PRAMANA
— journal of
physics

Vol. 47, No. 5,
November 1996
pp. 361–369

Poincaré sphere representation for three state systems

JOSEPH SAMUEL
Raman Research Institute, Bangalore, 560 080, India

MS received 8 August 1996; revised 15 October 1996

Abstract. We point out that the Poincaré sphere can be used to represent the rays of a three state quantum system. Those interested in geometric phase phenomena may find this representation a useful aid to visualize the global structure of ray space.

Keywords. Quantum mechanics; ray space; Poincaré sphere.

PACS No. 03·65

1. Introduction

The Poincaré sphere [1] provides a simple and elegant geometric representation of the rays of a two state system. It was originally introduced in optics, in the context of polarized light and has subsequently been widely used [2]. More recently, the Poincaré sphere has become popular in studies of the geometric phase [3–12] for two state systems. This article shows how the Poincaré sphere can also be used to represent the rays of three or N state systems. There appears to be some current interest [13–15] in this question and we hope that the following exposition will be useful. We will assume that the reader is familiar with the Poincaré sphere for two state systems and with Pancharatnam's work [16, 17, 6, 7, 18, 19] on the interference of polarized light.

This presentation makes use of spinors. While two component (Weyl) spinors are used in some areas of general relativity, their application to the present situation may not be known in the geometric phase community. The purpose of this article is to bridge this gap. In § 2 we briefly summarize the spinorial formulae that will be used in § 3 to show how spinors can be used to represent the ray space of a three state system. Readers desiring to learn more about spinors should consult Penrose and Rindler [20] and Wald [21].

2. Spinorial preliminaries

Let (V, ε) be a two dimensional complex vector space, where ε is an antisymmetric nondegenerate bilinear form on V. Elements of V are called spinors and written ξ^A, $A = 1, 2$. ε provides a natural isomorphism between V and its dual V^*. 'Lowering the index' on ξ^A yields the element $\xi_A := \xi^B \varepsilon_{BA}$ in V^*. Note that $\xi^A \xi_A = 0$. It is usual to pick a basis (ι^A, o^A) in V so that $\iota^A o_A = 1$. Such a frame is called a spin frame. The group of transformations that leaves the structure (V, ε) invariant is $SL(2, \mathbb{C})$, the double cover of the Lorentz group.

One can similarly consider $\bar{V}$, the complex conjugate space of V, whose elements are written $\bar{\xi}^{A'} \in \bar{V}$ and the dual of $\bar{V}$, $\bar{V}^*$ with elements $\bar{\xi}_{A'} \in \bar{V}^*$. The $SL(2, \mathbb{C})$ spinors defined above (while suitable for applications to general relativity) are too general for our purpose. We need to introduce more structure (see p. 376 of ref. [21] or ref. [22]) and reduce the group down to $SU(2)$. This is done by introducing a positive definite, Hermitian inner product $G_{AA'}$ on V. The group of linear transformations on V that preserves both ε and $G_{AA'}$ is $SU(2)$. We can choose our spin frame so that [21]

$$G_{AA'} = \iota_A \iota_{A'} + o_A o_{A'}$$

and use $G_{AA'}$ to define a $\dagger$ operation on spinors.

$$\xi^\dagger_A = \bar{\xi}^{A'} G_{AA'}.$$

We will sometimes use Dirac notation $|\xi)$ for the element ξ^A of V and $(\xi|$ for the element $\xi^\dagger_A$ of V^*. Note that $(\xi|$ is not ξ_A, for $(\xi|\xi)$ is positive definite, whereas $\xi_A \xi^A$ vanishes.

Spinors have some features which take getting used to. These stem from the fact that indices are raised and lowered by an antisymmetric tensor rather than a symmetric one. Here are a few properties that the readers may find useful to verify:

1. $\xi^A \eta_A = -\xi_A \eta^A$. (This is sometimes referred to as 'Penrose's seesaw'.)
2. $\xi^{\dagger\dagger} = -\xi$.
3. $\xi^\dagger$ is orthogonal to ξ: $(\xi^\dagger|\xi) = 0$.

