

CHAPTER IV

THE COUETTE FLOW OF NEMATICS

Introduction

In Couette flow the fluid occupies the annular space between two coaxial cylinders of different radii, rotating about their common axis with different angular velocities. In recent years some theoretical and experimental studies have been made on the Couette flow of nematics. Atkin and Leslie (1970) gave the analytical solution and scaling analysis for the field-free case and found qualitative agreement with the experiments of Porter and Johnson (1962) who found an increase in the apparent viscosity η with a decrease in the gap width. Currie (1970) investigated a number of possible solutions in the presence of magnetic fields. However he did not work out the apparent viscosity. In this chapter the simplest and most likely solutions for Couette flow are presented together with detailed calculations on the apparent viscosity as a function of shear rate (or equivalently the relative angular velocity between the cylinders), of radial and azimuthal magnetic fields and also of the geometry of the viscometer, for symmetric and asymmetric boundary conditions of

director alignment at the cylinders.

Theory

We shall first give a more simple and direct method for deriving the differential equations governing the Couette flow of nematics in the presence of magnetic fields than that used by Currie. The nematic is assumed to be confined in the annular space between two coaxial cylinders of radii R_1 and R_2 ($R_2 > R_1$) with their common axis along z . The cylinders are assumed to rotate with angular velocities Ω_1 and Ω_2 respectively. We consider only planar flows and seek steady state solutions for the director and velocity fields in cylindrical coordinates in the form

$$\begin{aligned} n_r &= \sin \theta(r) & n_\psi &= \cos \theta(r) & n_z &= 0 \\ v_r &= 0 & v_\psi &= r w(r) & v_z &= 0 \end{aligned} \quad (1)$$

The magnetic field components have to depend only on r so that they might be consistent with the symmetry of the director orientation and H_z should be zero. Maxwell's equations demand that the field components be of the form

$$H_r = \frac{A}{r}, \quad H_\psi = \frac{B}{r}, \quad H_z = 0 \quad (2)$$

where A and B are constants.

The surviving components of the stresses and body forces are

$$\begin{aligned}\pi_{rr} = & K_{11} \left[\cos \theta \frac{d\theta}{dr} + \frac{\sin \theta}{r} \right] \\ & + K_{33} \left[\sin^2 \theta \cos \theta \frac{d\theta}{dr} - \frac{\sin \theta \cos^2 \theta}{r} \right]\end{aligned}$$

$$\begin{aligned}\pi_{\psi\psi} = & K_{11} \left[\cos \theta \frac{d\theta}{dr} + \frac{\sin \theta}{r} \right] \\ & + K_{33} \left[\frac{\sin \theta \cos^2 \theta}{r} - \sin^2 \theta \cos \theta \frac{d\theta}{dr} \right]\end{aligned}$$

$$\pi_{r\psi} = K_{33} \left[\frac{\sin^2 \theta \cos \theta}{r} - \sin^3 \theta \frac{d\theta}{dr} \right]$$

$$\pi_{\psi r} = K_{33} \left[\sin \theta \cos^2 \theta \frac{d\theta}{dr} - \frac{\cos^3 \theta}{r} \right]$$

$$\begin{aligned}g_r = & K_{33} \left[\frac{\cos \theta}{r} \frac{d\theta}{dr} - \sin \theta \left(\frac{d\theta}{dr} \right)^2 \right] \\ & + \frac{(\lambda_1 + \lambda_2)}{2} r \frac{dw}{dr} \cos \theta + \gamma \sin \theta\end{aligned}$$

$$\begin{aligned}g_\psi = & -K_{33} \left[\frac{\cos \theta}{r^2} - \frac{\sin \theta}{r} \right] + \frac{(\lambda_2 - \lambda_1)}{2} r \frac{dw}{dr} \sin \theta \\ & + \gamma \cos \theta\end{aligned}$$

$$G_r = \frac{A(\Delta\chi)}{2} (A \sin \theta + B \cos \theta)$$

$$G_{\psi} = \frac{B(\Delta\chi)}{r^2} (A \sin \theta + B \cos \theta)$$

$$t_{rr} = -p - f(\theta) \left(\frac{d\theta}{dr}\right)^2 + \frac{1}{2r} \frac{df}{d\theta} \frac{d\theta}{dr} + r \frac{dw}{dr} g_1(\theta)$$

$$\equiv -p - \hat{t}_{rr}$$

$$t_{\psi\psi} = -p - \frac{f(\theta)}{r^2} + \frac{1}{2r} \frac{df}{d\theta} \frac{d\theta}{dr} + r \frac{dw}{dr} g_2(\theta) \equiv -p - \hat{t}_{\psi\psi}$$

$$t_{r\psi} = \frac{f(\theta)}{r} \frac{d\theta}{dr} - \frac{1}{2r^2} \frac{df}{d\theta} + g(\theta) r \frac{dw}{dr}$$

$$t_{\psi r} = \frac{1}{r} \frac{d\theta}{dr} \left[\frac{1}{2} \frac{d^2 f}{d\theta^2} + f \right] - \frac{1}{2} \frac{df}{d\theta} \left(\frac{d\theta}{dr}\right)^2 \\ + \frac{r}{2} \frac{dw}{dr} [2g(\theta) + \lambda_1 + \lambda_2 \cos 2\theta]$$

$$f_r = -\frac{\chi}{r^3} (A^2 + B^2) - \frac{(\Delta\chi)}{r^3} (A \sin \theta + B \cos \theta)^2$$

$$f_{\psi} = \frac{(\Delta\chi)}{r^3} [\sin \theta \cos \theta (A^2 - B^2) + AB \cos 2\theta] \quad (3)$$

$$\text{where } f(\theta) = K_{11} \cos^2 \theta + K_{33} \sin^2 \theta$$

$$2g(\theta) = 2\mu_1 \sin^2 \theta \cos^2 \theta + \mu_4 + (\mu_5 - \mu_2) \sin^2 \theta \\ + (\mu_6 + \mu_3) \cos^2 \theta$$

$$2g_1(\theta) = 2\mu_1 \sin^3 \theta \cos \theta + (\mu_2 + \mu_3 + \mu_5 + \mu_6) \sin \theta \cos \theta$$

$$2g_2(\theta) = 2\mu_1 \sin \theta \cos^3 \theta + (\mu_5 + \mu_6 - \mu_2 - \mu_3) \sin \theta \cos \theta .$$

The expressions for π_{jk} and g_m are not identical to those derived by Atkin and Leslie (1970). This is because these authors have used the expression for 'free energy density (W) proposed by Leslie (1968) which includes terms with coefficients K_{22} and K_{24} in addition to those containing K_{11} and K_{33} . But here Frank's expression for W has been used. The deformations in plane Couette flow do not involve twist. Hence no terms containing K_{22} are encountered. It has already been pointed out that K_{24} does not affect the equations of motion. In this Chapter one is concerned only with equations of motion whose solution yields the apparent viscosity, and not in the absolute values of π_{ji} and

Also the final equations governing the director orientation and velocity profile (equations (11) and (10)) reduce to the equations (3.17) and (3.16) of Atkin and Leslie (1970) for zero magnetic fields.

