

CHAPTER 4

HYDRODYNAMICS OF SMECTIC- C

4.1 Introduction

In the last two chapters, biaxial nematics were considered. Unfortunately, to date there does not exist a biaxial nematic which can be aligned to study many of its interesting elastic and dynamical properties of biaxial nematics. Though aligned lyotropic biaxial nematics exist, they are too fragile for experiments and further, being multicomponent systems they have a tendency to phase separate under large scale structural distortions. It is in this context that thermotropic smectic- C liquid crystals acquire significance. They can be easily synthesized in the laboratory. These are biaxial liquid crystals of monoclinic symmetry, *i.e.* it has a two-fold axis perpendicular to a mirror plane.

Smectic- C liquid crystals are novel since they incorporate the features of both nematics, and smectic- A liquid crystals. As a consequence, even in the linear regime, the theory is complex and theoretical investigations on macroscopic flows have been very limited. In 1972, Martin, Parodi and Pershan (RIPP) [1] formulated a linearized hydrodynamic theory of Sm- C which is successful only when the deformations are

small as in fluctuations. A more complete nonlinear version of this theory has been developed by others [2, 3] and the results follow along the lines of MPP. In the linearized hydrodynamic theory of MPP, the stress tensor is assumed to be *symmetric* and the phenomenon of *permeation* exists when there is flow in a direction normal to the layers. Nearly twenty years later, Leslie, Stewart and Nakagawa (LSN) [4, 5], following the approach of Ericksen and Leslie [6, 7] in the formulation of the hydrodynamics of uniaxial nematics, developed a theory which allows for nonlinear coupling between curvature of the layers, director orientation and macroscopic flow. They use an *asymmetric* stress tensor. But, the drawback of this theory is that permeation process and layer dilatation or compression are not included. Hence, these two approaches are different. The theory of MPP has been used to describe fluctuations and wave propagation; and, the theory of LSN has been employed recently in describing reorientation dynamics and the effects of “backflow” [S-11].

In this chapter, a complete nonlinear macroscopic theory of the isothermal hydrodynamics of smectic-C which we have formulated recently [12] will be presented. We assume an *asymmetric* stress tensor and incorporate the essential features of a hydrodynamic theory of a smectic phase, namely *permeation* and *variations in layer spacing*. It must be remembered that this theory can be applied even to chiral smectic-C, (Sm-C*). The main aim is to systematically derive the equations of motion by generalizing forces and torques. Using Onsager's reciprocity relations, it is found that the entropy production embodying dissipative torques and shear stresses is described by 16 viscosity coefficients, and a permeation constant arising due to the dissipative motion perpendicular to the layers of smectic-C. It may be mentioned in passing that LSN's theory has twenty viscosity coefficients for the incompressible case. The theory presented here incorporates coupling between the different hydrodynamic variables, viz. velocity, layer spacing and the $\mathbf{c}$ vector. It is a covariant

description in the sense that the constraints of the system have been incorporated and the theory is applicable for large scale distortions. The main problem regarding experimental and theoretical investigations of smectic-C has been the complexity arising from such couplings.

4.2 Hydrodynamic variables

As already described in the first chapter, Sm-C liquid crystal is a layered structure in which the director, $\mathbf{n}$, which represents the preferred direction of the local molecular orientation is everywhere inclined at the same angle, θ , with respect to the layer normal. These liquid crystals can be described by a density variation along the layer normal,

$$\rho = \rho_0 + [\delta\rho \exp i q_S \psi + c.c.] \quad (4.1)$$

where $q_S = (2\pi/d)$ with d the layer spacing and $\delta\rho$ is the amplitude of the density variations. The smectic layers are surfaces with $\psi(\mathbf{r}) = \text{constant}$. A deformed smectic structure, with Z-axis normal to the undeformed layers, is described by $\psi(\mathbf{r}) = z - u(\mathbf{r})$.

The hydrodynamic equations are always in terms of the hydrodynamic variables. These variables are characterized by slow decay times proportional to some power of their wavelengths. The number of such variables is determined by the sum of the number of conservation laws and the number of "continuous broken symmetries". As in ordinary fluids, here also we have the density, the components of the velocity and the energy density as five hydrodynamic variables associated respectively with the conservation of the mass, the linear momentum and the energy. In addition to these, we have two new variables describing the broken translational invariance in one dimension, as described by the layering, and the broken rotational

invariance of the director, $\mathbf{n}$, as described by a transverse two-fold axis. These new variables can also be respectively described by the scalar variable ψ and the vector $\mathbf{c}$ which is the projection of $\mathbf{n}$ onto the smectic layers. The vector $\mathbf{c}$ is subject to the constraints $\mathbf{c} \cdot \mathbf{c} = 1$ and $\mathbf{c} \cdot \mathbf{N} = 0$ where $\mathbf{N}$ is the local unit layer normal and, $\mathbf{N}$ can be given by $\nabla\psi/|\nabla\psi|$.

It must be remarked here that in the theory of LSN, the constraint $\mathbf{V} \times \mathbf{N} = \mathbf{0}$ is imposed on the structural deformations. This means that there cannot be bend or twist deformations in $\mathbf{N}$. It should be remembered that there exists bend deformations even in cases such as an undulation of the layers which does not involve variations in layer spacing. Thus, by itself the constraint $\mathbf{V} \times \mathbf{N} = \mathbf{0}$ will not allow for general layer deformations and the process of permeation. On the other hand, constraints on $\mathbf{c}$ and $\mathbf{N}$ assumed here allow for these. It should be noted that in this hydrodynamical description a variation in the tilt, θ , of $\mathbf{n}$ with respect to the layer normal is not permitted since θ is the thermodynamic order parameter of the liquid crystal. It is well-known that the macroscopic dynamics of any system does not depend on whether the stress tensor is symmetric or asymmetric since equations of motion involve only their gradients. Hence, we can have an asymmetric stress tensor as it is more appropriate to these systems which allow naturally for internal torques.

4.3 Conservation laws

We consider a compressible material and also assume that the system has reached statistical equilibrium locally. The orientation of the director, $\mathbf{n}$, is completely described by $\mathbf{c}$ and $\mathbf{N}$ since the tilt of $\mathbf{n}$ with respect to $\mathbf{N}$ is nothing but the order parameter, θ , of the Sm-C phase. In the approach of LSN, Sm-C is a biaxial system described by $\mathbf{c}$ and $\mathbf{N}$. Then, torques on the system can be considered to consist of two parts [13]. One part arises from the familiar moment of linear mo-

momentum of the fluid particle while the other part is due to torques that change c and N . Since N is completely described by $\nabla\psi/|\nabla\psi|$, generalized torques leading to changes in N can be equivalently described by generalized forces resulting in second-order gradients in ψ . An example of this is in the equations of equilibrium of smectic-A. In smectic-A, with ψ as the hydrodynamic variable, the stress tensor does not need to have an antisymmetric part because torques are totally due to moment of linear momentum of fluid particles, and the conservation of angular momentum is implicit. But, in Sm- C there is an additional internal degree of freedom due to c . The difference between the total angular momentum and the moment of linear momentum can be attributed to an internal angular momentum leading to changes in the c vector. Thus, in Sm- C , the equation for the conservation of angular momentum acquires an independent status. This leads to a new equation which is similar to the Oseen equation which describes the director dynamics in nematics.