These are all straightforward consequences of the spinorial formalism.

A spinor κ^A can be thought of as representing the state of a spin half system. The ray κ corresponding to the state

$$\kappa^A = z\iota^A + o^A$$

can be represented by z, a point on the extended complex plane $\mathbb{C}^*$. (We allow the possibility $z = \infty$, which is needed to represent $\kappa^A = \iota^A$.) The extended complex plane $\mathbb{C}^*$ can be stereographically mapped onto the Poincaré sphere. The formula

$$z = \cot(\theta/2)\mathrm{e}^{i\phi}$$

relates the stereographic coordinate z to the point p on the Poincaré sphere with polar coordinates, θ, ϕ. The group $SL(2, \mathbb{C})$, which acts on spinors, acts on the extended complex plane by Möbius transformations

$$z \rightarrow \frac{az + b}{cz + d}, \qquad (1)$$

where a, b, c, d are the entries of the $SL(2, \mathbb{C})$ matrix. Composing (1) with the stereographic projection, we get an action of $SL(2, \mathbb{C})$ on the Poincaré sphere [20]. These maps are conformal transformations and map circles to circles on the Poincaré sphere. In the physical interpretation used by relativists, spinors represent null vectors (the momentum four vectors of photons) and rays represent points on the sky [20]. The action of $SL(2, \mathbb{C})$ on the Poincaré sphere can be physically realized as the aberration of light from the sky by Lorentz transformations.

The action of $\dagger$ on the Poincaré sphere is easy to visualize. By explicit computation, one sees that $\iota^\dagger = o$, $o^\dagger = -\iota$ and so, if $\xi^A = \alpha\iota^A + \beta o^A$,

$$\xi^\dagger = \bar{\alpha} o^A - \bar{\beta}\iota^A.$$

Since $\xi^\dagger$ is clearly orthogonal to ξ, we see that $\dagger$ sends a point on the Poincaré sphere to its antipode. The subgroup of $SL(2,\mathbb{C})$ which preserves antipodality of points is $SU(2)$, which acts on the Poincaré sphere by rotations. With these preliminaries, we begin our discussion of three state systems.

3. Three state systems

For two state systems, the Poincaré sphere provides a representation of the ray space with the following properties:

P1. All rays of the system are represented exactly once.
P2. Neighboring rays are represented as neighboring points on the Poincaré sphere.

We would like to have a similar representation of the ray space of a three state system.

Since we are only interested in the geometry of the ray space (i.e., kinematic features of the Hilbert space, not dynamical ones like the Hamiltonian), we are free to make any convenient choice of system. The most convenient choice is a spin 1 system ($s = 1$). The physical idea underlying the following discussion is that a spin one ($s = 1$) particle can be thought of as a composite of two spin half ($s = 1/2$) particles. The antisymmetric tensor product of the states of a spin half system is (see p. 204 of ref. [23]) a singlet ($s = 0$) state and need not be further considered. We look at the symmetrized tensor product of spin half states and show how one represents these on the Poincaré sphere.

Let Ψ^{AB} be a symmetric spinor. $\mathscr{H} = V \otimes_s V$ is a three-dimensional complex vector space, whose elements we write in Dirac notation as $|\Psi\rangle$. Let us define

$$\Psi^\dagger_{AB} := \bar{\Psi}^{A'B'} G_{AA'} G_{BB'}.$$

We write elements $\Psi^\dagger_{AB}$ of $\mathscr{H}^*$ as $\langle\Psi|$ and $\Psi^\dagger_{AB}\Psi^{AB}$ as $\langle\Psi|\Psi\rangle$. The elements $(|e_1\rangle, |e_2\rangle, |e_3\rangle)$, where $e_1^{AB} := \iota^A \iota^B$, $e_2^{AB} := 1/\sqrt{2}(\iota^A o^B + \iota^B o^A)$ and $e_3 := o^A o^B$ form an orthonormal set in $\mathscr{H}$. Any element Ψ^{AB} of $\mathscr{H}$ can be expanded in this basis

