} [\mu_4 + (\mu_5 - \mu_2)s^2] . \tag{15}
\end{aligned}$$

Eliminating γ from equations (13) and (14) we obtain

$$\begin{aligned}
& \theta_{,sr}(K_{22} - K_{11})s + \frac{\theta_{,r}}{r}(K_{11} - K_{22})c + K_{22}\varphi_{,ss} \\
& + \frac{2\varphi_{,rs}}{r}(K_{33} - K_{11})sc + \varphi_{,rr}(K_{11}s^2 + K_{33}c^2) \\
& + \frac{\varphi_{,r}}{r^2}(K_{11}c^2 + K_{33}s^2) + \frac{\varphi_{,r}}{r^2}2(K_{11} - K_{33})sc \\
& + \theta_{,r}qc(K_{22} + K_{33}) + \frac{\theta_{,r}}{r}(K_{11}c^2 + K_{33}s^2) \\
& + \frac{\theta_{,r}}{r}(K_{22} + K_{33})qs + \lambda_1 vq = 0 \tag{16}
\end{aligned}$$

From (15) and (16) one gets

$$\begin{aligned}
& = \frac{v_{,rr}}{2} [\mu_4 + (\mu_5 - \mu_2)c^2] \\
& + \frac{v_{,r}}{2r} [\mu_4 + (\mu_5 - \mu_2)s^2] + \lambda_1 vq^2 .
\end{aligned}$$

As in the previous case here also the coefficients

of $v_{,rr}$ and $v_{,r}$ are replaced by their overage value $\bar{\eta}$ and one gets

$$v_{,rr} + \frac{1}{r}v_{,r} + \frac{\lambda_1 q^2 v}{\bar{\eta}} - \frac{p_{,z}}{\bar{\eta}} = 0.$$

This equation has a well behaved solution given by

$$\frac{\lambda_1 q^2 v}{\bar{\eta}} = \frac{p_{,z}}{\bar{\eta}} + A I_0(kr)$$

where A is a constant, I_0 the modified Bessel function of the first kind and zero order and $k^2 = -\lambda_1 q^2 / \bar{\eta}$.

Using the boundary condition $v(R) = 0$ one has, for the velocity field

$$v(r) = \frac{p_{,z}}{\lambda_1 q^2} \left[1 - \frac{I_0(kr)}{I_0(kR)} \right]$$

The volume of liquid crystal flowing out per second

$$Q = \int_0^R 2\pi r v dr = \frac{2\pi p_{,z}}{\lambda_1 q^2} \left[\frac{R^2}{2} - \frac{R I_1(kR)}{k I_0(kR)} \right]$$

where I_1 is the modified Bessel function of the first kind and first order. The apparent viscosity for this flow is

$$\eta = - \frac{\pi p_s R^4}{8Q} = - \frac{\lambda_1 q^2 R^2}{a \left[1 - \frac{2I_1(kR)}{(kR) I_0(kR)} \right]}$$

In practical situations $k \sim q \sim 10^5$ and $kR \sim qR \sim 10^3$ and we obtain

$$\eta \approx - \frac{\lambda_1 q^2 R^2}{8}$$

which is Helfrich's formula and in this approximation the velocity profile is flat over nearly 95% of the tube section.

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CHAPTER VI

FLOW OF CHOLESTERIC PERPENDICULAR TO THE HELICAL AXIS

Introduction

Theoretical studies by Leslie (1969) and experimental investigation by Candau et al. (1973) on this type of flow have led to some interesting features peculiar to this geometry. On the basis of the continuum theory Leslie (1969) showed that in the case of the flow between two parallel plates with one plate in motion relative to the other at a constant velocity V and with the helical axis normal to the plates the apparent viscosity η varies with shear rate and gap width and that at high values of the shear stress of the order of the square of the inverse pitch η approaches a value which is independent of the shear rate. Leslie also found that secondary flow (i.e., flow perpendicular to the plane of shear) exists in a cholesteric. This is a direct consequence of the intrinsic twist present in the structure. Leslie also proved that at low shear rates the twist persists throughout the gap.

Candau et al. studied the Poiseuille flow of a nematic (viz., Merck IV) twisted by adding cholesteryl propionate in different quantities to produce samples of differed pitch. Their optical observation showed that the helical axis of the structure was everywhere normal to the axis of the tube. (The different layers were folded around the axis of the tube to form coaxial cylinders.) They found that in general for a given pitch, with increasing shear rate η decreases from a constant value at low shear rates and approaches a lower limit at high shear rates. The initial constant value of η itself depends on the pitch. The critical shear rate at which η starts decreasing from its initial constant value is linearly dependent on this square of the inverse pitch. The fact that η drops to a lower limit at high shear rates is in qualitative agreement with Leslie's prediction that η should approach a constant lower limit. But Candau et al. could not go to high enough shear rates and so could not verify Leslie's prediction quantitatively.

Flow between parallel plates with one plate in motion

Following Leslie (1969) one has for isothermal flow

$$n_x = \cos \theta(z) \cos \varphi(z), \quad n_y = \sin \varphi(z) \cos \theta(z), \quad n_z = \sin \theta(z)$$

$$v_x = u(z), \quad v_y = v(z), \quad v_z = 0.$$

The plates are assumed to occupy the planes $z = \pm h/2$.

