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Asymptotic symmetries and subleading soft graviton theorem

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Motivated by the equivalence between the soft graviton theorem and Ward identities for the supertranslation symmetries belonging to the Bondi, van der Burg, Metzner and Sachs (BMS) group, we propose a new extension (different from the so-called extended BMS) of the BMS group that is a semidirect product of supertranslations and Diff(S^2). We propose a definition for the canonical generators associated with the smooth diffeomorphisms and show that the resulting Ward identities are equivalent to the subleading soft graviton theorem of Cachazo and Strominger.

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I. INTRODUCTION

It has been known since the 1960s that there is an infinite dimensional symmetry group underlying asymptotically flat spacetimes known as the Bondi, van der Burg, Metzner and Sachs (BMS) group [1,2]. The role of the BMS group in quantum theory was elucidated in a series of remarkable papers by Ashtekar *et al.* [3–5]. In [3] the radiative modes of the full nonlinear gravitational field were isolated and equipped with a symplectic structure, thus paving the way for (asymptotic) quantization of gravity. In [4], it was shown that the BMS group is a dynamical symmetry group of the radiative phase space and the corresponding Hamiltonians were obtained. The reasons behind the enlargement of the translation subgroup (of the Poincaré group) to supertranslations was clarified in [5], where it was shown that the space of “vacuum configurations” (i.e., points in phase space for which the fluxes of all BMS momenta vanish identically) are in one-to-one correspondence with supertranslations (modulo translations). This in turn led to the first detailed relation between the BMS supertranslations and the infrared issues in quantum gravity [6,7]. In particular, it clarified the need to use coherent states that lead to an S matrix free of infrared divergences [8,9].

In recent months there has been a renewed interest in analyzing these symmetries in the context of the quantum gravity S matrix. There are two reasons for this resurgence. First is a series of fascinating papers by Strominger *et al.* [10–12] where a precise relationship between Ward identities associated with supertranslation symmetries and Weinberg’s soft graviton theorem [13] was unraveled. The second reason is an extremely interesting proposal by Barnich and Troessaert [14–16] that this symmetry can be naturally extended to include the Virasoro group, which in turn may shed new light on duality between quantum gravity in the bulk and conformal field theory on the boundary. In the literature this group is referred to as the extended BMS group.

The two ideas mentioned above converged in [17] where it was shown that the Ward identities associated with precisely such Virasoro symmetries follow from the so-called subleading soft theorem for gravitons. This theorem, conjectured by Strominger, was proved at tree level in the so-called holomorphic soft limit in [18], where its validity was also checked in a number of examples. A more general proof for the theorem was later given in [19,20]. See [21–25] for earlier works on soft graviton amplitudes and [26–33] for an incomplete list of recent related works.

However, as noted in [17], whereas for the supertranslation symmetries the Ward identities are, in fact, equivalent to Weinberg’s soft graviton theorem, such an equivalence could not be established as far as the Virasoro symmetries and the subleading theorem were concerned. Motivated by the need to establish such an equivalence, in this paper we propose a different extension of the BMS group. Instead of extending the global conformal symmetries to the Virasoro symmetries as in [14], we extend them to smooth vector fields on the sphere. We refer to this group as the generalized BMS group and denote it by **G**. We show that **G** is the semidirect product of supertranslations with smooth diffeomorphisms of the conformal sphere [Diff(S^2)] and that it preserves the space of asymptotically flat solutions to Einstein’s equations. However, contrary to the BMS group, it does not preserve the leading order *kinematical* metric components, for instance, by generating arbitrary diffeomorphisms of the conformal sphere at infinity. We define charges associated with this symmetry [Diff(S^2)] in the radiative phase space of the gravitational field. Our definition of these charges is motivated by the charges one obtains for extended BMS symmetry. Although this definition is *ad hoc* and not derived by systematic analysis, we show its associated “Ward identities” are in one-to-one correspondence with the subleading soft graviton theorem. The analysis performed here is

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rather similar in spirit to the recent work by Lysov, Pasterski, and Strominger for massless QED [34]. Exactly as in that case, our charges do not form a closed algebra. We leave the interpretation of this nonclosure for future investigations.

The outline of this paper is as follows. In Sec. II we define **G** and show that it preserves asymptotic flatness. We show **G** can be characterized as the group of diffeomorphism that preserves null infinity and is asymptotically volume preserving. In Sec. III we review the radiative phase space formulation of Ashtekar and show how the action of extended BMS is Hamiltonian.[1] We emphasize using the radiative phase space framework carefully since, as illustrated in Appendix A, the weak nondegeneracy of the symplectic structure implies that certain seemingly natural Poisson bracket relations are ill defined and their use can lead to incorrect results.

Just as the BMS group can be defined purely in terms of structures available at null infinity without referring to spacetime, we present **G** from the perspective of null infinity in Sec. IV. In this section we also present our prescription for the Hamiltonian action of the generators of **G** on the radiative phase space of gravity. In Sec. V we analyze the Ward identities associated with this prescription and show their equivalence with the subleading soft graviton theorem.

II. SPACETIME PICTURE

A. Proposal for a generalization of the BMS group

Let us for concreteness focus on future null infinity $\mathcal{I}^+$. Following [11] we refer to the algebra of asymptotic symmetries at $\mathcal{I}^+$ as BMS^+. In the original derivation of BMS algebra, through an interplay between falloff conditions and Einstein equations, one arrives at the following form of asymptotically flat metrics (we take expressions from [11,15]):

$$\begin{aligned} ds^2 = {} & (1+O(r^{-1}))du^2 - (2+O(r^{-2}))dudr \\ & + (r^2 q_{AB} + rC_{AB}(u,\hat{x}) + O(1))dx^A dx^B \\ & + O(1)dx^A du. \end{aligned} \tag{1}$$

Here x^A are coordinates on the 2-sphere, q_{AB} is the round unit sphere metric (whose covariant derivative we denote by D_A), and $\hat{x}$ denotes points on the sphere. C_{AB} is trace-free and unconstrained by Einstein equations, whereas the remaining metric components are determined by C_{AB} through Einstein's equations. C_{AB} is referred to as the free gravitational radiative data.