The equations (23) of Chapter I can now be written as

$$\begin{aligned} \frac{d\pi_{xx}}{dr} + \frac{(\pi_{xx} - \pi_{\psi\psi})}{r} + G_x + g_x + \rho_1 w^2 \sin \theta &= 0 \\ \frac{d\pi_{x\psi}}{dr} + \frac{(\pi_{x\psi} + \pi_{\psi x})}{r} + G_\psi + g_\psi + \rho_1 w^2 \cos \theta &= 0 \end{aligned} \quad (4)$$

$$\frac{d\hat{t}_{rr}}{dr} + \frac{(t_{rr} - t_{\psi\psi})}{r} + f_r - \frac{\partial p}{\partial r} = -\rho r\omega^2 \quad (5)$$

$$\frac{dt_{r\psi}}{dr} + \frac{1}{r}(t_{r\psi} + t_{\psi r}) + f_{\psi} - \frac{\partial p}{\partial \psi} = 0 \quad (6)$$

$$\frac{\partial p}{\partial z} = 0 \quad (7)$$

From equations (4) one gets

$$\begin{aligned} 2f(\theta) \left[\frac{d^2\theta}{dr^2} + \frac{1}{r} \frac{d\theta}{dr} \right] r^2 + r^2 \frac{df}{d\theta} \left[\frac{1}{r^2} + \left(\frac{d\theta}{dr} \right)^2 \right] \\ + r^3 \frac{dw}{dr} (\lambda_1 + \lambda_2 \cos 2\theta) \\ + 2(\Delta\chi) [\sin \theta \cos \theta (A^2 - B^2) + AB \cos 2\theta] = 0 \quad (8) \end{aligned}$$

On separating variables equations (5) and (6) yield

$$p = E\psi + L(r)$$

$$\frac{dL}{dr} = \rho r\omega^2 + \frac{d\hat{t}_{rr}}{dr} + \frac{(t_{rr} - t_{\psi\psi})}{r} + f_r$$

where we equate the constant E to zero to ensure that the solution is single valued. Then equation (6) becomes

$$\frac{dt_{r\psi}}{dr} + \frac{(t_{r\psi} + t_{\psi r})}{r} + f_{\psi} = 0$$

or equivalently

$$\begin{aligned}
 & 2r(\theta) \left[\frac{d^2\theta}{dr^2} + \frac{1}{r} \frac{d\theta}{dr} \right] r^2 + \frac{df}{d\theta} \left[\frac{1}{r^2} + \left(\frac{d\theta}{dr} \right)^2 \right] r^2 \\
 & + r^3 \frac{dw}{dr} [\lambda_1 + \lambda_2 \cos 2\theta] \\
 & + 2(\Delta\chi) [\sin \theta \cos \theta (A^2 + B^2) + AB \cos 2\theta] \\
 & + 2r^3 \frac{d}{dr} \left[r \frac{dw}{dr} g(\theta) \right] + 4r^3 g(\theta) \frac{dw}{dr} = 0 \quad (9)
 \end{aligned}$$

From (9) and (8) one finds that

$$\frac{d}{dr} \left[r \frac{dw}{dr} g(\theta) \right] + 2 \frac{dw}{dr} g(\theta) = 0 .$$

On integrating this expression one gets

$$r^3 \frac{dw}{dr} g(\theta) = c = \text{a constant} \quad (10)$$

The total couple per unit length, acting on a cylinder of radius r is

$$\tau' = 2\pi(r^2 t_{r\psi} + r \tau_{rz}) = 2\pi c$$

where $\tau_{ji} = e_{ipq} n_p \pi_{jq}$ Thus c is a measure of the couple per unit length on either cylinder.

Using equations (10) and (8) one obtains

$$\begin{aligned}
& r(\theta) \left[\frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d\theta}{dr} \right] r^2 + r^2 \frac{df}{d\theta} \left[\frac{1}{r^2} + \left(\frac{d\theta}{dr} \right)^2 \right. \\
& \left. + 2(\Delta \chi) [\sin \theta \cos \theta (A^2 - B^2) + AB \cos 2\theta] \right. \\
& \left. + \frac{e}{g(\theta)} (\lambda_1 + \lambda_2 \cos 2\theta) \right] = 0 \quad (11)
\end{aligned}$$

To study η the equation (11) is solved for θ with boundary conditions

$$\theta(R_1) = \theta^{(1)} ; \quad \theta(R_2) = \theta^{(2)}$$

Thus θ becomes known as a function of r . From equation (10) the relative angular velocity between the two cylinders, viz., $\Delta \Omega = \Omega_2 - \Omega_1$ is determined

$$\Delta \Omega = \Omega_2 - \Omega_1 = e \int_{R_1}^{R_2} \frac{dr}{r^3 g(\theta)} .$$

One can now calculate η from the well known relation

$$\eta = \frac{e(R_2^2 - R_1^2)}{2(\Delta \Omega) R_1^2 R_2^2} .$$

It is of interest to note that η depends only on and not on Ω_1 Ω_2 individually.

Results

Computations have been carried out for PAA with the same values for material constants assumed earlier (see Chapter II). The differential equations have been solved by the orthogonal collocation method. As in Chapter II 16 collocation points have been used and results in the cases involving large deformations have been checked by using 24 collocation points. Since the two results agree to within 0.01% only the results for 16 collocation points have been presented. For further details see Appendix 1 of this Chapter.

Effects of Boundary Conditions: The radii have been chosen ($R_1 = 1$ cm and $R_2 = 1.005$ cm) so as to accommodate a gap width of 50 μ . Figure 1 exhibits η as a function of $\Delta\Omega$. Two boundary conditions of director orientation has been studied.

(1) $\theta^{(1)} = \pi/2$ $\theta^{(2)}$: At low shear rates η remains constant at the Miesowicz value $\eta_A = (\mu_4 + \mu_5 - \mu_2)/2 = 0.092$ poise for PAA (fig. 1a). As the shear rate increases η decreases approaching the Miesowicz value $= (\mu_3 + \mu_4 + \mu_6)/2$ ($= 0.024$ poise for PAA) at high shear rates. This is because at low shear rates the director orientation is nearly perpendicular to the flow over a

major section of the flow. With increasing shear rate the angle made by the director with the flow decreases till at high shear rates the director is oriented along the flow over a major section of the flow.