At any time t , the conservation (for a volume V of the fluid bounded by surface A) of mass, linear momentum, energy, and angular momentum respectively lead to the following equations:

$$\frac{D}{Dt} \int_V \rho dV = 0 \quad (4.2)$$

$$\frac{D}{Dt} \int_V \rho v_i dV = \int_V \rho f_i dV + \oint_A \sigma_{ji} dA_j \quad (4.3)$$

$$\begin{aligned} \frac{D}{Dt} \int_V \rho dV \left[\frac{1}{2} v_i v_i + U + \frac{1}{2} \rho_c \dot{c}_i \dot{c}_i \right] &= \int_V \rho dV [f_i v_i + G_i \dot{c}_i] + \\ &\quad \oint_A dA_j [\sigma_{ji} v_i + \pi_{ji} \dot{c}_i] \end{aligned} \quad (4.4)$$

$$\begin{aligned} \frac{D}{Dt} \int_V \rho dV e_{ijk} [x_j v_k + \rho_c c_j \dot{c}_k] &= \int_V \rho dV e_{ijk} [x_j f_k + c_j G_k] + \\ &\quad \oint_A dA_l e_{ijk} [x_j \sigma_{lk} + c_j \pi_{lk}] \end{aligned} \quad (4.5)$$

Further, for the $\mathbf{c}$ vector:

$$\frac{D}{Dt} \int_V \rho \rho_c \dot{c}_i dV = \int_V [\rho G_i + g_i] dV + \oint_A \pi_{ji} dA_j \quad (4.6)$$

Equation (4.6) can be called the Oseen equation for the $\mathbf{c}$ director. Here, ρ is the density, $\frac{D}{Dt}$ is the material derivative, dA_i is a vector representing the area element, and dV is the volume element. Further, v_i , f_i , σ_{ji} , G_i , g_i and π_{ji} are respectively the components of the velocity, force, stress tensor, external director body force, intrinsic director body force and director surface stress tensor. In equation (4.4), U is the internal energy per unit mass. And, ρ_c is the moment of inertia per unit mass associated with the vector $\mathbf{c}$. This moment of inertia depends only on the degree of orientational order and hence is a constant at a given temperature and pressure.

With Reynold's transport theorem and Gauss' theorem, equations (4.2) to (4.6) imply,

$$\frac{D\rho}{Dt} + \rho v_{k,k} = 0 \quad (4.7)$$

$$\rho \frac{Dv_i}{Dt} = \rho f_i + \sigma_{ji,j} \quad (4.8)$$

$$\rho \rho_c \frac{D^2 c_i}{Dt^2} = \rho G_i + g_i + \pi_{ji,j} \quad (4.9)$$

$$\rho \frac{DU}{Dt} = \sigma_{ji} d_{ij} + \pi_{ji} (\dot{c}_{i,j} - \omega_{ik} c_{k,j}) - g_i (\dot{c}_i - \omega_{ik} c_k) \quad (4.10)$$

$$\sigma_{jk} + c_{j,l} \pi_{lk} - c_j g_k = \sigma_{kj} + c_{k,l} \pi_{lj} - c_k g_j \quad (4.11)$$

where $d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ and $w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$. And, $\sigma_{ji,j} = \partial \sigma_{ji} / \partial x_j$, $\pi_{ji,j} = \partial \pi_{ji} / \partial x_j$, $v_{i,j} = \partial v_i / \partial x_j$, $c_{i,j} = \partial c_i / \partial x_j$, $\dot{c}_i = Dc_i / Dt$. The moment of inertia associated with $\mathbf{c}$ vector, ρ_c , is an extremely small quantity and is negligible. In other words, we ignore the inertia term due to ρ_c in all our future discussions.

If S is the entropy per unit mass at an absolute temperature, T , then the Helmholtz free energy per unit mass is $F = U - TS$. According to thermodynamics,

S , the rate of entropy generation is always positive. At constant temperature, this means

$$\rho T \dot{S} = \rho \dot{U} - \rho \dot{F} \geq 0 \quad (4.12)$$

4.4 Constitutive equations

We assume that F is a single valued function of the variables $p, \psi_{,i}, N_i, N_{i,j}, c_i, c_{i,j}$. Using the chain rule of differential calculus to expand $\dot{F}$, and making use of Langrange multipliers to express the constraints $\mathbf{c} \cdot \mathbf{c} = 1, \mathbf{c} \cdot \mathbf{N} = 0$ and the fact that $\mathbf{N} = \nabla \psi / |\nabla \psi|$, we have

$$\begin{aligned} \dot{F} = & \frac{\partial F}{\partial \rho} \frac{D\rho}{Dt} + \frac{\partial F}{\partial \psi_{,i}} \frac{D\psi_{,i}}{Dt} + \frac{\partial F}{\partial N_i} \frac{DN_i}{Dt} + \frac{\partial F}{\partial N_{i,j}} \frac{DN_{i,j}}{Dt} + \frac{\partial F}{\partial c_i} \frac{Dc_i}{Dt} + \frac{\partial F}{\partial c_{i,j}} \frac{Dc_{i,j}}{Dt} + \\ & \lambda N_i \frac{Dc_i}{Dt} + \lambda c_i \frac{DN_i}{Dt} + \gamma c_i \frac{Dc_i}{Dt} + \beta_i \frac{D}{Dt} (\psi_{,i} - |\psi_{,i}| N_i) \end{aligned}$$

where λ, γ, β_i are the Langrange multipliers. Then. it is easy to show that

$$\begin{aligned} \int_V \rho \dot{F} dV = & \oint_A dA_i \left[\Lambda_i \dot{\psi} + \rho \frac{\partial F}{\partial N_{j,i}} \dot{N}_j + \rho \frac{\partial F}{\partial c_{j,i}} \dot{c}_j \right] + \\ & \int_V dV \left[d_{ij} \left(-\rho^2 \frac{\partial F}{\partial \rho} \delta_{ij} - \psi_{,i} \Lambda_j - \rho N_{k,i} \frac{\partial F}{\partial N_{k,j}} - \rho c_{k,i} \frac{\partial F}{\partial c_{k,j}} \right) \right] - \\ & \int_V dV \left[\omega_{ij} \left(\psi_{,i} \Lambda_j + \rho N_{k,i} \frac{\partial F}{\partial N_{k,j}} - c_j h_i^c + \rho c_{k,i} \frac{\partial F}{\partial c_{k,j}} - \lambda \rho N_i c_j \right) \right] + \\ & \int_V dV \left[\dot{N}_i \left(h_i^N + \lambda \rho c_i - \beta_i \rho |\psi_{,m}| \right) - \dot{\psi} \Lambda_{i,i} \right] + \\ & \int_V dV \left[(\dot{c}_i - \omega_{ij} c_j) (h_i^c + \lambda \rho N_i + \gamma \rho c_i) \right] \end{aligned}$$

where

$|\psi_{,i}|$ is a measure of layer curvature and dilatation or contraction,

$$\Lambda_i = \rho \partial F / \partial \psi_{,i} + \rho \beta_j P_{ji} \quad ,$$

$$h_i^N = \rho \partial F / \partial N_i - (\rho \partial F / \partial N_{i,j})_{,j} \quad ,$$

$$h_i^c = \rho \partial F / \partial c_i - (\rho \partial F / \partial c_{i,j})_{,j} \quad ,$$

and $P_{ij} = \delta_{ij} - N_i N_j$. With

$$\pi_{ji} = \rho \partial F / \partial c_{i,j} \quad ,$$

$$N_i h_i^c = -\rho \lambda \quad ,$$

$$N_i g_i = -\rho N_i \partial F / \partial c_i \quad .$$

$$P_{ik} (g_k + \pi_{lk,l} + h_k^c) + \rho c_i = P_{ik} g'_k ;$$

the asymmetric stress tensor taken to be

$$\sigma_{ji} = \sigma'_{ji} - \rho^2 \frac{\partial F}{\partial \rho} \delta_{ij} - \rho \psi_{,i} \frac{\partial F}{\partial \psi_{,j}} - N_i P_{kj} h_k^N - \rho N_{k,i} \frac{\partial F}{\partial N_{k,j}} - \rho c_{k,i} \frac{\partial F}{\partial c_{k,j}}$$

and making use of $\dot{N}_i = (P_{ik}/|\psi_{,m}|) D\psi_{,k}/Dt$ the expression for entropy generation is as follows:

$$\begin{aligned} T \int_V \rho \dot{S} dV &= \oint_A dA_i \left[- \left(\rho \frac{\partial F}{\partial \psi_{,i}} + \frac{P_{ki}}{|\psi_{,m}|} h_k^N \right) \dot{\psi} - \rho \frac{\partial F}{\partial N_{j,i}} \dot{N}_j \right] + \\ &\quad \int_V dV \left[\sigma'_{ji} d_{ij} + \omega_{ij} W_{ij} + \dot{\psi} \left(\rho \frac{\partial F}{\partial \psi_{,i}} + \frac{P_{ki}}{|\psi_{,m}|} h_k^N \right)_{,i} \right] - \\ &\quad \int_V dV [P_{ik} g'_k (\dot{c}_i - \omega_{ij} c_j)] \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} W_{ij} &= \rho \psi_{,i} \frac{\partial F}{\partial \psi_{,j}} + \rho N_{k,i} \frac{\partial F}{\partial N_{k,j}} - \rho c_j \frac{\partial F}{\partial c_i} - \rho c_{j,k} \frac{\partial F}{\partial c_{i,k}} + \\ &\quad \rho c_{k,i} \frac{\partial F}{\partial c_{k,j}} + N_i P_{kj} h_k^N \end{aligned}$$

