The uncertainty product of position and momentum in classical dynamics

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(Received 26 October 2011; accepted 7 May 2012)

It is generally believed that the classical regime emerges as a limiting case of quantum theory. Exploring such quantum-classical correspondences provides a deeper understanding of foundational aspects and has attracted a great deal of attention since the early days of quantum theory. It has been proposed that since a quantum mechanical wave function describes an intrinsic statistical behavior, its classical limit must correspond to a classical ensemble—not to an individual particle. This idea leads us to ask how the uncertainty product of canonical observables in the quantum realm compares with the corresponding dispersions in the classical realm. In this paper, we explore parallels between the uncertainty product of position and momentum in stationary states of quantum systems and the corresponding fluctuations of these observables in the associated classical ensemble. We confine ourselves to one-dimensional conservative systems and show, with the help of suitably defined dimensionless physical quantities, that first and second moments of the canonical observables match with each other in the classical and quantum descriptions—resulting in identical structures for the uncertainty relations in both realms.
[http://dx.doi.org/10.1119/1.4720101]

I. INTRODUCTION

It is imperative to retrieve classical dynamics as a limiting case—in its domain of validity—from quantum theory. The generally prevailing notion is that classical mechanics emerges in the limit $\hbar \to 0$. The applicability of this limit is reviewed critically in Refs. 1 and 2.

Another quantum-classical correspondence discussed widely is Ehrenfest's theorem,[3] which states that the equations of motion for the expectation values of the position and momentum are the same as those obeyed by a classical particle under certain conditions. More specifically, for a system with Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, the equations of motion for the expectation values of the position and momentum operators are

$$\frac{d\langle \hat{x}\rangle}{dt} = \frac{\langle \hat{p}\rangle}{m}, \tag{1}$$

$$\frac{d\langle \hat{p}\rangle}{dt} = -\left\langle \frac{dV(\hat{x})}{dx}\right\rangle = \langle F(\hat{x})\rangle, \tag{2}$$

where $F(\hat{x}) = -dV(\hat{x})/dx$ is the force operator. Under the approximation $\langle F(\hat{x})\rangle \approx F(\langle \hat{x}\rangle)$ (which is exact for linear and quadratic potentials), the equation of motion for $\langle \hat{p}\rangle$ reduces to

$$\frac{d\langle \hat{p}\rangle}{dt} = F(\langle \hat{x}\rangle). \tag{3}$$

In other words, the quantum averages $\langle \hat{x}\rangle$ and $\langle \hat{p}\rangle$ satisfy the classical equations of motion (1) and (3). However, in order for these equations to lead to classical trajectories, the quantum wave function must be narrow compared to the typical length scale over which the force varies. Furthermore, for the stationary states of a Hamiltonian that is symmetric under $x \leftrightarrow -x$, both $\langle \hat{x}\rangle$ and $\langle \hat{p}\rangle$ are always zero and therefore Eherenfest's theorem does not yield any useful information.

The discussions in many textbooks on quantum mechanics are essentially confined to the limit $\hbar \to 0$ and the Eherenfest theorem in discussing the emergence of the classical regime. While both these quantum-classical correspondences operate in their own domains of applicability, it has been identified that they are not universally satisfactory.[4–10] In the absence of a commonly accepted notion of the classical limit, it is important to recognize the quantum features that are expected to leave their imprints in the classical regime.

It has been pointed out that the classical limit of a quantum state ought to correspond to an ensemble rather than a single particle.[1,6,11] The averages, variances, and higher-order moments of the quantum and classical probability distributions are therefore expected to agree in the limiting case. Interestingly, considerable attention has been evinced recently in exploring the borderline between classical and quantum worlds via uncertainty principles.[12] Conceptual advances in symplectic geometry and topology—followed by Gromov's discovery of *symplectic non-squeezing* phenomena[13]—shed light on the fact that there is an underlying uncertainty principle governing classical Hamiltonian phase flows too.[14]

In order to compare the statistical form of classical dynamics with the corresponding one in quantum dynamics, the phase space probability distribution of the classical ensemble (a counterpart of the corresponding quantum state) needs to be identified. The classical phase space probability distribution satisfies the Liouville equation and the phase space averages of the classical observables are shown to exhibit dynamical behaviour analogous to that of the corresponding quantum case—even when Ehrenfest's theorem breaks down.[6] More recently,[15] it has been shown that starting from the Ehrenfest theorem, either the Liouville equation (if the momentum and coordinate commute) or the

Schrödinger equation (if the momentum and coordinate obey the canonical commutation relation) would ensue.

