Of Connections and Fields – II

Chern's Mathematical Ideas in Physics

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In the first part of this article[1] we gave an elementary introduction to Chern's ideas and their impact on modern physics. In this concluding article we describe some more advanced applications of Chern's ideas. This second part is somewhat more demanding than the first part and is addressed to students with some background in mathematics and physics.

In the first part we described some elementary occurrences of Chern's ideas in physics. We now treat a few more advanced topics from elementary particle physics. The standard model of elementary particle physics relies heavily on 'gauge theories'. Classically, gauge theories are just connections on fibre bundles, a structure that mathematicians like Chern have studied. To see the relation between the mathematics and the physics, one needs to view gauge theories from a slightly advanced point of view. Let us do this starting with electrodynamics. Recall that electrodynamics is described by a vector potential $A_\mu(x)$, where $\mu = 0, 1, 2, 3$ labels the components of the 1 form $A = A_\mu dx^\mu$ and x^μ represent the four coordinates of a space-time point. Let $\psi(x)$ be the wave function of a particle with charge q. The theory enjoys a symmetry called 'gauge invariance'. The Lagrangian of electrodynamics is invariant under the transformations: $A_\mu \to A_\mu + u^{-1}\partial_\mu u$ and $\psi \to u\psi$, where $u = \exp iq\chi(x)$ and $\chi(x)$ is an arbitrary real function of x. $u(x)$ is a complex number of modulus 1, $i.e$ an element of the group $U(1)$. In modern language, electrodynamics is a $U(1)$ gauge theory. Note that there is no gauge invariant meaning to comparing the wave function at different space-time points. The ordinary derivative $\partial_\mu\psi$ is

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Keywords

Box 1. Local Versus Global Description

Although in the main article, we have used coordinates x^i, these need not be globally defined all over space time. It is enough if the spacetime manifold (which is the same as the base space $\mathcal{B}$) can be covered by charts, each of which admits local coordinates. When charts overlap, we require that the coordinate transformation connecting different systems of coordinates be compatible. Similarly, the vector potential A may only be defined in local patches. These patches can be 'sewn together' like a quilt to produce the global picture. In the overlap of patches, we require that there is a gauge transformation connecting different vector potentials. Mathematicians have a global way of describing this structure: one thinks of a fibre bundle with $U(1)$ as fibre and $\mathcal{B}$ as the base. (The fibre is $U(1)$ because we are concerned here with $U(1)$ connections as in electromagnetism. More generally, the fibre is some group manifold.) Replacing the fibre with some other group such as $SU(2), SU(3)$ of $SU(N)$ leads to non-Abelian gauge theories. In the mathematical description, a connection is a rule for horizontally lifting curves in $\mathcal{B}$ to $\mathcal{E}$. In general the horizontal lift of closed curves in $\mathcal{B}$ may be open in $\mathcal{E}$. One returns to the same fibre but to a different point on the fibre. This means that the connection is not integrable. A local measure of the non-integrability of the connection is the curvature, which physicists identify with generalised electric and magnetic fields.

not gauge invariant. However, the covariant derivative $D_\mu\psi = \partial_\mu\psi - A_\mu\psi$ is a gauge invariant object. This requires the use of additional structure, the vector potential A_μ. Mathematically this additional structure is called a *connection* (See *Box* 1). This is one of the most important mathematical ideas to have entered physics and this is an important focus of this article.

Mathematically, we view the complex-valued wave function of the charged particle as taking values in a fibre, a vector space of one complex dimension. A connection gives us a rule for comparing wavefunctions on fibres attached to different points. This rule is in general not integrable: given three points (a, b, c) in $\mathcal{B}$, with fibres $\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c$, comparing $\mathcal{F}_a$ with $\mathcal{F}_b$ and $\mathcal{F}_b$ with $\mathcal{F}_c$ is not the same as directly comparing $\mathcal{F}_a$ with $\mathcal{F}_c$. This lack of integrability is locally captured by the curvature F of the connection, which is in local coordinates, the commutator of the covariant derivative: $F_{\mu\nu} = D_\mu D_\nu - D_\nu D_\mu$. F is an antisymmetric tensor and therefore a 2 form (see *Box* 2).

