An information theoretic study of number-phase complementarity in a four level atomic system

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We study number-phase uncertainty in a laser-driven, effectively four-level atomic system under electromagnetically induced transparency (EIT) and coherent population trapping (CPT). Uncertainty is described using (entropic) knowledge of the two complementary variables, namely, number and phase, where knowledge is defined as the relative entropy with respect to a uniform distribution. In the regime where the coupling and probe lasers are approximately of equal strength, and the atom exists in a CPT state, there is coherence between the ground states, and correspondingly large phase knowledge and lower number knowledge. The situation is the opposite in the case where coupling laser is much stronger and the atom exists in an EIT state. We study these effects also in the presence of a higher-order nonlinear absorption, which is seen to produce a dephasing effect.

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I. INTRODUCTION

The uncertainty relation in number and phase is

$$\Delta n \Delta \phi \geq 1/2, \tag{1}$$

which holds accurately when the number n is large. When the number uncertainty is small, $\Delta\phi$ should be large but the limit on ϕ is 2π and the uncertainty relation breaks down. Also, it is not possible to introduce a hermitian phase operator because n is bounded below [1]. One way to get around the difficulty is to introduce a probability distribution $\mathcal{P}(\phi)$ of phase, where the operator corresponding to phase is not a conventional projective measurement, but a positive operator-valued measure (POVM) [2], in particular, given by projectors to the atomic coherent states $|\theta, \phi\rangle$. As the expectation value of the dipole moment operator for these states is $j(\sin\theta)e^{-i\phi}$, where j is the total angular momentum of the electron, measurement of this POVM yields phase information. Phase ϕ is an important quantity in atomic coherences and in the interferometry based on such coherences [3, 4].

In this paper, number-phase complementarity in an atomic system interacting with a laser field is studied via an information theoretic treatment. A particular context in which this could be studied is in spin squeezed atomic ensembles, which can be created either by transfer of squeezing from squeezed light fields or by the presence of nonlinearities [5]. Here we consider an atom (e.g., Rb) with a four-level electronic hyperfine manifold under the action of two laser fields. This system exhibits several interesting nonlinear phenomena such as electromagnetically induced transparency (EIT), coherent population transfer (CPT) and three-photon absorption (TPA), prompting us to study the number-phase complementarity in these regimes.

The plan of the paper is: Section II describes the atom laser sytem. Section III briefly introduces the concept of phase distribution; in Section IV, the information theoretic formulation of complementarity is presented in terms of an upper bound to a (convex) sum of knowledge of the complementary observables, number and phase. In Section V, the entropic uncertainty relations are used to study number-phase complementarity in EIT and CPT systems. In Section V B, the effect of higher order nonlinearity on the complementaristic behavior is studied. In Section VI, we make our conclusions.

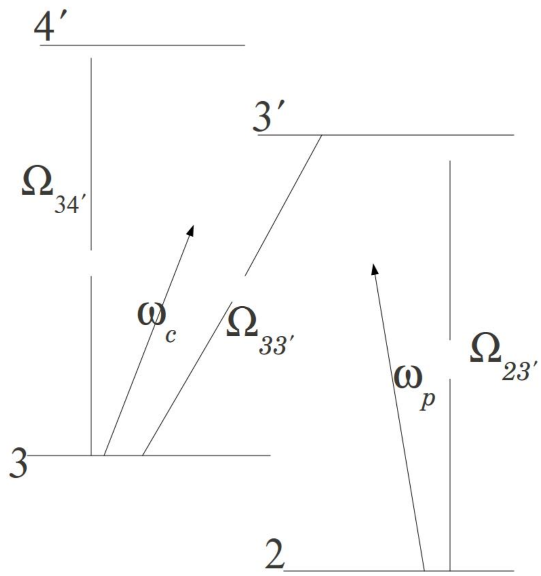

FIG. 1: The atom-laser system: couplings provided by the lasers. The coupling laser gives rise to two Rabi frequency couplings ($3-3'$ and $3-4'$) and the probe to one ($2-3'$). The coupling that does not participate in EIT formation ($3-4'$) gives rise to third order nonlinearity in probe absorption. There is nonlinear absorption in the coupling $\Omega_{34'}$ also. This coupling is called the signal beam in the XPM scheme.

