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Effect of an axial magnetic field on the Poiseuille flow of a nematic liquid crystal

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Abstract. The effect of an axial magnetic field on the Poiseuille flow of nematic *p*-azoxyanisole (PAA) has been computed using the Ericksen-Leslie continuum theory. The apparent viscosity decreases appreciably in the presence of the magnetic field. Orientation and velocity profiles for different shear rates and magnetic fields are presented.

Keywords. Liquid crystal; p-azoxyanisole; Ericksen-Leslie continuum theory.

1. Introduction

The flow properties of a nematic liquid crystal are very much different from those of an ordinary fluid. For example, the viscosity of an ordinary fluid is independent of shear rate, while the coefficient of apparent viscosity of a nematic (defined as the ratio of shear stress to average velocity gradient) may rise by as much as 100% as the shear rate is decreased. These unusual properties are now well explained in terms of Ericksen-Leslie theory (Ericksen 1962, Leslie 1968). Atkin (1970) developed the analytical theory of Poiseuille flow of a nematic liquid crystal on the basis of the Ericksen-Leslie equations. Recently Tseng *et al* (1972) numerically solved the differential equation obtained by Atkin and presented detailed calculations for the case in which the molecules are oriented perpendicular to the wall. These results were found to be in good accord with the experimental observations of Fisher and Fredrickson (1969). In the present paper we investigate the effect of an axial magnetic field on Poiseuille flow. It turns out that the apparent viscosity is quite strongly influenced by the magnetic field.

2. Theory

In the continuum theory of liquid crystals we describe the anisotropy of the structure by a dimensionless unit vector $\boldsymbol{n}$ called the director, and set up differential equations both for the velocity $\boldsymbol{v}$ and for the director $\boldsymbol{n}$. They are

$$\rho \dot{v}_i = F_i + t_{ji,j}$$

$$\rho_1 \ddot{n}_i = G_i + g_i + \pi_{ji,j} \qquad (1)$$

ρ is the density of the fluid, ρ_1 the moment of inertia of the director, v_i the velocity, F_i the external body force per unit volume, G_i the external director body force per unit volume, t_{ji} the stress tensor, π_{ji} the director surface stress and g_i the intrinsic director body force;*

$$t_{ji} = -p\delta_{ji} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + \hat{t}_{ji} \tag{2}$$

$$\hat{t}_{ji} = \mu_1 n_k n_p d_{kp} n_j n_i + \mu_2 n_j N_i + \mu_3 n_i N_j + \mu_4 d_{ji} + \mu_5 n_j n_k d_{ki} + \mu_6 n_i n_k d_{kj}$$

$$\pi_{ji} = \frac{\partial W}{\partial n_{i,j}} \tag{3}$$

$$g_i = -\frac{\partial W}{\partial n_i} + \gamma n_i + \hat{g}_i \tag{4}$$

$$\hat{g}_i = \lambda_1 N_i + \lambda_2 n_j d_{ji}$$

with

$$N_i = \dot{n}_i - \omega_{ik} n_k, \quad d_{ki} = (v_{k,i} + v_{i,k})/2,$$

$$\omega_{ki} = (v_{k,i} - v_{i,k})/2, \quad \lambda_1 = \mu_2 - \mu_3 \quad \text{and} \quad \lambda_2 = \mu_5 - \mu_6 \tag{5}$$

Here μ_1 to μ_6 are the viscosity coefficients introduced by Leslie (1968). W is the Frank elastic energy per unit mass given by Leslie (1968) [Frank (1958)].

$$2\rho W = K_{22} n_{i,j} n_{i,j} + (K_{11} - K_{22} - K_{24})\, n_{i,i} n_{j,j} + (K_{33} - K_{22})\, n_i n_j n_{k,i} n_{k,j} + K_{24} n_{i,j} n_{j,i} \tag{6}$$

with K_{11}, K_{22}, K_{33} and K_{24} as the elastic constants of a nematic liquid crystal. In (2) and (4) p and γ are arbitrary constants, which arise from the constraints that the fluid is incompressible and that the director is of constant magnitude.