It is pertinent to mention here another approach towards the classical limit, where one considers only stationary state solutions of the quantum Hamiltonian and graphically compares the probability density function $P^{(n)}_{\rm QM}(x)=|\psi_n(x)|^2$ with the corresponding classical probability distribution $P_{\rm CL}(x)$ of an ensemble; it is then recognized that the *envelope* of the quantum probability density approaches the classical one in the large-n limit.[16]

In this paper, we show that the first and second moments of suitably defined *dimensionless* canonical variables evaluated for the stationary states of one-dimensional conservative quantum systems match with those associated with the corresponding classical ensemble. This, in turn leads to *identical* structure for uncertainty relations of the dimensionless position and momentum variables in both classical and quantum domains—bringing out the underlying unity of the two formalisms—irrespective of their structurally different mathematical and conceptual nature.

II. CLASSICAL PROBABILITY DISTRIBUTIONS CORRESPONDING TO QUANTUM MECHANICAL STATIONARY STATES

We begin by reviewing the classical probability distributions[16] for an ensemble of particles bound in a one-dimensional potential $V(x)$. The probability density function for the position x of a *single* particle, whose initial position and velocity are specified, is given by

$$P^{\rm single}_{\rm CL}(x,t)=\delta[x-x(t)], \tag{4}$$

where $x(t)$ denotes the deterministic trajectory of the particle at any instant of time t. However, the quantum mechanical probability density $P^{(n)}_{QM}(x)=|\psi_n(x)|^2$ associated with the stationary-state solution $\psi_n(x)$ is not expected to approach—in the classical limit—the single-particle probability density of Eq. (4). Rather, the locally averaged quantum probability density does approximate a probability distribution $P_{\rm CL}(x)$ of a classical ensemble of particles (of fixed energy E) in the large n limit.[16]

Classical particles of fixed energy E in a statistical ensemble (microcanonical ensemble) are confined to move on a surface of constant energy E in the phase space and the associated phase space probability distribution $P_{\rm CL}(x,p)$ obeys the stationary state Liouville equation

$$\frac{dP_{\rm CL}(x,p)}{dt}=\{P_{\rm CL}(x,p),H\}=0, \tag{5}$$

where $\{P_{\rm CL}(x,p),H\}$ is the Poisson bracket of $P_{\rm CL}(x,p)$ with the Hamiltonian $H=(p^2/2m)+V(x)$. In other words, the phase space distribution $P_{\rm CL}(x,p)$ is a function of the Hamiltonian H itself. The constant energy assumption then corresponds to

$$P_{\rm CL}(x,p)\propto\delta\left[\frac{p^2}{2m}+V(x)-E\right]. \tag{6}$$

The position probability function is then obtained by integrating over the momentum variable p

$$\begin{aligned}P_{\rm CL}(x)&=\int dpP_{\rm CL}(x,p)\\&=\text{Constant}\cdot\int dp\delta\left[\frac{p^2}{2m}+V(x)-E\right].\end{aligned} \tag{7}$$

Using the properties $\delta(ax)=\delta(x)/|a|$ and $\delta(x^2-a^2)=[\delta(x+a)+\delta(x-a)]/2|a|$ of the Dirac delta function, the classical probability distribution reduces to

$$\begin{aligned}P_{\rm CL}(x)&=\text{Constant}\cdot\int dp2m\delta\big(p^2+2m[V(x)-E]\big)\\&=\text{Constant}\cdot\sqrt{\frac{2m}{[E-V(x)]}}\\&\quad\times\int dp\Big[\delta\Big(p+\sqrt{2m[E-V(x)]}\Big)\\&\quad+\delta\Big(p-\sqrt{2m[E-V(x)]}\Big)\Big]=\frac{\mathcal{N}}{\sqrt{E-V(x)}},\end{aligned} \tag{8}$$

where $\mathcal{N}$ denotes the normalization factor, such that $\int_{x_1}^{x_2}dxP_{\rm CL}(x)=1$ (the integration is taken between the classical turning points (x_1,x_2) as the probability distribution $P_{\rm CL}(x)$ vanishes outside the domain $x_1\le x\le x_2$).

It may be readily seen that, by substituting $E=m\omega^2A^2/2$ and $V(x)=m\omega^2x^2/2$ in the familiar example of the harmonic oscillator, the classical position probability distribution (8) reduces to the well-known expression $P_{\rm CL}(x)=1/\pi\sqrt{A^2-x^2}$.