Since ω_{ij} can be varied arbitrarily by superposed rigid rotations. $W_{ij} = W_{ji}$. Hence, from equation (4.11), we get the relation

$$\sigma'_{ji} - P_{ik} c_j g'_k = \sigma'_{ij} - P_{jk} c_i g'_k \quad (4.14)$$

At this point, we recapitulate the interpretation of Kléman and Parodi [14] regarding the surface terms which appear in equation (4.13). With layers parallel to a plane surface, the first term describes the effects due to an imposition of a dilatation (or contraction) of layers. When the dilatation is large, this process leads to creation of edge dislocations and then the number of layers in the system will not be conserved. When the layers are normal to a plane surface, the second term leads to depinning of layers along the surface in order to relax their bending. The last term plays the role of a surface torque. We shall not deal with these surface terms any further since we are interested in problems wherein the number of layers is conserved, the surface torques and layer depinning are considered to be absent due to boundary conditions.

The entropy generation in the bulk satisfies the following inequality :

$$\rho T \dot{S} = \sigma'_{ji} d_{ij} + \dot{\psi} \left(\rho \frac{\partial F}{\partial \psi_{,i}} + \frac{P_{ki} h_k^N}{|\psi_{,m}|} \right)_{,i} - P_{ik} g'_k (\dot{c}_i - \omega_{ij} c_j) \geq 0 \quad (4.15)$$

The first term is associated with shape and volume changes as in normal hydrodynamics; the second describes dissipative dynamics of the layered structure; and the third is due to dissipative torques along the layer normal and acting on the $\mathbf{c}$ vector. It should be noted that torques on the $\mathbf{c}$ vector that distort the layer is described by the second term and such contributions have been removed from the last term (not to count the same thing twice, of course) by making use of the constraint $\mathbf{c} \cdot \mathbf{N} = 0$. It must be remarked that while this work was in progress, we came across the work of Weinan [15] who, following Kléman and Parodi, had developed at the same time a nonlinear continuum theory of smectic-A liquid crystals. His derivation and interpretation of the permeation process is similar to that presented in this chapter.

4.4.1 Free energy and dissipative part of the stress tensor

The free energy density and the dissipative part of the stress tensor should be invariant under the transformation: $\mathbf{N} \rightarrow -\mathbf{N}$ and $\mathbf{c} \rightarrow -\mathbf{c}$. The free energy density, $F' = \rho F$, satisfying this constraint has been worked out by Nakagawa [16]. This has the form:

$$\rho F = F_s + F_a + F_c + F_{sc} \quad (4.16)$$

The first two terms are associated with layer distortions. F_s is given by

$$F_s = \frac{A_{11}}{2} (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{b})^2 + \frac{A_{21}}{2} (\mathbf{b} \cdot \nabla \mathbf{a} \cdot \mathbf{b})^2 + \frac{A_{12}}{2} (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{c})^2 + \frac{B}{2} (\mathbf{a} \cdot \mathbf{N} - 1)^2$$

Here, the A_{11} , A_{21} and A_{12} are elastic constants involved with layer curvature. The B term is the elastic constant describing layer dilatation and compression. In smectic- C , there are additional contributions to F due to couplings between layer curvature and compression or dilatation, and they are given by F_a .

$$F_a = \frac{L_1}{2} (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{N})^2 + \frac{L_2}{2} (\mathbf{b} \cdot \nabla \mathbf{a} \cdot \mathbf{N})^2 + \frac{L_3}{2} (\mathbf{N} \cdot \nabla \mathbf{a} \cdot \mathbf{N})^2 + L_{13} (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{N})(\mathbf{N} \cdot \nabla \mathbf{a} \cdot \mathbf{N}) - M_1 (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{N})(\mathbf{b} \cdot \nabla \mathbf{a} \cdot \mathbf{b}) + M_2 (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{N})(\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{c}) + N_1 (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{N})(\mathbf{b} \cdot \nabla \mathbf{c} \cdot \mathbf{b}) + N_2 (\mathbf{b} \cdot \nabla \mathbf{a} \cdot \mathbf{N})(\mathbf{N} \cdot \nabla \mathbf{c} \cdot \mathbf{b})$$

The terms in F_c describe the distortion in the orientation of the $\mathbf{c}$ vector

$$F_c = \frac{B_1}{2} (\mathbf{c} \cdot \nabla \mathbf{c} \cdot \mathbf{b})^2 + \frac{B_2}{2} (\mathbf{b} \cdot \nabla \mathbf{c} \cdot \mathbf{b})^2 + \frac{B_3}{2} (\mathbf{N} \cdot \nabla \mathbf{c} \cdot \mathbf{b})^2 + B_{13} (\mathbf{c} \cdot \nabla \mathbf{c} \cdot \mathbf{b})(\mathbf{N} \cdot \nabla \mathbf{c} \cdot \mathbf{b})$$

Finally, F_{sc} has terms that couple the layer distortions to the $\mathbf{c}$ vector deformations.

$$F_{sc} = -C_1 (\mathbf{c} \cdot \nabla \mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \nabla \mathbf{c} \cdot \mathbf{b}) - C_2 (\mathbf{b} \cdot \nabla \mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \nabla \mathbf{c} \cdot \mathbf{b})$$

with C_1 and C_2 as coupling elastic constants. Here, $\mathbf{a} = \nabla \psi$ and, $\mathbf{a} = \mathbf{N}$ when there is no layer dilatation or contraction, and $\mathbf{b} = \mathbf{N} \times \mathbf{c}$. When the variations are assumed to be small, this reduces to the free energy density of the Orsay Group [5]. It should be mentioned in passing that in this model of Nakagawa [16], instead of allowing $\nabla \times \mathbf{N} = 0$, one allows $\nabla \times \mathbf{a} = 0$. This condition also precludes the existence of dislocations and in addition allows for a more general description of layer distortions. We are adopting here the procedure of Nakagawa.

To satisfy the entropy generation inequality (4.15), we assume the first constitutive relation as

$$\dot{\psi} = \lambda_p \left(\rho \frac{\partial F}{\partial \psi_{,i}} + \frac{P_{ki} h_k^N}{|\psi_{,m}|} \right)_{,i} \quad (4.17)$$

where A_p is a constant called the permeation constant first introduced by Helfrich [3] for layered systems. Equation (4.17) has the form of a continuity equation. Within the context of continuum mechanics, equation (4.17) describes a situation in which any relative motion of the fluid particles with respect to the layers along the layer normal results in a distortion of the layers. Further, as Helfrich pointed out [3], with the layers fixed at the boundaries, any flow normal to the layers would experience a large viscosity. And, the velocity in that direction would be proportional to the pressure gradient just as in a flow through porous media. With the layers fixed rigidly at the walls, a part of the pressure gradient distorts the layers and the remaining part leads to fluid flow through a "fixed layered structure". The permeation constant, A_p , can be roughly estimated to be $\lambda_p \eta \sim d^2$, where η and d are respectively the viscosity coefficient and the smectic layer separation. Generally, $\lambda_p \sim 10^{-13} - 10^{-14} \text{ gm}^{-1} \text{ cm}^3 \text{ sec}$.

Now, we take up the dissipative hydrodynamic stress, σ'_{ji} , and the dissipative director torque, g'_i . Following the Ericksen-Leslie formalism [6, 7], we can write down the most general expressions for σ'_{ji} and g'_i as follows :

$$\sigma'_{ji} = A_{jik} M_k + A_{jikm} d_{km} \quad (4.18)$$

$$g'_i = B_{ij} M_j + B_{ijk} d_{jk} \quad (4.19)$$

where $M_i = P_{ik} (\dot{c}_k - \omega_{kj} c_j)$ describes the relative rotation of $\mathbf{c}$ with respect to the fluid due to torques along $\mathbf{N}$ and has the properties $M_i c_i = M_i N_i = 0$.