$$\Psi^{AB} = \Psi^1 e_1^{AB} + \Psi^2 e_2^{AB} + \Psi^3 e_3^{AB}.$$

The inner product of $|\Psi\rangle$ with itself is

$$\langle\Psi|\Psi\rangle = \Psi^\dagger_{AB}\Psi^{AB} = \overline{\Psi^1}\Psi^1 + \overline{\Psi^2}\Psi^2 + \overline{\Psi^3}\Psi^3.$$

Thus $\mathscr{H}$ is a three dimensional complex vector space with a positive definite inner product and can be identified with the Hilbert space of any three state quantum system.

Let us now represent rays in $\mathscr{H}$ on the Poincaré sphere. Let Ψ^{AB} be an (unnormalized, nonzero) symmetric spinor. Consider the quadratic polynomial

$$f(z) = \Psi^{AB} \kappa_A \kappa_B, \tag{2}$$

where $\kappa_A = z\iota_A + o_A$. $f(z)$ has exactly two roots z_1 and z_2 on the extended complex plane and can be factorized as

$$f(z) = C(z - z_1)(z - z_2),$$

where C is a nonzero constant. Defining [21] (for $i = 1, 2$)

$$\kappa_{iA} = z_i \iota_A + o_A, \tag{3}$$

Figure 1. A general state of a three state system represented on the Poincaré sphere.

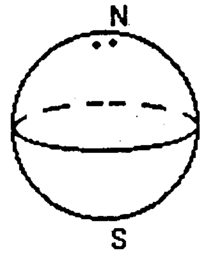

Figure 2. The basis vector $|e_1\rangle$.

we see that

$$z - z_i = \kappa_{iA}\kappa^A.$$

It follows that

$$f(z) = C(\kappa_1^A \kappa_2^B)\kappa_A \kappa_B$$

for any κ_A. Therefore Ψ^{AB} admits the decomposition

$$\Psi^{AB} = C(\kappa_1^A \kappa_2^B + \kappa_1^B \kappa_2^A). \tag{4}$$

(This is the decomposition used in the Petrov classification of gravitational fields [21].) The ray Ψ to which the vector Ψ^{AB} belongs uniquely determines a pair of rays κ_1, κ_2, which can be represented on the Poincaré sphere by points p_1, p_2. Interchanging p_1 and p_2 does not affect Ψ. So a ray of a three state system is represented (figure 1) by an unordered pair (p_1, p_2) (identified with (p_2, p_1)) of points on the Poincaré sphere. The ray space of a three state system is the same as the configuration space of two indistinguishable particles on the surface of a sphere. This is the main observation that we wish to make here.

Note that all rays of the three state system can be thus represented. Neighboring rays are represented by neighboring pairs of points on the Poincaré sphere. This representation correctly captures the global structure of CP^2 and is quite easy to visualize.

We now give a few examples of the general considerations above to gain familiarity with the use of the Poincaré sphere. Let us start with the rays containing the basis vectors: $|e_1\rangle = e_1^{AB} = \iota^A \iota^B$ is represented by two points at the north pole of the sphere. (see figure 2, which shows the points slightly separated for reasons of visibility). Similarly $|e_3\rangle = e_3^{AB} = o^A o^B$ is represented by two points at the south pole of the sphere (figure 3). $|e_2\rangle = 1/(\sqrt{2})(\iota^A o^B + \iota^B o^A)$ is represented by a point at each of the poles (figure 4). One also sees that a general linear combination $1/(\sqrt{2})(\sqrt{2}z|e_1\rangle + |e_2\rangle)$ of $|e_1\rangle$ and $|e_2\rangle$ is

$$\iota^A(z\iota^B + o^B) + \iota^B(z\iota^A + o^A).$$

The ray containing this state is shown in figure 5. The ray containing the state

$$o^A(z\iota^B + o^B) + o^B(z\iota^A + o^A) = 2(e_3^{AB} + \sqrt{2}ze_2^{AB})$$

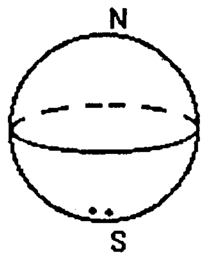

Figure 3. The basis vector $|e_2\rangle$.