(ii) $\theta^{(1)} = 0, \theta^{(2)} = \pi/2$: The variation of η with shear rate (fig. 1b) is similar to that in case (i). But at low shear rate the constant value of η is about $0.5 \eta_A$ in the absence of magnetic fields. This can be interpreted as being due to a distortion being present in the structure even in the absence of flow; the projection of the director field along the flow is substantial even in the absence of flow and consequently η is lower than η_A .

Effects of Magnetic Fields: A radial field ($H_r = A/r$) in general tends to enhance the value of η whereas an azimuthal field ($H_\psi = B/r$) tends to diminish it. This is because a radial field opposes the viscous torques while an azimuthal field aids them in distorting the director orientation from the equilibrium values.

(i) $\theta^{(1)} = \pi/2 = \theta^{(2)}$: At low shear rates a radial field ($H_r = A/r$) does not make its presence felt

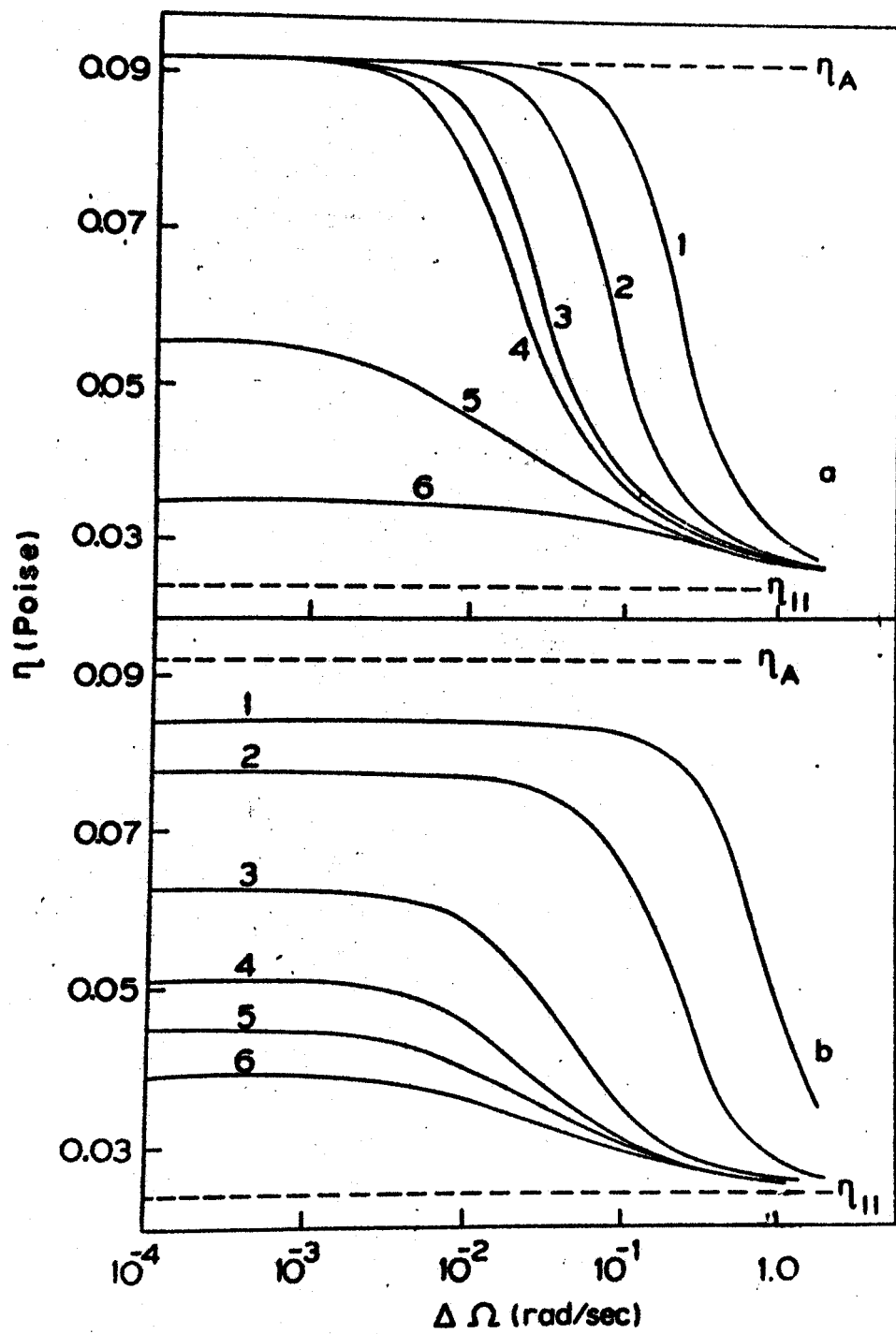
Figure 1. Variation of apparent viscosity with shear rate and magnetic field for $R_2/R_1 = 1.005$.

(a) $\theta_1 = \pi/2 = \theta_2$;

Radial magnetic field $A =$ (1) 5000, (2) 2500,
(3) 0 gauss cm; Azimuthal magnetic field $B =$
(4) 1000, (5) 2000, (6) 3000 gauss cm;

(b) $\theta_1 = 0$; $\theta_2 = \pi/2$

$A =$ (1) 10000, (2) 5000, (3) 2000, (4) 1000,
(5) 0 gauss cm, (6) $B = 1000$ gauss cm.



on η since the director is everywhere aligned almost normal to the flow. But on increasing the shear rate η decreases, though more slowly than in the field-free case - the larger the value of A the slower being the decrease of η (fig. 1a) till at high shear rate η tends to $\eta_{||}$. Now if the shear rate is kept constant at a high value (such that $\eta \approx \eta_{||}$) and if A is increased η remains nearly a constant and then increases rapidly tending to the value η_A at large values of A (fig. 2b).

the effect of an azimuthal field ($H_\psi = B/r$) is more interesting. At low shear rates on increasing B (fig. 2a) η remains nearly constant at η_A till B attains a value $B_c \approx \left\{ \pi R_1 / (R_2 - R_1) \right\} (K_{33} / \Delta\chi)^{\frac{1}{2}}$ (see Appendix 2 of this Chapter) which in the static case corresponds to the Fredericksz threshold, below which no deformation is possible. On increasing B further η decreases, attaining $\eta_{||}$ at large values of B . At low shear rates (fig. 1a) the value of η depends upon whether B is greater than or less than B_c . Put on increasing the shear rate η decreases approaching the value $\eta_{||}$ at high shear rates regardless of the value of B .

(11) $\theta^{(1)} = 0, \theta^{(2)} = \pi/2$: In this case, even

at low shear rates η changes, increasing with increasing value of A when a radial field $H_r = A/r$ is applied. At very values of A , η approaches η_A (fig. 2b). At any value of A , on varying the shear rate η remains constant at a value η_A' (higher than $0.5 \eta_A$ and depending upon the value of A) at low shear rates and then decreases on further increasing the shear rate attaining the value $\eta_{||}$ at high shear rates (fig. 1b).