The most general expression for σ'_{ji} and g'_i consistent with the monoclinic symmetry of Sm-C, the equation (4.14), and the Onsager's reciprocity relations, are:

$$\begin{aligned} \sigma'_{ji} = & -\frac{\gamma_2}{2} (N_j M_i + N_i M_j) - \frac{\gamma_3}{2} (c_j M_i - c_i M_j) + \frac{\gamma_1}{2} (c_j M_i - c_i M_j) + \\ & \alpha_1 \delta_{ij} d_{kk} + \alpha_2 d_{ij} + \alpha_3 (N_k N_m d_{km} \delta_{ij} + N_i N_j d_{kk}) + \\ & \alpha_4 (N_j N_m d_{im} + N_i N_m d_{jm}) + \alpha_5 (c_k c_m d_{km} \delta_{ij} + c_i c_j d_{kk}) + \\ & \alpha_6 (c_j c_m d_{im} + c_i c_m d_{jm}) + \frac{\gamma_1}{2} (c_j c_m d_{im} - c_i c_m d_{jm}) + \\ & \alpha_7 (N_k c_m d_{km} \delta_{ij} + N_i c_j d_{kk} + N_j c_i d_{kk}) + \\ & \alpha_8 (N_j c_m d_{im} + N_i c_m d_{jm} + N_m c_j d_{im} + N_m c_i d_{jm}) + \\ & \frac{\gamma_2}{2} (N_m c_j d_{im} - N_m c_i d_{jm}) + \\ & \alpha_9 (N_i c_j + N_j c_i) N_k c_m d_{km} + \frac{\gamma_3}{2} (N_j c_i - N_i c_j) N_k c_m d_{km} + \\ & \alpha_{10} (N_i c_j c_k c_m + N_j c_i c_k c_m + c_i c_j N_k c_m) d_{km} + \\ & \alpha_{11} (c_i N_j N_k N_m + c_j N_i N_k N_m + N_i N_j c_k N_m) d_{km} + \\ & \frac{\gamma_2}{2} (c_i N_j - c_j N_i) N_k N_m d_{km} + \alpha_{12} N_i N_j N_k N_m d_{km} + \\ & \alpha_{13} c_i c_j c_k c_m d_{km} \end{aligned} \quad (4.20)$$

$$g'_i = \gamma_1 M_i + \gamma_2 N_k d_{ik} + \gamma_3 c_k d_{ik} \quad (4.21)$$

These expressions can now be compared with those obtained by de Gennes [5] for smectic-C. In principle, there is full agreement between our and de Gennes' approaches.

4.4.2 Linearized equations

Before we tackle the various situations of macroscopic flow, we will choose a suitable parametrization of the variables and present the hydrodynamic equations in a tractable form that is convenient to make a more complete comparison with the theory of MPP. For perturbations of a planar structure with the layer normal initially aligned along Z -axis, it is assumed that

$$\psi = z - u(x, y, z) \quad (4.22)$$

Then, a suitable parametrization of the layer normal and the $\mathbf{c}$ vector is

$$\mathbf{N} \simeq \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1 \right) \quad (4.23)$$

$$\mathbf{c} \simeq \left(\cos \phi, \sin \phi, \frac{\partial u}{\partial x} \cos \phi + \frac{\partial u}{\partial y} \sin \phi \right) \quad (4.24)$$

where ϕ is the angle made by the $\mathbf{c}$ vector with the X -axis, the direction of the undistorted $\mathbf{c}$ vector. We take the free energy density up to linear terms in the gradients of u , and ignore terms of the kind $\partial^2 u / \partial x_i \partial z$ since the term $\partial u / \partial z$ would be of the highest order. If linearized in ϕ also, the expression reduces to that of the Orsay group.

Finally, the hydrodynamic equations become :

$$\frac{D\rho}{Dt} + \rho v_{k,k} = 0 \quad (4.25)$$

$$\rho \frac{Dv_x}{Dt} = -P_{,x} + \sigma'_{jx,j} \quad (4.26)$$

$$\rho \frac{Dv_y}{Dt} = -P_{,y} + \sigma'_{jy,j} \quad (4.27)$$

$$\rho \frac{Dv_z}{Dt} = -P_{,z} + \sigma'_{jz,j} + \frac{\delta F}{\delta u} \quad (4.28)$$

$$\frac{\partial u}{\partial t} - v_z = \lambda_p \frac{\delta F}{\delta u} \quad (4.29)$$

$$g'_y c_x - g'_x c_y = \frac{\partial F}{\partial \phi} - \left(\frac{\partial F}{\partial \phi_{,j}} \right)_{,j} \quad (4.30)$$

where $P (= \rho^2 \partial F / \partial \rho)$ is the pressure, and

$$\frac{\delta F}{\delta u} = \left(\frac{\partial F}{\partial u_z} \right)_{,z} - \left(\frac{\partial F}{\partial u_{xx}} \right)_{,xx} - \left(\frac{\partial F}{\partial u_{yy}} \right)_{,yy} - 2 \left(\frac{\partial F}{\partial u_{xy}} \right)_{,xy}$$

Written in this form, our theory is exactly similar to that of MPP apart from the fact that we have used an asymmetric stress tensor. To summarize, the main features of our theory are:

- We make use of an asymmetric stress tensor, and show that our theory does correspond to that of de Gennes when the hydrodynamic equations are linearized. This was not possible with the theory of LSN.
- The theory is covariant in the sense that the constraint $\mathbf{N} \cdot \mathbf{c} = 0$ is imposed whereby simultaneous rotations of $\mathbf{N}$ and $\mathbf{c}$ do not cost energy. This allows for a general description of large distortions of the $\mathbf{c}$ vector and the layers. Our theory also allows for variations in layer spacing.
- Permeation process is included in the theory.

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CHAPTER 5

DYNAMICS OF THE UNIFORM STATE OF SMECTIC- C

5.1 Introduction

In the previous chapter, a complete nonlinear theory of the macroscopic hydrodynamics of smectic- C was formulated. The theory incorporates nematic-like dynamics of the c vector and the essential features of smectic hydrodynamics. In this chapter, we will apply this theory to highlight certain unique features of the dynamics of the uniform state of smectic- C . Most of the results presented in this chapter have been published elsewhere [1, 2].

5.2 Inlet section: effects of permeation

In most studies to date, flow along the layers of smectics is studied as in normal fluids. In the classical Poiseuille flow, liquid flows into a capillary or the space between two parallel plates from a pressure head. The liquid enters with nearly a flat profile. A boundary layer starts to develop at the walls at the entry

point. As the liquid flows down, the boundary layer thickness continuously increases. A point downstream will be reached where the boundary layers with their nearly parabolic velocity profiles meet or crossover at the centre. It is only after this point the velocity profile is steady and becomes a symmetric parabola. Over the distance from the entry point to the crossover point the velocity profile over the cross-section continuously changes as we go down the tube. This distance is called the *inlet section*. Hence, in the inlet section, the liquid in the central region is continuously accelerated and that at the walls is always at rest. Earlier investigations had virtually ignored this in their analysis of the flows of smectics. But, this can lead to serious error in the analysis of experimental data.

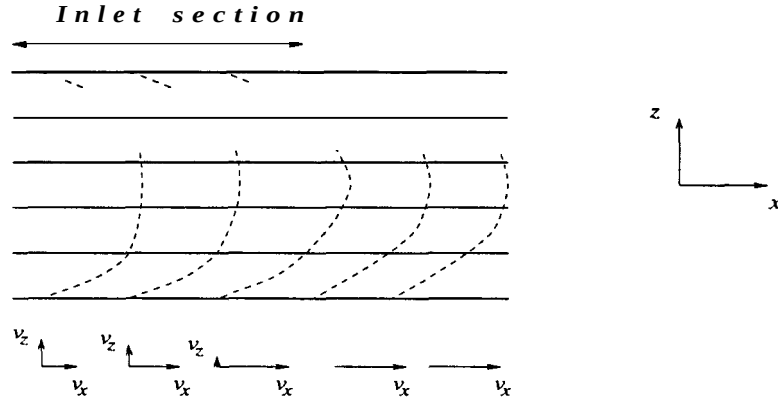


Figure 5.1: Flow geometry of Poiseuille flow along the layers in smectics. The driving pressure gradient is along the x -axis. In the inlet section, the velocity distribution (shown by the dotted line) develops from a flat profile at the inlet to the steady state parabolic distribution of Poiseuille flow. In the inlet section, there is a velocity component, v_z , normal to the layers. Owing to viscous friction, boundary layers will be formed on both walls and their width will increase in the downstream direction.