II. BRIEF DISCUSSION OF THE ATOM LASER SYSTEM

In this study we investigate the complementarity in atomic states driven by unsqueezed light. The isotope under study, ^{85}Rb, has a nuclear spin of 5/2. The electronic levels of interest in conventional atomic spectroscopy notation, are the ground hyperfine levels $5S_{1/2}$ $F=2,3$ and the excited states $5P_{3/2}$ $F=3',4'$. The selection rules permit electric dipole transitions that satisfy $\Delta L = \pm 1$ and ΔF=0, ± 1. Two lasers, *coupling* and *probe*, at frequency ω_C and ω_P, detuned by $\delta_{Coupling}$ (δc) and δ_{Probe} (δp) from the two transitions $5S_{1/2}F=3 \rightarrow 5P_{3/2}F=4'$ and $5S_{1/2}F=2 \rightarrow 5P_{3/2}F=3'$ with Rabi frequencies $\Omega_{33'}$, $\Omega_{34'}$ and $\Omega_{23'}$, are incident on the atom, see Fig. (1).

A three-level Λ system is formed by the levels $5S_{1/2}$ $F=2,3$ and $5P_{3/2}$ $F=3'$. The electron may make a transition between the two ground hyperfine states via the excited level $F=3'$. However, depending on the Rabi frequencies of the two lasers, either a cancellation of transition amplitudes leading to EIT or the formation of a superposition dark state leading to CPT, is expected when the detunings of the lasers satisfy $\delta c = \delta p$. Due to the Rabi frequency coupling $\Omega_{34'}$, there is a three photon absorption taking place and the probe laser sees a $\chi^{(3)}$ nonlinearity [7, 8]. There is nonlinear absorption in the signal beam also [6, 9].

In [10], the four level density matrix equations in the presence of the two beams and three Rabi frequency couplings were solved. There the Lindblad terms corresponding to spontaneous decay were used.

To make a study of the four level system in number-phase variables we recast the relevant 4 levels of the atom in terms of a pseudospin of spin 3/2. The projections of the spin can take any of the four values $s=-3/2,-1/2,1/2,3/2$, which we take to represent the four electronic levels, and are mapped to $F=2,3,3',4'$, respectively. It can be shown that a different assignment merely constitutes a re-labelling of vectors, and does not qualitatively alter the conclusions that follow.

Following the notation of [11] we evaluate the phase distribution function P(ϕ), from which we obtain phase knowledge R[ϕ] and number knowledge R[m], defined below. The level populations are reflected in the angular momentum basis (number, m) while the coherence between levels is reflected in the phase variable ϕ.

III. PHASE DISTRIBUTION

The quantum description of phases [12, 13] has a long history [1, 14–17]. Pegg and Barnett [16], following Dirac [14], carried out a polar decomposition of the annihilation operator and defined a hermitian phase operator in a finite-dimensional Hilbert space. In their scheme, the expectation value of a function of the phase operator is first carried out in a finite-dimensional Hilbert space, and then the dimension is taken to the limit of infinity. However, it is not possible to interpret this expectation value as that of a function of a hermitian phase operator in an infinite-dimensional Hilbert space [2, 18]. To circumvent this problem, the concept of phase distribution for the quantum phase has been introduced [18, 19]. In this scheme, one associates a phase distribution to a given state such that the average of a function of the phase operator in the state, computed with the phase distribution, reproduces the results of Pegg and Barnett.