The phase space averages of any arbitrary function *F(x,p)* of position and momentum variables get reduced to those evaluated with the position probability distribution function $P_{\rm CL}(x)$ as follows:

$$\begin{aligned}\langle F(x,p)\rangle_{\rm CL}&=\int dx\int dpP_{\rm CL}(x,p)F(x,p)\\&=\text{Constant}\cdot\int dx\int dp\delta\left(\frac{p^2}{2m}+V(x)-E\right)F(x,p)\\&=\text{Constant}\cdot\int dx\sqrt{\frac{2m}{E-V(x)}}\\&\quad\times\int dp\Big[\delta\Big(p+\sqrt{2m[E-V(x)]}\Big)\\&\quad+\delta\Big(p-\sqrt{2m[E-V(x)]}\Big)\Big]F(x,p)\\&=\text{Constant}\cdot\int dx\sqrt{\frac{2m}{E-V(x)}}\\&\quad\times\Big[F(x,-\sqrt{2m[E-V(x)]})\\&\quad+F(x,\sqrt{2m[E-V(x)]})\Big]\\&=\frac{1}{2}\int dxP_{\rm CL}(x)\Big[F\Big(x,-\sqrt{2m[E-V(x)]}\Big)\\&\quad+F\Big(x,\sqrt{2m[E-V(x)]}\Big)\Big].\end{aligned} \tag{9}$$

We define dimensionless (scaled) position and momentum variables,

$$X = \frac{x}{A}, \qquad P = \frac{p}{\sqrt{2mE}}, \tag{10}$$

such that $|X|, |P| \leq 1$ in a bounded system.

In Sec. III, we compute the first and second moments $\langle X\rangle_{\rm CL}, \langle X^2\rangle_{\rm CL}, \langle P\rangle_{\rm CL}$, and $\langle P^2\rangle_{\rm CL}$ of the classical probability distribution in three specific examples of one-dimensional bound systems. We then compare these classical averages with the quantum expectation values $\langle\hat{X}\rangle_{\rm QM}, \langle\hat{X}^2\rangle_{\rm QM}, \langle\hat{P}\rangle_{\rm QM}$, and $\langle\hat{P}^2\rangle_{\rm QM}$, evaluated for the stationary states $\psi_n(x)$, and show that they agree remarkably with each other in the classical limit.

III. COMPARISON OF FIRST AND SECOND MOMENTS OF THE CLASSICAL DISTRIBUTION WITH THE STATIONARY STATE QUANTUM MOMENTS

We focus now on three specific examples of one-dimensional bound systems: the harmonic oscillator, the infinite well, and the bouncing ball. We evaluate the first and second moments of the dimensionless position and momentum variables [Eq. (10)] and show that the quantum moments—evaluated for stationary eigenstates of the Hamiltonian—match their classical counterparts.

A. One-dimensional harmonic oscillator

As shown in Sec. II, the classical probability density for finding a system of harmonic oscillators—all with the same amplitude A—between position x and $x + dx$ is given by

$$P_{\rm CL}(x) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{A^2 - x^2}} & \text{for } |x| \leq A, \\ 0 & \text{for } |x| > A. \end{cases} \tag{11}$$

We use scaled canonical variables $X = x/A$ and $P = p/\sqrt{2mE} = p/(m\omega A)$, and evaluate the averages of X, X^2, P, and P^2, making use of Eqs. (9) and (11)

$$\langle X\rangle_{\rm CL} = \frac{1}{A}\int dx P_{\rm CL}(x)x = \frac{1}{A\pi}\int_{-A}^{A} dx\frac{x}{\sqrt{A^2 - x^2}} = 0, \tag{12}$$

$$\langle X^2\rangle_{\rm CL} = \frac{1}{A^2}\int dx P_{\rm CL}(x)x^2 = \frac{1}{A^2\pi}\int_{-A}^{A} dx\frac{x^2}{\sqrt{A^2 - x^2}} = \frac{1}{2}, \tag{13}$$

$$\langle P\rangle_{\rm CL} = \frac{1}{2m\omega A}\int_{-A}^{A} dx P_{\rm CL}(x)\left(-\sqrt{2m\left[E - \frac{1}{2}m\omega^2x^2\right]} + \sqrt{2m\left[E - \frac{1}{2}m\omega^2x^2\right]}\right) = 0, \tag{14}$$

$$\langle P^2\rangle_{\rm CL} = \frac{1}{m^2\omega^2A^2}\int_{-A}^{A} dx P_{\rm CL}(x)2m\left[E - \frac{1}{2}m\omega^2x^2\right] = \frac{1}{A^2\pi}\int_{-A}^{A} dx\sqrt{A^2 - x^2} = \frac{1}{2}. \tag{15}$$

The variances of X and P are given by

$$(\Delta X)^2_{\rm CL} = \langle X^2\rangle_{\rm CL} - \langle X\rangle^2_{\rm CL} = \frac{1}{2}, \quad (\Delta P)^2_{\rm CL} = \langle P^2\rangle_{\rm CL} - \langle P\rangle^2_{\rm CL} = \frac{1}{2}, \tag{16}$$

and hence the product of variances is

$$(\Delta X)^2_{\rm CL}(\Delta P)^2_{\rm CL} \equiv \frac{1}{4} \tag{17}$$

in a classical ensemble [characterized by the probability distribution (11)] of harmonic oscillators.

