

Classical orbital magnetic moment in a dissipative stochastic system

N. Kumar

Raman Research Institute, Bangalore 560080, India

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We present an analytical treatment of the dissipative-stochastic dynamics of a charged classical particle confined biharmonically in a plane with a uniform static magnetic field directed perpendicular to the plane. The stochastic dynamics gives a steady state in the long-time limit. We have examined the orbital magnetic effect of introducing a parametrized deviation ($\eta - 1$) from the second fluctuation-dissipation relation that connects the driving noise and the frictional memory kernel in the standard Langevin dynamics. The main result obtained here is that the moving charged particle generates a finite orbital magnetic moment in the steady state, and that the moment shows a crossover from para- to diamagnetic sign as the parameter η is varied. It is zero for $\eta = 1$ that makes the steady state correspond to equilibrium, as it should. The magnitude of the orbital magnetic moment turns out to be a nonmonotonic function of the applied magnetic field, tending to zero in the limit of an infinitely large as well as an infinitesimally small magnetic field. These results are discussed in the context of the classic Bohr–van Leeuwen theorem on the absence of classical orbital diamagnetism. Possible realization is also briefly discussed.

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I. INTRODUCTION

The Bohr–van Leeuwen (BvL) theorem [1–3] on the stated absence of orbital diamagnetism for a classical system of charged particles in equilibrium has been one of the surprises of physics [4]. The static external magnetic field exerts a Lorentz force on the moving charged particle, acting at a right angle to its instantaneous velocity ($\mathbf{v}$). While such a gyroscopic force does no work on the particle, it does induce an orbital cyclotron motion that subtends an amperean current loop. The magnetic field associated with this current loop is expected to be nonzero and directed oppositely to the externally applied magnetic field—the Lenz law. Hence the expectation of a finite orbital diamagnetic moment $q/2c(\mathbf{r} \times \mathbf{v})$ [5]. Yet, as is well known, the partition function for a classical system in equilibrium turns out to be independent of the applied magnetic field, thus giving a zero orbital magnetic moment. And this has been the surprise [4]. [The field independence of the classical partition function derives simply from the fact that the classical partition function involves integration of the canonical momentum over an infinite range for any given value of the conjugate coordinate, and thus the magnetic vector potential ($\mathbf{A}$) entering the Hamiltonian through minimal coupling ($\mathbf{p} \rightarrow \mathbf{p} - \frac{q}{c}\mathbf{A}$) gets eliminated through a trivial shift of the momentum variable. This shift is, however, not allowed for a quantum system because of the canonical noncommutation involved there; hence, the stated quantum origin of the orbital magnetic moment in equilibrium: the Landau orbital diamagnetism [6].] A remarkably heuristic real-space explanation for the vanishing of the classical orbital moment was first suggested by Bohr [1] in terms of a cancellation of the orbital diamagnetic moment of the completed amperean orbits (Maxwell cycles) in the bulk interior by the paramagnetic moment subtended by the incompleting orbits skipping the bounding surface of the system in the opposite sense. This *edge current* has a large arm length, or leverage, and therefore can effectively cancel out the bulk diamagnetic moment. The cancellation has indeed been demonstrated graphically for a

simple planar geometry [7]. This real space-time picture is consistent with the zero-orbital magnetic moment following from an exact analytical solution of the classical Langevin dynamics with a white noise describing the motion of the charged particle confined harmonically in two dimensions, with a uniform static magnetic field applied perpendicular to the plane [8]. Here, the steady state ($t \rightarrow \infty$) orbital moment indeed vanishes for the given potential confinement (owing to the spring constant k of the harmonic potential, providing a soft boundary). Interestingly, this null result persists in the limit $k \rightarrow 0$, provided it is taken after taking the limit $t \rightarrow \infty$. This suggests that in this case the stochastic particle dynamics has had time enough to sense (i.e., be influenced by) the confinement (k). On the other hand, the moment survives to a nonzero value if the order of the two limits is interchanged. (The effect of these so-called Darwinian limiting processes is also manifest in the case of the quantum version of the above Langevin treatment [9]). It is to be noted, however, that in the case of the quantum Langevin equation the orbital moment tends to zero as the Planck constant is formally reduced to zero, i.e., in the classical limit, for which the noise term reduces to a classical white noise that is consistent with a local Stokes friction constant—in accord with the second fluctuation-dissipation (II-FD) theorem [10,11]. This reasonably suggests to us that it may well be the constraint of the second fluctuation-dissipation relation that forces the orbital magnetic moment to vanish in the classical case. This is further supported by our recent numerical simulation [12]. Motivated by these observations, we have carried out an exact analytical calculation of the orbital magnetic moment of a charged particle confined biharmonically in two dimensions with a uniform static magnetic field applied normal to the xy plane, but now with the proviso that the stochastic driving force (noise) is a sum of two uncorrelated noise terms—an exponentially correlated term and a δ -correlated term—and there is a parametrized ($\eta - 1$) deviation from the II-FD relation. Interestingly, we do obtain a nonzero orbital magnetic moment in the infinite time limit: in the steady state.