For an atomic system, the phase distribution $\mathcal{P}(\phi)$, ϕ being related to the phase of the dipole moment of the system, is given by [3]

$$\mathcal{P}(\phi) = \frac{2j+1}{4\pi} \int_0^{\pi} d\theta \sin(\theta) Q(\theta, \phi), \tag{2}$$

where $\mathcal{P}(\phi) \geq 0$ and is normalized to unity, i.e., $\int_0^{2\pi} d\phi \mathcal{P}(\phi) = 1$. In the above, j is the pseudo angular momentum of the atom. Here $Q(\theta, \phi)$ is defined as

$$Q(\theta, \phi) = \langle \theta, \phi | \rho^s | \theta, \phi \rangle, \tag{3}$$

where $|\theta, \phi\rangle$ are the atomic coherent states [20, 21] defined by Eq. (4) in terms of Wigner-Dicke states [22], which are the simultaneous eigenstates of the angular momentum operators J^2 and J_z;

$$|\theta, \phi\rangle = \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}} (\sin(\theta/2))^{j+m} (\cos(\theta/2))^{j-m} |j, m\rangle e^{-i(j+m)\phi}. \tag{4}$$

It can be shown that the angular momentum operators J_ξ, J_η and J_ζ (obtained by rotating the operators J_x, J_y and J_z through an angle θ about an axis $\hat{n} = (\sin\phi, -\cos\phi, 0)$), being mutually non-commuting, obey an uncertainty relationship of the type $\langle J_\xi^2 \rangle \langle J_\eta^2 \rangle \geq \frac{1}{4} \langle J_\zeta^2 \rangle$. Atomic coherent states (Eq. 4) are precisely those states that saturate this bound, hence the name, in analogy with radiation fields [3]. For two level systems, they exhaust all pure states, whereas for larger dimensions, this is no longer true. Using Eq. (3) in Eq. (2), with insertions of partitions of unity in terms of the Wigner-Dicke states, we can write the phase distribution function as [11]

$$\mathcal{P}(\phi) \ = \ \frac{2j+1}{4\pi} \int_0^{\pi} d\theta \sin\theta \sum_{n,m=-j}^{j} \langle \theta, \phi | j, n \rangle \langle j, n | \rho^s(t) | j, m \rangle \langle j, m | \theta, \phi \rangle. \tag{5}$$

The phase distribution $\mathcal{P}(\phi)$, taking into account the environmental effects, has been studied in detail for QND as well as dissipative systems in [11, 23] for physically interesting initial conditions of the system S, i.e., (a) Wigner-Dicke state, (b) atomic coherent state and (c) atomic squeezed state.

In our mapping scheme, Wigner-Dicke or excitation states are thought of as 'number states', thereby making J_z the 'number observable', whose distribution $p(m)$, given below as

$$p(m) = \langle j, m | \rho^s(t) | j, m \rangle, \tag{6}$$

is considered as complementary to $\mathcal{P}(\phi)$ [3].

IV. INFORMATION THEORETIC REPRESENTATION OF COMPLEMENTARITY

Two observables A and B of a d-level system are called complementary in quantum mechanics if measurement of A disturbs B, and vice versa [24, 25]. Complementarity is related to the Heisenberg uncertainty principle, which says that for any state ψ, the probability distributions obtained by measuring A and B cannot both be simultaneously peaked if A and B are non-commuting. Heisenberg uncertainty is traditionally expressed by the relation

$$\triangle_\psi A \triangle_\psi B \geq \frac{1}{2} |\langle [A, B] \rangle_\psi|, \tag{7}$$

where $(\triangle_\psi A)^2 = \langle A^2\rangle_\psi - (\langle A\rangle_\psi)^2$. However, this representation of the Heisenberg uncertainty relation has the disadvantage that the right hand side of Eq. (7) is not a fixed lower bound but is state dependent. Further, the form of Eq. (7) is not invariant when A or B is scaled by some numerical factor, though physically we would want the measure of non-commutativity to be scale-invariant.

The information theoretic (or "entropic") version of the Heisenberg uncertainty relationship [24–26], which uses Shannon entropy of measurement outcomes, instead of variance, as a measure of uncertainty [27, 28], overcomes both these problems.