The plate $z = h/2$ moves along x axis with a constant velocity V . Equations (6) of Chapter V become

$$[H_1(\theta) + H_2(\theta) \cos^2 \varphi] \frac{du}{dz} + H_2(\theta) \sin \varphi \cos \varphi \frac{dv}{dz} = 2a \quad (1)$$

$$[H_1(\theta) + H_2(\theta) \sin^2 \varphi] \frac{dv}{dz} + H_2(\theta) \sin \varphi \cos \varphi \frac{du}{dz} = 0 \quad (2)$$

$$\begin{aligned} f_1(\theta) \frac{d^2 \theta}{dz^2} + \frac{1}{2} \frac{df_1}{d\theta} \left(\frac{d\theta}{dz} \right)^2 - \frac{1}{2} \frac{df_2}{d\theta} \left(\frac{d\varphi}{dz} \right)^2 - 2K_2 \sin \theta \cos \theta \frac{d\varphi}{dz} \\ + \frac{1}{2} (\lambda_1 + \lambda_2 \cos 2\theta) \left[\frac{du}{dz} \cos \varphi + \frac{dv}{dz} \sin \varphi \right] = 0 \end{aligned} \quad (3)$$

$$\begin{aligned} f_2(\theta) \frac{d^2 \varphi}{dz^2} + \frac{df_2}{d\varphi} \frac{d\theta}{dz} \frac{d\varphi}{dz} + 2K_2 \sin \theta \cos \theta \frac{d\theta}{dz} \\ + \frac{(\lambda_1 - \lambda_2) \sin \theta}{2} \cos \theta \left(\frac{du}{dz} \sin \varphi - \frac{dv}{dz} \cos \varphi \right) = 0 \end{aligned} \quad (4)$$

where

$$f_1(\theta) = K_{11} \cos^2 \theta + K_{33} \sin^2 \theta$$

$$f_2(\theta) = (K_{22} \cos^2 \theta + K_{33} \sin^2 \theta) \cos^2 \theta$$

$$H_1(\theta) = \mu_4 + (\mu_5 - \mu_2) \sin^2 \theta$$

$$H_2(\theta) = (2\mu_1 \sin^2 \theta + \mu_3 + \mu_6) \cos^2 \theta$$

and a is the shear stress applied to the fluid.

On solving equations (1) to (4) one finds that θ and v are even functions while φ and $u - \frac{v}{2}$ are odd functions of z . Leslie (1969) has suggested boundary conditions which are such that

$$u\left(\frac{h}{2}\right) = v, \quad u\left(-\frac{h}{2}\right) = v\left(-\frac{h}{2}\right) = v\left(\frac{h}{2}\right) = \theta\left(-\frac{h}{2}\right) = 0$$

$$\frac{d\varphi}{dz}\left(z = +\frac{h}{2}\right) = \frac{d\varphi}{dz}\left(z = -\frac{h}{2}\right) = \frac{K_2}{K_{22}} = q = \frac{2\pi}{P} \quad 5(a)$$

The twist angle φ is not fixed at the plates, but only the derivative of φ is equated to q , the twist in the unperturbed cholesteric. This implies that at any shear rate the cholesteric retains its twist in the vicinity of the plates. Experiments have not so far indicated precisely the boundary condition for the director that one can get in practical situations involving cholesteric flow. In experiments like those performed by Candau et al. a twisted nematic is used as a substitute for the cholesteric. As is well known in the case of a nematic, it is the director orientation that is anchored at the boundary and hence a more reasonable assumption with regard to the boundary conditions would be:

$$u\left(\frac{h}{2}\right) = v, \quad u\left(-\frac{h}{2}\right) = v\left(-\frac{h}{2}\right) = v\left(\frac{h}{2}\right) = \theta\left(\frac{h}{2}\right) = \theta\left(-\frac{h}{2}\right) = 0$$

$$\varphi\left(z = \frac{h}{2}\right) = \frac{h}{2}q = -\varphi\left(z = -\frac{h}{2}\right) \quad 5(b)$$

This now means fixing the angle φ at the boundary rather than its derivative. Though at low shear rates $\frac{d\varphi}{ds} = q$ throughout the gap, at high shear rates $\frac{d\varphi}{ds}$ will no longer be q even in the vicinity of the plates.

A model calculation on the shear flow of twisted MBBA has been performed with boundary condition 5(b). Equations (1) to (4) have been solved for different shears 'a' by the collocation method (using the zeroes of the Legendre polynomials P_{24} and double precision arithmetic). For every value of 'a' there corresponds a velocity V of the moving plate, from which the apparent viscosity η at that shear rate can be calculated from the relation $\eta = ah/V$. A plot of η as a function of V is shown in figure 1 for $h = 50 \mu$, $P = 10 \mu$. The data of Gähwiller (1971) for the viscosity coefficients of MBBA and those of Haller (1972) for the elastic constants of MBBA have been used. At low shear rates (or equivalently at low velocities V of the moving plate) η remains almost constant at $\eta_H (\approx 0.303 \text{ poise})$. On increasing V above a certain critical value η decreases and for high values of V approaches a limiting lower value $\eta_L (\approx 0.262 \text{ poise})$. This behaviour of η is in agreement with the experimental