Moreover, the sign of the orbital magnetic moment turns out to show a dia- to paramagnetic crossover as the parameter η is tuned through $\eta = 1$, where the moment vanishes. In the following, we present an exact analytical treatment of this dissipative-stochastic system and discuss the results that follow.

II. STOCHASTIC DISSIPATIVE MODEL

Consider the classical dissipative-stochastic dynamics of a particle, of charge $-e$ and mass m , which is confined biharmonically in the xy plane in the presence of a uniform static magnetic field B applied perpendicular to the plane. The governing stochastic (Langevin) equations are

$$m\ddot{x}(t) = -kx(t) - \int_0^t \left(\frac{\Gamma}{t_c} e^{-(t-t')/t_c} + \Gamma_0 \delta(t-t') \right) \dot{x}(t') dt' - \frac{eB}{c} \dot{y}(t) + \xi_x(t), \quad (1a)$$

$$m\ddot{y}(t) = -ky(t) - \int_0^t \left(\frac{\Gamma}{t_c} e^{-(t-t')/t_c} + \Gamma_0 \delta(t-t') \right) \dot{y}(t') dt' - \frac{eB}{c} \dot{x}(t) + \xi_y(t), \quad (1b)$$

where $\xi_i(t)$ is the noise term with $\langle \xi_i(t) \rangle = 0$ and

$$\langle \xi_i(t) \xi_j(t') \rangle = \delta_{i,j} m k_B T \left(\frac{\gamma}{t_c} e^{-|t-t'|/t_c} + \eta \gamma_0 \delta(t-t') \right), \quad (2)$$

and we are interested in the long-time limit $t \rightarrow \infty$. Here $i = x, y$, and the angular brackets $\langle \dots \rangle$ denote average over realizations of the two uncorrelated noise terms: one a δ -correlated (white) noise and the other an exponentially correlated noise with a correlation time t_c . This sum of a white noise and an exponentially correlated noise, we believe, is the simplest non-Markovian Gaussian process allowed by Doob's theorem [13]. Further, $(\eta - 1)$ parametrizes deviation from the II-FD relation as noted above.

III. ANALYTICAL TREATMENT

It is convenient to introduce here the quantities $\Omega_0 = \sqrt{k/m}$ (harmonic oscillator circular frequency), $\frac{eB}{mc} = \Omega_c$ (cyclotron circular frequency), and $\frac{\Gamma}{m} = \gamma$ (the frictional relaxation frequency). We further define the following dimensionless parameters $\tau = \gamma t$ (dimensionless time) and $\omega_0 = \frac{\Omega_0}{\gamma}, \omega_c = \frac{\Omega_c}{\gamma}$ (dimensionless circular frequencies).

Following the “Landau trick,” the two coupled Langevin equations for the real displacements $x(\tau)$ and $y(\tau)$ as functions of the dimensionless time τ can be conveniently combined into a single Langevin equation for the complex displacement $z(\tau) \equiv x(\tau) + iy(\tau)$, giving

$$\ddot{z}(\tau) = -\omega_0^2 z(\tau) - \int_0^\tau \left(\frac{1}{\tau_c} e^{-(\tau-\tau')/\tau_c} + \frac{\gamma_0}{\gamma} \delta(\tau - \tau') \right) \dot{z}(\tau') d\tau' + i\omega_c \dot{z}(\tau) + g(\tau) \quad (3)$$

with $\langle g(\tau) \rangle = 0$ and $\langle g(\tau) g^*(\tau') \rangle = \frac{2k_B T}{(m\gamma^2)} \left(\frac{1}{\tau_c} e^{-(\tau-\tau')/\tau_c} + (\eta - 1) \frac{\gamma_0}{\gamma} \delta(\tau - \tau') \right)$.