The relative entropy associated with a discrete distribution $f(j)$ with respect to a distribution $g(j)$ defined over the same index set, is given by

$$S(f||g) = \sum_j f(j)\log\left(\frac{f(j)}{g(j)}\right). \tag{8}$$

$S(f||g) \geq 0$ can be thought of as a measure of 'distance' of distribution f from distribution g , where the equality holds if and only if $f(j) = g(j)$ [27]. Consider a random variable F with probability distribution f. We will define $R(F)$ as the relative entropy of f with respect to the uniform distribution which is $\frac{1}{d}$, for a system of dimension d (as the system has equal probability of being in any of the d states), i.e.,

$$R(F) \equiv R[f(j)] = \sum_j f(j)\log(df(j)). \tag{9}$$

As a measure of distance from a uniform distribution, which has maximal entropy, $R(F)$ can be interpreted as a measure of *knowledge*, as against uncertainty, of the random variable described by distribution f. Following [29], we recast the Heisenberg uncertainty principle in terms of relative entropy as

$$R(A) + R(B) \leq \log d, \tag{10}$$

where d is the (finite) dimension of the system. The Hermitian observables A and B are said to correspond to mutually unbiased bases (MUB-s) if any eigenstate of one of the observable can be written as an equal amplitude superposition of all the eigenstates of the other observable [29]. Physically, Eq. (10) expresses the fact that simultaneous knowledge of A and B is bounded above by $\log d$, and that the probability distributions obtained by measuring A and B on several identical copies of a given state cannot both peak simultaneously.

Eq. (8) has a natural extension to the continuous case, given by

$$S(f||g) = \int dp\; f(p)\log\left(\frac{f(p)}{g(p)}\right). \tag{11}$$

As in the discrete case, we define $R(f)$ as relative entropy setting $g(p)$ to a continuous constant function. In particular, the relative entropy of $\mathcal{P}(\phi)$ with respect to a uniform distribution $\frac{1}{2\pi}$ [11, 23], corresponding to the case where phase is completely randomized, over ϕ is given by the functional

$$R[\mathcal{P}(\phi)] = \int_0^{2\pi} d\phi\; \mathcal{P}(\phi)\log[2\pi\mathcal{P}(\phi)], \tag{12}$$

where the $\log(\cdot)$ refers to the binary base.

For the phase POVM measure ϕ, because of the non-orthogonality of the states $|\theta,\phi\rangle$, the concept of an eigenstate of an observable is weakened to that of a minimum uncertainty state, which corresponds to maximum phase knowledge. On the other hand, number is a regular, Hermitian observable. This leads to the concept of MUB being replaced by that of a *quasi-MUB*.

Two variables A and B form a quasi-MUB if the minimum uncertainty state of A is a maximum uncertainty state of B, but the minimum uncertainty states of B are not necessarily maximally uncertain in A. The number-phase complementarity in our four-level system indeed shows such quasi-MUB character [29]. This can be seen by noting that for the Wigner-Dicke states $|j,\tilde{m}\rangle$, the phase distribution is [11]

$$\mathcal{P}(\phi) = \frac{2j+1}{2\pi}\begin{pmatrix} 2j \\ j+\tilde{m}\end{pmatrix}\mathcal{B}\left[j+\tilde{m}+1, j-\tilde{m}+1\right] = \frac{1}{2\pi}, \tag{13}$$

where $\mathcal{B}$ stands for the Beta function. Thus, it follows via Eq. (12) that the knowledge R_ϕ vanishes. On the other hand, it can be shown that in the present case the states which minimize R_ϕ are the not Wigner-Dicke states. However, it can be shown that they are number states in the $d = 2$ case. To see this, we observe that if $\mathcal{P}(\phi)$ is constant, then in Eq. (5), each term in the summation, which is proportional to $e^{i(m-n)\phi}$, must individually be independent of ϕ. Since ϕ is arbitrary, this is possible only if $m = n$, i.e., the state ρ^s is diagonal in the Wigner-Dicke basis.

Thus J_z and ϕ form a quasi-MUB because of the POVM nature of ϕ [27]. For a POVM, knowledge R of the minimum uncertainty state need not be $\log(d)$ bits, essentially because outcomes are non-sharp owing to non-orthogonality [30] of the corresponding measurement operators [29]. Hence, the plain summation over knowledges in the entropic formulation of the uncertainty relation is replaced by a suitable convex sum.