For Poiseuille flow in the presence of a homogeneous axial magnetic field H the steady state equations are

$$\frac{d^2\theta}{dr^2} + \frac{1}{2f}\frac{df}{d\theta}\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r}\frac{d\theta}{dr} = \frac{K_{11}\sin 2\theta}{2f(\theta)\, r^2} + ar\,\frac{(\lambda_1 + \lambda_2)\cos 2\theta}{4f(\theta)\, g(\theta)} + \frac{(\Delta\chi)\, H^2 \sin\theta\cos\theta}{f(\theta)} \tag{7}$$

$$g(\theta)\frac{dv}{dr} = -\frac{1}{2}ar \tag{8}$$

$$f(\theta) = K_{11}\cos^2\theta + K_{33}\sin^2\theta$$

$$g(\theta) = \mu_1\cos^2\theta\sin^2\theta + \tfrac{1}{2}(\mu_4 + \mu_5 - \mu_2)\sin^2\theta + \tfrac{1}{2}(\mu_4 + \mu_6 + \mu_3)\cos^2\theta \tag{9}$$

Also from Parodi's relation (Parodi 1970) $\mu_2 + \mu_3 = \mu_6 - \mu_5$. In the above

* In reality G_i and g_i are torques per unit volume but in conformity with standard usage [Leslie (1968)] we refer to them as forces; similarly π_{ji} is the director surface torque.

equations $a = \mathrm{d}p/\mathrm{d}z$ the pressure gradient, θ the angle between the tube axis and the director and $(\triangle\chi)$ the anisotropy of diamagnetic susceptibility. Equations (7) and (8) are solved subject to the boundary conditions

$$\theta\,(0) = 0, \quad \theta\,(R) = \theta_b \quad \text{and} \quad v\,(R) = 0$$

where R is the tube radius and θ_b the orientation at the wall.

3. Results

To simplify the computational procedure we transform (7) and (8) using a different variable $\xi = a^{1/3}r$.

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \frac{1}{2f}\frac{\mathrm{d}f}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^2 + \frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = \frac{K_{11}\sin 2\theta}{2\xi^2 f(\theta)} + \frac{(\triangle\chi)\, H^2 \sin\theta\cos\theta}{f(\theta)\, a^{2/3}} + \frac{\lambda_1 + \lambda_2\cos 2\theta}{4f(\theta)\, g(\theta)} \tag{10}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = -\frac{a^{1/3}\xi}{2g\,(\theta)} \tag{11}$$

Tseng *et al* (1972) solved the above equations for $H = 0$ and $\theta_b = -\frac{1}{2}\pi$. One can employ Runge-Kutta method to solve eq. (10) numerically. The procedure is to assume a value of $\theta' = \mathrm{d}\theta/\mathrm{d}\xi$ at $\xi = 0$ and to integrate eq. (10) upto $\theta = \theta_b$, at which $\xi = \xi_m$ so that $R = \xi_m a^{-1/3}$.

If we simultaneously integrate eq. (11) we can get Q, the rate of flow

$$Q = -\,2\pi a^{-1/3}\left[\left\{\int_0^{\xi_m} w\,(\xi)\,\xi\mathrm{d}\xi\right\} - \tfrac{1}{2}\, w\,(\xi_m)\,\xi_m^{\,2}\right] \tag{12}$$

where

$$w\,(\xi) = -\int_0^{\xi}\frac{\xi\mathrm{d}\xi}{2g\,(\theta)} \quad \text{with} \quad w\,(0) = 0.$$

The velocity profile is given by

$$v\,(\xi) = [w\,(\xi) - w\,(\xi_m)]\, a^{1/3} \tag{13}$$

Then the apparent viscosity is given by

$$\eta = \frac{\pi R^4\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)}{8Q} = \frac{\xi_m^{\,3}}{8\left(\frac{Q}{\pi R}\right)} \tag{14}$$

Table 1. Viscosity and elastic constants of PAA

μ's (poise)		K's (dyne)
$\mu_1 = -0{\cdot}038$	$\mu_4 = 0{\cdot}068$	$K_{11} = 4{\cdot}9 \times 10^{-7}$
$\mu_2 = -0{\cdot}068$	$\mu_5 = 0{\cdot}048$	$K_{33} = 10{\cdot}5 \times 10^{-7}$
$\mu_3 = 0{\cdot}000$	$\mu_6 = -0{\cdot}02$	
$\lambda_1 = -\lambda_2$		