BMS^+ is defined as the algebra of vector fields that preserve the falloffs (1). It is generated by vector fields of the (asymptotic) form,

$$\xi_+^a \partial_a = V_+^A \partial_A + u\alpha_+ \partial_u - r\alpha_+ \partial_r + f_+ \partial_u + \lambda_+^a \partial_a, \tag{2}$$

where V_+^A is a conformal Killing vector field (CKV) of the sphere, $\alpha_+ = (D_A V_+^A)/2$, and $f_+ = f_+(\hat{x})$ is any smooth function on the sphere. $\lambda_+^a \partial_a$ denotes the next terms in the $1/r$ expansion [15],

$$\lambda_+^a \partial_a = -\frac{u}{r} D^A \alpha \partial_A + \frac{u}{2} D_B D^B \alpha \partial_r + \cdots. \tag{3}$$

One can similarly define the algebra BMS^- of asymptotic symmetries associated with past null infinity $\mathcal{I}^-$.

In [11] Strominger introduced the remarkable notion of $\mathrm{BMS}^0 \subset \mathrm{BMS}^+ \times \mathrm{BMS}^-$, which he argued to be a symmetry of quantum gravity S matrix. This group maps incoming scattering data, characterized by fields on $\mathcal{I}^-$, to outgoing scattering data, characterized by fields on $\mathcal{I}^+$, while conserving total energy. Identifying the null generators of $\mathcal{I}^+$ and $\mathcal{I}^-$ through $\mathcal{I}^+|_{u=-\infty} = \mathcal{I}^-|_{v=+\infty} = i^0$, the group is defined by the conditions [11]

$$V_+^A(\hat{x}) = V_-^A(\hat{x}), \qquad f_+(\hat{x}) = f_-(\hat{x}). \tag{4}$$

We now consider the scenario where in ξ_+^a given in (2), V_+^A is not CKV. A simple computation reveals that under the diffeomorphisms generated by such vector fields, the metric coefficients whose falloffs are violated are

$$\mathcal{L}_{\xi^+} g_{AB} = O(r^2), \mathcal{L}_{\xi_+} g_{uu} = O(1). \tag{5}$$

Thus, relaxing the CKV condition forces us to consider metrics where the $O(r^2)$ part of g_{AB} is not necessarily the round metric and such that $g_{uu} = O(1)$. We are thus led to consider metrics of the form[2]

$$\begin{aligned} ds^2 = {} & O(1)du^2 - (2+O(r^{-2}))dudr \\ & + (r^2 q_{AB} + O(r))dx^A dx^B + O(1)dx^A du, \end{aligned} \tag{6}$$

with q_{AB} no longer the standard metric on S^2. We can now ask if these spacetimes with more general falloffs of the metric coefficients are asymptotically flat. As shown in [7] the answer is in the affirmative. This can most easily be seen from the conformal description of asymptotic flatness. In this description, asymptotic flatness is captured by the existence of a conformal factor Ω such that $\Omega^2 ds^2$ has a well defined limit at null infinity and satisfies a number of properties. It can be shown that such spacetimes admit coordinates in a neighborhood of null infinity for which the metric falloffs include those of the form (6), with $\Omega \sim 1/r$ [7,35].

We refer to this group of asymptotic symmetries at future null infinity as the generalized BMS^+ group and denote it

[1] Modulo certain subtleties related to the IR sector.

[2] The form (6) is of the type of metrics considered in [15] except that we take q_{AB} to be u independent and we do not require q_{AB} to be a conformal rescaling of the unit round metric.

by $\mathbf{G}^+$. $\mathbf{G}^+$ is a semidirect product of supertranslations and $\mathrm{Diff}(S^2)$, with supertranslations being a normal Abelian subgroup exactly as in the case of the BMS group.

One can similarly define a corresponding group associated with $\mathcal{I}^-$, and we refer to it as $\mathbf{G}^-$. Following the strategy used for the BMS [11] and extended BMS [17] cases, we define the subgroup $\mathbf{G}^0$ of $\mathbf{G}^+ \times \mathbf{G}^-$ by the identification (4) for generators of $\mathbf{G}^+$ and $\mathbf{G}^-$. It then follows that $\mathbf{G}^0$ reduces to BMS^0 when V^A is CKV.

From now on we drop the labels $+,-$ and parametrize the generalized BMS vector fields by (V^A, f).

B. Characterization of G

We now ask if there is any geometrical characterization of the generalized BMS vector fields. Recall that BMS vector fields can be characterized as asymptotic Killing vector fields,

$$\nabla_{(a}\xi_{b)} \to 0 \quad \text{as } r \to \infty. \tag{7}$$

Whereas generalized BMS clearly do not satisfy this condition, it turns out they are asymptotic divergence-free vector fields,

$$\nabla_a \xi^a \to 0 \quad \text{as } r \to \infty. \tag{8}$$

Indeed, a simple calculation shows

$$\nabla_a \xi^a = \partial_u \xi^u + D_A V^A + \frac{1}{r^2}\partial_r(r^2 \xi^r) + O(r^{-1}) \tag{9}$$

$$= \alpha + 2\alpha - 3\alpha + O(r^{-1}) \tag{10}$$

$$= O(r^{-1}). \tag{11}$$

We now show the converse, namely that generalized BMS vector fields are characterized by (8) and the preservation of the falloffs (6). A general vector field preserving $\mathcal{I}$ has the following general form as $r \to \infty$:

$$\xi^a = \overset{\circ}{\xi}{}^A(u,\hat{x})\partial_A + \overset{\circ}{\xi}{}^u(u,\hat{x})\partial_u + r\overset{\circ}{\xi}{}^r(u,\hat{x})\partial_r + \cdots, \tag{12}$$

where the dots indicate terms of the form $O(r^{-1})\partial_A + O(r^{-1})\partial_u + O(1)\partial_r$. We only focus on the leading terms in the $1/r$ expansion. Subleading terms are determined by requiring preservation of the falloffs (6), and their forms depend on the specific metric coefficients in (6). Equation (8) gives

$$\nabla_a \xi^a = O(r^{-1}) \Leftrightarrow D_A \overset{\circ}{\xi}{}^A + \partial_u \overset{\circ}{\xi}{}^u + 3\overset{\circ}{\xi}{}^r = 0. \tag{13}$$

The components of (6) leading to restrictions on the leading part of (12) are

$$\mathcal{L}_\xi g_{ur} = O(r^{-1}) \Leftrightarrow \partial_u \overset{\circ}{\xi}{}^u + \partial_r \overset{\circ}{\xi}{}^r = 0, \tag{14}$$

$$\mathcal{L}_\xi g_{uA} = O(r), \quad \mathcal{L}_\xi g_{uu} = O(1) \Leftrightarrow \partial_u \overset{\circ}{\xi}{}^A = 0, \quad \partial_u \overset{\circ}{\xi}{}^r = 0. \tag{15}$$

It is easy to verify that the most general solution to Eqs. (13), (14), and (15) is given by

$$\overset{\circ}{\xi}{}^A(u,\hat{x}) = V^A(\hat{x}), \tag{16}$$

$$\overset{\circ}{\xi}{}^u(u,\hat{x}) = u\alpha(\hat{x}) + f(\hat{x}), \tag{17}$$

$$\overset{\circ}{\xi}{}^r(u,\hat{x}) = -\alpha(\hat{x}), \tag{18}$$

with $V^A(\hat{x})$ and $f(\hat{x})$ undetermined and $\alpha = (D_A V^A)/2$ as before. Thus, we recover the leading term of (2) with the CKV condition on V^A dropped. This precisely represents the proposed generalized BMS vector fields. The preservation of (12) for the remaining metric components impose conditions on the subleading terms of ξ^a indicated by the dots in (12).