Thus, for a POVM in the four level case such as phase in our case, we have that for minimum uncertainty states $R < 2$ bits. To compensate for smaller values of R_ϕ and to be able to define coherent states in a four level system that are not Wigner-Dicke states, we introduce the parameter $\mu_2 \geq 1$, to obtain inequality:

$$R_S(\mu_2) \equiv \mu_2 R_\phi + R_m \leq 2 \tag{14}$$

over all states in $\mathbf{C}^4$. Our strategy is to numerically search over all states in this space, other than the Wiger-Dicke states, where $R_m = 2$ and $R_\phi = 0$, saturating the inequality (14) trivially– in order to determine the largest value of μ_2 such that this inequality is *just* satisfied. By this method, we find $\mu_2 = 1.973$.

An alternative formulation of Eq. (14), which we do not use here, is to rewrite the summation in the right hand side as a convex sum:

$$pR_\phi + (1-p)R_m < c, \tag{15}$$

where $p \equiv \mu_2(1+\mu_2)^{-1}$ and $c \equiv 2(1+\mu_2)^{-1}$, which can now be considered as a generalized uncertainty relation in the sense of [31]. For the present case, $p = 0.66$ and $c = 0.67$.

V. NUMBER-PHASE COMPLEMENTARITY IN CPT AND EIT SYSTEMS

By coherences, we mean the off-diagonal elements of ρ in the number representation, while by populations we mean the diagonal elements. Because, as we saw, a number state has a flat distribution of phase, so does a mixture of number states, which is represented by a purely diagonal density operator. Thus, non-vanishing phase knowledge implies that there is coherence in the state.

In this and the following section, we apply the ideas and tools of the preceding sections to the atomic system described in Section II. The density matrix calculations and the evaluation of the entropies was carried out for a wide range of values of parameters δc, δp, $\Omega_{33'}$, $\Omega_{23'}$, $\Omega_{34'}$.

A. The coupling-probe Λ-system

Initially we discuss the case of CPT (both lasers of equal strengths) and then EIT (lasers of very dissimilar strengths). To begin with, we omit the coupling $\Omega_{34'}$ and set coupling $\Omega_{23'} = \Omega_{33'} = 5$. The phase distribution $\mathcal{P}(\phi)$ for the case of probe and coupling being on resonance with their respective transitions, is non-uniform and peaks at $\phi = \pi$. Thus phase knowledge is non-vanishing, as expected. On the other hand, with the coupling laser at resonance, but with the probe detuned away from resonance (no CPT or coherence in ρ_{23}), the number knowledge remains roughly the same but the phase distribution tends to become uniform, suggestive of dephasing. By contrast, making the coupling laser much stronger at resonance produces a uniform phase distribution by creation of a number state under EIT conditions. These ideas are systematically presented in Figure 2.

Figure 2 is an array plot depicting the degree of mixedness $(1 - \text{Tr}\rho^2)$ (left column), phase knowledge (center column) and number knowledge (right column) as we proceed from CPT (top row) to EIT (bottom row). Each plot has detuning of the probe laser on the abscissa.

In each regime (row), we find that the system is in a pure state on resonance, and thus amenable to an interpretation in terms of complementarity. The following discussion pertains to the system under resonant condition. Comparing the different regimes, we find that coherence (phase knowledge) is largest in CPT and number knowledge the least, while the opposite is true in the EIT case. Further, the number knowledge being 2 bits in the latter case implies that the atom exists in a definite number state. This state can be readily identified with level $F = 2$, which is 'dark' under the stronger $3 - 3'$ laser. On the other hand, the number knowledge being 1 bit in the former case, with phase knowledge being large, implies that the atom exists in an equal weight state $\frac{1}{\sqrt{2}}(|2\rangle + e^{i\phi}|3\rangle)$.