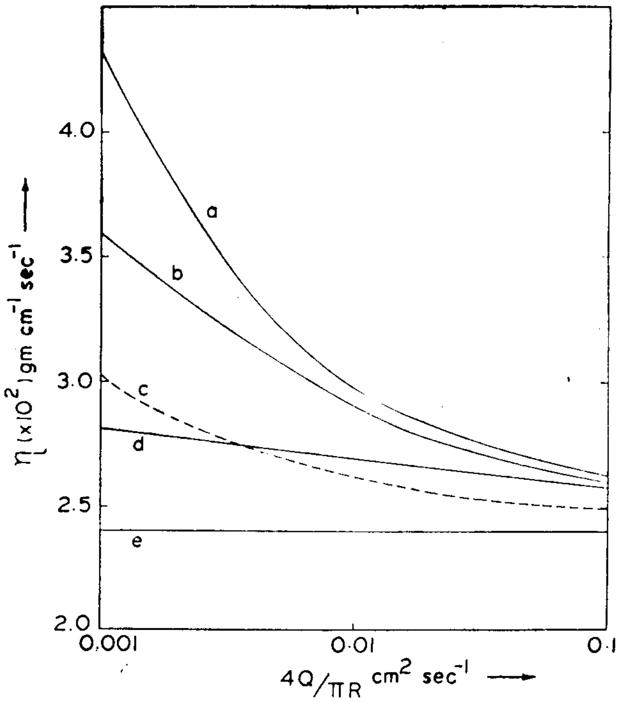

Figure 1. Apparent viscosity η of PAA *versus* $4Q/\pi R$ for a tube radius of $R = 55\cdot 5\,\mu$. The values of $(\Delta\chi)\,H^2$ and θ_b are (*a*) 0, $-\frac{1}{2}\pi$; (*b*) 1·23, $-\frac{1}{2}\pi$; (*c*) 0, $-\frac{1}{4}\pi$; (*d*) 24·2, $-\frac{1}{2}\pi$; (*e*) ∞, $-\frac{1}{2}\pi$ or 0, 0.

We have computed the apparent viscosity for PAA for two values of $(\Delta\chi)\,H^2$, 24·2 and 1·23 CGS respectively, with a homeotropic (perpendicular) wall alignment for a tube radius $= 55\cdot 5\,\mu$. Figure 1 shows the results. In our calculations we have used μ's and K's which give a very good fit of η (*versus* $4Q/\pi R$) with the experimental results of Fisher and Fredrickson (Tseng *et al* 1972). Table 1 gives the values of these parameters. Here $\lambda_1 = -\lambda_2$ and in addition Parodi's relation is satisfied.

The curves for $(\Delta\chi)\,H^2 \neq 0$ in figure 1 are not universal in that they cannot be used for any value of R. This is so because the scaling laws (which leave the differential equation invariant) first obtained by Ericksen (1969) and later applied to Poiseuille flow by Atkin (1970) do not hold in the presence of an external perturbation like a magnetic field*. For this reason magnetic field results are presented for a tube of finite radius (in our case $= 55\cdot 5\,\mu$).

If there is weak anchoring at the wall and the homeotropic alignment is disturbed, there can of course be considerable errors in the experimental determinations. To

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* Recently B A Finlayson [*Liquid crystals and ordered fluids*, eds J F Johnson and R S Porter, Plenum Press (1974), Vol. 2, p. 211] has computed the viscosity and transport properties for a flow between parallel plates, in the presence of a magnetic field. Also, it has been shown in this paper that by using a dimensionless group containing the parameter $(\Delta\chi)\,H^2/d$ (d being the gap width) one can get 'common' curves for apparent viscosity *versus* shear rate, *i.e.*, if the gap width is doubled and the field is increased $2^{1/2}$ times the solutions remain unaltered for this parameter. We are grateful to a referee for bringing this work to our notice.