C. Difficulties in extracting a map on radiative data

We recall that BMS vector fields have a well defined action on the unconstrained radiative data characterized by C_{AB}. For ξ^a as in (2) with $f = 0$ the action is given by [11]

$$\delta_V C_{AB} = \mathcal{L}_V C_{AB} - \alpha C_{AB} + \alpha u \partial_u C_{AB}. \tag{19}$$

Although generalized BMS vector fields map an asymptotically flat spacetime to another one, they do not induce any obvious map on the free radiative data. As they change the zeroth order structure, the linear in r coefficients of g_{AB} do not represent all free data.

To bring out the differences with the BMS case, consider the action of the generalized BMS vector field on the g_{AB} metric components [again we consider the case $f = 0$ and g_{ab} as in (1)],

$$\begin{aligned}\mathcal{L}_\xi g_{AB} = {}& r^2(\mathcal{L}_V q_{AB} - 2\alpha q_{AB}) \\ &+ r(\mathcal{L}_V C_{AB} - \alpha C_{AB} + \alpha u \partial_u C_{AB}) + urs_{AB},\end{aligned} \tag{20}$$

where

$$s_{AB} = -2D_A D_B \alpha + D_C D^C \alpha q_{AB}. \tag{21}$$

Since the zeroth order structure changes, the action of generalized BMS encodes the physical transformations (i.e., change in the radiative data) as well as "gauge transformations" induced by the change in the zeroth order structure. It is not clear to us how to extract out the gauge-invariant change in the news from this action. This point will be important in defining the action of the generalized BMS operator in quantum theory. We return to this issue in Sec. IV C.

III. RADIATIVE PHASE SPACE

A. Review of Ashtekar formulation

In this section we recall Ashtekar's description of the radiative phase space of gravity following mostly Refs. [4,7]. We will present only the end result of the description and encourage the reader to look at [3,4,7] for its motivation from the spacetime perspective, as well as Ref. [36] for its relation with canonical phase space.

The idea is to start with $\mathcal{I}$ (which will eventually stand for either future or past null infinity) as an abstract 3-manifold, topologically $S^2 \times \mathbb{R}$, and ruled by preferred directions or "rays" so that there is a canonical projection $\mathcal{I} \to \hat{\mathcal{I}} \sim S^2$ with $\hat{\mathcal{I}}$ the space of rays. Next, one endows $\mathcal{I}$ with a "universal structure" that plays the role of a kinematical arena. This universal structure is given by an equivalence class of pairs (q_{ab}, n^a) where n^a is a vector field tangent to the rays and q_{ab} a $(0,+,+)$ degenerate metric that is given by the pullback of a 2-metric $\hat{q}_{ab}$ on $\hat{\mathcal{I}}$, so that $q_{ab}n^b = 0$ and $\mathcal{L}_n q_{ab} = 0$. Each pair is referred to as a "frame." The equivalence is given by

$$(q_{ab}, n^a) \sim (\omega^2 q_{ab}, \omega^{-1} n^a), \quad \forall\ \omega : \mathcal{I} \to \mathbb{R} : \mathcal{L}_n \omega = 0, \tag{22}$$

and the corresponding equivalence class $[(q_{ab}, n^a)]$ gives the universal structure. The BMS group discussed in the previous section arises in this context as the group of diffeomorphism of $\mathcal{I}$ that preserve this universal structure [7].

We now describe the dynamical degrees of freedom and associated phase space. The description uses a fixed frame $(q_{ab}, n^a) \in [(q_{ab}, n^a)]$, so that, strictly speaking, one arrives at a family of phase spaces parametrized by the frames $(q_{ab}, n^a) \in [(q_{ab}, n^a)]$. One then shows that there exists a natural isomorphism between the different phase spaces associated with the different frames. Below we present the phase space associated with a given frame. The isomorphism, crucial for the implementation of boosts in phase space, is described in Appendix B.

A derivative operator D_a on $\mathcal{I}$ is said to be compatible with a frame (q_{ab}, n^a) if it satisfies

$$D_c q_{ab} = 0, \qquad D_a n^b = 0,$$
$$2D_{(a}V_{b)} = \mathcal{L}_V q_{ab} \quad \text{if } V_a n^a = 0, \tag{23}$$

where the Lie derivative is along any vector V^a satisfying $V_a = q_{ab}V^b$. Introduce the following equivalence relation on derivative operators satisfying (23)[3]:

$$D'_a \sim D_a \quad \text{if } D'_a k_b = D_a k_b + f n^c k_c q_{ab} \tag{24}$$

for some function f. The phase space, denoted by Γ, is the space of equivalence classes $[D_a]$ of (torsion-free) derivative operators satisfying (23). A parametrization of this space is obtained as follows. Fix a derivative $\mathring{D}_a$ satisfying (23). It can be shown that any other derivative D_a satisfying (23) is given by

$$D_a k_b = \mathring{D}_a k_b + (\Sigma_{ab} n^c) k_c, \tag{25}$$

where Σ_{ab} is symmetric and satisfies $\Sigma_{ab} n^b = 0$. Such tensors parametrize the space of connections D_a compatible with (q_{ab}, n^a). From (24) it follows that

$$\sigma_{ab} \coloneqq \Sigma_{ab} - \frac{1}{2} q_{ab} q^{cd} \Sigma_{cd} = ((D_a - \mathring{D}_a) k_b)^{\mathrm{TF}}$$
$$\text{for any } k_b : n^b k_b = 1 \tag{26}$$

can be used to parametrize the space of equivalence classes $[D_a]$. We recall that q^{ab} is defined up to $v^{(a} n^{b)}$ so that the trace-free symbol "TF" is only well defined on tensors annihilated by n^a. In terms of this parametrization the symplectic structure reads

$$\Omega(\sigma_1, \sigma_2) = \int d^3 V q^{ac} q^{bd} (\sigma^1_{ab} \mathcal{L}_n \sigma^2_{cd} - \sigma^2_{ab} \mathcal{L}_n \sigma^1_{cd}), \tag{27}$$

where $d^3 V = \epsilon_{abc}$, with $\epsilon_{ab} = \epsilon_{abc} n^c$ the area form of q_{ab}.