Note the complex conjugation $(*)$ that we have introduced in $\langle g(\tau) g^*(\tau') \rangle$ above, as the same will be needed in subsequent calculations. Also, we have changed over to the dimensionless time parameter (τ) , but have retained the same symbols for the dynamical variables without the risk of confusion. Inasmuch as the particle motion along the uniform magnetic field normal to the xy plane decouples from that in the xy plane, the present model equally well describes a three-dimensional system. The orbital magnetic moment can now be rewritten as

$$\langle M(\tau) \rangle = \frac{e\gamma}{2c} \text{Im} \langle z(\tau) \dot{z}^*(\tau) \rangle. \quad (4)$$

It is convenient now to introduce the Laplace transform

$$\tilde{z}(s) = \int_0^\infty e^{-s\tau} z(\tau) d\tau \text{ and } \tilde{\dot{z}}(s) = s\tilde{z}(s), \quad (5)$$

with the initial conditions $z(0) = 0 = \dot{z}(0)$ without loss of generality, as we are interested in the steady state (long-time limit $\tau \rightarrow \infty$). We obtain straightforwardly

$$\tilde{z}(s) = \frac{\tilde{g}(s)(1 + \tau_c s)/\tau_c}{\left[s^3 + \left(\frac{1}{\tau_c} + \frac{\gamma_0}{\gamma} - i\omega_c \right) s^2 + \left(\omega_0^2 + \frac{1}{\tau_c} + \frac{\gamma_0}{\gamma\tau_c} - \frac{i\omega_c}{\tau_c} \right) s + \frac{\omega_0^2}{\tau_c} \right]} \quad (6)$$

and

$$\tilde{\dot{z}}(s) = \frac{s(1 + \tau_c s)\tilde{g}(s)/\tau_c}{\left[s^3 + \left(\frac{1}{\tau_c} + \frac{\gamma_0}{\gamma} - i\omega_c \right) s^2 + \left(\omega_0^2 + \frac{1}{\tau_c} + \frac{\gamma_0}{\gamma\tau_c} - \frac{i\omega_c}{\tau_c} \right) s + \frac{\omega_0^2}{\tau_c} \right]}, \quad (7)$$

where we have used $\tilde{\dot{z}}(s) = s\tilde{z}(s)$, with the initial conditions $z(0) = \dot{z}(0) = 0$ as noted above. In order to inverse-Laplace transform the expressions above, it is convenient to introduce the factorized denominator

$$D(s) = \left[s^3 + \left(\frac{1}{\tau_c} + \frac{\gamma_0}{\gamma} - i\omega_c \right) s^2 + \left(\omega_0^2 + \frac{1}{\tau_c} + \frac{\gamma_0}{\gamma\tau_c} - \frac{i\omega_c}{\tau_c} \right) s + \frac{\omega_0^2}{\tau_c} \right] \equiv (s - s_1)(s - s_2)(s - s_3), \quad (8)$$

where $s_i (i = 1, 2, 3)$ are the three roots of the cubic denominator $D(s)$. These roots are readily obtained following the Cardano procedure. We can then write $\tilde{z}(s)$ and $\tilde{\dot{z}}(s)$ as partial fractions

$$\tilde{z}(s) = \frac{1}{\tau_c} \sum_i \frac{a_i}{s - s_i} \tilde{g}_z(s) \quad (9)$$

and

$$\tilde{z}(s) = \frac{1}{\tau_c} \sum_i \frac{A_i}{s - s_i} \tilde{g}_z(s), \quad (10)$$

where $\{a_i\}$ are given as solutions of the set of equations

$$\begin{aligned} a_1 + a_2 + a_3 &= 0, \\ a_1(s_2 + s_3) + a_2(s_3 + s_1) + a_3(s_1 + s_2) &= -1, \\ a_1 s_2 s_3 + a_2 s_3 s_1 + a_3 s_1 s_2 &= \frac{1}{\tau_c}, \end{aligned} \quad (11)$$

and similarly, $\{A_i\}$ are given by

$$A_1 + A_2 + A_3 = 1,$$

$$A_1(s_2 + s_3) + A_2(s_3 + s_1) + A_3(s_1 + s_2) = -\frac{1}{\tau_c},$$

$$A_1 s_2 s_3 + A_2 s_3 s_1 + A_3 s_1 s_2 = 0. \quad (12)$$

In terms of the above, we now have the inverse Laplace transforms as

$$z(\tau) = \frac{1}{\tau_c} \sum_i a_i \int_0^\tau e^{s_i(\tau-\tau')/\tau_c} g(\tau') d\tau' \quad (13a)$$

and

$$\dot{z}(\tau) = \frac{1}{\tau_c} \sum_j A_j \int_0^\tau e^{s_j(\tau-\tau'')/\tau_c} g(\tau'') d\tau''. \quad (13b)$$