A simple analysis is presented to estimate the length of this inlet section. We consider the case where the smectic layers are parallel to the plates which are in the xy -plane. Further, the velocity field is described by $\mathbf{v} = (v_x, 0, v_z)$. The flow geometry is shown in Figure 5.1. The hydrodynamic equations that are relevant in

the boundary layers are now mentioned. The equation of continuity is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \quad (5.1)$$

The liquid flows in the smectic layers which are assumed to be stationary. From equations (4.28) and (4.29), and ignoring higher order terms

$$\frac{\partial p}{\partial z} \simeq -\frac{v_z}{\lambda_p} \quad (5.2)$$

where λ_p is the permeation constant. Finally, the boundary layer equation for the flow along the layers obeys equation (4.26). It is identical to that of ordinary fluids,

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 v_x}{\partial z^2} \quad (5.3)$$

Here η is the effective coefficient of viscosity.

Owing to viscous friction, boundary layers will be formed on both walls, and their width will increase in the downstream direction. The resulting velocity profile will consist of two boundary layer profiles on the two walls joined in the centre by a line of constant velocity. We can estimate from the above equations the boundary layer thickness, δ , up to which the transverse pressure gradient extends. The boundary layer thickness, δ , is a function of the distance x from the leading edge of the plate. Let the flow along x -axis be V_0 at the centre. Since the volume of flow must be the same for every section, the decrease in the rate of flow near the walls which is due to friction must be compensated by a corresponding increase near the axis. From the continuity equation (5.1), a rough estimate gives

$$v_z = V_0 \delta/x$$

From equation (5.2), the pressure, p , can be taken to be approximately

$$p = V_0 \delta^2/(x \lambda_p)$$

The equation (5.3) simplifies to:

$$x^2 - 2r\delta x - (\delta^4/\kappa^2) = 0 \quad (5.4)$$

where $r = V_0\rho\delta/\eta$, and $\kappa = \sqrt{\lambda_p\eta}$, the length scale introduced by permeation effects. In the absence of permeation, $\lambda_p \rightarrow \infty$, *i.e.*, $\kappa \rightarrow \infty$. In smectics, it can be assumed [3] that $\lambda_p \sim d^2/\eta$, d being the smectic layer thickness; hence, $\kappa \sim d$ in this case. The distance x at which the boundary layer thickness equals half of the separation between the plates is the length of the inlet section, L . Using this condition and equation (5.4), the length of the inlet section is

$$L = l \left[R + \frac{l}{\kappa} \sqrt{1 + \rho^2 V_0^2 \frac{\kappa^2}{\eta^2}} \right] \quad (5.5)$$

where $R = \rho V_0 l / \eta$ is the Reynold's number, and $2l$ the separation between the plates. From this it can be seen that we get the length of inlet section as $2lR$ for ordinary fluids in the limit $\kappa \rightarrow \infty$. Thus, in ordinary liquids, the inlet section is comparable to l and depends on main flow velocity, V_0 . On the other hand, in smectics, as already said, κ is very small being of the order of the layer thickness. Then, the inlet section is given by

$$L \simeq \frac{l^2}{\kappa} \sim \frac{l^2}{d} \quad (5.6)$$

This is invariably very large compared to l and is independent of main flow velocity, V_0 . For smectics $d \sim 10^{-7}$ cm. Hence, for $l \sim 10^{-3}$ cm, $L \sim 10$ cm. Therefore, actual experimental results in short tubes are not really representative of the steady Poiseuille flow. This result is true of all smectics in general. The effects of permeation in the inlet section will essentially be the same whether the system be smectic-A or smectic-C. But, the inherent biaxial nature of smectic-C would produce

differences. In our simple analyses, we have assumed the smectic layer structure to be undistorted. But, as mentioned in the previous chapter, permeation would result in distortions of the layered structure. The force required to produce a curvature of the layers with an undistorted configuration of the vector $\mathbf{c}$ is generally less than that with distortions of the $\mathbf{c}$ vector. This fact immediately implies that the effects of permeation with the vector $\mathbf{c}$ parallel or perpendicular to the shear plane would be quantitatively different. Hence, the length of the inlet section in the case of Poiseuille flow along the layers of smectic-C will depend upon the orientation of the $\mathbf{c}$ vector with respect to the shear plane.

5.3 Smectic Hall effect

An interesting case of macroscopic flow in nematics is Poiseuille flow in the presence of a strong external electric or magnetic field applied at an angle to the shear plane [4]. In this case, the director is aligned along the external field and at an angle to the flow lines. The flow geometry is shown in Figure 5.2.

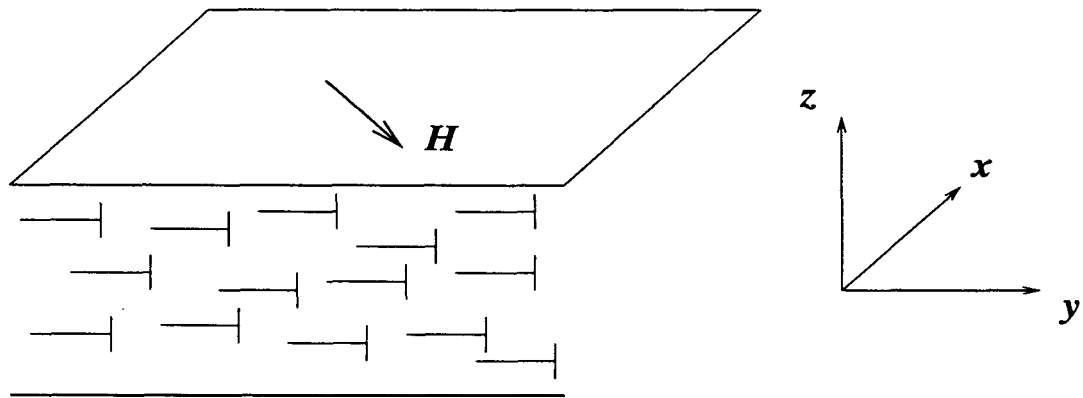


Figure 5.2: Flow geometry of the nematic Hall effect. A driving pressure gradient is applied along x -axis. A strong field is applied in the xy -plane at an angle to the shear plane, *i.e.*, xz -plane. To maintain the flow a transverse pressure gradient develops along the y -axis. The nail-heads indicate that $\mathbf{n}$ comes out of the plane to be aligned along the field.

A driving pressure gradient is applied along the flow lines. From nematodynamics, it

is found that to maintain this flow a *transverse pressure gradient* must be maintained. This has been termed as the *nematic Hall effect*. The very same hydrodynamic effect can be expected in smectic-*C* in two geometries. In the first geometry, $\mathbf{c}$ is fixed at an angle to the shear plane which happens to be perpendicular to the layers and the situation is exactly similar to the nematic Hall effect. In addition, in smectic-*C*, we get it in another configuration also.