Let us now make contact with the spacetime picture of Sec. II. For concreteness we focus on future null infinity. For spacetime metrics as in (1), $\mathcal{I}$ is described by the coordinates (u, x^A) with $n^a \partial_a = \partial_u$ and $q_{ab} dx^a dx^b = q_{AB} dx^A dx^B$. One can verify that the nonzero components of σ_{ab} are $\sigma_{AB} = (1/2) C_{AB}$. The news tensor is then given by[4]

$$N_{AB}(u, \hat{x}) = -2 \dot{\sigma}_{AB}(u, \hat{x}), \tag{28}$$

where $\dot{\sigma}_{AB} \equiv \mathcal{L}_n \sigma_{AB} \equiv \partial_u \sigma_{AB}$.

We conclude by describing the falloffs of radiative phase space. In (u, x^A) coordinates they are given by [4]

$$\sigma_{AB}(u, \hat{x}) = \sigma^{\pm}_{AB}(\hat{x}) + O(u^{-\epsilon}) \quad \text{as } u \to \pm\infty, \tag{29}$$

where $\epsilon > 0$ and the limiting values $\sigma^{\pm}_{AB}(\hat{x})$ are kept unspecified (but smooth in $\hat{x}$). These falloffs ensure the convergence of the integral defining the symplectic structure (27).[5]

[3]This equivalence relation is unrelated to the one in (22). From the spacetime perspective, (22) arises from different values the conformal factor Ω can take at $\mathcal{I}$, whereas (23) arises from different values the derivative of the conformal factor (along directions off $\mathcal{I}$) can take at $\mathcal{I}$.

[4]Our convention for the news tensor, taken from [4], differs by a sign from that used in [12,17].

[5]The falloffs used by Strominger based on the analysis of CK spaces corresponds to $\epsilon = 1/2$ [11]. It thus seems that the range $0 < \epsilon < 1/2$ is not relevant for gravitational scattering. We nevertheless keep ϵ general as all we need in our analysis is $\epsilon > 0$.

1. Poisson brackets subtleties

We comment on a subtlety associated with the Poisson brackets (PBs) that was noticed in [12]. From the radiative phase space perspective the symplectic form (27) is the fundamental structure, whereas Poisson brackets are derived quantities. We recall that in this approach the Hamiltonian vector field (HVF) X_F of a phase space function F is defined as the solution to the equation

$$\Omega(X_F, \cdot) = dF, \tag{30}$$

and that, given two phase space functions F and G admitting HVFs, their Poisson bracket is defined by $\{F, G\} \coloneqq \Omega(X_F, X_G) = X_G(F)$. In [4] it is shown that Ω is weakly nondegenerate; that is, Ω considered as a map from $T\Gamma$ to $T^*\Gamma$ is injective but not necessarily surjective. Thus, there is no guarantee that one can always solve Eq. (30) (but if there is a solution, it is unique). As discussed in Appendix A, an example of a function not admitting a HVF is given by $F[\sigma] \coloneqq \int_{\mathcal{I}} d^3V F^{AB}(u,\hat{x})\sigma_{AB}(u,\hat{x})$ with $\int_{-\infty}^{\infty} du F^{AB}(u,\hat{x}) \neq 0$. In particular, one cannot define PBs between $\sigma_{AB}(u,\hat{x})$ and $\sigma_{AB}(u',\hat{x}')$. Fortunately, these "undefined PBs" are nowhere needed in the analysis.

B. (Extended) BMS action on Γ

Let D_a be a connection as in (23) with $[D_a]$ the corresponding element in radiative phase space. Under the action of a BMS vector field ξ^a the connection changes by $\delta_\xi D_a = [\mathcal{L}_\xi, D_a]$. If ξ^a preserves the frame (case of supertranslations and rotations), the transformed connection $D'_a \approx D_a + \delta_\xi D_a$ is compatible with the frame and one can directly read off the phase space action from $\delta_\xi D_a$. For boosts, however, the transformed connection is compatible with the frame $(q'_{ab}, n'^a) \approx ((1+2\alpha)q_{ab}, (1-\alpha)n^a)$. One thus needs to use the isomorphism between the phase spaces associated with the different frames in order to obtain the phase space action. The resulting action reads (see Appendix B for its derivation)

$$(X_\xi)_{ab} = ([\mathcal{L}_\xi, D_a]k_b + 2k_{(a}\partial_{b)}\alpha)^{\mathrm{TF}}, \tag{31}$$

where k_a is any covector satisfying $n^a k_a = 1$.

In (u, x^A) coordinates, for a "pure rotation/boost" vector field

$$\xi^a\partial_a = V^A\partial_A + u\alpha\partial_u, \tag{32}$$

the expression takes the form

$$(X_V)_{AB} = \mathcal{L}_V\sigma_{AB} - \alpha\sigma_{AB} + u\alpha\dot{\sigma}_{AB} - u(D_A D_B\alpha)^{\mathrm{TF}}. \tag{33}$$

Following [12,17], we refer to the piece linear in σ as the "hard term" and the σ-independent, linear in u piece as the "soft term." The soft term appears to violate the falloffs (29). However, the CKV nature of V^A implies $(D_A D_B\alpha)^{\mathrm{TF}}$ vanishes.

The above analysis goes through if we replace V^a by a local CKV so that ξ^a represents a generator of the extended BMS group. In this case, however, the soft term does not vanish. In $(z, \bar{z})$ coordinates the action takes the form

$$(X_V)_{zz} = \mathcal{L}_V\sigma_{zz} - \alpha\sigma_{zz} + u\alpha\dot{\sigma}_{zz} - \frac{u}{2}D_z^3 V^z, \tag{34}$$

where we used the fact that $D_z D_z(D_{\bar{z}}V^{\bar{z}}) = 0$ for local CKV. A similar expression holds for the $\bar{z}\bar{z}$ component. In quantum theory, the action (34) is generated by the charge $Q = Q_H + Q_S$ given in Eq. (5.10) of [17].