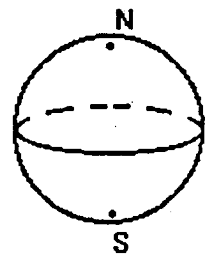

Figure 4. The basis vector $|e_3\rangle$.

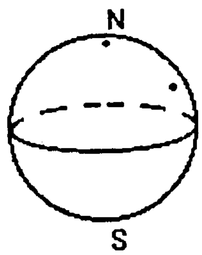

Figure 5. A general vector in the 1–2 plane.

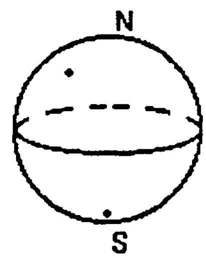

Figure 6. A general vector in the 2–3 plane.

which is in the 2–3 plane is shown in figure 6. Finally, the ray containing a state

$$2(-z^2 e_1^{AB} + e_3^{AB}) = (z\iota^A + o^A)(-z\iota^B + o^B) + (z\iota^B + o^B)(-z\iota^A + o^A)$$

in the 1–3 plane is represented by a pair of points (shown in figure 7) with the same lattitude but opposite longitudes.

4. Orthogonal rays

Let (p_1, p_2) and (p_3, p_4) be pairs of points representing two rays of a three state system. When are these rays orthogonal? The condition for orthogonality is

$$\kappa^{\dagger}_{3A}\kappa^{\dagger}_{4B}(\kappa_1^A\kappa_2^B + \kappa_1^B\kappa_2^A) = 0, \tag{5}$$

which can be written in Dirac notation as

$$(\kappa_3|\kappa_1)(\kappa_4|\kappa_2) + (\kappa_3|\kappa_2)(\kappa_4|\kappa_1) = 0. \tag{6}$$

To extract the geometrical interpretation of (6) multiply by $(\kappa_2|\kappa_3)(\kappa_1|\kappa_4)$ to rewrite (6) as

$$(\kappa_1|\kappa_4)(\kappa_4|\kappa_2)(\kappa_2|\kappa_3)(\kappa_3|\kappa_1) = -(\kappa_3|\kappa_2)(\kappa_2|\kappa_3)(\kappa_1|\kappa_4)(\kappa_4|\kappa_1). \tag{7}$$

The phase of such a chain of inner products is the geometric phase acquired by a system traversing the path 13241 on the Poincaré sphere. From Pancharatnam's rule [16] this is equal to the solid angle subtended by the broken geodesic curve 13241. The value of say $|\langle\kappa_1|\kappa_4\rangle|$ is $\cos(\delta(\kappa_1,\kappa_4)/2)$, where $\delta(\kappa_1,\kappa_4)$ is the angle between rays 1 and 4. Geometrically, $|\langle\kappa_1|\kappa_4\rangle|$ is the length of the perpendicular bisector dropped from the

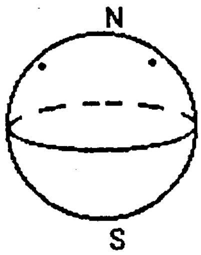

Figure 7. A general vector in the 1–3 plane.