The stationary-state solutions of the quantum Hamiltonian $\hat{H} = [\hat{p}^2 + m^2\omega^2\hat{x}^2]/2m$ are given by

$$\psi_n(x) = \left(\frac{\sqrt{m\omega/\pi\hbar}}{2^n n!}\right)^{1/2} H_n(\sqrt{m\omega/\hbar}x)e^{-m\omega x^2/\hbar}, \tag{18}$$

where H_n are the Hermite polynomials of degree n. These states correspond to the energy eigenvalues $E_n = (n + 1/2)\hbar\omega$. The *classical turning points* associated with energy E_n are readily identified to be $A_n = \sqrt{2E_n/m\omega^2} = \sqrt{(2n+1)\hbar/m\omega}$.

We use scaled position and momentum operators, $\hat{X} = \hat{x}/A_n = \hat{x}\sqrt{m\omega/(2n+1)\hbar}, \hat{P} = \hat{p}/\sqrt{2mE_n} = \hat{p}/\sqrt{(2n+1)\hbar m\omega}$ (corresponding to their classical counterparts above), and evaluate the expectation values of $\hat{X}, \hat{X}^2, \hat{P}$, and $\hat{P}^2$ for the stationary states $\psi_n(x)$

$$\langle\hat{X}\rangle_{QM} = \sqrt{\frac{m\omega}{(2n+1)\hbar}}\int_{-\infty}^{\infty} dx|\psi_n(x)|^2x = 0, \tag{19}$$

$$\langle\hat{X}^2\rangle_{QM} = \frac{m\omega}{(2n+1)\hbar}\int_{-\infty}^{\infty} dx|\psi_n(x)|^2x^2 = \frac{1}{2}, \tag{20}$$

$$\langle\hat{P}\rangle_{QM} = -i\sqrt{\frac{\hbar}{(2n+1)m\omega}}\int_{-\infty}^{\infty} dx\psi_n^*(x)\frac{d\psi_n(x)}{dx} = 0, \tag{21}$$

$$\langle\hat{P}^2\rangle_{QM} = \frac{-\hbar}{(2n+1)m\omega}\int_{-\infty}^{\infty} dx\psi_n^*(x)\frac{d^2\psi_n(x)}{dx^2} = \frac{1}{2}. \tag{22}$$

Clearly, these quantum expectation values match the classical ones given in Eqs. (12)–(15) and we obtain the uncertainty product, for *all* stationary-state solutions of the quantum oscillator,

$$(\Delta\hat{X})^2_{\rm QM}(\Delta\hat{P})^2_{\rm QM} \equiv \frac{1}{4}. \tag{23}$$

It is pertinent to point out here that the commutator relation

$$[\hat{X}, \hat{P}] = \left[\sqrt{\frac{m\omega}{(2n+1)\hbar}}\hat{x}, \frac{\hat{p}}{\sqrt{\hbar m\omega(2n+1)}}\right] = \frac{1}{(2n+1)\hbar}[\hat{x}, \hat{p}] = \frac{i}{2n+1} \tag{24}$$

leads to the uncertainty relation

$$(\Delta\hat{X})^2_{QM}(\Delta\hat{P})^2_{QM} \geq \frac{1}{4(2n+1)^2}. \tag{25}$$

In the large-n limit the right-hand side goes to zero, which would be expected in the classical regime. However Eq. (23) for the uncertainty product holds for *all* the stationary-state solutions; and strikingly, this result matches that of a classical ensemble of oscillators with fixed energy E [see Eq. (17)].

B. One dimensional infinite potential box

We consider a symmetric infinite potential well defined by

$$V(x) = \begin{cases} 0 & \text{for } -L/2 \le x \le L/2, \\ \infty & \text{for } |x| > L/2. \end{cases} \tag{26}$$

The particles move with a constant velocity within the box and get reflected back and forth. The position probability distribution for an ensemble of classical particles confined to move within the box is a constant [as can be readily seen by substituting Eq. (26) in Eq. (8)] and is given by[16]

$$P_{\mathrm{CL}}(x) = \begin{cases} 1/L & \text{for } |x| \le L/2, \\ 0 & \text{for } |x| > L/2. \end{cases} \tag{27}$$

This distribution obeys $\int_{-L/2}^{L/2} P_{\mathrm{CL}}(x)dx = 1$.