C. Mode functions

In this section we describe the classical functions in radiative phase space that correspond to the standard creation/annihilation operators of gravitons in quantum theory. These are essentially given by the zz and $\bar{z}\bar{z}$ components of the Fourier transform of σ_{AB},

$$\sigma_{AB}(\omega, \hat{x}) \coloneqq \int_{-\infty}^{\infty} \sigma_{AB}(u,\hat{x})e^{i\omega u}du. \tag{35}$$

As long as $\omega \neq 0$, (35) admits a HVF,[6] and hence we can find their PBs. The nonvanishing PBs are found to be[7]

$$\{\sigma_{zz}(\omega, z, \bar{z}), \sigma_{\bar{z}\bar{z}}(\omega', z', \bar{z}')\} = \frac{\pi}{i\omega}\sqrt{\gamma}\delta(\omega+\omega')\delta^{(2)}(z-z'). \tag{36}$$

For later purposes, we note that the relation of the mode functions (35) with the Fourier transform of the news tensor (28) is given by

$$\sigma_{AB}(\omega, \hat{x}) = (2i\omega)^{-1}N_{AB}(\omega, \hat{x}). \tag{37}$$

Following Secs. 5 of [12] and 5.3 of [17] (see also [36–38]), we can find the relation of (35) with the creation/annihilation functions from standard perturbative gravity. Following [17] we take coordinates in past null infinity that are antipodally related to those of future null infinity. In that case the expressions relating "in" quantities take the same form as the expressions relating "out" quantities. The following discussion thus applies to either case.

The "annihilation function" $a_\pm(\omega, \hat{x})$, $\omega > 0$, of a helicity ± 2 graviton is found to be given by

[6] In a distributional sense; strictly speaking one should integrate (35) with a smearing function in $(\omega, \hat{x})$ with support outside $\omega = 0$.

[7] In the present subsection as well as in Sec. V, $\gamma_{z\bar{z}} \equiv q_{z\bar{z}} = \sqrt{\gamma} = 2(1+z\bar{z})^{-2}$.

$$a_+(\omega,\hat{x}) = \frac{4\pi i}{\sqrt{\gamma}}\sigma_{zz}(\omega,\hat{x}), \qquad a_-(\omega,\hat{x}) = \frac{4\pi i}{\sqrt{\gamma}}\sigma_{\bar{z}\bar{z}}(\omega,\hat{x}). \tag{38}$$

Since $\bar{\sigma_{zz}}(\omega) = \bar{\sigma}_{\bar{z}\bar{z}}(\omega) = \sigma_{\bar{z}\bar{z}}(-\omega)$, the relations for the "creation functions" have the opposite relation between helicity and holomorphic components,

$$a_+(\omega,\hat{x})^\dagger = -\frac{4\pi i}{\sqrt{\gamma}}\sigma_{\bar{z}\bar{z}}(-\omega,\hat{x}),$$
$$a_-(\omega,\hat{x})^\dagger = -\frac{4\pi i}{\sqrt{\gamma}}\sigma_{zz}(-\omega,\hat{x}), \tag{39}$$

where in the present classical context, the dagger just means complex conjugation. The Poisson bracket (36) implies

$$\{a_h(\omega,\hat{x}), a_{h'}(\omega',\hat{x}')^\dagger\} = \frac{2(2\pi)^3}{i\omega\sqrt{\gamma}}\delta_{hh'}\delta(\omega-\omega')\delta(\hat{x},\hat{x}'), \tag{40}$$

and corresponds to the Poisson brackets the functions have from the perspective of perturbative gravity: $\{a^h_{\vec{p}}, (a^{h'}_{\vec{q}})^\dagger\} = -i2E_{\vec{p}}\delta_{hh'}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$, with $\vec{p} = \omega\hat{x}$ and $\vec{q} = \omega'\hat{x}'$.

D. Action of BMS on mode functions

The action of BMS on the mode functions can be obtained by taking the Fourier transform of (31). Here we are interested in rotations and boost, so we focus attention on the action of a BMS vector field of the form of (32). Taking the Fourier transform of (33) one finds

$$X_V(\sigma_{AB}(\omega,\hat{x})) = \mathcal{L}_V\sigma_{AB}(\omega,\hat{x}) - 2\alpha\sigma_{AB}(\omega,\hat{x}) - \alpha\omega\partial_\omega\sigma_{AB}(\omega,\hat{x}). \tag{41}$$

From (38) and (39) one can verify that the corresponding action on the creation/annihilation functions is given by the differential operator

$$J^h_V \coloneqq V^z\partial_z + V^{\bar{z}}\partial_{\bar{z}} - \frac{1}{2}(D_zV^z + D_{\bar{z}}V^{\bar{z}})\omega\partial_\omega + \frac{h}{2}(\partial_zV^z - \partial_{\bar{z}}V^{\bar{z}}), \tag{42}$$

according to

$$X_V(a_h(\omega,z,\bar{z})) = J^h_V a_h(\omega,z,\bar{z});$$
$$X_V(a_h(\omega,z,\bar{z})^\dagger) = J^{-h}_V a_h(\omega,z,\bar{z})^\dagger. \tag{43}$$

In quantum theory, J^h_V represents the total angular momentum of a helicity $h = \pm 2$ graviton.

IV. GENERALIZED BMS AND RADIATIVE PHASE SPACE

A. Intrinsic characterization of generalized BMS group

From the perspective of null infinity, the proposed generalized BMS vector fields ξ^a are given by supertranslations and vector fields of the form (32) with the CKV condition on V^A dropped. The dropping of the CKV condition implies that ξ^a does not preserve the universal structure $[(q_{ab}, n^a)]$ described in Sec. III A. It is natural to ask whether there is any other geometrical structure that is kept invariant under the action of generalized BMS. As we now show, such a geometrical structure is given by equivalence classes of pairs $[(\epsilon_{abc}, n^a)]$ with n^a as before, ϵ_{abc} the volume form satisfying $\mathcal{L}_n\epsilon_{abc} = 0$, and the equivalence relation given by

$$(\epsilon_{abc}, n^a) \sim (\omega^3\epsilon_{abc}, \omega^{-1}n^a), \quad \forall\ \omega: \mathcal{I} \to \mathbb{R}: \mathcal{L}_n\omega = 0. \tag{44}$$

First, we notice that any generalized BMS vector field still satisfies $\mathcal{L}_\xi n^a = -\alpha n^a$, whereas its action on the volume form is [4,7]

$$\mathcal{L}_\xi\epsilon_{abc} = 3\alpha\epsilon_{abc}; \tag{45}$$

hence it keeps the pair (ϵ_{abc}, n^a) in the same equivalence class (44). Conversely, one can verify that the group of symmetries of $[(\epsilon_{abc}, n^a)]$ is given by the generalized BMS group. This can be shown along the same lines as the proof given for the BMS case [7]. One finds that supertranslations are again a normal subgroup, and the quotient group is now the group of diffeomorphisms on the sphere.