Going off resonance, the situation changes considerably in the CPT case in that phase knowledge vanishes and mixedness becomes maximal, implying a transition from a coherent superposition to a statistical mixture. Note

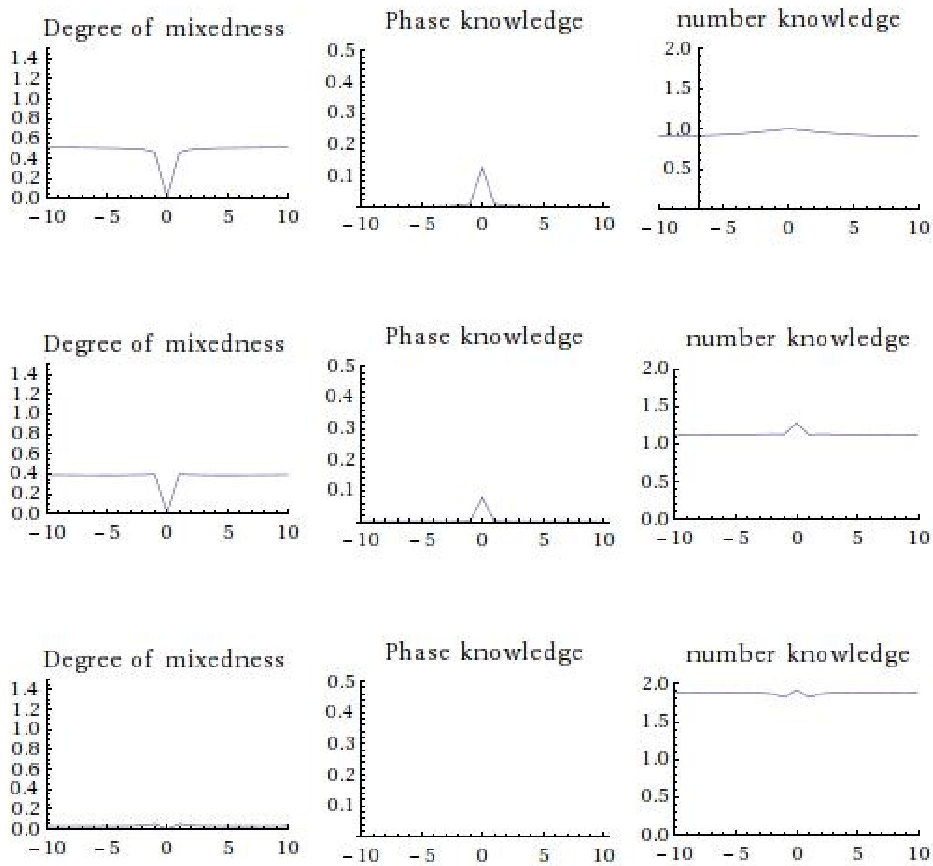

FIG. 2: The parameters are $\Omega_{33'}$ (coupling) = $\Omega_{23'}$ (probe) = 5 MHz for the first row, corresponding to a CPT state. For the second and third rows, the probe is 2.5 and 0.5 MHz, while coupling beam strength remains the same, corresponding to an intermediate and an EIT state, respectively. In all these figures, the x-axis represents ramping of the probe by means of detuning from level $3'$. The CPT position is at detuning δp= 1MHz of the Probe laser. On resonance, the figures show complementary behavior between number and phase. Going off resonance, phase knowledge is affected, but not number knowledge, implying a dephasing effect. This also explains why there is loss of purity in the CPT and intermediate case (first and second rows, where there is non-vanishing phase knowledge) but not in the EIT state (third row), where there is complete number knowledge (2 bits) and no phase knowledge.

however that the number knowledge remains the same, which signifies that the noise is of a dephasing kind, i.e., the phase ϕ gets randomized while populations remain unchanged.

Thus, in going down the central column of the figure, the mechanism of loss of phase knowledge on-resonance is the CPT-to-EIT transition, whereas off-resonance it is dephasing when the coupling and probe laser strengths are comparable. In the purely EIT case (bottom row), the off-resonance purity is attributed to optical pumping into the level 2.