In this example, the dimensionless position and momentum variables are

$$X = \frac{x}{L/2}, \qquad P = \frac{p}{\sqrt{2mE}} = \frac{p}{|p|}, \tag{28}$$

and the classical averages $\langle X\rangle_{\mathrm{CL}}, \langle X^2\rangle_{\mathrm{CL}}, \langle P\rangle_{\mathrm{CL}}$, and $\langle P^2\rangle_{\mathrm{CL}}$ are readily evaluated using the probability distribution (27)

$$\langle X\rangle_{\mathrm{CL}} = \int dx P_{\mathrm{CL}}(x)\frac{x}{L/2} = \frac{2}{L^2}\int_{-L/2}^{L/2} x dx = 0, \tag{29}$$

$$\langle X^2\rangle_{\mathrm{CL}} = \int dx P_{\mathrm{CL}}(x)\frac{x^2}{L^2/4} = \frac{4}{L^3}\int_{-L/2}^{L/2} dx x^2 = \frac{1}{3}, \tag{30}$$

$$\langle P\rangle_{\mathrm{CL}} = 0, \tag{31}$$

$$\langle P^2\rangle_{\mathrm{CL}} = 1. \tag{32}$$

So, the variances of X and P are $(\Delta X)^2_{\mathrm{CL}} = 1/3$ and $(\Delta P)^2_{\mathrm{CL}} = 1$ for the classical ensemble of particles of fixed energy E, confined within the infinite well. The product of the variances is

$$(\Delta X)^2_{\mathrm{CL}}(\Delta P)^2_{\mathrm{CL}} \equiv \frac{1}{3}. \tag{33}$$

The quantum mechanical stationary-state solutions (even and odd parity) for a particle confined in the one-dimensional infinite potential well are

$$\begin{aligned} \psi_n^{(+)}(x) &= \sqrt{\frac{2}{L}}\cos(n\pi x/L), \quad n = 1,3,5,\ldots, \\ \psi_n^{(-)}(x) &= \sqrt{\frac{2}{L}}\sin(n\pi x/L), \quad n = 2,4,6,\ldots, \end{aligned} \tag{34}$$

and the corresponding energy eigenvalues are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}. \tag{35}$$

The scaled dimensionless position and momentum operators are

$$\hat{X} = \frac{\hat{x}}{L/2}, \qquad \hat{P} = \frac{\hat{p}}{\sqrt{2mE_n}} = \frac{\hat{p}}{n\pi\hbar/L}. \tag{36}$$

The expectation values of $\hat{X}, \hat{X}^2, \hat{P}$, and $\hat{P}^2$ are evaluated in the stationary states (both even and odd) to obtain

$$\langle\hat{X}\rangle_{QM} = \frac{1}{L/2}\int_{-L/2}^{L/2} dx|\psi_n^{(+/-)}(x)|^2 x = 0, \tag{37}$$

$$\langle\hat{X}^2\rangle_{QM} = \frac{1}{L^2/4}\int_{-L/2}^{L/2} dx|\psi_n^{(+/-)}(x)|^2 x^2 = \frac{1}{3} - \frac{2}{n^2\pi^2}, \tag{38}$$

$$\langle\hat{P}\rangle_{QM} = -i\frac{L}{n\pi}\int_{-L/2}^{L/2} dx\psi_n^{(+/-)}(x)\frac{d\psi_n^{(+/-)}(x)}{dx} = 0, \tag{39}$$

$$\langle\hat{P}^2\rangle_{QM} = -\frac{L^2}{n^2\pi^2}\int_{-L/2}^{L/2} dx\psi_n^{(+/-)}(x)\frac{d^2\psi_n^{(+/-)}(x)}{dx^2} = 1. \tag{40}$$

It may be seen that $\langle\hat{X}^2\rangle_{\mathrm{QM}}$ approaches the classical result, $\langle X^2\rangle_{\mathrm{CL}} = 1/3$, in the large-$n$ limit. In this limit, the uncertainty product becomes

$$\lim_{n\to\infty}(\Delta\hat{X})_{\mathrm{QM}}(\Delta\hat{P})_{\mathrm{QM}} = \frac{1}{3}. \tag{41}$$