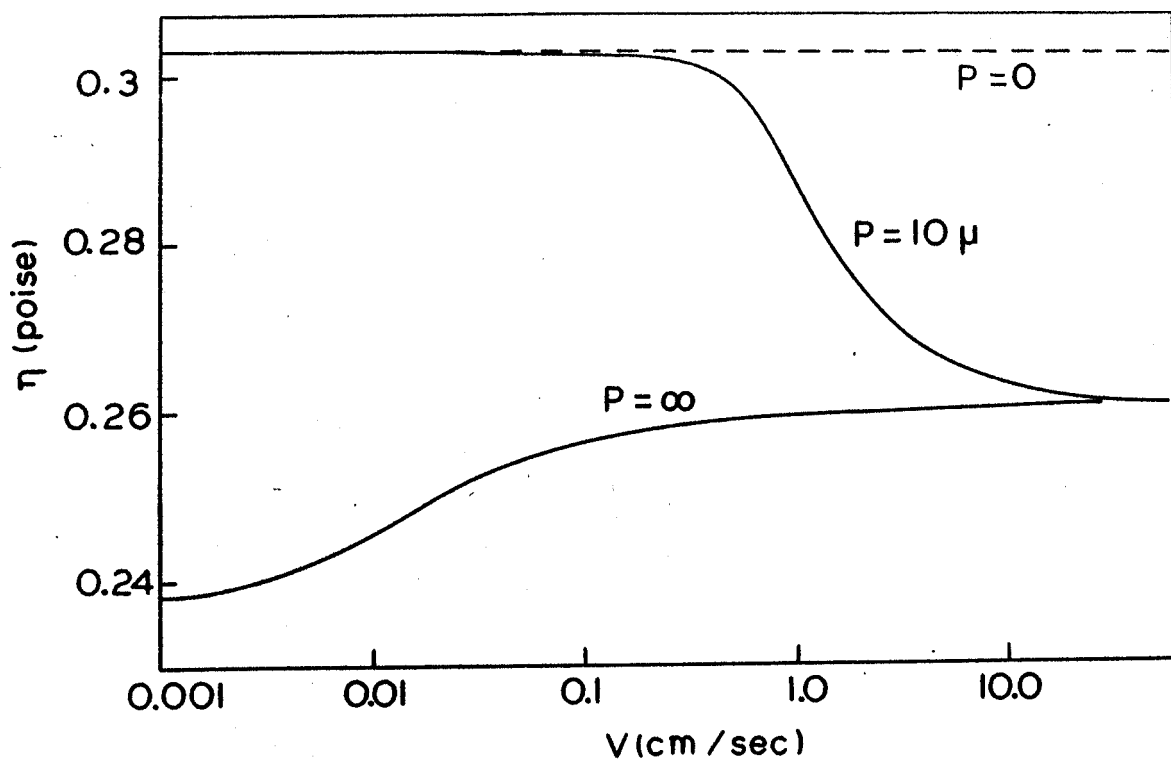


Figure 1

Theoretical variation of apparent viscosity η of twisted MBBA with shear rate for three different pitches.

results of Candau et al. (figure 2). Also η_L corresponds to the viscosity of a nematic in which the director is oriented at an angle $\theta_0 = \frac{1}{2} \cos^{-1}(-\frac{\lambda_1}{\lambda_2})$ with the direction of flow. This result is in agreement with Leslie's predictions concerning the flow at high shear rates.

If the pitch is very small ($P \rightarrow 0$) considerable energy is required to deform the structure and therefore one may expect the structure to be undeformed with $\varphi \approx qz$ and $\theta \approx 0$ even upto very high shear rates. Using equations (1) and (2) one finds that

$$u(z) = \frac{a(2H_1 + H_2)}{H_1(H_1 + H_2)} \left(\frac{h}{2} + z\right) - \frac{aH_2}{2qH_1(H_1 + H_2)} (\sin 2qz + \sin qh)$$

$$v(z) = \frac{aH_2}{2qH_1(H_1 + H_2)} (\cos 2qz - \cos qh).$$

The velocity V of the moving plate is

$$V = \frac{ah(2H_1 + H_2)}{H_1(H_1 + H_2)} - \frac{aH_2}{H_1(H_1 + H_2)} \frac{\sin qh}{q}$$

so that the apparent viscosity

$$\eta = \frac{H_1(H_1 + H_2)}{2H_1 + H_2 - H_2 \frac{\sin qh}{qh}} \rightarrow \frac{a_1(H_1 + H_2)}{2H_1 + H_2} \quad \text{for } q \rightarrow a.$$

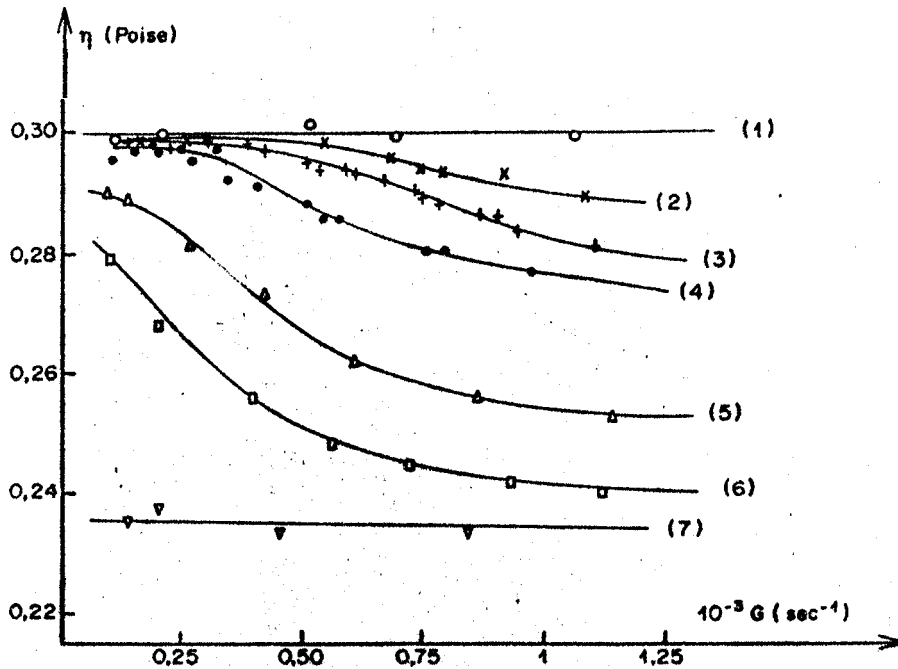


Figure 2: Apparent viscosity in Poiseuille flow as a function of shear rate. Flow normal to the helical axis. Pitch (μm) (1) 1.9, (2) 2.6, (3) 3, (4) 3.9, (5) 6, (6) 9.1 and (7) ∞ . (After Candau, Martinoty and Debeauvais 1973).

Thus η is independent of the shear rate (which for MBBA is approximately 0.3027 poise). This is indicated by the horizontal dotted line in figure 1.