An azimuthal field ($H_\psi = B/r$) does not exhibit the type of critical behaviour discussed in case (i). At low shear rates (fig. 2a) η decreases from its value at $0.5 \eta_A$ on increasing B attaining the value $\eta_{||}$ at high values of B . At any value of B , on varying the shear rate η remains constant at a value η_A'' (lower than $0.5 \eta_A$ and depending on the value of B) at low shear rates. On increasing the shear rate (fig. 1b) η decreases attaining the value $\eta_{||}$ at high shear rates regardless of the value of B .

Orientation profiles: Figure 3 exhibits the dependence of the director orientation profiles on boundary condition and magnetic fields for $R_1 = 1$ cm and $R_2 = 1.005$ cm.

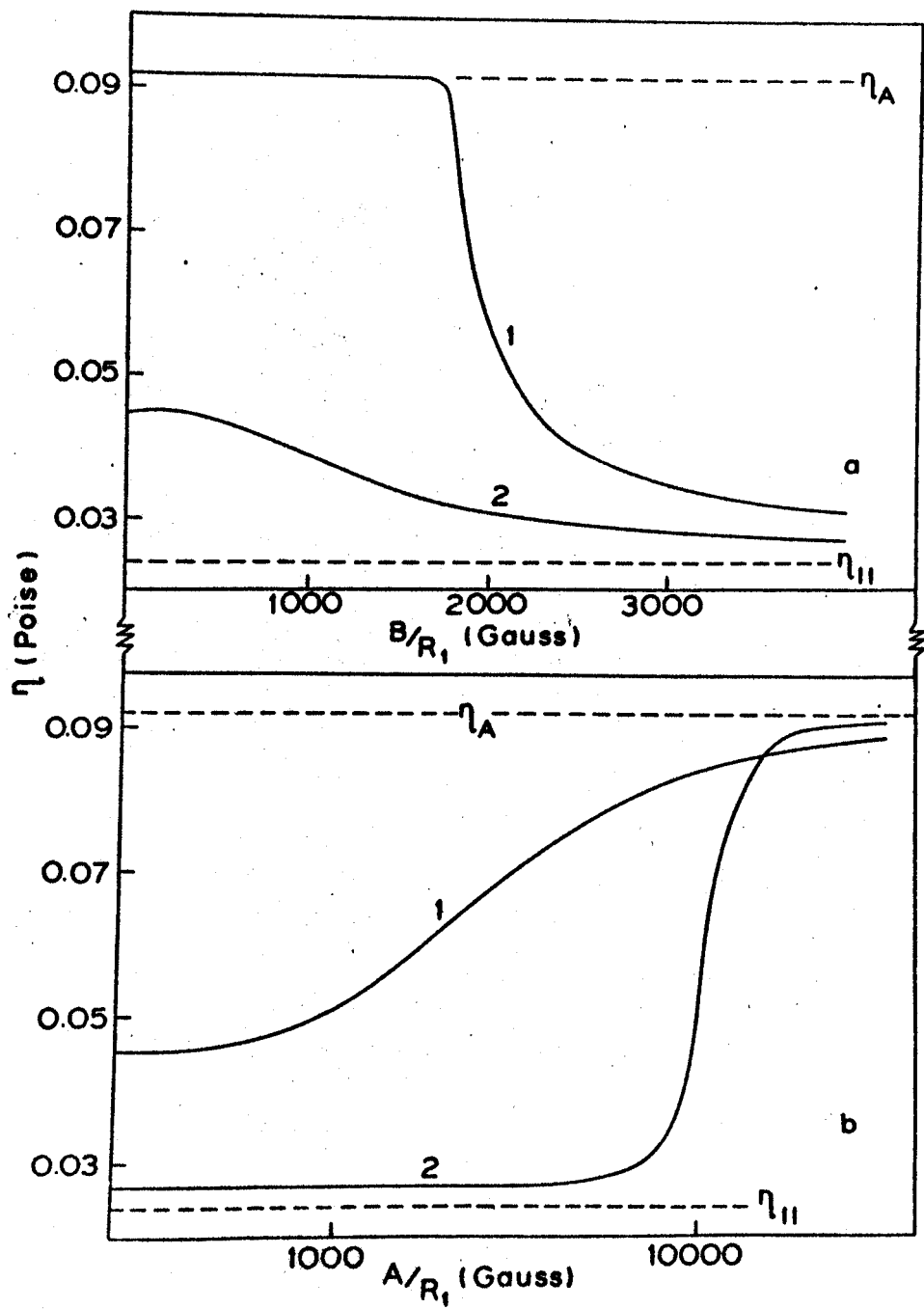
Figure 2

Variation of apparent viscosity with magnetic field for $R_2/R_1 = 1.005$.

(a) Aximuthal field $c = 10^{-3}$ dyne

(1) $\theta_1 = \pi/2 = \theta_2$, (2) $\theta_1 = 0$, $\theta_2 = \pi/2$;

(b) Radial field, (1) $\theta_1 = 0$, $\theta_2 = \pi/2$, $c = 10^{-3}$ dyne, (2) $\theta_1 = \theta_2 = \pi/2$; $c = 10.0$ dyne.