B. Effect of higher order nonlinearity on phase knowledge

Thus far we had considered an atom under the action of two light fields, $\Omega_{2,3'}$ and $\Omega_{33'}$. However, the atom under consideration is a multi-level system with additional closely spaced excited levels– $4'$ and $2'$. The transition $3-4'$ is an order of magnitude more probable than $3-2'$. We therefore introduce a term in $\Omega_{34'}$ representing the off-resonant coupling $3-4'$ of the coupling laser to a nearby excited line. This results in a higher order nonlinearity, the effects of which are discussed below.

In the CPT case, the coupling $\Omega_{34'}$ disrupts the dark coherent superposition state formed during CPT by bringing about induced transitions to the state $4'$. This lowers the phase information as strength of the signal beam is increased

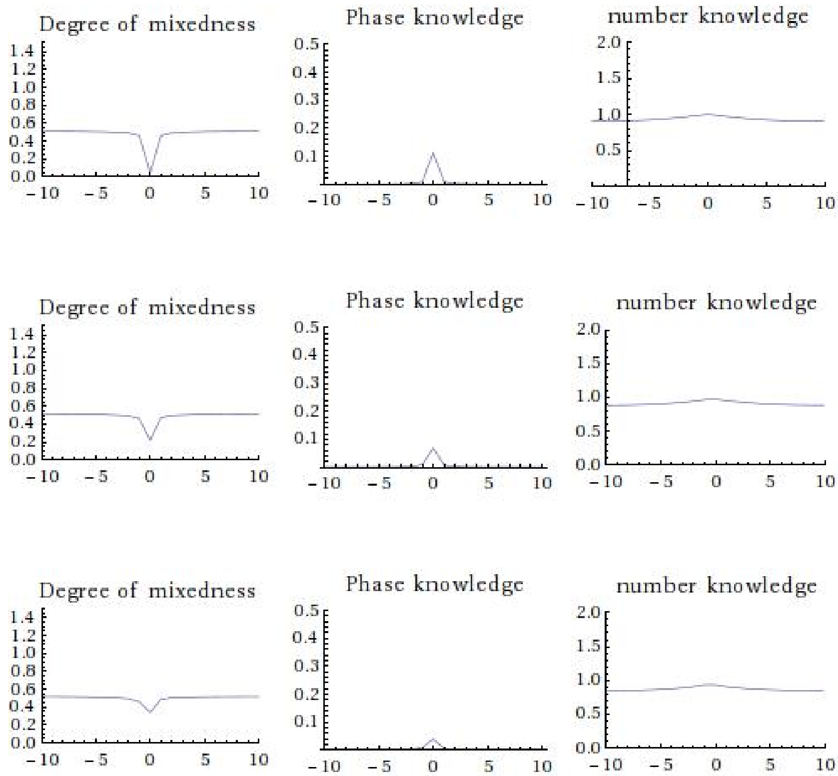

FIG. 3: The effect of the off-resonant coupling $3-4'$ (signal beam) on the CPT state of Figure 2. The coupling and probe beams, for all three rows, are taken at strengths 5 MHz. The signal beam strength is taken to be 0.5, 2.5 and 5 MHz respectively for the top, middle and bottom rows, respectively.

(second column of Fig. 3) as where there is a possibility of spontaneous emission. This is equivalent to measuring the CPT state in the basis $\{|F=2\rangle, |F=3\rangle\}$ with a probability determined by $\Omega_{34'}$, and thus equivalent to an application of a phase damping channel of strength determined by $\Omega_{34'}$. This has the effect of randomizing the phase in the CPT state, thereby decreasing phase knowledge but not affecting the number knowledge (third column of Fig. 3).

VI. RESULTS AND CONCLUSIONS

EIT and CPT states present two contrasting nonlinear phenomena in optics that illustrate the complementary behavior of number and phase in atomic systems. A conventional description of this complementarity, based on the non-commutativity of these two variables is not possible, as lower-boundedness prevents the possibility of phase as a Hermitian observable. One way out is to represent number by a continuous-variable POVM, described by a probability distribution for a given state. Complementarity can then be quantified, among other ways, by expressing the spread in the respective distributions by the entropy generated by measurement. The issue of employing the discrete-valued Pegg-Barnett phase operator instead of the POVM used here will be discussed elsewhere.