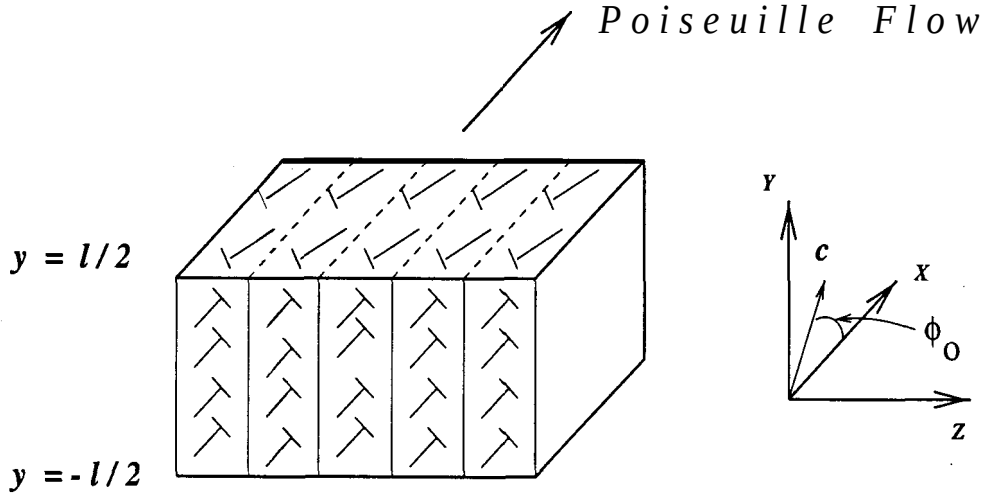


Figure 5.3: A schematic representation of the geometry with the Poiseuille flow along the layers which are perpendicular to the walls. The shear plane is parallel to the layers.

Consider the flow geometry depicted in Figure 5.3. The state $\mathbf{N} = (0, 0, 1)$, $\mathbf{c} = (\cos \phi, \sin \phi, 0)$, and $\mathbf{v} = (2v_x(y), 0, 0)$ is stable below a threshold shear if we assume that a strong magnetic field aligns $\mathbf{c}$ at a particular angle, ϕ . Then, equations (4.26) and (4.28) reduce to

$$P_{,x} = \frac{d^2 v_x}{dy^2} \left[\alpha_2 + \alpha_6 + 2\alpha_{13} \cos^2 \phi \sin^2 \phi - \frac{\gamma_1}{2} - \gamma_3 \cos 2\phi \right] \quad (5.7)$$

$$P_{,z} = \cos \phi \frac{d^2 v_x}{dy^2} \left[\alpha_8 - \frac{\gamma_2}{2} + 2\alpha_{10} \sin^2 \phi \right] \quad (5.8)$$

It can be seen that there exists a transverse pressure gradient, P_z , whenever the $\mathbf{c}$ vector is such that $\phi \neq \frac{1}{2}\pi$ or at a particular angle $\phi \neq \sin^{-1} \left[\sqrt{(\gamma_2 - 2\alpha_3)/4\alpha_{10}} \right]$. The $\mathbf{c}$ vector dynamics is often like nematic director dynamics but there are subtle differences. The additional features in this geometry compared to the first geometry is that there are three possibilities in view of equations (5.7) and (5.8):

- A pressure gradient develops along the Z direction without any lattice dilatation or flow along Z ;
- Pressure gradient is accompanied with layer dilatation but without any macroscopic flow along the layer normal; and
- Pressure gradient leads to layer dilatation and flow along the layer normal.

In the first geometry, the Hall effect will not exist when there is no aligning field. But, in the second geometry, the effect exists even when there is no field. Even in the absence of an external field, the $\mathbf{c}$ vector will align but along the angle, ϕ_0 , given by $\cos 2\phi_0 = -\gamma_1/\gamma_3$. Hence, the transverse pressure gradient exists even in such a situation, unless ϕ_0 accidentally happens to be equal to $\frac{1}{2}\pi$ or $\sin^{-1} \left[\sqrt{(\gamma_2 - 2\alpha_3)/4\alpha_{10}} \right]$. This type of smectic Hall effect is peculiar to Sm-C and does not exist in Sm-A or nematics.

5.4 Uniform state in a rotating field

In the last two sections, we have discussed the dynamics of smectic-C in two cases where there is only macroscopic flow and does not involve the dynamics of the director. Here, we investigate a case which involves only the dynamics of the $\mathbf{c}$ vector. The geometry is schematically depicted in Figure 5.4.

In the cartesian coordinate system, the layer normal is taken along the **z-axis**,

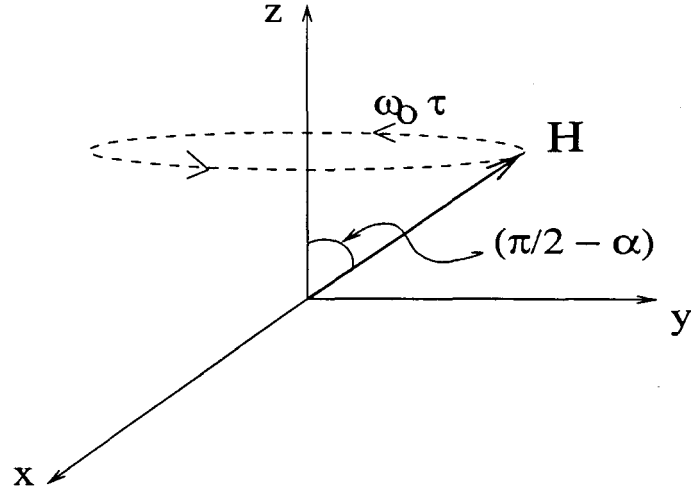


Figure 5.4: A magnetic field, $\mathbf{H}$, tilted at a constant angle α with respect to the layers rotating uniformly about the z -axis, the layer normal.

that is, $\mathbf{N} = (0, 0, 1)$, and the director $\mathbf{n}$ is

$$\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

where $\phi = \phi(z, t)$ is the angle that the $\mathbf{c}$ vector makes with the x -axis. The free energy density is

$$F = -\frac{1}{2}\chi_a(\mathbf{n} \cdot \mathbf{H})^2 \quad (5.9)$$

where χ_a is the diamagnetic anisotropy of the Sm-C in the local uniaxial approximation. Thus, in the case of the dynamics of the uniform state, the Oseen equation (4.30) reduces to an ordinary differential equation given by

$$\lambda \frac{\partial \phi}{\partial t} = -\frac{\delta F}{\delta \phi} \quad (5.10)$$

where $\lambda = -\gamma_1$ is the twist viscosity coefficient. We consider a spatially homogeneous magnetic field rotating about $\mathbf{N}$. It is described by

$$\mathbf{H} = H_0 (a + b \cos \omega t) (\cos \alpha \cos \omega t, \cos \alpha \sin \omega t, \sin \alpha) \quad (5.11)$$

where γ is the frequency of the oscillating field, a is the angle that the field makes with the layers, and w is the angular frequency of rotation about the layer normal, N . In terms of the dimensionless quantity, $\tau = (\chi_a H_0^2 / \lambda) t$, the dynamical equation (5.10) can be written in terms of the azimuthal angle, $u = (\phi - \omega_0 \tau)$, between the director and the field:

$$\frac{du}{d\tau} = -\omega_0 - (a + b \cos \gamma_0 \tau)^2 [\sin \theta \cos \theta \sin a \cos a \sin u + \sin^2 \theta \cos^2 a \sin u \cos u] \quad (5.12)$$

where $\gamma_0 = \gamma t / \tau$ and $\omega_0 = \omega t / \tau$. In this chapter, time and frequencies (of rotation and oscillation) have been expressed with these dimensionless quantities. Hence, $\tau = t / t_0$, $\omega_0 = \omega t_0$ and $\gamma_0 = \gamma t_0$. The characteristic relaxation time, t_0 , is $\lambda / \chi_a H_0^2$. Since the results to be presented are independent of the phase of the oscillating field, it has been taken to be zero.

In a non-oscillating field ($\gamma_0 = 0$) with $a = 0$, equation (5.12) yields analytical solutions. but, with $a \neq 0$, the analytical solution is a complicated function. Numerical integration with a fourth-order Runge-Kutta method reveals that the general features are nearly the same in both these cases. The major difference is that in the case of $a \neq 0$, the symmetry of the system makes the states u and $(u + 2\pi)$ equivalent with regard to their potential energy; while, in the case of $a = 0$, the states u and $(u + \pi)$ are equivalent. The second case is analogous to a similar problem in nematics. But whether $a = 0$ or $a \neq 0$, the behaviour of the system falls in two regimes first pointed out for nematics [5]. In the present case, this is described by: (i) the *synchronous* regime where the c vector follows the field H at an angle constant in time, and (ii) the *asynchronous* regime where there is slip between the field and the c vector. These two regimes are demarcated by a critical rotational frequency of the field, $\omega_0 = w_c$.