For very large pitches ($P \rightarrow \infty$, $q \rightarrow 0$) the permanent twist in the structure reduces to zero so that one can put $\varphi = 0$, $K_2 = 0$. Equation (2) indicates that the secondary velocity cannot exist. Equation (4) is identically satisfied. Equations (1) and (3) reduce to

$$[H_1(\theta) + H_2(\theta)] \frac{du}{dz} = 2a$$

and

$$r_1(\theta) \frac{d^2\theta}{dz^2} + \frac{1}{2} \frac{dr_1}{d\theta} \left(\frac{d\theta}{dz} \right)^2 + \frac{a(\lambda_1 + \lambda_2 \cos 2\theta)}{H_1(\theta) + H_2(\theta)} = 0$$

which have to be solved with the boundary conditions

$$\theta(-\frac{h}{2}) = \theta(\frac{h}{2}) = u(-\frac{h}{2}) = 0, \quad u(\frac{h}{2}) = v.$$

At $\text{Pow} \sim h$ a writes $\theta = 0$ so that the apparent viscosity

$$\eta_{11} = \frac{(\mu_3 + \mu_4 + \mu_6)}{2} \quad (= 0.238 \text{ poise for MBBA})$$

At very high shear rates $\theta \approx \theta_0 \equiv \frac{1}{2} \cos^{-1}(-\frac{\lambda_1}{\lambda_2})$
(= 0.1658 radians) and

$$\eta_{\theta_0} = \frac{H_1(\theta_0) + H_2(\theta_0)}{2} \quad (= 0.262 \text{ poise for MBBA}).$$

Thus on increasing the shear rate η increases from η_{11} attaining η_{θ_0} at high shear rates. This is illustrated by the curve at the bottom of figure 1. (It may be pointed out that if $\lambda_1 = -\lambda_2$ as is the case for PAA, η would not show an increase with increase of shear rate.)

The variation of η with pitch is also quite interesting to facilitate analytical discussion the region of very low shear rates has been chosen. At low shears since the cholesteric is not distorted much one can write $\varphi = qs + \theta(z)$ where θ is small. Second and higher powers of θ , u , v and θ are assumed to be negligible. Noting that $\cos \varphi \approx \cos qs - \theta \sin qs$, $\sin \varphi \approx \sin qs + \theta \cos qs$ we can write eqns.(1) and (2) as

$$[H_1 + H_2 \cos^2 qs] \frac{du}{dz} + H_2 \sin qs \cos qs \frac{dv}{dz} = a \quad (6)$$

$$H_2 \sin qs \cos qs \frac{du}{dz} + [H_1 + H_2 \sin^2 qs] \frac{dv}{dz} = 0 \quad (7)$$

solving which we obtain

$$\frac{du}{dz} = \frac{2a(H_1 + H_2 \sin^2 qs)}{H_1(H_1 + H_2)} \quad (8)$$

$$\frac{dv}{dz} = \frac{2a H_2 \sin qs \cos qs}{H_1(H_1 + H_2)} \quad (9)$$

where

$$H_1 = \mu_4, \quad H_2 = \mu_3 + \mu_6$$

Solving (8) and (9) with the boundary conditions (5b) one gets

$$u(z) = \frac{a(2H_1 + H_2)}{H_1(H_1 + H_2)} \left(\frac{h}{2} + z \right) - \frac{aH_2}{2qH_1(H_1 + H_2)} (\sin 2qs + \sin qh)$$

$$v(z) = \frac{aH_2}{2qH_1(H_1 + H_2)} (\cos 2qs - \cos qh) .$$

The velocity V of the plat. $z = \frac{h}{2}$ is

$$V = \frac{ah(2H_1 + H_2)}{H_1(H_1 + H_2)} - \frac{aH_2}{H_1(H_1 + H_2)} \frac{\sin qh}{q}$$

and therefore the apparent viscosity is given by

$$\eta = \eta(q) = \frac{H_1(H_1 + H_2)}{2H_1 + H_2 - H_2 \frac{\sin qh}{qh}} \quad (10)$$

and is independent of the shear.

Observe that where as $v(z)$ is an even function of z ,

$u(z) - \frac{V}{2}$ is an Odd function

$$u(z) - \frac{V}{2} = - \frac{a(2H_1 + H_2)z}{H_1(H_1 + H_2)} - \frac{aH_2 \sin 2qs}{2qH_1(H_1 + H_2)}$$

When $q \rightarrow 0$ (the case of a nematic) $\frac{\sin qh}{qh} \rightarrow 1$ and

$$\eta(0) \rightarrow \frac{H_1 + H_2}{2} = \frac{\mu_3 + \mu_4 + \mu_6}{2}$$

which is the η value for a nematic which is flowing with all molecules oriented along the flow direction. On the other hand when q becomes extremely large (the case of a highly twisted cholesteric) $\sin qh/qh \approx 0$ and hence

$$\eta(\infty) \approx \frac{H_1(H_1 + H_2)}{2H_1 + H_2} = \frac{\mu_4(\mu_3 + \mu_4 + \mu_6)}{2\mu_4 + \mu_3 + \mu_6}$$

A plot of $\eta(q)/\eta(0)$ as a function of the pitch $P = 2\pi/q$ is given in fig. 3 for PAA and for MBBA whose viscosity coefficients were obtained by Gahwiller (1971). A sample thickness of 100μ D₀ used. The curve tends to the value 1 for large pitches P . In the region of small pitches the function oscillates around the value $\eta(\infty)/\eta(0)$ as pitch changes. As pitch increases the extrema get more widely spaced and differ more and more from $\eta(\infty)/\eta(0)$. Then when the pitch attains a value P_M the function shows the highest maximum. On increasing the pitch further the function decreases, attaining the value 1 at very large pitches. The