Box 2. Differential Forms

Forms are generally found just under integral signs. In a multiple integral (say d dimensional), one slices up the region of integration into parallelopiped cells. The integral can be split up into contributions, one from each cell and this is proportional to the d volume of the cell. The d volume of a cell is naturally expressed as a determinant. For example, the volume of a three dimensional parallelopiped with sides $(\vec{a}, \vec{b}, \vec{c})$ is $\vec{a}.(\vec{b} \times \vec{c})$ or

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Determinants are antisymmetric objects and switch sign when two rows (say, a and b above) are exchanged. p-forms are antisymmetric tensors of rank p. p-forms can be integrated over p dimensional manifolds. For example, 0-forms are just ordinary functions and 'integrating' a 0 form over a 0 dimensional manifold (a point) just consists of evaluating the function at that point. 1-forms are like the vector potential in electrodynamics and the line integral $\int A_\text{i} dx^\text{i}$ is a familiar object from electromagnetic theory. An example of a 2-form is a magnetic field and $\int_\mathsf{S} \vec{B}.d\vec{S}$ is the integral over a two dimensional manifold $\mathcal{S}$. More usually we convert the vector $\vec{B}$ into a second rank antisymmetric tensor $F = B^\mathsf{i}\epsilon_{\mathsf{ijk}}$ and write $\int_\mathsf{S} F$. Forms provide us with a particularly powerful language for expressing physical ideas. Unlike other tensors, differential forms can be integrated (that's how we introduced them) and also differentiated. Simply differentiate the p-form in local coordinates and then antisymmetrise with respect to the $p+1$ indices. If α is a p form, then $d\alpha$ is a $p+1$ form. Antisymmetrisation results in the identity $dd\alpha = 0$, which is called Poincare's lemma. This identity includes familiar identities like **curl**.**grad** $= 0$ and **div**.**curl** $= 0$. Forms can be multiplied together: just multiply the p and q forms and then antisymmetrise in all $p+q$ indices. They can be contracted with vectors to produce lower rank forms, $p-1$ forms. These manipulations do not require any metric on the manifold. E Cartan was a great advocate of the use of differential forms. S S Chern came into contact with Cartan early in his life and was very much influenced by him. The use of forms has been particulary fruitful in physics. For instance, electrodyanamics lends itself easily to a formulation in terms of differntial forms. The formulation of supergravity takes the Einstein-Cartan theory (which is general relativity souped up with forms) as a starting point.

Gauge theories are generalisations of electrodynamics and play an important role in the standard model of elementary particle physics and also in gravity. Quantum Chromodynamics (QCD), the gauge theory of the strong nuclear interactions is a non-Abelian gauge theory based on the non-Abelian group $SU(3)$ and the Weinberg-Salam model of the electroweak interactions is based on the gauge group $SU(2) \times U(1)$. Gravity also can be fomulated as a gauge theory (based on a non-compact group) but applying quantum mechanics to it is notoriously difficult and presently an open problem. For a gauge theory based on the group G, the vector potential takes values in the adjoint representation of the group (or more simply, it is a matrix valued object). Gauge transformations are written as $A \rightarrow u^{-1}Au - u^{-1}du$, $\psi \rightarrow u\psi$, where the 'wave function' ψ is now a vector in a representation of G. The 'curvature' (or field strength to physicists) is given by

$$F = dA + A \wedge A \tag{1}$$

which also takes values in the adjoint representation of G. (Like A, F is also matrix valued.) In studying gauge theories, we often find that we have to deal with connections with globally nontrivial properties. In the first already seen one, the magnetic monopole, which is an example of the first Chern class c_1. We now describe a few more globally nontrivial connections.

The Instanton

The instanton is a topologically non-trivial gauge field configuration which represents tunnelling between states of Quantum Chromodynamics (QCD) the theory of the strong nuclear interactions based on the group $SU(3)$. To simplify matters consider instead an $SU(2)$ gauge theory, without Fermionic matter. An important state in the quantum theory is the vacuum state, defined as the state which minimises the Hamiltonian (or energy). It is plausible that one can learn about the quantum vacuum by considering small quantum fluctuations about classical static configurations (independent of the time coordinate x^0) which minimise energy. These are given by $F = 0$ or $A = -U^{-1}dU$ and thus define a map from $\mathbb{R}^3$ to G. $\vec{x} \in \mathbb{R}^3$ to $U(\vec{x}) \in G$. Boundary conditions at infinity of $\mathbb{R}^3$ require that $U \rightarrow Id$ as $|\vec{x}| \rightarrow \infty$. So we really have a map from S^3 (the one point compactification of $\mathbb{R}^3$) to $SU(2)$, which also has global topology S^3.