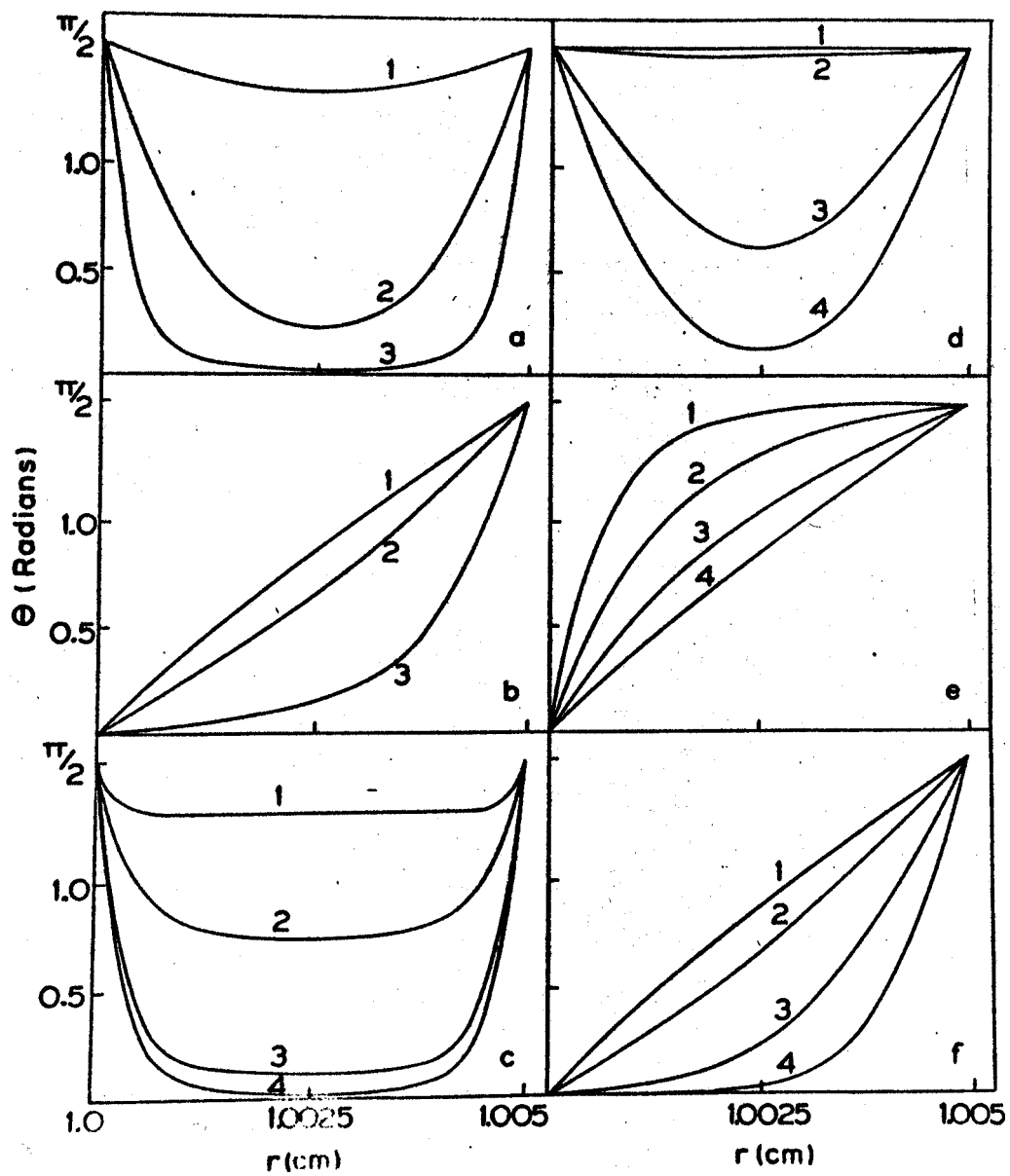
$\theta^{(1)} = \pi/2 = \theta^{(2)}$: The orientation profile is symmetric. At higher shear rates θ differs more from its boundary values over a larger section of the gap till at high shear rates θ is nearly zero over most of the gap increasing to $\pi/2$ in a narrow distance near the boundaries (fig. 3a). The dependence of the θ profile on the azimuthal field (fig. 56) is similar. But a radial field $H_r = A/r$ tends to bring the director alignment normal to the flow. At high values of A (fig. 3c) θ does not differ much from its boundary values in the rest of the sample.

$\theta_1 = 0, \quad \pi/2$ a In the field free case the θ profile is almost a straight line, starting from zero from the lower plate and increasing to $\pi/2$ at the moving plate. At low shear rates and increasing applied field $H_\psi = B/r$ the director orientation tends to a lower value over a larger portion of the gap width (fig. 31) going to $\pi/2$ over a small distance. On increasing the shear rate one finds a similar behaviour even in the field free case (fig. 3b). In both cases the θ profile is found to be concave upward. But when a radial field $H_r = A/r$ is applied θ increases from zero at the lower plate, more rapidly than in the field free case. As the value of A is increased the director

Figure 5

Orientation profiles for various magnetic fields and shear rates for $R_2/R_1 = 1.005$.

- (a) $H = 0$, $\theta_1 = \theta_2 = \pi/2$, $c =$ (1) 0.1, (2) 1.0, (3) 10.0 dynes;
- (b) $H = 0$, $\theta_1 = 0$, $\theta_2 = \pi/2$, $c =$ (1) 0.01, (2) 0.1, (3) 1.0 dynes;
- (c) $c = 10.0$ dyne, $\theta_1 = \pi/2 = \theta_2$, $A =$ (1) 15000, (2) 10000, (3) 5000, (4) 0 gauss cm;
- (d) $c = 10^{-3}$ dyne, $\theta_1 = \pi/2 = \theta_2$, $B =$ (1) 0, (2) 1700, (3) 2000, (4) 3000 gauss cm;
- (e) $c = 10^{-3}$ dyne, $\theta_1 = 0$, $\theta_2 = \pi/2$, $A =$ (1) 4000, (2) 2000, (3) 1000, (4) 0 gauss cm;
- (f) $c = 10^{-3}$ dyne, $\theta_1 = 0$, $\theta_2 = \pi/2$, $B =$ (1) 0, (2) 1000, (3) 2000, (4) 3500 gauss cm.



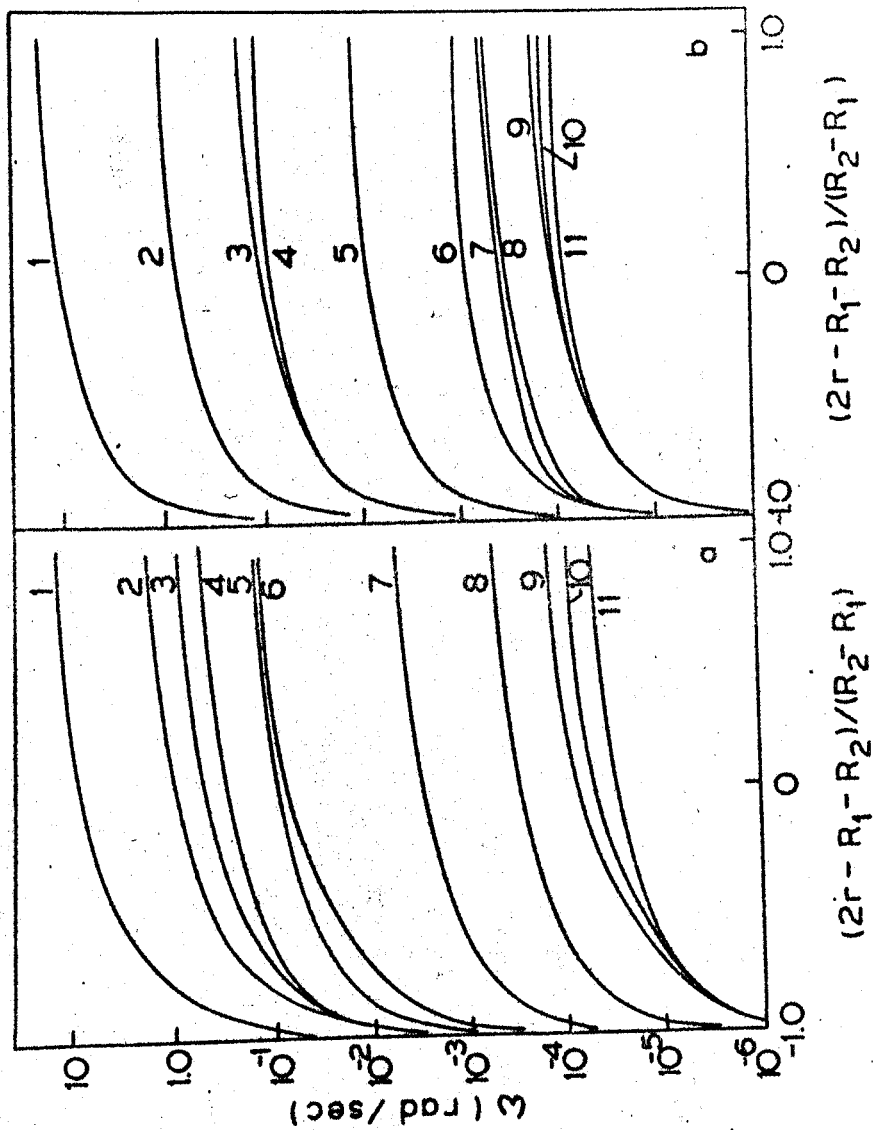
orientation tends to be normal to the flow over a larger section of the gap width; the θ profile becomes concave downward.