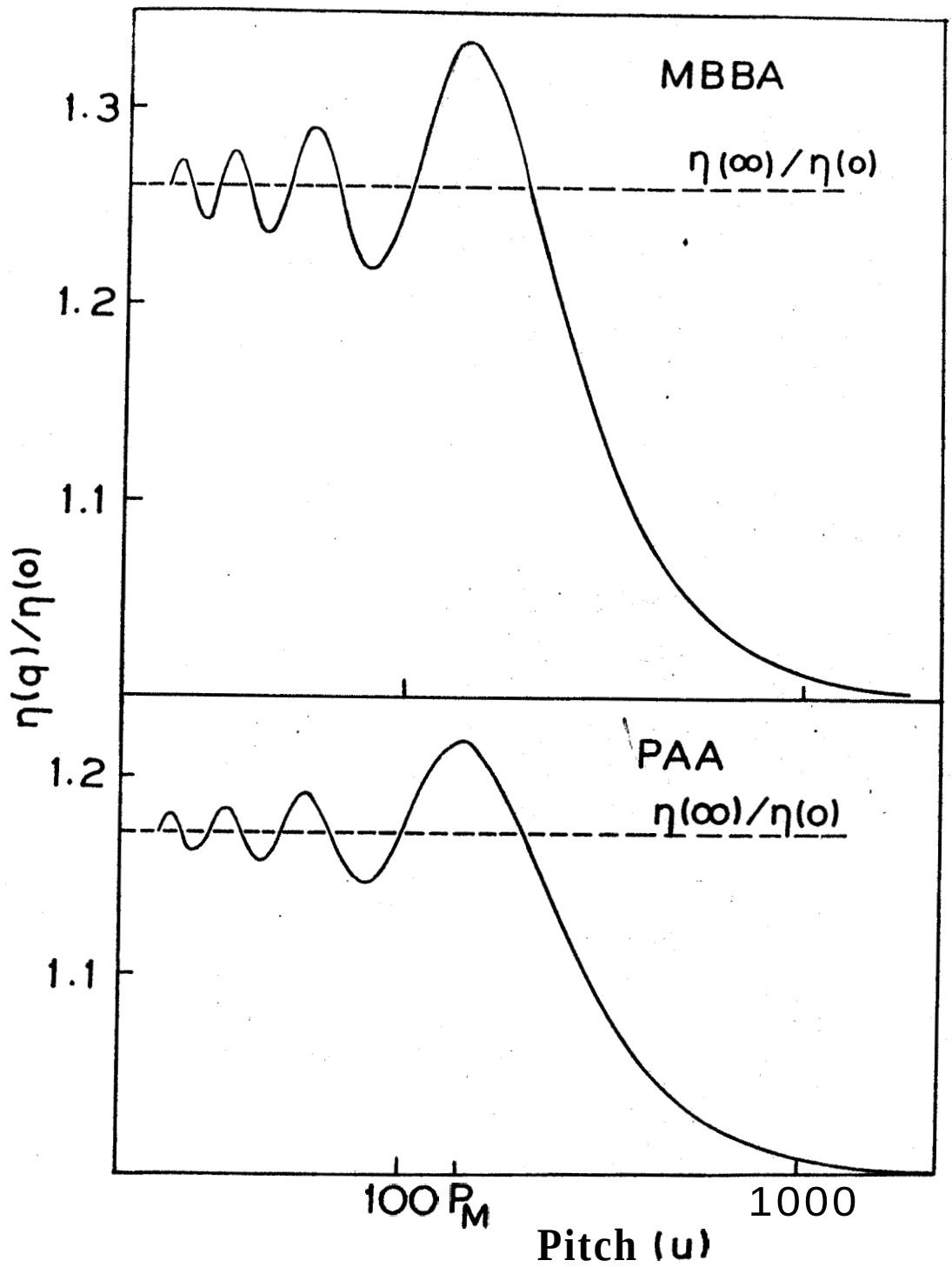


Figure 3: Variation of $\eta(q)/\eta(0)$ with pitch in shear flow.

extrema of $\eta(q)$ can be found from (10) and are the roots of $\tan qh = qh$. The trivial solution $q = 0$ corresponds to the case of an untwisted nematic. The next higher root occurs when $qh = 4.485$ and corresponds to the highest maximum. Thus $P_M = 2\pi h/4.485$. If η is determined as a function of the pitch at low shear rates one can see whether these interesting effects do indeed take place. This could serve as a good test for the theory. Also, by knowing η as a function of the pitch the values of μ_4 and $\mu_3 + \mu_6$ can be worked out.

The deformations θ and ϕ can be determined by solving (3) and (4). To first approximation one finds that $\phi = 0$ and that

$$\theta = \frac{a(\lambda_1 + \lambda_2)}{(k^2 + q^2)K_{11}(\mu_3 + \mu_4 + \mu_6)} \left[\cos qz - \frac{\cosh kz \cos qh/2}{\cosh kh/2} \right]$$

where $k^2 = K_{33}q^2/K_{11}$.

Poiseuille flow: In this case solutions for the director and the velocity field are sought in cylindrical polar coordinates in the form

$$n_x = \sin \theta(r) \sin \varphi(r), \quad n_\varphi = \cos \theta(r) \sin \varphi(r),$$

$$n_z = \cos \varphi(r)$$

$$v_r = 0, \quad v_\psi = r w(r), \quad v_z = v(r).$$

The flow is assumed to take place in a pipe of radius R under a constant pressure gradient $p_{,z} (=A)$. w is the angular velocity corresponding to the secondary flow. From equations (6) of Chapter V one obtains the following set of exact equations (in obvious notation)