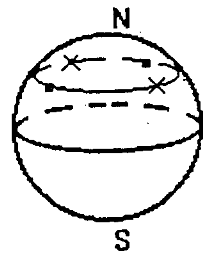

Figure 8. The figure shows the points $(p_1, \tilde{p}_3, p_2, \tilde{p}_4)$ on the Poincaré sphere, when (p_1, p_2) and (p_3, p_4) represent orthogonal rays. (p_1, p_2) are represented by dots and $(\tilde{p}_3, \tilde{p}_4)$ by crosses.

center of the Poincaré sphere to the chord 1–4. Separately equating the phases and amplitudes of these two complex numbers we find:

O1. The curve with geodesic segments connecting $(p_1, p_3, p_2, p_4, p_1)$ on the Poincaré sphere must enclose a solid angle of 2π.

O2. The products of the lengths of the perpendicular bisectors from the origin of the Poincaré sphere to the opposite sides of the rectangle $(p_1, p_3, p_2, p_4, p_1)$ (joined by chords, not along the surface of the sphere) must be equal.

These are necessary and sufficient conditions for the orthogonality of the rays of the three state system. (O1 is irrelevant (and also ambiguous, since there is no unique geodesic in this case) if any of the lengths of the perpendicular bisectors vanish; i.e. the successive vertices are antipodal.)

An alternative characterization of orthogonal rays obtained are as follows. Let us replace (p_3, p_4) by their antipodes $(\tilde{p}_3, \tilde{p}_4)$. Equation (6) implies

$$\kappa_3^{\dagger A} \propto (\kappa^{\dagger}_{4B}\kappa_2^B)\kappa_1^A + (\kappa^{\dagger}_{4B}\kappa_1^B)\kappa_2^A, \tag{8}$$

$$\kappa_4^{\dagger A} \propto (\kappa^{\dagger}_{3B}\kappa_2^B)\kappa_1^A + (\kappa^{\dagger}_{3B}\kappa_1^B)\kappa_2^A, \tag{9}$$

where $\propto$ means equal but for an unimportant (but nonzero) constant.

From (6), (8), (9) it follows that $\kappa_3^{\dagger}, \kappa_4^{\dagger}$ are linear combinations of κ_1 and κ_2 with the same relative amplitude and opposite relative phase. From this one may conclude (see below) that

O′1. The points $(p_1, \tilde{p}_3, p_2, \tilde{p}_4)$ lie on a small circle and (see figure 8) therefore form a rectangle in a plane passing through this circle.

O′2. The products of the lengths (measured along the chord, not along the surface of the sphere) of the opposite sides of this rectangle are equal to each other.

Clearly, any three points of such a set determine the fourth. The space of rays orthogonal to a given ray is two-dimensional.

To see how condition O′1 emerges, recall that the locus of points $\chi(r)$ on the Poincaré sphere traced by the states $\chi(r)^A = \kappa_1^A + r\kappa_2^A$, where r is real, is a small circle passing through the rays p_1 and p_2. This is proved in ref. [2] (see their figure on page 9). An easy way to see this result is to notice that it is true if p_1 and p_2 are antipodal (in fact, in this case, $\chi(r)$ traces out a great circle through p_1 and p_2). One deduces the result by performing an $SL(2,\mathbb{C})$ transformation, which in general maps antipodal points to non antipodal points and great circles to small circles. As r increases from 0 to infinity, the ray $\chi(r)$ goes along one arc of the circle from p_1 to p_2. As r decreases to minus infinity, $\chi(r)$ traces the other arc of the circle. O′2 is an elementary consequence of condition 2 above and replacing (p_3, p_4) by their antipodes $(\tilde{p}_3, \tilde{p}_4)$. Thus, either set of conditions (O1, O2) or (O′1, O′2) above provides a characterization of orthogonal states.