With this, the orbital magnetic moment in Eq. (4) turns out to be

$$\langle M(\tau) \rangle = \left(\frac{e}{mc} \right) \left(\frac{k_B T}{\gamma \tau_c^3} \right) \text{Im} \sum_{i,j=1,2,3} a_i A_j^* \iint_0^\tau \left[e^{s_i(\tau-\tau')/\tau_c} e^{s_j^*(\tau-\tau'')/\tau_c} e^{-|\tau'-\tau''|/\tau_c} + \eta \frac{\gamma_0 \tau_c}{\gamma} e^{s_i(\tau-\tau')/\tau_c} e^{s_j^*(\tau-\tau'')/\tau_c} \delta(\tau' - \tau'') \right] d\tau' d\tau'', \quad (14)$$

where Im denotes the imaginary part. Straightforward integration gives the $\tau \rightarrow \infty$ (steady-state) limit for the orbital magnetic moment as

$$M(\infty) = \left(\frac{e}{mc} \right) \left(\frac{k_B T}{\gamma \tau_c^3} \right) \text{Im} \sum_{i,j=1,2,3} a_i A_j^* \left[\left(\frac{1}{1+s_j^*} \right) \left\{ \frac{2}{\tau_c(s_j^* - \frac{1}{\tau_c})(s_i + s_j^*)} + \frac{1}{(s_i - \frac{1}{\tau_c})} \right\} - 2\eta \frac{\gamma_0}{\gamma} \tau_c \left(\frac{1}{s_i + s_j^*} \right) \right]. \quad (15)$$

After some simplification, the above expression for the orbital magnetic moment $M(\infty)$ reduces to

$$\mu \equiv \frac{M(\infty)}{\left(\frac{ek_B T}{mc\gamma \tau_c^3} \right)} = -2(\eta - 1) \frac{\gamma_0 \tau_c}{\gamma} \text{Im} \sum_{i,j=1,2,3} a_i A_j^* \left(\frac{1}{s_i + s_j^*} \right). \quad (16)$$

IV. RESULTS AND DISCUSSION

In Fig. 1, we have plotted the limiting steady-state value of the dimensionless orbital magnetic moment μ against the dimensionless applied magnetic field $\beta (= \frac{eB}{mc\gamma})$, where η parametrizes the II-FD violation and is varied over the range 0.5–2.0. As we are interested here mainly in the matter of principles, we have made a simple choice for the dimensionless parameters involved, namely, $\frac{1}{\tau_c} = \omega_0 = 1$ and $\frac{\gamma_0}{\gamma} = 0.5$ (the strength of the white noise relative to the exponentially correlated noise).

As we observe the field-induced orbital magnetic moment is clearly nonzero in the steady state. There is a crossover from the paramagnetic to the diamagnetic sign as the parameter η is tuned from $\eta = 0.5$ to $\eta = 2.0$. This crossover is a surprise. The magnetic moment is zero for $\eta = 1$, which corresponds to the canonical II-FD consistent (equilibrium) state. Hence there is no violation of the BvL theorem. The orbital magnetic moment is obviously zero for zero magnetic field; but not so obviously, it tends to zero for large magnetic field as well. The latter behavior can, however, be understood from the following, namely, that the radius/frequency of the cyclotron orbit tends to zero/infinity as the applied magnetic field is made

infinitely large [4]. The fact that the orbital magnetic moment can be paramagnetic in a certain range of the II-FD deviation parameter η is significant in that unlike diamagnetism, it leads to a positive feedback for a collection of charged particles in