B. An example: Action on radiative phase space of a massless scalar field

One example of a radiative phase space where the underlying kinematical structure is provided by the (equivalence class) of pairs $[(\epsilon_{abc}, n^a)]$ is that of a massless scalar field [4]. As we show below, in this case it is indeed true that the generalized BMS group has a symplectic action.

The symplectic structure of the radiative phase space Γ_ϕ of a massless scalar field ϕ is given by [4]

$$\Omega_\phi(\phi_1,\phi_2) = \int d^3V(\phi_1\mathcal{L}_n\phi_2 - \phi_2\mathcal{L}_n\phi). \tag{46}$$

The symplectic structure (46) is defined in terms of the pair (ϵ_{abc}, n^a), and there is a canonical isomorphism between different choices of pairs in the class (44) given by [4]

$$(\epsilon_{abc}, n^a) \to (\omega^3\epsilon_{abc}, \omega^{-1}n^a), \qquad \phi \to \omega^{-1}\phi. \tag{47}$$

The action of a generalized BMS vector field ξ^a on Γ_ϕ can be obtained as in the BMS case for gravity discussed in Sec. III B and Appendix B: First compute the variation of ϕ under ξ^a, and then use the canonical isomorphism (47) to express the "transformed field" in the original frame. The result is

$$X^\phi_\xi = \mathcal{L}_\xi \phi + \alpha\phi. \tag{48}$$

The form (48) is the same as the one given in [4] for the action of BMS. It is easy to verify that (48) is symplectic and that $[X_\xi, X_{\xi'}] = X_{[\xi,\xi']}$.

C. The case of gravitational radiative phase space

Since generalized BMS does not preserve the universal structure $[(q_{ab}, n^a)]$, and there is no (known to us) natural isomorphism between the various universal structures that generalized BMS can map to (namely those compatible with $[(\epsilon_{abc}, n^a)]$), we lack a geometrical framework from which we can attempt to derive an action of generalized BMS on the radiative phase space of gravity. Thus, the strategy followed in Secs. III B and IV B is not available. This problem is the phase space counterpart of the issue discussed in Sec. II C: As generalized BMS vector fields change the leading order metric at $\mathcal{I}$, it is not clear how to deduce an action of $\mathbf{G}$ on the free data.

We shall limit ourselves to present an *ad hoc* HVF X_ξ. The interest in this proposal lies in the fact that the associated Ward identities will be shown to be in one-to-one correspondence with the Cachazo-Strominger (CS) soft theorem.

There are, however, two shortcomings of our proposal that we hope to address in the future investigations.

(1) The HVFs do not represent an action of generalized BMS since in general $[X_\xi, X_{\xi'}] \neq X_{[\xi,\xi']}$.[8]

(2) The HVFs do not respect the falloff behavior of the radiative data and hence strictly speaking are not well defined on the entire phase space. (This infrared divergence is also present when the underlying vector fields are local CKVs.)

Our definition for the HVF is exactly the same as in (34), where V^A is an arbitrary (smooth) vector field on the sphere and $\alpha = (D_A V^A)/2$. It is the sum of hard and soft terms,

$$X_V = X_V^{\rm hard} + X_V^{\rm soft}, \tag{49}$$

where

$$\begin{aligned}(X_V^{\rm hard})_{zz} &= \mathcal{L}_V \sigma_{zz} - \alpha\sigma_{zz} + u\alpha\dot\sigma_{zz};\\ (X_V^{\rm soft})_{zz} &:= -\frac{u}{2} D_z^3 V^z,\end{aligned} \tag{50}$$

and corresponding $z \to \bar z$ expressions for $(X_V)_{\bar z\bar z}$. It can be seen that $X_V^{\rm hard}$ preserves the falloffs (29). Further, as shown in Appendix C, it is symplectic,

$$\Omega(X_V^{\rm hard}(\sigma_1), \sigma_2) + \Omega(\sigma_1, X_V^{\rm hard}(\sigma_2)) = 0 \quad \forall\ \sigma_1, \sigma_2 \in \Gamma. \tag{51}$$

Being linear in σ, its Hamiltonian can be found by

$$H_V^{\rm hard}(\sigma) := \frac{1}{2}\Omega(X_V^{\rm hard}(\sigma), \sigma), \tag{52}$$

which leads to the same expression as the Hamiltonian for boosts (with the CKV condition on V^A dropped).

Unless $D_z^3 V^z = D_{\bar z}^3 V^{\bar z} = 0$, $X_V^{\rm soft}$ diverges linearly in u and hence is not well defined on Γ. At a formal level $X_V^{\rm soft}$ is, however, symplectic since it is just a c number vector field. We can make sense of the "would be" Hamiltonian on the subspace $\Gamma_0 \subset \Gamma$ given by

$$\begin{aligned}\Gamma_0 := \{\sigma_{AB} \in \Gamma : \sigma_{AB}(u,\hat x) = \sigma^{+}(\hat x) + O(u^{-1-\epsilon})\\ \text{as } u \to \pm\infty\}.\end{aligned} \tag{53}$$

For $\sigma \in \Gamma_0$ we define

$$\begin{aligned}H_V^{\rm soft}(\sigma) &:= \Omega(X_V^{\rm soft}, \sigma)\\ &= -\int d^3V u(\dot\sigma^{zz} D_z^3 V^z + \dot\sigma^{\bar z\bar z} D_{\bar z}^3 V^{\bar z}).\end{aligned} \tag{54}$$

Finally, for $\sigma \in \Gamma_0$ the total Hamiltonian is defined by

$$H_V(\sigma) := H_V^{\rm hard}(\sigma) + H_V^{\rm soft}(\sigma). \tag{55}$$

We will use these expressions to define the hard and soft operators in quantum theory. In [17] X_V is derived directly from the action of V^A on C_{AB} as given in Eq. (20). If we follow this prescription here, it will lead to an expression for X_V different from the one given above. However, as the action of $\xi^a\partial_a = V^A\partial_A + u\alpha\partial_u$ changes the leading order metric at $\mathcal{I}$, this procedure is not applicable in this case.