Meanwhile, from the commutator relation,

$$[\hat{X},\hat{P}] = \left[\frac{2}{L}\hat{x}, \frac{L}{n\pi\hbar}\hat{p}\right] = \frac{2i}{n\pi}, \tag{42}$$

it is clear that the uncertainty product obeys

$$(\Delta\hat{X})^2_{QM}(\Delta\hat{P})^2_{QM} \ge \frac{1}{n^2\pi^2}, \tag{43}$$

and in the large-n limit one recovers the expected result $(\Delta\hat{X})^2_{QM}(\Delta\hat{P})^2_{QM} \ge 0$. However, the stationary-state uncertainty product, Eq. (41), does not vanish in the limit $n \to \infty$. Instead, it approaches the value 1/3, which coincides *exactly* with that associated with the classical ensemble [see Eq. (33)].

C. Bouncing ball

We now consider the example of a particle bouncing vertically up and down in a uniform gravitational field, which is described by the confining potential

$$V(z) = \begin{cases} \infty & \text{for } z < 0, \\ mgz & \text{for } z \ge 0. \end{cases} \tag{44}$$

A classical particle of total energy E, subject to this potential, bounces back and forth between $z = 0$ and a maximum height $z = A$, where $A = E/mg$.

An ensemble of bouncing balls of energy E is characterized by the classical position probability distribution[16]

$$P_{\mathrm{CL}}(z) = \begin{cases} \dfrac{1}{2A}\dfrac{1}{\sqrt{1-(z/A)}} & \text{for } 0 \le z \le A, \\ 0 & \text{otherwise}. \end{cases} \tag{45}$$

This expression follows from substituting Eq. (44) in Eq. (8).

Employing dimensionless position and momentum variables

$$Z = \frac{z}{A}, \qquad P = \frac{p}{\sqrt{2mE}} = \frac{p}{\sqrt{2m^2gA}} \tag{46}$$

(so that $0 \le Z \le 1$ and $-1 \le P \le 1$ for the bouncing particles), we obtain the classical moments

$$\langle Z\rangle_{\rm CL} = \frac{1}{A}\int dz P_{\rm CL}(z) z = \frac{1}{2A^2}\int_0^A dz \frac{z}{\sqrt{1-(z/A)}} = \frac{2}{3}, \tag{47}$$

$$\langle Z^2\rangle_{\rm CL} = \frac{1}{A^2}\int dz P_{\rm CL}(z) z^2 = \frac{1}{2A^3}\int_0^A dz \frac{z^2}{\sqrt{1-(z/A)}} = \frac{8}{15}, \tag{48}$$

$$\begin{aligned}\langle P\rangle_{\rm CL} &= \frac{1}{2\sqrt{2m^2gA}}\int_0^A dz P_{\rm CL}(z)(-\sqrt{2m(E-mgz)} \\ &\quad + \sqrt{2m(E-mgz)}) = 0,\end{aligned} \tag{49}$$

$$\begin{aligned}\langle P^2\rangle_{\rm CL} &= \frac{1}{2m^2gA}\int_0^A dz P_{\rm CL}(z) 2m(E-mgz) \\ &= \frac{1}{2A}\int_0^A dz \sqrt{1-(z/A)} = \frac{1}{3}.\end{aligned} \tag{50}$$

Thus, the variances of Z and P are $(\Delta Z)^2_{\rm CL} = 4/45$ and $(\Delta P)^2_{\rm CL} = 1/3$, leading to

$$(\Delta Z)^2_{\rm CL}(\Delta P)^2_{\rm CL} \equiv 4/135. \tag{51}$$

Stationary-state solutions of a quantum bouncing particle[17] are obtained by solving the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n(z)}{dz^2} + mgz\psi_n(z) = E_n\psi_n(z), \tag{52}$$

with the boundary condition

$$\psi_n(0) = 0. \tag{53}$$

In terms of the characteristic *gravitational length*[17]

$$l_g = \left(\frac{\hbar^2}{2m^2g}\right)^{1/3}, \tag{54}$$

it is convenient to define dimensionless quantities

$$E'_n = \frac{E_n}{mgl_g}, \qquad z' = \frac{z}{l_g} - E'_n, \tag{55}$$

so that the Schrödinger equation (52) takes the standard form

$$\frac{d^2\psi_n(z')}{dz'^2} = z'\psi_n(z'), \tag{56}$$

which is the Airy differential equation. The solutions of Eq. (56) are two linearly independent sets of Airy functions, ${\rm Ai}(z')$ and ${\rm Bi}(z')$. However, the function ${\rm Bi}(z')$ diverges as its argument increases, and so it is not a physically admissible solution. The stationary state solutions of a quantum bouncer are thus given by

$$\psi_n(z') = N_n {\rm Ai}(z'), \quad z' \ge -E'_n, \quad n = 1, 2, 3, \ldots, \tag{57}$$

where N_n is a normalization constant. From the boundary condition (53), one obtains ${\rm Ai}(-E'_n) = 0$, indicating that the (scaled) energy eigenvalue E'_n is minus the nth zero of the Airy function. (The zeros of the Airy function are all negative). The first few energy eigenvalues E'_n of the quantum bouncing ball are given in Table I.