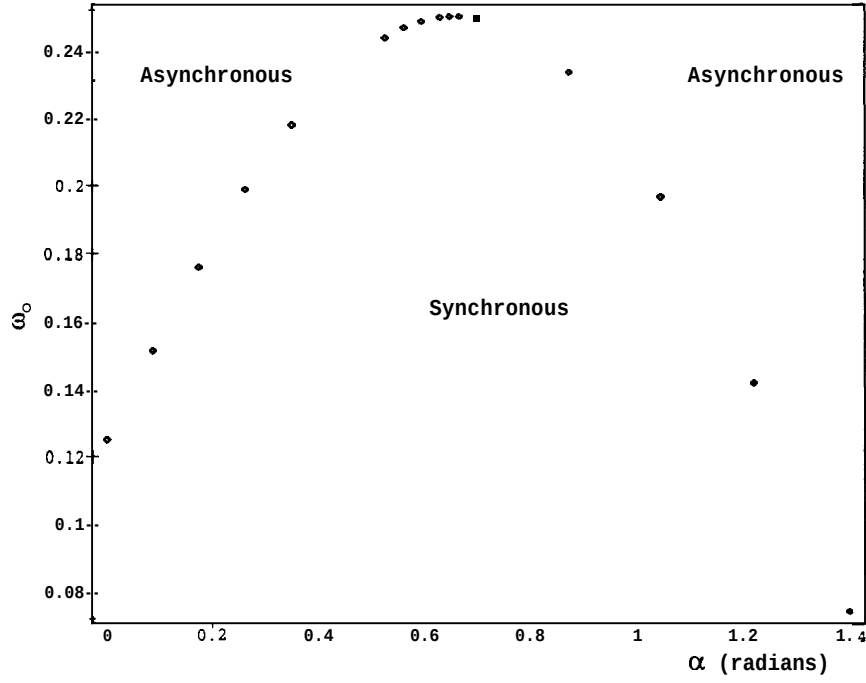


Figure 5.5: ω_0 - α diagram depicting asynchronous to synchronous transition in a constant field in the geometry shown in Figure 5.4. Below a threshold rotational frequency, asynchronous to synchronous and back to asynchronous transitions are seen as α varies from 0 to $\frac{\pi}{2}$ radians. Here, $a = 1, b = 0$.

In Figure 5.5, the dependence of ω_c on α , the angle made by the field $\mathbf{H}$ with respect to the layers, is shown. Interestingly, at an ω_0 below a threshold, as the tilt angle, α , increases from zero, there is a transition from an asynchronous to a synchronous state and at a still higher α a transition back to an asynchronous state. This reentrant behaviour is characteristic of SmC.

In a tilted non-rotating field, $\phi = 0$ is always a stable state and in addition, a metastable state at $\phi = \pi$ exists for $\alpha < \theta$, the tilt of $\mathbf{n}$ with respect to $\mathbf{N}$. This

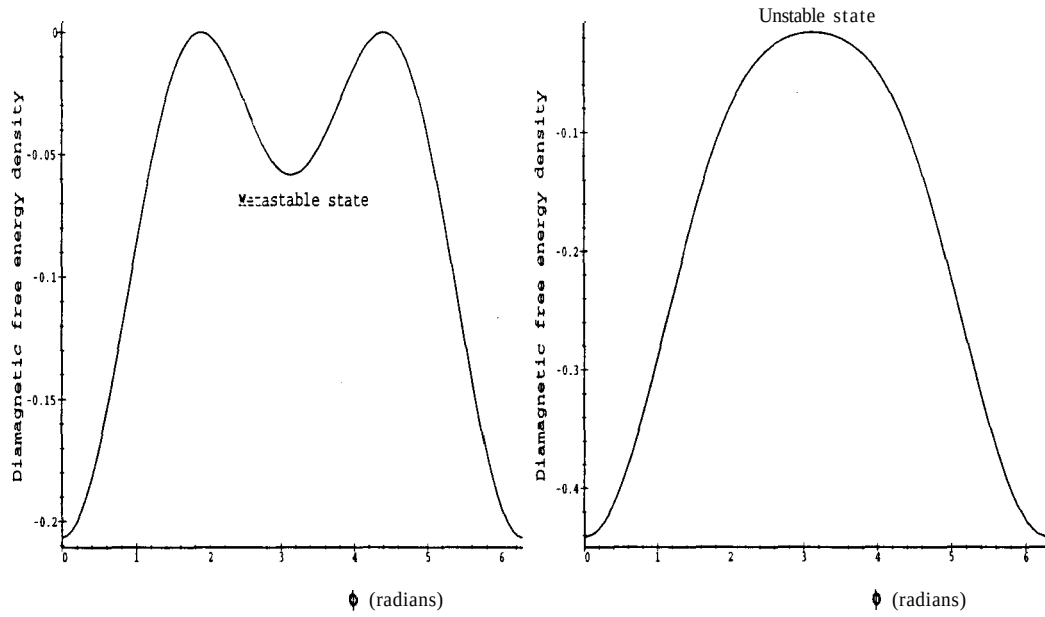


Figure 5.6: The diamagnetic free energy density, is shown as a function of ϕ . Here, we have taken $\chi_a H^2 = 1 \text{ ergs/cm}^3$. And, the diamagnetic free energy density $F = -\frac{1}{2}(\sin \theta \cos a \cos \phi + \cos \theta \sin a)^2$ is plotted for (i) $a = 10^\circ$ (with a metastable state at $\phi = \pi$) and (ii) $a = 40^\circ$ (with an unstable state at $\phi = \pi$). In this case, $\theta = 30^\circ$, and $a, = 30^\circ$.

becomes obvious if we compute the magnetic energy as a function of orientation. This is shown in Figure 5.6. When a field at $\alpha < \theta$ rotates about the layer normal, the relative orientation of both these states changes. The stable state shows a monotonic variation. But, the metastable state gradually shifts towards the stable state, that is, lower values of ϕ until at a critical rotational frequency it switches to the stable state. This critical frequency is less than the critical frequency at which asynchronous regime sets in. Figure 5.7 shows these features.

In the asynchronous regime, the uniform stable state follows the field with an angle which is a nonlinear function of time. At a given rotational frequency of the field, the uniform state changes its orientation as depicted in Figure 5.8. For long intervals of time, the uniform state nearly 'sticks' to the unperturbed state but, at

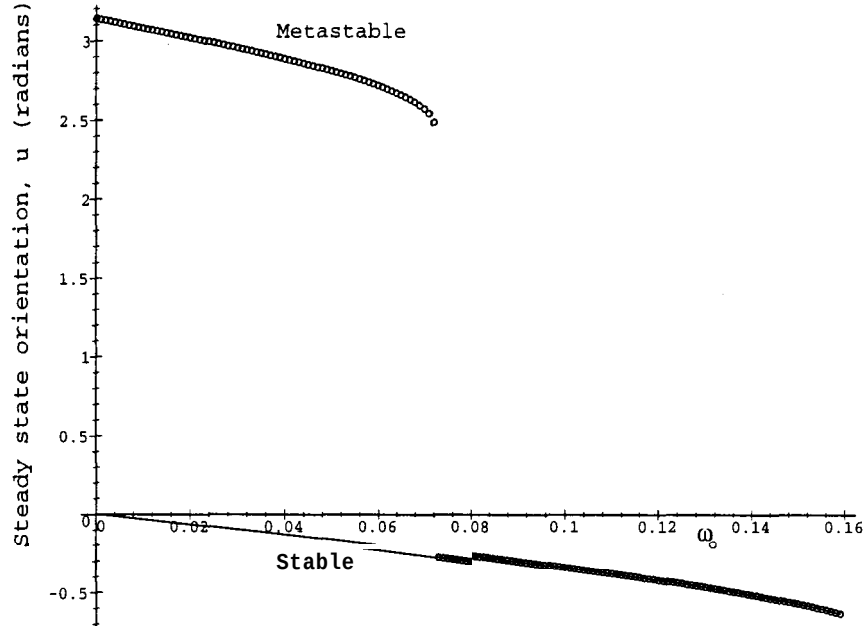


Figure 5.7: The variation of the steady state angle, u , (of a uniformly oriented sample) between c and H with rotational frequency. The circles and continuous line correspond to the initial metastable $u(\tau = 0) = \pi$ and stable $u(\tau = 0) = 0$ states respectively. The metastable state becomes unstable and collapses to the stable state at a critical rotational frequency. Here, $a = 10^\circ, \theta = \frac{\pi}{6}$ radians.

regular intervals, it suffers a 'slip' leading to a phase slip of 2π for $a \neq 0$ and for $a = 0$, this phase slip is π . This interval is a continuous function of ω_0 . This process can be called a “stick-slip” process.