These maps fall into topological classes which cannot be continuously deformed into one another. The classes are characterised by an integer n, the number of times U 'winds' around $SU(2)$ as $\vec{x}$ winds around S^3. There appear to be multiple classical vacua as would happen even in a simple quantum mechanical problem if the potential energy function is periodic like $\cos x$. Classically these vacua are degenerate (they all have the same energy). But quantum mechanically, there is tunnelling between them. This tunnelling lifts the degeneracy and reveals the true vacuum state as a particular superposition (characterised by θ the vacuum angle) of the classical vacua. This is entirely analogous to Bloch states in a periodic potential. The phenomenon of tunnelling is described in Euclidean $\mathbb{R}^4$ by analytically continuing to imaginary time. In the path integral formulation of quantum field theory, one regards physical amplitudes as an integral over all classical field configurations ('sum over histories'). The amplitude for tunnelling is dominated by the saddle points, the solutions to the classical equations of motion. These are called 'instantons'. They describe tunnelling between topologically distinct vacua. These gauge field configurations are non-trivial bundles on S^4 (the one point compactification of $\mathbb{R}^4$). The instanton solution connecting vacua n and $n+k$ is characterised by its 'topological charge' or winding number

$$1/\mathcal{N}\int_{\mathcal{B}} \mathrm{tr} F\wedge F = k, \qquad (2)$$

where $\mathcal{N}$ is a normalisation constant. Notice again that the left hand side is the integral of a geometrical quantity, the Chern density $\mathrm{tr}F\wedge F$, but the right hand side is an integer and topological! The instanton is an example of the second Chern class. The monopole and the instanton illustrate the first two Chern classes. There are higher order invariant polynomials in F describing the other Chern classes.

The Chiral Anomaly

Another context where the Chern density $\mathrm{Tr} F \wedge F$ appears is the Chiral anomaly, which arises when one couples a gauge theory to massless fermions. At a classical level, massless fermions have two conserved currents $j^\mu = \bar{\psi}\gamma^\mu\psi$, the electric charge current and $j_5^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi$, the chiral current. Both currents are conserved: they satisfy the equation $\partial_\mu j^\mu = \partial_\mu j_5^\mu = 0$. The conservation of these currents follows from Nöther's theorem from symmetries of the classical theory. However, quantum field theories have an infinite number of degrees of freedom and need regularisation in order to produce finite answers. The regularisation procedure results in a loss of chiral symmetry. One finds that the chiral current is no longer conserved, but

$$\partial_\mu j_5^\mu = C\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr} F_{\mu\nu}F_{\alpha\beta}, \quad (3)$$

where C is a constant depending on the couplings and the number and chirality of the fermions and $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric tensor. The reader will recognise the right hand side of (3), the chiral anomaly, as the Chern density $\mathrm{Tr} F \wedge F$. The chiral anomaly was first discovered in perturbation theory and only later was its global topological significance realised. There are clear physical consequences resulting from the chiral anomaly. The decay of the π meson into two photons ($\pi^0 \to \gamma\gamma$) is forbidden by classical symmetry, but is in fact observed in nature and understood using the chiral anomaly.

We just saw that an anomaly in the chiral current has physical consequences. However if an anomaly occurs in a current corresponding to a gauge symmetry, the theory becomes inconsistent. This can be used to impose constraints on the allowed matter fields. The matter fields must be so chosen that the anomaly cancels out (i.e. the analogue of the constant C in equation (3) is zero) and doesn't spoil the conservation of charge. Anomaly can-

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cellation is an important principle in present day particle physics and used in string theory to restrict the allowed matter content.