Velocity profile: The angular velocity Ω has been plotted as a function of the radial distance (figs. 4a and 4b) with the assumption that Ω_1 the angular velocity of the inner cylinder is zero. Because of the large range of values involved it has been necessary to plot $\log w$. The maximum value of w gives $\Delta\Omega$. One finds in general that (i) $\Delta\Omega$ increases with increasing shear rate, (ii) at a given shear rate $\Delta\Omega$ increases when an azimuthal field $H_\psi = B/r$ is applied, the larger the value of B the greater being the increase in $\Delta\Omega$ ((lit) at a given shear rate $\Delta\Omega$ decreases when a radial field $H_r = A/r$ is applied, the larger the value of A the greater being the decrease (iv) in general, at a given shear rate the $\Delta\Omega$ value for the symmetric boundary condition is less than that for the asymmetric boundary condition.

At this stage it is interesting to note that all the above results for Couette flow, which have been obtained for a gap width ($= 50 \mu$) small compared with the average radius, are almost identical to the corresponding results for shear flow for the same gap width (rrr Chapter II).

Figure 4. Velocity profiles for various magnetic fields and shear rates.

- (a) $\theta_1 = \pi/2 = \theta_2$. (1) $c = 10^{-3}$ dyne, $H = 0$, $R_2/R_1 = 1.25$; $c = 10.0$ dyne, $R_2/R_1 = 1.005$, $A = (2) 0$, (3) 10000, (4) 15000 gauss cm; (5) $H = 0$, $c = 10^{-3}$ dyne, $R_2/R_1 = 1.083$; $H = 0$, $R_2/R_1 = 1.005$, $c = (6) 1.0$, (7) 0.1, (8) 10^{-2} dyne; $c = 10^{-3}$ dyne, $R_2/R_1 = 1.005$, $B = (9) 3000$, (10) 2000, (11) 0 gauss cm.
- (b) $\theta_1 = 0$, $\theta_2 = \pi/2$, $c = 10^{-3}$ dyne, $H = 0$, $R_2/R_1 = (1) 1.25$, (2) 1.1, (3) $c = 1.0$ dyne, $H = 0$, $R_2/R_1 = 1.005$; (4) $c = 10^{-3}$ dyne, $H = 0$, $R_2/R_1 = 1.05$; $R_2/R_1 = 1.005$, $H = 0$, $c = (5) 0.1$, (6) 0.01 dyne; $c = 10^{-2}$ dyne, $R_2/R_1 = 1.005$, $A = (7) 5000$, (8) 10000 gauss cm; $c = 10^{-3}$ dyne, $R_2/R_1 = 1.005$, $B = (9) 3000$, (10) 1500, (11) 0 gauss cm.