Identifying the classical turning points A_n associated with the energy eigenvalues E_n of the quantum bouncer to be

$$A_n = \frac{E_n}{mg} = l_g E'_n, \tag{58}$$

we define appropriately scaled position and momentum operators [which are quantum counterparts of Z and P defined in Eq. (46)] as

$$\hat{Z} = \frac{\hat{z}}{A_n} = \frac{\hat{z}}{l_g E'_n}, \qquad \hat{P} = \frac{\hat{p}}{\sqrt{2mE_n}} = \frac{l_g\hat{p}}{\hbar\sqrt{E'_n}}. \tag{59}$$

Further, substituting Eqs. (54) and (55) in Eq. (59), we may express the configuration representation of the operators $\hat{Z}$ and $\hat{P}$ in terms of z' and E'_n as

$$\hat{Z} \to \frac{1}{E'_n}(z' + E'_n), \qquad \hat{P} \to \frac{-i}{\sqrt{E'_n}}\frac{d}{dz'}. \tag{60}$$

The expectation values $\langle\hat{Z}\rangle_{QM}$ and $\langle\hat{Z}^2\rangle_{QM}$ can be evaluated analytically[18] for the eigenstates (57) of the quantum bouncing ball

$$\begin{aligned}\langle\hat{Z}\rangle_{QM} &= \frac{1}{E'_n}\int_{-E'_n}^{\infty} dz'(z' + E'_n)|\psi_n(z')|^2 \\ &= \frac{N_n^2}{E'_n}\int_{-E'_n}^{\infty} dz'(z' + E'_n){\rm Ai}^2(z') = \frac{2}{3},\end{aligned} \tag{61}$$

$$\begin{aligned}\langle\hat{Z}^2\rangle_{QM} &= \frac{1}{E'^2_n}\int_{-E'_n}^{\infty} dz'(z' + E'_n)^2|\psi_n(z')|^2 \\ &= \frac{N_n^2}{E'^2_n}\int_{-E'_n}^{\infty} dz'(z' + E'_n)^2{\rm Ai}^2(z') = \frac{8}{15}.\end{aligned} \tag{62}$$

These agree exactly with the corresponding moments in a classical ensemble of bouncing balls [see Eqs. (47) and (48)].

The expectation value $\langle\hat{P}\rangle_{QM}$ is

$$\begin{aligned}\langle\hat{P}\rangle_{QM} &= \frac{-i}{\sqrt{E'_n}}\int_{-E'_n}^{\infty} dz'\psi_n^*(z')\frac{d\psi_n(z')}{dz'} \\ &= \frac{-iN_n^2}{\sqrt{E'_n}}\int_{-E'_n}^{\infty} dz'{\rm Ai}(z')\frac{d{\rm Ai}(z')}{dz'} = 0,\end{aligned} \tag{63}$$

Table I. The first few scaled energy eigenvalues E'_n of the quantum bouncing ball.

n	E'_n
1	2.3381
2	4.0879
3	5.5205
4	6.7867
5	7.9441

where the last step follows from integration by parts. Finally, we evaluate the expectation value $\langle\hat{P}^2\rangle_{QM}$ as follows:

$$\begin{aligned}\langle\hat{P}^2\rangle_{QM} &= -\frac{1}{E'_n}\int_{-E'_n}^{\infty} dz'\psi_n^*(z')\frac{d^2\psi_n(z')}{dz'^2} \\ &= -\frac{1}{E'_n}\int_{-E'_n}^{\infty} dz'\psi_n^*(z')z'\psi_n(z') \\ &= -\frac{N_n^2}{E'_n}\int_{-E'_n}^{\infty} dz' z'\mathrm{Ai}^2(z') \\ &= 1-\langle\hat{Z}\rangle_{QM} = \frac{1}{3}, \end{aligned} \tag{64}$$

where we have used Eq. (56) in the second line and Eq. (61) in the fourth line.

The expectation values $\langle\hat{P}\rangle_{QM}$ and $\langle\hat{P}^2\rangle_{QM}$ match identically with the corresponding moments (49) and (50) of scaled momentum variables in an ensemble of classical bouncing balls. This is indeed a novel identification, bringing forth the deep-rooted unifying features of the classical and quantum realms.