We have used entropic knowledge, rather than variances, to describe uncertainty in n and ϕ. We find that in CPT, the coherence between the ground states participating in the dark state is reflected in large phase knowledge and about 1 bit of number knowledge. In EIT, where the dark state is a number state, number knowledge is maximal (2 bits), while phase knowledge vanishes. Thus on-resonance, we see a clear manifestation of number-phase uncertainty. Off-resonance, there is a general reduction in coherence, and, in the CPT case, a reduction in phase knowledge due to dephasing. A similar phase damping effect is seen also when a higher order nonlinearity is introduced by allowing for $3 - 4'$ transitions.

[1] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, (1968) 411.
[2] M. J. W. Hall, Quantum Opt. **3**, (1991) 7.
[3] G. S. Agarwal and R. P. Singh, Phys. Lett. A **217**, (1996) 215.
[4] Pg 47, Section 3.5, Hans-A. Bachor and Timothy C.Ralph, *A guide to experiments in Quantum Optics* (Wiley-VCH, 2004).
[5] A. Dantan *et.al.*, Phys. Rev. A **67** , 045801 (2003).
[6] A. Narayanan, A. Sharma, T. M. Preethi, H. Abheera, and H. Ramachandran, Can. J. Phys.,**87** (7), pg 843-850 (2009).
[7] H. Schmidt and A. Imamoglu, Optics Letters **21**, 1936 (1996).
Press-1989).
[8] H. Kang and Y. Zhu, Phys. Rev. Lett. **91** ,093601 (2003).
[9] Archana Sharma, Multiphoton process due to XPM generated nonlinearities (to be published).
[10] A. Narayanan, R. Srinivasan, U. K. Khan, A. Vudaygiri and H. Ramachandran, Eur. Phys. J. D **31**,107-112 (2004).
[11] S. Banerjee, J. Ghosh and R. Ghosh, Phys. Rev. A **75**, (2007) 062106; eprint quant-ph/0703055.
[12] *Quantum Phase and Phase Dependent Measurements*, Eds. W. P. Schleich and S. M. Barnett, Phys. Scr. (Special issue) **T48**, (1993) 1-144.
[13] V. Perinova, A. Luks and J. Perina, *Phase in Optics* (World Scientific, Singapore 1998).
[14] P. A. M. Dirac, Proc. R. Soc. Lond. A **114**, (1927) 243.
[15] L. Susskind and J. Glogower, Physics **1**, (1964) 49.
[16] D. T. Pegg and S. M. Barnett, J. Mod. Opt. **36**, (1989) 7; Phys. Rev. A **39**, (1989) 1665.
[17] J. H. Shapiro, S. R. Shepard and N. C. Wong, Phys. Rev. Lett. **62**, (1989) 2377.
[18] J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, (1991) 3795.
[19] G. S. Agarwal, S. Chaturvedi, K. Tara and V. Srinivasan, Phys. Rev. A **45**, (1992) 4904.
[20] M. A. Rashid, J. Math. Phys. **19**, (1978) 1391.
[21] G. S. Agarwal and R. R. Puri, Phys. Rev. A **41**, (1990) 3782.
[22] F. T. Arecchi, E. Courtens, R. Gilmore and H. Thomas, Phys. Rev. A **6**, (1972) 2211.
[23] S. Banerjee and R. Srikanth, Phys. Rev. A.**76**, (2007) 062109; eprint arXiv:0706.3633.
[24] K. Kraus, Phys. Rev. D **35**, (1987) 3070.
[25] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, (1988) 1103.
[26] D. Deutsch, Phys. Rev. Lett. **50**, (1983) 631.
[27] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge 2000).
[28] A. Galindo, M. A. Martin-Delgado, Rev. Mod. Phys. **74**, (2000) 347.
[29] R. Srikanth and S. Banerjee, Eur. Phys. J. D. **53**, (2009) 217; eprint arXiv:0711.0875.
[30] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North Holland 1982).
[31] S. Wehner and A. Winter, Eprint arXiv:0907.3704.