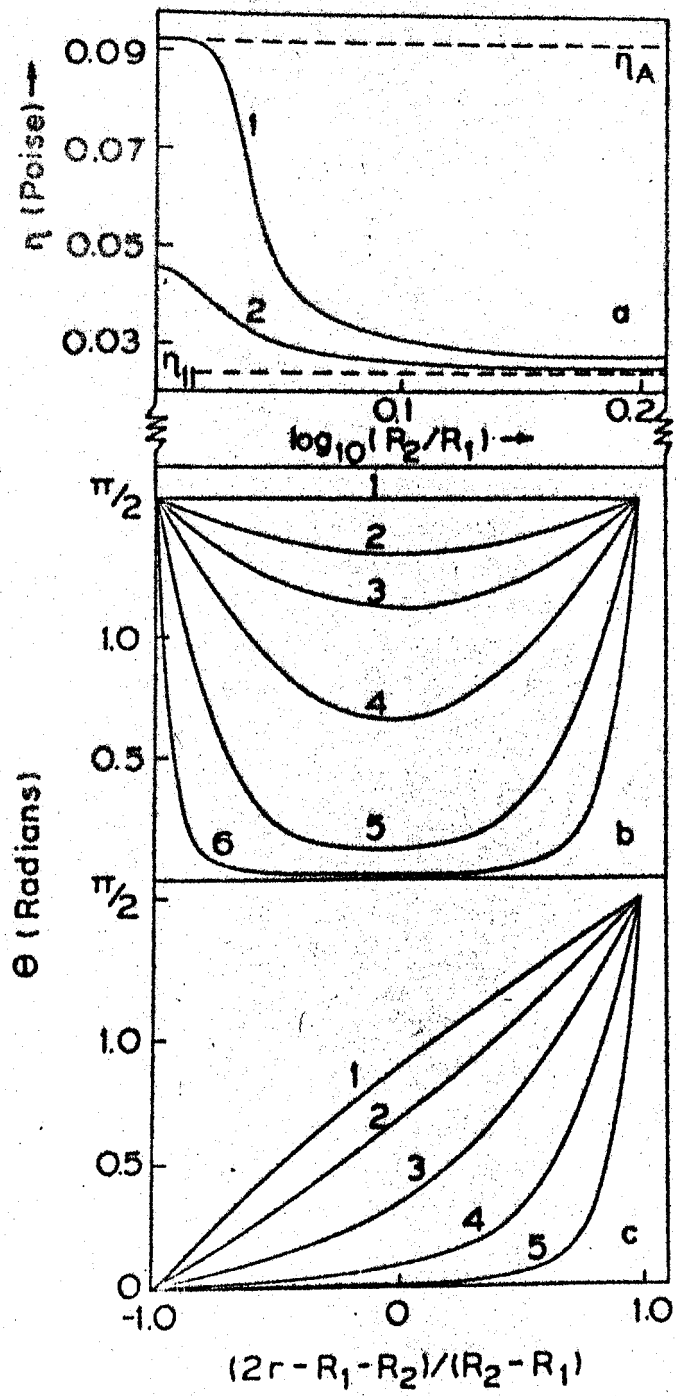


Effect of geometry^a If the gap width is of the same order of magnitude as the cylinder radii certain of the asymmetries which are smoothed out for the case of the small gap width grow into prominence. On increasing the ratio R_2/R_1 one finds that for either boundary condition η decreases, even at low shear rates, attaining the value $\eta_{||}$ at large values of R_2/R_1 (fig. 5a). This is in qualitative agreement with the observations of Porter and Johnson who found an increase in η when the gap width is decreased. Atkin and Leslie (1970) predicted that in the field free case the Θ profile for symmetric boundary conditions of Θ should show an extremum at a point $x_m = (R_1 R_2)^{1/2}$. When $(R_1 + R_2) \gg (R_2 - R_1)$, $x_m \approx (R_1 + R_2)/2$ a fact which has already been demonstrated for $R_2/R_1 = 1.005$. If however the value of this ratio is increased (fig. 5b) the Θ profile shows asymmetry the amount of asymmetry being more pronounced the higher the value of R_2/R_1 . Also for both boundary conditions, at high values of R_2/R_1 the director field is found to be oriented along the flow in most of the gap (figs. 5b and 5c). This is a natural consequence of the high distortion (splay and bend) present in the director field, even in the static limit, when the gap width is of the same order of magnitude as the radii of the cylinders and Pa also

Figure 5

Variation of apparent viscosity and orientation profile with geometry.

- (a) Variation of apparent viscosity with $\log_{10}(R_2/R_1)$; $c = 10^{-3}$ dyne, [1) $\theta_1 = \pi/2 = \theta_2$, (2) $\theta_1 = 0$; $\theta_2 = \pi/2$;
- (b) Orientation profile $\theta_1 = \pi/2 = \theta_2$, $c = 10^{-3}$ dyne, $R_2/R_1 =$ (1) 1.005, (2) 1.05, (3) 1.071, (4) 1.1, (5) 1.25, (6) 2.0;
- (c) Orientation profile $\theta_1 = 0$; $\theta_2 = \pi/2$, $c = 10^{-3}$ dyne, $R_2/R_1 =$ (1) 1.005, (2) 1.05, (3) 1.1, (4) 1.25, (5) 2.0.



the reason for η becoming less than that for smaller R_2/R_1 . This is also borne out by the velocity profile. (figs. 4a and 4b). In general $\Delta\Omega$, at a given shear rate is found to be larger for a larger value of R_2/R_1 . This fact is again in agreement with the previous results.

Appendix 1

A brief description of the collocation method has been given earlier [Appendix of Chapter II]. The transformations to which equations (10) and (11) have to be subjected before being solved are indicated below:

We effect a change of variable from r to $\bar{r}$ such that

$$\bar{r} = \frac{2r - R_1 - R_2}{(R_2 - R_1)}$$

At the boundaries $r = R_1, R_2$ $\bar{r}$ takes the values $-1, +1$ respectively. Equations (11) and (10) can now be written as

$$\begin{aligned} f(\theta) \left[\frac{d^2 \theta}{d\bar{r}^2} \frac{1}{M_1^2} + \frac{1}{M_1 [\bar{r} M_1 + M_2]} \frac{d\theta}{d\bar{r}} \right] [\bar{r} M_1 + M_2]^2 \\ + \frac{c}{g(\theta)} [\lambda_1 + \lambda_2 \cos 2\theta] + \frac{df}{d\theta} \left[1 + \left\{ \bar{r} + \frac{M_2}{M_1} \right\}^2 \left(\frac{d\theta}{d\bar{r}} \right)^2 \right] \\ + 2(\Delta \chi) [\sin \theta \cos \theta (A^2 - B^2) + AB \cos 2\theta] = 0 \quad (A1) \end{aligned}$$

$$\frac{dw}{d\bar{r}} = \frac{c M_1}{g(\theta) [\bar{r} M_1 + M_2]^3} \quad (A2)$$

$$\text{where } M_1 = \frac{R_2 - R_1}{2} \quad \text{and} \quad \frac{R_1 + R_2}{2}$$

As before these equations are evaluated at the collocation points using the matrices A and B and one gets

$$\begin{aligned}
 f(\theta_j) & \left[\sum_{k=1}^{N+2} B_{jk} \theta_k \frac{1}{M_1^2} + \frac{1}{M_1 [\bar{r}_j M_1 + M_2]} \sum_{k=1}^{N+2} A_{jk} \theta_k \right] [\bar{r}_j M_1 + M_2]^2 \\
 & + \frac{c}{g(\theta_j)} [\lambda_1 + \lambda_2 \cos 2\theta_j] + \frac{df(\theta_j)}{d\theta_j} \left[1 + \left\{ \bar{r}_j + \frac{M_2}{M_1} \right\}^2 \right] \times \\
 & \left\{ \sum_{k=1}^{N+2} A_{jk} \theta_k \right\}^2 + (A^2 - B^2) \sin 2\theta_j + 2AB \cos 2\theta_j = 0
 \end{aligned}$$

[j = 2, 3, ..., N+1] (A3)

$$\sum_{k=1}^{N+2} A_{jk} w_k = \frac{c M_1}{g(\theta_j) [\bar{r}_j M_1 + M_2]^3}$$

[j = 2, 3, ..., N+2] (A4)

Here $\theta_1 = \theta(\bar{r} = -1) = \theta^{(1)}$, $\theta_{N+2} = \theta(\bar{r} = +1) = \theta^{(2)}$. It is also assumed that $w_1 = w(\bar{r} = -1) = \omega_1 = 0$ so that $w_2 = w(\bar{r} = +1) = \omega_2 = \Delta\omega = \omega_2 - \omega_1$. The N equations (A3) are solved for a given value of c by the Newton-Raphson method. This yields the θ profile. The linear equations (A4) are now solved and

the complete velocity profile obtained.

The collocation method was used in preference to the finite difference method since the latter was found not to yield results for $\sigma > 1.0$ in all cases. But the agreement between η values obtained from the two methods is extremely good (about 1 in 10^4). About 20 hours of computer time were used for the Couette flow calculations.