D. Action of generalized BMS generators on mode functions

For $\omega \neq 0$, the action of X_V on the mode functions $\sigma_{AB}(\omega, \hat x)$ is fully determined by the term $X_V^{\rm hard}$. By taking the Fourier transform of (50) we arrive at the analogue of Eq. (41) (with an additional "trace-free" symbol on the Lie derivative term). The corresponding action on the functions $a_\pm(\omega, \hat x)$ is given by the same equations as in the boost/rotation case, Eq. (43), with the CKV condition on V^A dropped. We thus find

$$\begin{aligned}\{a_h(\omega, z, \bar z), H_V\} &= J_V^h a_h(\omega, z, \bar z),\\ \{a_h(\omega, z, \bar z)^\dagger, H_V\} &= J_V^{-h} a_h(\omega, z, \bar z)^\dagger,\end{aligned} \tag{56}$$

[8]The situation is thus analogous to the recently proposed symmetries for massless QED that follow from the subleading soft photon theorem [34].

with J_V^h the same differential operator given in Eq. (42),

$$J_V^h = V^z\partial_z + V^{\bar z}\partial_{\bar z} - \frac{1}{2}(D_z V^z + D_{\bar z}V^{\bar z})\omega\partial_\omega + \frac{h}{2}(\partial_z V^z - \partial_{\bar z}V^{\bar z}). \tag{57}$$

The nonclosure of the HVFs X_V manifests in a particular simple form through the nonclosure of the commutator of operators J_V^h for general smooth vector fields. A simple calculation reveals

$$[J_V^h, J_W^h]a_h = J^h_{[V,W]}a_h + h(\partial_{\bar z}V^z\partial_z W^{\bar z} - \partial_z V^{\bar z}\partial_{\bar z}W^z)a_h. \tag{58}$$

Thus, the "nonclosure" is proportional to the helicity. This is in accordance with the discussion of Sec. IV B: The action of generalized BMS on the mode functions of a massless scalar field lacks a helicity contribution, and the nonclosure term is absent there.

V. GENERALIZED BMS AND SUBLEADING SOFT THEOREM

In this section we show the equivalence between CS soft theorem and generalized BMS symmetries. After summarizing the content of the soft theorem in Sec. V A, in Sec. V B we propose the Ward identities for smooth vector fields belonging to the generalized BMS algebra. Although our derivation is simply a repeat of the derivation given in [17], we express it in a slightly different form that facilitates the proof of the equivalence.

We then argue, in Sec. V C, that the derivation of Ward identities associated with the CS soft theorem as given in [17] goes through for smooth vector fields on the sphere. In Sec. V D we show that using Ward identities for generalized BMS algebra, we can obtain the CS soft theorem. This derivation parallels the derivation for the case of supertranslations as mentioned in [12]. We conclude in Sec. V E with a comparison of this equivalence with the equivalence between Ward identities for supertranslations and Weinberg's soft graviton theorem.

In the following we work with the Fock space $\mathcal{H}^{\mathrm{out}}$ generated by the standard creation/annihilation operators with nontrivial commutators given by i times the PBs (40),

$$[a_h^{\mathrm{out}}(\omega,\hat x), a_{h'}^{\mathrm{out}}(\omega',\hat x')^\dagger] = \frac{2(2\pi)^3}{\omega\sqrt{\gamma}}\delta_{hh'}\delta(\omega-\omega')\delta(\hat x,\hat x'), \tag{59}$$

and with the analogue $\mathcal{H}^{\mathrm{in}}$ Fock space. The nature of the present section is rather formal. In particular, we do not construct the operator associated with H_V but rather assume that (i) it is normal ordered so that its action on the vacuum is determined by the soft term; (ii) its commutator with creation/annihilation operators is given by i times the PBs (56). Below we consider in and out states of the form

$$\langle \mathrm{out}| \coloneqq \langle 0|\prod_{i=1}^{n_{\mathrm{out}}} a_{h_i}^{\mathrm{out}}(E_i^{\mathrm{out}}, \hat k_i^{\mathrm{out}}),$$
$$|\mathrm{in}\rangle \coloneqq \prod_{i=1}^{n_{\mathrm{in}}} a_{h_i}^{\mathrm{in}}(E_i^{\mathrm{in}}, \hat k_i^{\mathrm{in}})^\dagger|0\rangle. \tag{60}$$

The subleading soft operator that acts on asymptotic Fock states can be read off from Eq. (54), and it precisely matches with the operator Q_S^{out} as given in [17],

$$(H_V^{\mathrm{soft}})^{\mathrm{out}} = \tfrac{1}{2}\int_{\mathcal{I}^+} du d^2 z D_z^3 V^z N_{\bar z}^z = Q_S^{\mathrm{out}}. \tag{61}$$

A. CS soft theorem

In this section we summarize the content of the CS soft theorem. We express the soft factor in terms of a vector field on the sphere appearing in Eq. (6.6) of [17]. This will facilitate the discussions in the subsequent sections. The CS subleading soft theorem for an outgoing soft graviton of helicity h_s and momentum q^μ parametrized by $(\omega, z_s, \bar z_s)$ can be written as [17][9]

$$\lim_{\omega\to 0^+}(1+\omega\partial_\omega)\langle \mathrm{out}|a_{h_s}^{\mathrm{out}}(\omega, z_s, \bar z_s)S|\mathrm{in}\rangle = \sum_i S_i^{(1)h_s}\langle \mathrm{out}|S|\mathrm{in}\rangle, \tag{62}$$

where the sum runs over all ingoing and outgoing particles. For an outgoing particle of momentum k^μ and helicity h the soft factor is given by [18]

$$S_{(k,h)}^{(1)h_s} = (q\cdot k)^{-1}\epsilon_{\mu\nu}^{h_s}(q)k^\mu q_\rho J^{\rho\nu}, \tag{63}$$

where $\epsilon_{\mu\nu}^{h_s}(q) = \epsilon_\mu^{h_s}(q)\epsilon_\nu^{h_s}(q)$ is the polarization tensor of the soft graviton and $J^{\rho\nu}$ the total angular momentum of the (k,h) particle. Following Strominger and collaborators, we seek to express (63) in holomorphic coordinates. Let $(E, z, \bar z)$ be the parametrization of the 4-momentum k^μ. As discussed in Sec. III D, the total angular momentum can be expressed in terms of the differential operator J_V^h given in Eq. (42). The six CKVs corresponding to the (μ,ν) components are

$$V_{i0}^A \coloneqq D^A\hat k_i, \qquad V_{ij}^A \coloneqq \hat k_i D^A\hat k_j - \hat k_j D^A \hat k_i,$$
$$i,j = 1,2,3, \qquad A = z,\bar z, \tag{64}$$

so that

$$J_{\mu\nu} = J^h_{V_{\mu\nu}}, \qquad \mu,\nu = 0,1,2,3, \tag{65}$$

[9]The subsequent analysis can easily be extended to the case of incoming soft gravitons.