$$\begin{aligned} & \varphi_{,rr}(K_{11}-K_{22})s_\theta c_\theta s_\varphi c_\varphi + \theta_{,rr}(K_{11}c_\theta^2 s_\varphi^2 + K_{22}s_\theta^2 c_\varphi^2 + K_{33}s_\varphi^4 s_\theta^2) \\ & + \varphi_{,r}^2(2K_{22}-K_{11}-K_{33})s_\theta c_\theta s_\varphi^2 - \theta^2(I_{33}s_\varphi^2 + K_{22}c_\varphi^2 - K_{11})s_\theta c_\theta s_\varphi^2 \\ & + \theta_{,r}\varphi_{,r}(2K_{11}c_\theta^2 + K_{22}s_\theta^2 + K_{22}s_\theta^2 c_\varphi^2 - 3K_{22}s_\theta^2 s_\varphi^2 + 4K_{33}s_\theta^2 s_\varphi^2)s_\varphi a_\varphi \\ & + \frac{\theta_{,r}}{r}(K_{11}c_\theta^2 s_\varphi^2 + K_{22}s_\theta^2 c_\varphi^2 + K_{33}s_\theta^2 s_\varphi^4) + \frac{1}{2}(-K_{11} + K_{22}c_\varphi^2 \\ & + K_{33}s_\varphi^2)s_\theta c_\theta s_\varphi^2 + \frac{\varphi_{,r}}{r}(K_{11}c_\theta c_\varphi + 3K_{22}c_\theta c_\varphi s_\varphi^2 - 2K_{22}c_\theta c_\varphi^3 \\ & + 2K_{22}c_\varphi s_\varphi - 4K_{33}c_\theta c_\varphi s_\varphi^2)s_\theta s_\varphi + \frac{r}{2} \frac{dw}{dr}[(\lambda_1 + \lambda_2)c_\theta^2 \\ & + (\lambda_1 - \lambda_2)s_\theta^2]s_\varphi^2 + \frac{(\lambda_1 + \lambda_2)}{2} \frac{dv}{dr}c_\theta s_\varphi c_\varphi = 0 \end{aligned} \quad (11)$$

$$\begin{aligned} & \varphi_{,rr}(K_{11}s_\theta c_\varphi^2 + K_{22}s_\theta c_\theta^2 s_\varphi^2 + K_{33}s_\theta^3 s_\varphi^2) + \theta_{,rr}[K_{11}-K_{22}s_\theta^2 s_\varphi^2 \\ & + K_{33}s_\theta^2 s_\varphi^2]c_\theta s_\varphi c_\varphi + \frac{(\lambda_1 + \lambda_2)}{2} r \frac{dw}{dr} c_\theta s_\varphi c_\varphi \\ & + \varphi_{,r}^2(-K_{11} + 2K_{22}c_\theta^2 - K_{33} + 2K_{33}s_\theta^2)s_\theta s_\varphi c_\varphi + \frac{\theta_{,r}}{r}[K_{11}c_\theta s_\varphi c_\varphi \\ & - 2K_{22}c_\varphi s_\theta^2 s_\varphi^2 + (5K_{33} - 5K_{22})s_\theta^2 s_\varphi^3 c_\theta c_\varphi] + \theta_{,r}^2[-K_{11} - K_{11}c_\theta^2 s_\varphi^2 \end{aligned}$$

$$\begin{aligned}
& + 2K_{22}s_{\theta}^2s_{\varphi}^2 - K_{33}s_{\theta}^2s_{\varphi}^2 + 2K_{33}c_{\theta}^2s_{\varphi}^2 - K_{33}s_{\theta}^2] s_{\theta}s_{\varphi}c_{\varphi} \\
& + \theta_{,r}\varphi_{,r}[2K_{11}c_{\theta}^2c_{\varphi}^2 + K_{22}s_{\theta}^2c_{\theta}s_{\varphi}^4 - 3K_{22}s_{\theta}^2c_{\theta}s_{\varphi}^2 \\
& - 3K_{22}s_{\theta}^2c_{\theta}s_{\varphi}^2c_{\varphi}^2 + 2K_{33}s_{\theta}^2c_{\theta}s_{\varphi}^4 + 6K_{33}s_{\theta}^2c_{\theta}s_{\varphi}^2c_{\varphi}^2] \\
& + \frac{\varphi_{,r}}{r}[K_{11}s_{\theta}c_{\varphi}^2 + 2K_{22}c_{\theta}c_{\varphi}s_{\theta}s_{\varphi} - K_{22}s_{\theta}c_{\theta}s_{\varphi}^4 \\
& + 3K_{22}c_{\theta}^2s_{\varphi}^2c_{\varphi}^2s_{\theta} + 2K_{22}s_{\theta}c_{\theta}^2s_{\varphi}^2 - 4K_{33}s_{\theta}c_{\theta}^2s_{\varphi}^2c_{\varphi}^2 + K_{33}s_{\theta}^3s_{\varphi}^2] \\
& + \frac{1}{r^2}[-K_{11}s_{\theta}s_{\varphi}c_{\varphi} + 2K_{22}c_{\theta}c_{\varphi}s_{\theta}s_{\varphi}^2 + K_{22}c_{\varphi}^2s_{\varphi}^3s_{\theta} - K_{33}s_{\theta}c_{\theta}^2s_{\varphi}^3c_{\varphi}] \\
& + \frac{1}{2} \frac{d\varphi}{dr} [(\lambda_1 + \lambda_2)c_{\varphi}^2 + (\lambda_1 - \lambda_2)s_{\theta}^2s_{\varphi}^2] = 0 \quad (12)
\end{aligned}$$

$$r \frac{dw}{dr} g_2(\theta, \varphi) + \frac{d\varphi}{dr} g_5(\theta, \varphi) = \frac{Ar}{2} + \frac{B}{r} \quad (13)$$