Appendix 2

In this appendix an expression for the critical value of B encountered earlier is derived* Leslie (1970) considered a nematic confined in the annular space between two cylinders of radii R_1 and R_2 ($R_2 > R_1$) in the presence of a radial field $H_r = A/r$, $H_\varphi = 0$, $H_z = 0$. Seeking a solution for the director in the form $n_r = \sin \theta(r)$, $n_\varphi = \cos \theta(r)$, $n_z = 0$, he found that for boundary conditions $\theta(R_1) = 0 = \theta(R_2)$ there exists a critical value of A given by

$$A_0 = \left[\left\{ \frac{K_{11}\pi^2}{[\ln(R_2/R_1)]^2} + K_{11} - K_{33} \right\} \frac{1}{\Delta\chi} \right]^{\frac{1}{2}}$$

below which there is NO deformation. If the value of R_2/R_1 exceeds a value R_0 given by $R_0 = \exp \left[\left\{ \frac{K_{33}-K_{11}}{\Delta\chi} \right\}^{\frac{1}{2}} \pi \right]$ A_0 can become imaginary. This is the direct consequence of K_{11} being smaller than K_{33} for nematics.

A similar problem is considered below, for an azimuthal field $H_r = 0$, $H_\varphi = B/r$, $H_z = 0$ but for boundary conditions $\theta(R_1) = \pi/2 = \theta(R_2)$. Equation (11) simplifies, for the static case, to yield

$$2r^2 f(\theta) \frac{d^2 \theta}{dr^2} + r^2 \frac{df}{d\theta} \left(\frac{d\theta}{dr} \right)^2 + 2r f(\theta) \left(\frac{d\theta}{dr} \right) + \frac{df}{d\theta} - 2(\Delta\chi)B^2 \sin \theta \cos \theta = 0$$

Multiplication by $\frac{d\theta}{dr}$ and subsequent integration yields

$$f(\theta) + r^2 f'(\theta) \left(\frac{d\theta}{dr} \right)^2 - (\Delta\chi) B^2 \sin^2 \theta = C' \quad (A1)$$

where C' is a constant. An analysis similar to that given by Atkin and Leslie (1970) shows that $\frac{d\theta}{dr}(r_m) = 0$ and $\theta(r_m) = \theta_m$ an extremum where $r_m = (R_1 R_2)^{\frac{1}{2}}$. From (A1) we get $C' = f(\theta_m) - (\Delta\chi) B^2 \sin^2 \theta_m$. On integrating (A1) one finds that

$$\int_{R_1}^{r_m} \frac{dr}{r} = \log \frac{r_m}{R_1} = \int_{\pi/2}^{\theta_m} \frac{[f(\varphi)]^{\frac{1}{2}} d\varphi}{[\Delta\chi B^2 (\sin^2 \varphi - \sin^2 \theta_m) + f(\theta_m) - f(\varphi)]^{\frac{1}{2}}} \quad (A2)$$

Using the Legendre transformation $\cos \varphi = \cos \theta_m \sin \lambda$ in (A2) one gets

$$\log \frac{r_m}{R_1} = - \int_0^{\pi/2} \left\{ \frac{K_{33} + (K_{11} - K_{33}) \sin^2 \lambda \cos^2 \theta_m}{[(\Delta\chi) B^2 + K_{11} - K_{33}] [1 - \sin^2 \lambda \cos^2 \theta_m]} \right\}^{\frac{1}{2}} d\lambda$$

Taking limits as $\theta_m \rightarrow \pi/2$ one can obtain the critical

value B_0 of B given by

$$B_0 = \left[\frac{1}{\Delta\chi} \left\{ \frac{K_{33}\pi^2}{[\log(R_1/R_2)]^{\frac{1}{2}}} + K_{33} - K_{11} \right\} \right]^{\frac{1}{2}}$$

It is of interest to note that since $K_{33} \geq K_{11}$ for nematics B_0 can never become either zero or imaginary for reasonable values of R_1 and R_2 (i.e., $R_1 \neq 0$, $R_2 \neq 0$, $R_1 \neq R_2$).

If R_1 and R_2 are large compared to $d = R_2 - R_1$ one finds that

$$\frac{B_0'}{R_1} \approx \frac{\pi}{d} [K_{33}/\Delta\chi]^{\frac{1}{2}} \quad (= 1750 \text{ for PAA if } d = 50 \mu).$$

Figure 6 shows a plot of $2B_0/(R_1 + R_2)$ as a function of R_1/R_2 for a constant gap width of 50μ .

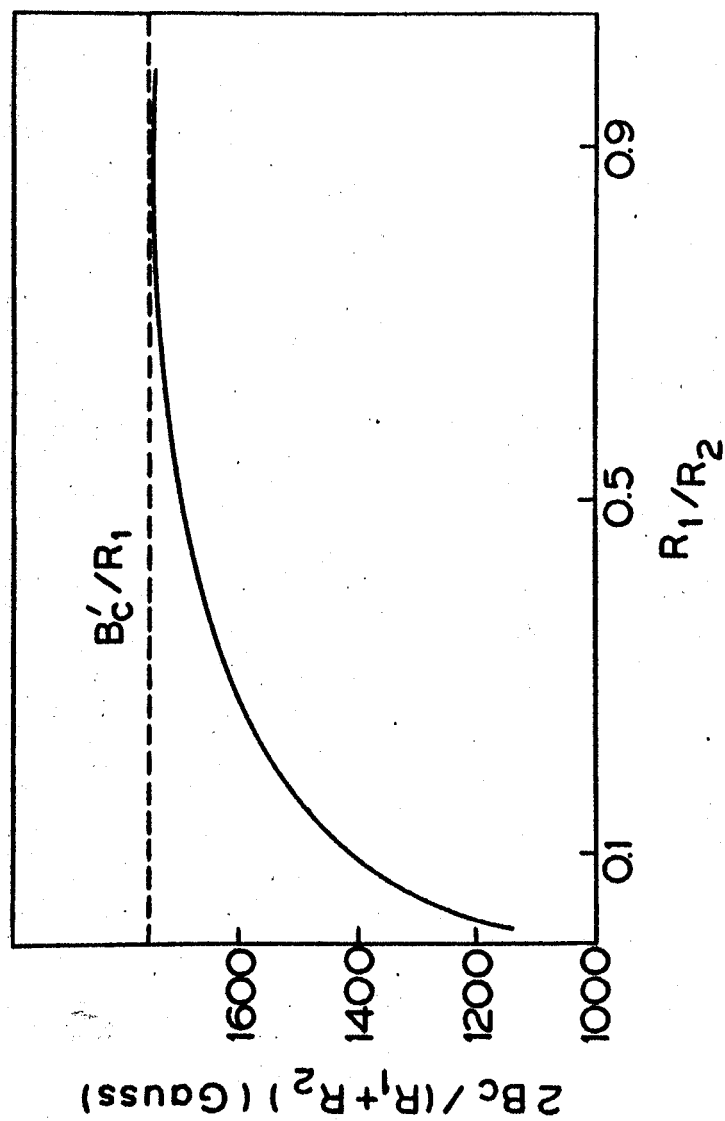


Figure 6. Variation of $2B_c / (R_1 + R_2)$ as a function of R_1/R_2 for PAA, with $R_2 - R_1 = 50 \mu$.

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