From Eqs. (61)–(64), we obtain the variances of $\hat{Z}$ and $\hat{P}$ for the stationary states to be $(\Delta\hat{Z})^2_{\mathrm{QM}} = 4/45$ and $(\Delta\hat{P})^2_{\mathrm{QM}} = 1/3$. Hence, the uncertainty product is

$$(\Delta\hat{Z})^2_{\mathrm{QM}}(\Delta\hat{P})^2_{\mathrm{QM}} \equiv \frac{4}{135}, \tag{65}$$

which exactly matches that of the classical ensemble of bouncing balls [see Eq. (51)].

It may be noted that the commutation relation

$$[\hat{Z},\hat{P}] = \left[\frac{\hat{z}}{l_g E'_n}, \frac{l_g\hat{p}}{\hbar\sqrt{E'_n}}\right] = \frac{i}{(E'_n)^{3/2}} \tag{66}$$

would lead to the uncertainty relation $(\Delta\hat{Z})^2_{\mathrm{QM}}(\Delta\hat{P})^2_{\mathrm{QM}} \geq 1/4(E'_n)^3$. In the large-$n$ limit $1/E'_n \to 0$ (as the energy eigenvalues obey the scaling relation[16] $E'_n \propto n^{2/3}$ for large n), thus resulting in the classical limit on the variance product $(\Delta\hat{Z})^2_{\mathrm{QM}}(\Delta\hat{P})^2_{\mathrm{QM}} \geq 0$. Equation (65), on the other hand, is exact for the energy eigenstates.

IV. CONCLUSIONS

Emergence of classical behaviour from the corresponding quantum world has remained an enigmatic topic ever since the inception of quantum theory. It is shown here that in three specific examples of one-dimensional bound systems—harmonic oscillator, infinite square well, and bouncing ball—the uncertainty products of position and momentum evaluated for stationary quantum states agree with those of the corresponding constant-energy classical ensembles. This identification points towards a deep underlying connectivity between the two formalisms, despite their mathematical and conceptual differences.

The uncertainty principle is one of the intrinsic trademarks of quantum theory and is not a feature of the classical deterministic motion of single particle. However, recent investigations[12]—motivated by Gromov's non-squeezing theorem[13]—have shown that there indeed are intrinsic uncertainties governed by the symplectic geometry of Hamiltonian phase space flows associated with classical ensembles. Our work establishes a remarkable agreement between the uncertainty product for quantum stationary states and the classical microcanonical ensemble of constant energy, for the three specific examples considered here. This agreement could be a reflection of subtle aspects of symplectic toplogy. It would be interesting to investigate the nature of quantum-classical uncertainties from a unifying point of view based on phase space topology.[12]

According to the Eherenfest theorem, the dynamical equations of motion for the average values of the position and momentum coincide with the classical equations for linear and quadratic potentials. The three specific examples analyzed here focused on stationary-state solutions associated with linear and quadratic potentials, and this raises the question of whether the agreement between classical and quantum uncertainty products happens to be an indirect reflection of Ehrenfest theorem itself.[19] Yet another reason why the classical and quantum uncertainty relationships coincide might be because the quasi-classical (WKB) approximation is exact for the potentials considered.[20] It is therefore important to extend our results to the case of non-quadratic potentials, which we will take up in a forthcoming communication.

ACKNOWLEDGMENTS

The authors thank Professor A. K. Rajagopal, Dr. Sudha Shenoy, and Professor B. A. Kagali for insightful comments. The authors also thank the referees for their insightful suggestions, in the light of which this paper has been revised.

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[19] Irrespective of the condition $\langle F(\hat{x})\rangle = F(\langle \hat{x}\rangle)$ being satisfied in the case of linear and quadratic potentials, it is crucial to consider an appropriate wave packet, the centroid of which results in the corresponding classical trajectory. In this work, we have focused on stationary-state solutions for which the above condition of localization is not met and the Eherenfest theorem leads to a redundant equation $0 = 0$.
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Nörrenberg Doubler

The Nörrenberg doubler dates from 1858 and is a generalized apparatus for viewing transparent objects between crossed polarizing filters. A light beam reflects at Brewster's angle from the glass plate, passes through the object before and after reflecting from the mirror at the bottom, and then passes through the glass plate and is examined by the eye through a Nicol prism (a device that produces linearly polarized light). This example is in the collection of historical apparatus at Creighton University. (Notes by Thomas B. Greenslade, Jr., Kenyon College, photograph by Vacek Miglus of Wesleyan University)