The condition for orthogonality (6) has a simple interpretation in terms of spinors. Given four rays $(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ one can form an $SL(2,\mathbb{C})$ invariant, their cross ratio [20]

$$\chi(\kappa_1, \kappa_2; \kappa_3, \kappa_4) := \frac{(\kappa_3 \cdot \kappa_1)(\kappa_4 \cdot \kappa_2)}{(\kappa_3 \cdot \kappa_2)(\kappa_4 \cdot \kappa_1)}, \tag{10}$$

where κ are representative elements from κ and $\kappa_3 \cdot \kappa_1 = \kappa_{3A} \kappa_1^A$. Clearly, (10) depends only on the rays $(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ and not on the representatives chosen. By $SL(2,\mathbb{C})$ transformations, one can make any set of four rays coincide with any other set if and only if the two sets have the same cross ratio. The condition (6) states that the rays $(\kappa_1, \kappa_2; \kappa_3^\dagger, \kappa_4^\dagger)$ have cross ratio -1. In particular, they can be brought to the vertices of a square on the equatorial plane of the Poincaré sphere by $SL(2,\mathbb{C})$ transformations. Such a set of rays with cross ratio -1 is called harmonic [20]. Conversely, boosting four points in the sky equally spaced on a great circle will generate all sets of points $(p_1, p_2, \tilde{p}_3, \tilde{p}_4)$ which describe orthogonal rays of a three state system.

5. Conclusion

It is easy to generalize the Poincaré sphere representation above to N state systems. $\mathscr{H}$ is then the space of symmetric spinors of rank $N-1$. Such a spinor determines $N-1$ principal spinors $(\kappa_1, \kappa_2, \ldots, \kappa_{N-1})$. These determine $N-1$ indistinguishable points on the Poincaré sphere. As N gets larger, the value of such a visualizable representation of the ray space diminishes because of its complexity. It may be simpler to work directly on the Hilbert space by picking representative elements as in [18] and ensuring gauge invariance explicitly.

Notice that the representation above captures the global properties of the ray space. It represents each ray exactly once and maps neighboring rays to neighboring points. Thus it satisfies (P1, P2) just like the Poincaré sphere. One could introduce a local chart as done for example in ref. [14] on the ray space to label its points. One drawback with this is that neighboring points on the ray space are sometimes widely separated in the representation. Some rays are left out entirely. In other words, a local chart misses some important global information. If one is following the evolution of a system in a local chart, it could happen that the ray ‘falls off the edge’ of the chart. The main point of this article is that there exists an easily visualizable, globally faithful representation of the rays of a three state system.

The Poincaré sphere representation for three state systems does have a drawback: it does not reflect the full symmetry of the ray space. For two state systems, the Poincaré sphere representation has the desirable feature that the representation displays the full $SU(2)$ symmetry of the ray space. (Recally that unitary transformations of the Hilbert space are represented as rotations of the Poincaré sphere). The $SU(3)$ symmetry of the three state problem is obscured in the Poincaré sphere representation. Note that some rays are represented by coincident points on the Poincaré sphere. These might appear to be 'different' from other states, which are represented by two distinct points. The difference is illusory. By $SU(3)$ transformations any state can be mapped to any other state. The coincidence of points is not an $SU(3)$ invariant notion. A representation in which $SU(3)$ invariance is manifest is given in [14]. The ray space then appears as a four dimensional subset of S^7. This however has the drawback of not being easy to visualize.

It is very important to remember that the two points (p_1, p_2) are indistinguishable. Else one might think that there is an apparent loss of differentiability at coincidence points. For example, if p_1 and p_2 pass through each other, one could see a 'kink' in their trajectories if one ignored their indistinguishability. Mathematically, one sees that z_1 and z_2 are roots of a quadratic equation, whose coefficients vary smoothly over the ray space. Smooth symmetric functions of (z_1, z_2) (i.e. those which satisfy $f(z_1, z_2) = f(z_2, z_1)$) are smooth functions on CP^2. Examples of symmetric functions are $(z_1 + z_2)$ and $(z_1 z_2)$, both of which can be expressed in terms of the coefficients of the quadratic polynomial (2), and are therefore smooth. Notice that antisymmetric functions like $z_1 - z_2$ involve the square root of the coefficients and are therefore not differentiable at coincidence points.