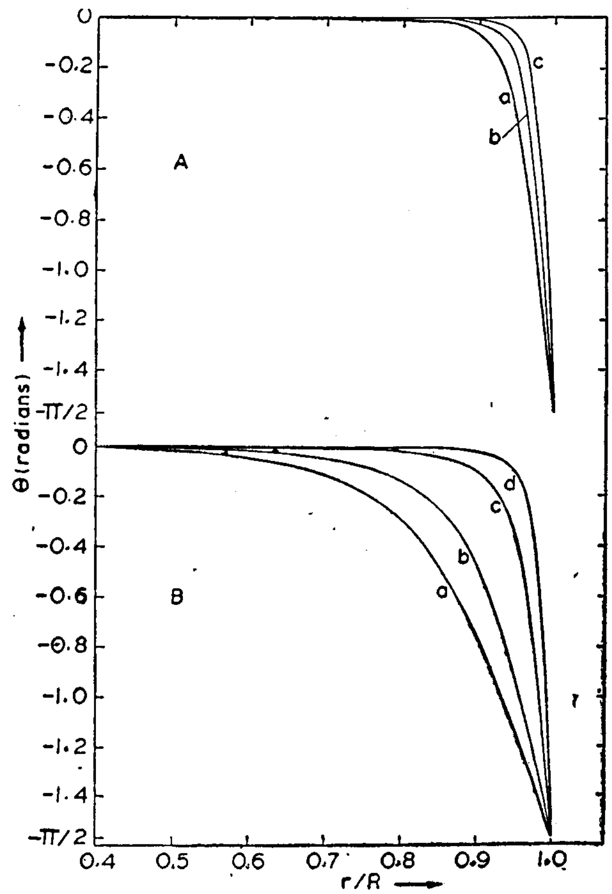

Figure 2. Orientation profiles at different shear rates for $(\Delta\chi) H^2 = 24{\cdot}2$ (A) and for $(\Delta\chi) H^2 = 1{\cdot}23$ (B), with homeotropic wall alignment. The values of $4Q/\pi R$ in (A) are (*a*) 0·003045; (*b*) 0·03354; (*c*) 0·1345 and in (B) (*a*) 0·001245; (*b*) 0·004167 (*c*) 0·032438; and (*d*) 0·1372.

illustrate this point, we have computed the apparent viscosity for a wall orientation $= -\frac{1}{4}\pi$ for $H = 0$ (dashed line in figure 1). A non-homeotropic wall alignment decreases the apparent viscosity. This is in qualitative agreement with the experimental observation of Fisher and Fredrickson who found a decrease in η when they attempted to effect parallel alignment at the wall.

For $H = \infty$, $\theta_b = -\frac{1}{2}\pi$ or $H = 0$, $\theta_b = 0$ the liquid crystal behaves like the anisotropic fluid of Ericksen which has neither a surface stress nor an elastic contribution to the intrinsic director body force. Figure 1 also includes this case. In all our calculations R is accurate to $\pm 0{\cdot}5\,\mu$ and η to 0·2%.

In figures 2 (A) 2 (B) we have given the orientation profile at different shear rates for $(\Delta\chi\, H^2 = 24{\cdot}2$ and $1{\cdot}23$ (or $H = 13{,}340$ and 3,000 G) respectively. At the same shear rate the transition of the director orientation from $\theta = 0$ to $\theta = -\frac{1}{2}\pi$ becomes more and more abrupt as the magnetic field increases. The same phenomenon is observed when the shear rate is increased keeping the field a constant. The effect of the magnetic field on the velocity profile is shown in figure 3. The

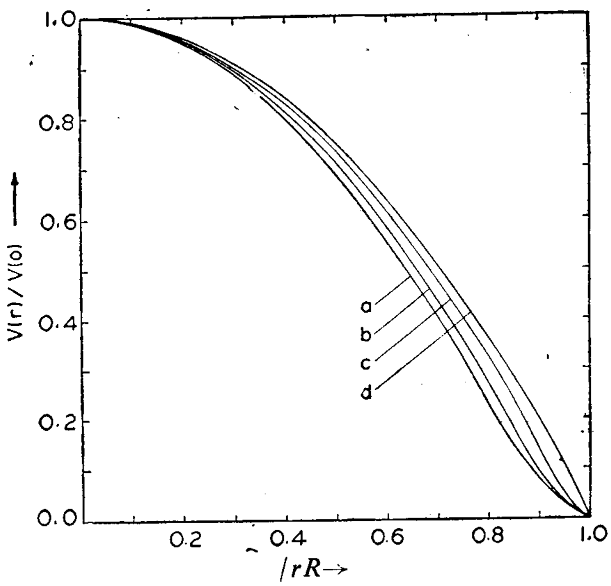

Figure 3. Velocity profiles at different values of $4Q/\pi R$ and same field (curves *a* and *b*) and nearly same value of $4Q/\pi R$ and different fields (curves *b* and *c*). Values of $(\Delta\chi)H^2$ and $4Q/\pi R$ are (*a*) 1·23, 0·001245; (*b*) 1·23, 0·004167; (*c*) 24·2, 0·003045; and (*d*) $(\Delta\chi)H^2 = \infty$ or $4Q/\pi R = \infty$ (truly parabolic).

profile which is non-parabolic at low fields (or low shear rates) tends to become truly parabolic only when the field or the shear rate is extremely large.

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