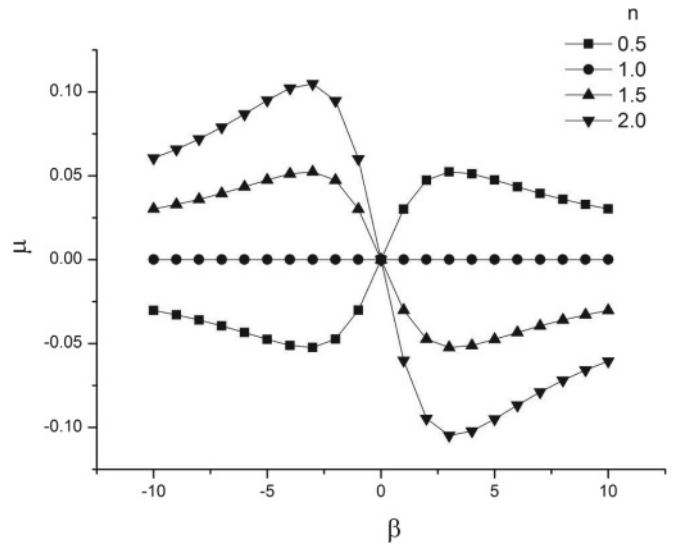


FIG. 1. Plot of dimensionless magnetic moment μ against dimensionless magnetic field β for four different values of η that parametrizes deviation from the II-FD relation. It clearly shows a dia- to paramagnetic crossover as η is varied through $\eta = 1$. Also, the moment tends to zero in the limit of zero as well as large magnetic field β .

such a classical system—it can give an enhancement of the orbital paramagnetism.

Physical realization of such a classical system in the laboratory is admittedly somewhat demanding. One needs to create a dilute (highly nondegenerate) gas of charged particles (e.g., electrons and holes) at sufficiently high temperatures, and confined on a mesoscopic scale in the presence of a static magnetic field. The temperature has to be high enough so as to wash out the quantum effects, namely, the discreteness of the quantized level spacings owing to the mesoscopic confinement. Now, the II-FD violating parameter ($\eta - 1$) necessarily requires a nonequilibrium steady-state condition. This is the real problem for an experimental realization. One is tempted to think that such a nonequilibrium steady state may be induced through a noisy laser excitation, e.g., the Kubo-Anderson non-Markovian noise [14,15], of charged particles confined in an optical tweezer [16]. There is, however, a problem here involving the energy injected by the laser, its dissipation in the system and the associated rise of temperature. In fact, one necessarily needs to have a two-temperature configuration. Thus, e.g., a possible physical realization of our model can, in principle, have charged particles in contact with a thermal reservoir at one temperature, while another type of neutral particles is in contact with the reservoir at a different temperature. Then particle collisions will ensure a stationary (steady-state) nonequilibrium state with flow of energy between the reservoirs. This,

according to our model calculation, should give a nonzero orbital magnetic moment in an externally applied magnetic field.

Given that classically a static magnetic field does no work on a moving charged particle, our model calculation giving a nonequilibrium steady-state solution in the presence of a static magnetic field could lead to some insightful molecular dynamical simulations when appropriately thermostatted [17]. Also, it has a significant bearing on the work related to generalized fluctuation-dissipation theorem for steady-state systems [18].

In conclusion, we have presented an exact analytical treatment of a classical dissipative-stochastic model system in a uniform static magnetic field, which is found to give a finite orbital magnetic moment in the steady state. Interestingly, we find that there is a crossover from the diamagnetic to the paramagnetic sign of the magnetic moment as function of a parametrized deviation from the second fluctuation-dissipation relation. We think that these results do complement, rather than violate, the classic Bohr–van Leeuwen theorem.

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**Erratum: Classical orbital magnetic moment in a dissipative stochastic system
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N. Kumar*

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I have noticed a few errors in some of the intermediate steps in our above published paper. The corrections duly listed out and numbered are given below.

I would like to emphasize that all results and conclusions of the paper remain totally unaffected by these errors.

(1) Eq. (1b), second line: Minus “−” sign in front of first term to be replaced by plus “+” sign.

(2) Eq. (2), right-hand side: Greek symbol “ η ” to be replaced by “ 2η .”

(3) Eq. (3), last line: “ $(\eta - 1)$ ” to be replaced by “ 2η .”

(4) Eqs. (9), (10), (13a), and (13b), right-hand side: Premultiplier “ $\frac{1}{\tau_c}$ ” to be deleted.

(5) Eq. (14), right-hand side: Greek symbol “ η ” to be replaced by “ 2η .”

(6) Eq. (15), right-hand side: Prefactor “ $(\frac{1}{1+s_j^*})$ ” multiplying { . . . } to be replaced by “ $(\frac{1}{\frac{1}{\tau_c} + s_j^*})$.”

(7) Eq. (16), left-hand side: “ τ_c^3 ” to be replaced by “ τ_c .”

*nkumar@rri.res.in