It would have been entirely possible to make do with $SU(2)$ spinors rather than the more general $SL(2, \mathbb{C})$ ones used in relativity. Our reason for using $SL(2, \mathbb{C})$ spinors is that the extra power of $SL(2, \mathbb{C})$ transformations is sometimes handy for deducing results even about $SU(2)$ spinors! See also the interesting remarks due to Nityananda quoted in [25].

There is another route to the Poincaré sphere representation for N state systems, which does not make use of spinors. One associates the polynomial

$$f(z) = \Sigma_{n=0}^{N-1} \Psi^n z^n / \sqrt{n!} \tag{11}$$

to the state $|\Psi\rangle$ with components $(\Psi^1, \Psi^2, \ldots, \Psi^{N-1}, \Psi^0)$. The roots of this polynomial (which depend only on the ray to which $|\Psi\rangle$ belongs) are $N-1$ points on $\mathbb{C}^*$, which are stereographically mapped to the Poincaré sphere. It is amusing to note that in the Bargmann representation [24] for the simple harmonic oscillator, the Hilbert space of coherent superpositions of the first N energy levels *is* the set of polynomials (11).

Acknowledgements

The author is grateful to Rajaram Nityananda for useful discussions and Supurna Sinha for making suggestions for improving the manuscript.

References

[1] H Poincaré, *Theorie Mathematique de la Lumiere* edited by George Carre (Paris, 1892) p. 275
[2] G N Ramachandran and S Ramaseshan, in *Handbuch der Phys.* (Springer, Berlin, 1961) vol. 25, part I
[3] M V Berry, *Proc. R. Soc. (London)* **A392**, 45 (1984)
[4] B Simon, *Phys. Rev. Lett.* **51**, 2167 (1983)

[5] Y Aharonov and J Anandan, *Phys. Rev. Lett.* **58**, 1593 (1987)
[6] S Ramaseshan and R Nityananda, *Curr. Sci. (India)* **55**, 1225 (1986)
[7] M V Berry, *J. Mod. Opt.* **34**, 1401 (1987)
[8] R Bhandari and J Samuel, *Phys. Rev. Lett.* **60**, 211 (1988)
[9] A Shapere and F Wilczek, *Geometric phases in physics* (World Scientific, Singapore, 1989)
[10] R Simon, H J Kimble, and E C G Sudarshan, *Phys. Rev. Lett.* **61**, 19 (1988)
[11] T H Chyba, L J Wang, L Mandel and R Simon, *Opt. Lett.* **13**, 562 (1988)
[12] P Hariharan, Hema Ramachandran, K S Suresh and J Samuel, *The Pancharatnam phase as a strictly geometric phase: A demonstration using pure projections*, RRI-96-18, to appear in *J. Mod. Opt.*
[13] G Khanna, S Mukhopadhyay, R Simon and N Mukunda, *Ann. Phys.* (in press)
[14] Arvind, K S Mallesh and N Mukunda, Archives quant-ph/9605042
[15] N Mukunda, IISc preprint (1996)
[16] S Pancharatnam *Proc. Indian Acad. Sci.* **A44**, 247 (1956)
[17] S Pancharatnam, *Collected works of S. Pancharatnam* (Oxford University Press, London, 1975)
[18] J Samuel and R Bhandari, *Phys. Rev. Lett.* **60**, 2339 (1988)
[19] N Mukunda and R Simon, *Ann. Phys. (NY)* **228**, 205 (1993)
[20] R Penrose and W Rindler, *Spinors and spacetime* (Cambridge University Press, Cambridge, 1984) vol. I
[21] R M Wald, *General Relativity* (University of Chicago Press, Chicago, 1984)
[22] A Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991)
[23] L D Landau and E M Lifshitz, *Quantum mechanics* (Pergamon Press, Oxford, 1976)
[24] V Bargmann, *Commun. Pure Appl. Math.* **14**, 187 (1961)
[25] See the figure in the article by S Ramaseshan, in *Essays on particles and fields* edited by R R Daniel and B V Sreekantan (Indian Academy of Sciences, Bangalore, 1989)