With an oscillating rotating field, *i.e.*, $\gamma_0 \neq 0$ and $\omega_0 \neq 0$, it is found that the dynamics of the uniform state is very different from what we got in a non-oscillating rotating field, *i.e.*, $\gamma_0 = 0$ and $\omega_0 \neq 0$. In an oscillating field, there is a regime very similar to the synchronous regime. Here, the c vector follows the field but oscillating

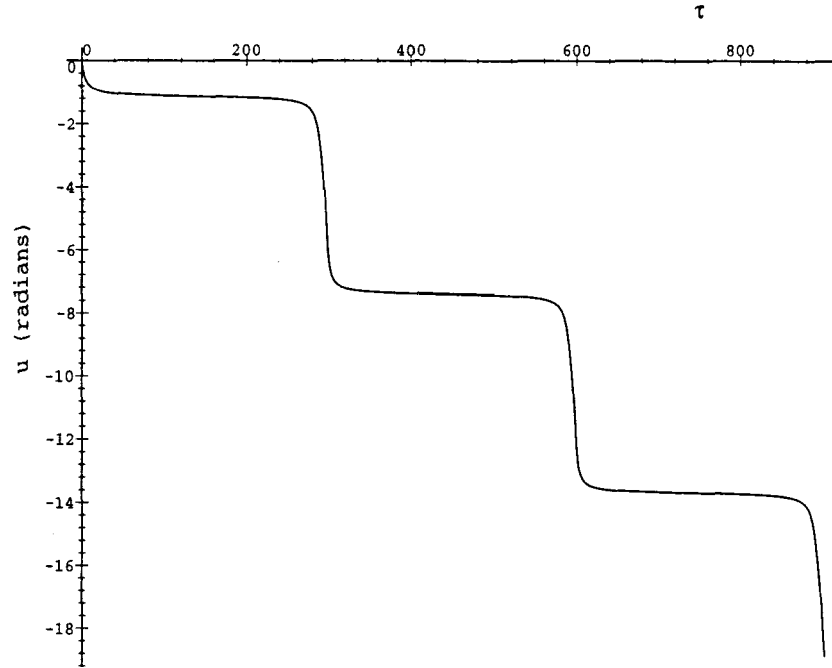


Figure 5.8: Asynchronous regime in a non-oscillating field. Variation of the angle, u , between $\mathbf{c}$ and $\mathbf{H}$ as a function of time in the asynchronous regime. There is a sudden jump by 2π at regular intervals. Here, $a = 1, b = 0, \omega_0 = 0.25, \alpha = 40^\circ$.

about a fixed direction relative to the field. These oscillations are at the frequency of the oscillating field. This is called the *phase-oscillating* regime. The relative mean orientation of the director with respect to the field direction gradually increases as ω_0 increases. Above a critical value of the rotational frequency, ω'_c , the behaviour of the uniform state is rather similar to the already described asynchronous regime in that there exists a phase-slip. This is called the *phase-slipping* regime.

In the case of the asynchronous regime in non-oscillating field, there is a frequency for phase-slip to an equivalent state which varies continuously as one varies

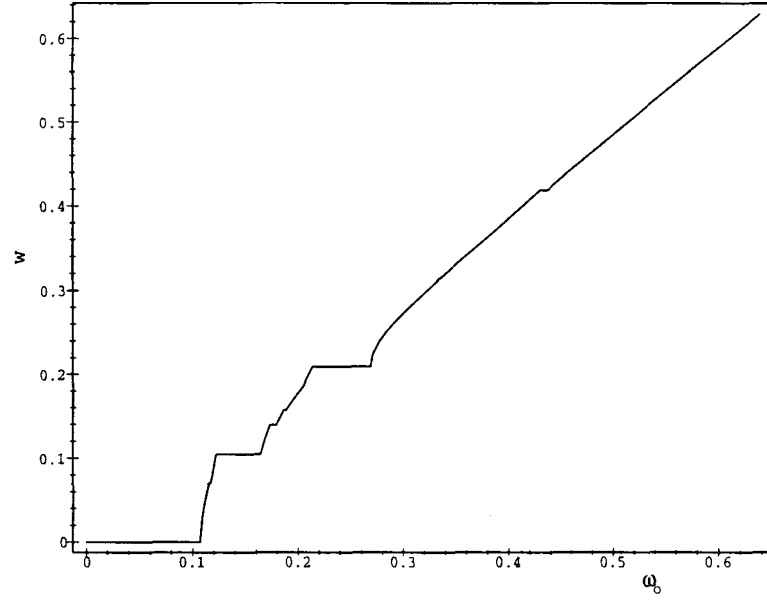


Figure 5.9: The phase-slip frequency, w as a function of the rotational frequency of the field, ω_0 . Below a critical $\omega_0 = \omega'_c$ there is the phase-oscillating regime. Here, $a = 0.8, b = 0.3, \gamma_0 = \frac{2\pi}{30}, \alpha = 10^\circ, \theta = \frac{\pi}{2}$ radians.

the rotational frequency. But, with an oscillating field, this is not the case. It is found that the uniform state always responds in such a way that in n cycles of the oscillating field, the uniform state would have rotated, in the frame of reference of the rotating field, by $2\pi m$ (or πm) for $a \neq 0$ (or $a = 0$). In this case, a phase-slip frequency, w , can be defined as

$$w = \frac{u(\tau + nT) - u(\tau)}{nT} = \frac{2\pi m}{nT} \quad (5.13)$$

where T is the period of the oscillating field.² Incidentally, $w = 0$ for the phase-oscillating regime. Numerical analysis indicates that w is always a rational fraction

²It may be recalled here that Jung, Kissner, and Hänggi [6] describe the motion of an over-damped particle in an asymmetric "ratchet" potential in a similar way.

of the frequency, γ_0 , of the oscillating field with w varying with the rotational frequency, ω_0 , as shown in Figure 5.9. And, in Table 5.1 the values of m and n for different values of ω_0 are given. It clearly shows the extreme sensitivity of m and n to values of ω_0 . The essential features for any $a, \gamma_0, a, b, \theta$ are: (i) the step-like behaviour and, (ii) for large γ_0 , w varies nearly linearly with ω_0 and the length of steps, if present, is too small to show any departure from linearity. In view of the eternal presence of finer and finer steps at all values of the parameter, the process is rather like a 'Devil's staircase'. It should be noted that the integers m and n are extremely dependent on the parameters of the problem.

Table 5.1: The values of m and n of w for different ω_0 . They are very sensitive to values of ω_0 . The fixed parameters are $a = 0.8, b = 0.3, \theta = \frac{\pi}{6}$ radians, $\alpha = 10^\circ, \gamma = \frac{2\pi}{30}$.

ω_0	m	n	ω_0	m	n
$0 \rightarrow 0.107$	0	1	0.271	63	59
0.109	1	7	0.277	305	267
0.111	2	9	0.284	169	141
0.112	1	4	0.29159359	5	4
$0.115 \rightarrow 0.117$	1	3	0.299715606	13	10
0.119	311	800	0.305363042	4	3
$0.122 \rightarrow 0.164$	1	2	$0.335 \rightarrow 0.336$	3	2
0.169	3	5	0.34930185	11	7
$0.173 \rightarrow 0.179$	2	3	0.367657157	5	3
$0.186 \rightarrow 0.188$	3	4	0.368	337	202
0.194	4	5	0.38395962	7	4
0.198	5	6	$0.430 \rightarrow 0.437$	2	1
0.201	6	7	0.438	549	274
0.205	8	9	0.637	665	222
$0.214 \rightarrow 0.269$	1	1	0.638	3	1

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