$$\begin{aligned}
& K_{22}s_{\theta}^2s_{\varphi}^3c_{\varphi} \theta_{,r} \varphi_{,r} + \varphi_{,r}^2 (K_{33}s_{\varphi}^2 - K_{22}) s_{\theta}c_{\theta}s_{\varphi}^2 \\
& + \frac{\varphi_{,r}}{r}[K_{22}c_{\theta}c_{\varphi}^3s_{\theta}s_{\varphi} - K_{22}c_{\theta}c_{\varphi}s_{\varphi}^3s_{\theta} - K_{22}c_{\theta}r s_{\theta}s_{\varphi}^2] \\
& + \frac{1}{r} \left[\frac{d}{dr} r^3 \frac{dw}{dr} g_1(\theta, \varphi) + r^2 \frac{d\varphi}{dr} g_2(\theta, \varphi) \right] = 0 \quad (14)
\end{aligned}$$

where

$$\begin{aligned}
g_1(\theta, \varphi) &= \mu_1 s_{\theta}^2 c_{\theta}^2 s_{\varphi}^4 + \frac{\mu_4}{2} + \frac{(\mu_5 - \mu_2)}{2} s_{\theta}^2 s_{\varphi}^2 \\
&\quad + \frac{(\mu_3 + \mu_6)}{2} c_{\theta}^2 s_{\varphi}^2, \\
g_2(\theta, \varphi) &= \mu_1 s_{\theta}^2 c_{\theta} s_{\varphi}^3 c_{\varphi} + \frac{(\mu_3 + \mu_6)}{2} c_{\theta} s_{\varphi} c_{\varphi},
\end{aligned}$$

$$g_5(\theta, \varphi) = \mu_1 s_\theta^2 c_\varphi^2 s_\varphi^2 + \frac{(\mu_5 - \mu_2)}{2} s_\theta^2 s_\varphi^2 + \frac{\mu_4}{2} + \frac{(\mu_3 + \mu_6)}{2} c_\varphi^2.$$

The constant B in (13) is assumed to be zero to ensure that the velocity gradients and hence the velocities do not attain infinite values. In order to ensure a single valued pressure $\partial p / \partial \varphi$ has been assumed to be equal to zero. The form of the equations leads to the director orientation being singular at the origin. Candau et al. (1973) have observed the singularity along the axis of the flow. Also equations (11) to (14) are exactly satisfied by the equilibrium configuration of $v = w = 0 = \theta$ and $\varphi = qr$.

At very low shear rates the cholesteric structure will be hardly deformed. Therefore one can assume that $\varphi = qr + \theta(r)$ where θ, θ, u, v are ^{all} small (so that higher products of these quantities and of their derivatives can be ignored). Using equations (11) and (12) and putting $S = \sin(qr)$ and $C = \cos(qr)$ one finds that

$$\left[-\frac{9}{r}(2K_{22}S^2 + K_{22}C^4) + \frac{K_{22}S^3C + 2K_{33}S^3C - K_{22}S^3C}{r^2} - \frac{2K_{22}qS^2}{r} \right] \theta = 0$$

which implies that θ is zero. Hence eqn. (13) yields

$$\frac{d}{dr} \left(r^3 \frac{dw}{dr} g_1(qr) + r^2 \frac{dv}{dr} g_2(qr) \right) = 0$$

or

$$r \frac{dw}{dr} \varepsilon_1(qr) + \frac{dv}{dr} \varepsilon_2(qr) = \frac{\text{const}}{r^2}.$$

Again, in order to ensure finite velocity gradients we put $\text{const} = 0$. Thus the velocities v and w are determined from the equations

$$r \frac{dw}{dr} \varepsilon_1(qr) + \frac{dv}{dr} \varepsilon_2(qr) = 0 \quad (15)$$

$$r \frac{dw}{dr} \varepsilon_2(qr) + \frac{dv}{dr} \varepsilon_5(qr) = \frac{Ar}{2} \quad (16)$$

where

$$\varepsilon_1(qr) = \frac{\mu_4 + (\mu_3 + \mu_6)s^2}{2}$$

$$\varepsilon_2(qr) = \frac{(\mu_3 + \mu_6)so}{2}$$

$$\varepsilon_5(qr) = \frac{\mu_4 + (\mu_3 + \mu_6)o^2}{2}$$

Hence

$$\frac{dv}{dr} = \frac{A[\mu_4 + (\mu_3 + \mu_6)\sin^2 qr]r}{\mu_4(\mu_3 + \mu_4 + \mu_6)} \quad (17)$$

$$\frac{dw}{dr} = -\frac{A(\mu_3 + \mu_6)\sin 2qr}{2\mu_4(\mu_3 + \mu_4 + \mu_6)} \quad (18)$$

At the wall of the tube v and w vanish. Hence

$v(R) = 0 = w(R)$ (18) yields

$$w(r) = \frac{\Lambda(\mu_3 + \mu_6)}{4\mu_4(\mu_3 + \mu_4 + \mu_6)} (\cos 2qr - \cos 2qR)$$

From (19) one gets,

$$\begin{aligned} v(r) = & \frac{\Lambda(2\mu_4 + \mu_3 + \mu_6)(r^2 - R^2)}{4\mu_4(\mu_3 + \mu_4 + \mu_6)} \\ & + \frac{\Lambda(\mu_3 + \mu_6)}{2\mu_4(\mu_3 + \mu_4 + \mu_6)} \left[\frac{R \sin 2qR - r \sin 2qr}{2q} \right. \\ & \left. + \frac{\cos 2qR - \cos 2qr}{4q^2} \right] \end{aligned}$$

The volume rate of fluid flow $A_L = 2\pi \int_0^a r v(r) dr$

is determined and the apparent viscosity η is calculated from the Poiseuille formula

$$\eta = - \frac{\pi \Lambda R^4}{8A_L}$$

$$\eta(q) = \frac{\mu_4(\mu_3 + \mu_4 + \mu_6)}{\left\{ 2\mu_4 + \mu_3 + \mu_6 + (\mu_3 + \mu_6) \left[-\frac{2\sin 2qR}{qR} - \frac{3\cos 2qR}{q^2 R^2} \right. \right.} \\ \left. \left. + \frac{3\sin 2qR}{q^3 R^3} + \frac{3(\cos 2qR - 1)}{2q^4 R^4} \right] \right\}}$$

which is again independent of the applied shear.