Chern-Simons Theory

The Chern density Tr$F \wedge F$ can locally be written as an exterior derivative of a 3 form. Let us restrict to Abelian $U(1)$ gauge theory where $F = dA$. We have Tr$F \wedge F = $ Tr$dA \wedge dA = d[$Tr$A \wedge dA]$, where we have used Poincare's lemma $dd = 0$. The three form Tr$A \wedge dA$ is called the Chern-Simons form. Its integral over a closed three manifold $\int$ Tr$A \wedge dA$ is gauge invariant. The Chern-Simons invariant can be used as a Lagrangian to describe 3 dimensional field theories. Such field theories have found application in the Quantum Hall effect. The Chern-Simons action describes a gauge field theory with no local degrees of freedom, a 'topological field theory'. Such quantum field theories have proved useful, not only in physics, but also in mathematics as they lead to a better understanding of knots!

The Geometric Phase

In the first part of this article, we described the geometric phase and showed how it leads to a natural $U(1)$ connection. Berry's phase also leads to non-Abelian connections. These occur if the eigenspace for each eigenvalue is of dimension more than one. Examples are sytems with Kramers'degeneracy, which arises in fermionic systems with time-reversal symmetry. In such systems, the time reversal operator $\mathcal{T}$ squares to -1. We can think of $i, j = \mathcal{T}$ and $k = ij$ as generators of Hamilton's quaternions and we now have quaternionic Hilbert spaces rather than complex ones. Since quaternions do not commute, we get a nonabelian connection. *Box* 3 brings out globally nontrivial connections within the context of the geometric phase. This part of the article is more technical and needs working through.

Suggested Reading

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Box 3. Instantons and Berry's Phase

The geometric phase provides us with examples of globally non-trivial bundles in physics. To see instantons, we need to consider a four state system with pair wise degenerate energy levels. The 'phase' here is not an Abelian $U(1)$ phase but a non-Abelian $U(2)$ phase. Consider the five dimensional Clifford algebra generated by $\{\Gamma_i, i = 1...5\}$ $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\delta_{ij}$. These generators can be realised as 4×4 Dirac matrices (γ_1, γ_5). We consider the system described by the Hamiltonian $H = x^i\Gamma_i$, where the x^i now span a five-dimensional parameter space $\mathbb{R}^5 - \{0\}$. Such a system can be realised experimentally in Nuclear Quadrupole Resonance (NQR). We need only restrict our attention to the unit sphere in parameter space. The positive energy subspace of H defines a $\mathbb{C}^2$ bundle over S^4. Choosing an orthonormal frame in each eigenspace gives an $U(2)$ bundle over S^4. Just as before, we notice that $H(x) = h(x)\Gamma_5 h^{-1}(x)$, where $h(x)$ is now defined by

$$h(x) = \frac{1 + H\Gamma_5}{\sqrt{2(1 + x^5)}},$$

at all points of S^4 except the south pole, where $x^5 = -1$. If we pick an orthonormal pair of positive energy states $|v_\alpha^N>, \alpha = 1, 2$ at the north pole, which satisfy $\Gamma_5|v_\alpha^N> = |v_\alpha^N>$, the states $|v_\alpha(x)> := h(x)|v_\alpha^N>$ are orthonormal positive energy states all over the sphere, except for the south pole, where $h(x)$ is ill-defined. The Berry potential is now a 2×2 Hermitian matrix

$$A_{\alpha\beta} = < v_\alpha(x)|d|v_\beta(x) > = < v_\alpha^N|h^{-1}dh|v_\beta^N > .$$

A is in fact, traceless and so is really an $SU(2)$ connection. Its field strength is given by $F = dA + A \wedge A$ and represents an instanton of charge $k = 1$. Globally the instanton bundle is S^7 and the fibres are S^3 and the base is S^4.

Conclusion

It often happens that mathematicians study structures, which turn out to be exactly right for describing the real world. One instance of this is Riemannian geometry, which was developed before Einstein used it in General Relativity. Another example is the idea of a connection. This idea too was developed by mathematicians quite independently of the real world. After many efforts to understand the world of atoms, nucleii and quarks, physicists have realised that connections provide the right description. Are there other such mathematical objects waiting in the wings to enter the stage of theoretical physics? Only time (whatever that is!) will tell.

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