For small pitches ($q \rightarrow \infty$)

$$\eta(\infty) = \frac{\mu_4(\mu_3 + \mu_4 + \mu_6)}{(2\mu_4 + \mu_3 + \mu_6)}$$

For large pitches ($q \rightarrow 0$)

$$\eta(0) = \frac{(\mu_3 + \mu_4 + \mu_6)}{2}$$

The values of $\eta(\infty)$ and $\eta(0)$ are the same as in the case of flow between parallel plates. $\eta(q)/\eta(0)$ has been plotted as a function of the pitch in figure 4, for twisted PAA and MBBA.

The extrema of $\eta(q)$ are in this case determined by the equation

$$x \sin 2x(8x^2 - 12) + \cos 2x(-4x^4 + 12x^2 - 6) + 6 = 0$$

where $x = qR$. $x = 0$ corresponds to the untwisted nematic. The next higher root $x = 1.892$, $q = q_M$ corresponds to the highest maximum. It is in principle possible to obtain μ_4 and $\mu_3 + \mu_6$ if one can measure η as a function of q at very low shear rates.

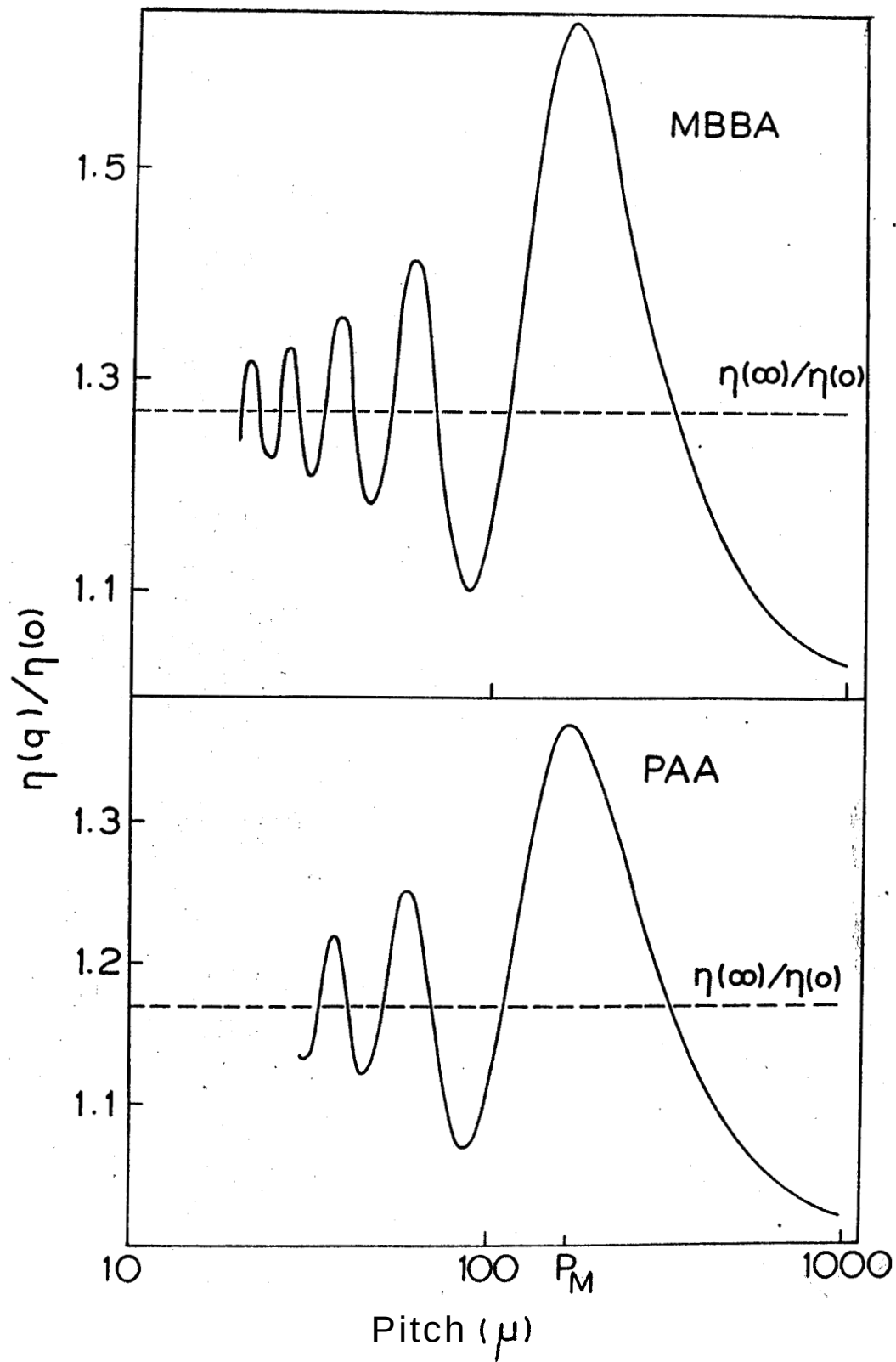


Figure 4: Variation of $\eta(q)/\eta(0)$ with pitch in Poiseuille flow

Before concluding this Chapter it may be mentioned that the apparent viscosity at low shear rates behaves in a markedly similar manner in both the geometries as a function of pitch. However the more pronounced variation occurs only for the flow through a pipe, where η attains nearly 1.6 times the nematic value for MBBA. Another observation is that in general $\eta(q_M)/\eta(0)$ has a higher value for MBBA than for PAA. This appears to be due to the viscosity coefficients of MBBA being higher than those of PAA by nearly an order of magnitude.

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