where $\hat{k}$ is the unit direction on the sphere parametrized by $(z,\bar{z})$. For the polarization tensor we follow [12,17] and take

$$\epsilon^{+}(q)^{\mu}=\frac{1}{\sqrt{2}}(\bar{z}_s,1,-i,-\bar{z}_s),\qquad \epsilon^{-}(q)^{\mu}=\frac{1}{\sqrt{2}}(z_s,1,i,-z_s). \tag{66}$$

Notice that (63) takes the form of a function of $(z,\bar{z})$ times a linear combination of boosts and rotations (with coefficients depending on $z_s,\bar{z}_s$, and h_s). Thus, all $(z,\bar{z})$-independent factors multiplying $J^{\rho\nu}$ can be realized as linear combinations of CKVs. For instance,

$$\epsilon^{+}_{\nu}(q)q_{\rho}J^{\rho\nu}=J^{h}_{\epsilon^{+}_{\nu}(q)q_{\rho}V^{\rho\nu}}. \tag{67}$$

Taking this into account, Eq. (63) can be written as

$$S^{(1)+}_{(k,h)}=(z-z_s)^{-1}J^{h}_{(\bar{z}-\bar{z}_s)^2\partial_{\bar{z}}},\qquad S^{(1)-}_{(k,h)}=(\bar{z}-\bar{z}_s)^{-1}J^{h}_{(z-z_s)^2\partial_{z}}. \tag{68}$$

We finally show that (68) can, in fact, be written in terms of the vector fields,

$$K_{(+,z_s,\bar{z}_s)}\coloneqq(z-z_s)^{-1}(\bar{z}-\bar{z}_s)^2\partial_{\bar{z}},\qquad K_{(-,z_s,\bar{z}_s)}\coloneqq(\bar{z}-\bar{z}_s)^{-1}(z-z_s)^2\partial_{z}, \tag{69}$$

according to

$$S^{(1)+}_{(k,h)}=J^{h}_{K_{(+,z_s,\bar{z}_s)}},\qquad S^{(1)-}_{(k,h)}=J^{h}_{K_{(-,z_s,\bar{z}_s)}}. \tag{70}$$

Let us discuss the "−", case, the "+" one being analogous. From the definition of J^h_V, Eq. (57), one can verify

$$J^{h}_{(\bar{z}-\bar{z}_s)^{-1}(z-z_s)^2\partial_z}=(\bar{z}-\bar{z}_s)^{-1}J^{h}_{(z-z_s)^2\partial_z}+\frac{1}{2}(-E\partial_E+h)(z_s-z)^2\partial_z\frac{1}{(\bar{z}-\bar{z}_s)}. \tag{71}$$

The second term is proportional to $(z_s-z)^2\delta^{(2)}(z,z_s)$. As long as (62) is understood as a distribution to be smeared against a smooth function on the sphere, this term vanishes and we obtain (70).

B. Proposed Ward identities

In this section we motivate a proposal for the "Ward identities."[10] This proposal is a straightforward generalization of the Ward identities proposed for the local CKVs associated with the extended BMS algebra. We repeat the derivation here in the interest of pedagogy and for introducing notation for later use.

Consider the analogue of the Virasoro symmetry proposed in [17], but with V^A a smooth vector field on the sphere rather than a local CKV,

$$H^{\rm out}_V S=SH^{\rm in}_V. \tag{72}$$

The evaluation of (72) between the states (60) is obtained by using the commutators [see Eq. (56)],

$$[a^{\rm out}_h(\omega,\hat{x}),H^{\rm out}_V]=iJ^h_V a^{\rm out}_h(\omega,\hat{x}),\qquad [a^{\rm in}_h(\omega,\hat{x})^{\dagger},H^{\rm in}_V]=iJ^{-h}_V a^{\rm in}_h(\omega,\hat{x}), \tag{73}$$

together with the action $H^{in({\rm out})}_V$ on the in (out) vacuum. This action is determined by the soft part of $H^{\rm in(out)}_V$ (54). Following [17], we express (54) in terms of the Fourier transform of the news tensor so that (the prescription for the $\omega\to 0$ limit is described below)

$$H^{\rm in}_V|0\rangle=-\frac{i}{2}\lim_{\omega\to0}\partial_\omega\oint d^2V(N^{\rm in}_{zz}(\omega,\hat{x})D^zD^zD_{\bar{z}}V^{\bar{z}})+N^{\rm in}_{\bar{z}\bar{z}}(\omega,\hat{x})D^{\bar{z}}D^{\bar{z}}D_zV^z|0\rangle, \tag{74}$$

and similar expression for $\langle0|H^{\rm out}_V$. The matrix element of (72) between the in and out states implies then

$$\frac{1}{2}\lim_{\omega\to0}\partial_\omega\oint d^2VD^zD^zD_{\bar{z}}V^{\bar{z}}(\langle{\rm out}|N^{\rm out}_{zz}(\omega,\hat{x})S|{\rm in}\rangle-\langle{\rm out}|SN^{\rm in}_{zz}(\omega,\hat{x})|{\rm in}\rangle)+z\leftrightarrow\bar{z}=\sum_i J^{h_i}_{V_i}\langle{\rm out}|S|{\rm in}\rangle. \tag{75}$$

The sum runs over all in and out particles, with the convention that for an in particle one takes $J^{h_i}_{V_i}=J^{-h^{\rm in}_i}_{V_i}$ according to (73).

We now focus on the left-hand side (LHS) of (75). First we need to specify how the $\omega\to0$ limit is taken. We take $\omega\to0^+$ in (75) so that only the out term survives. This prescription is slightly different from the one given in [17]. However, it leads to the same form of Ward identities as given in [17].[11]

With this prescription, and using Eqs. (37) and (38), the LHS of (75) takes the form

$${\rm LHS}=\frac{1}{4\pi}\lim_{\omega\to0^+}(1+\omega\partial_\omega)\int d^2z(D^3_{\bar{z}}V^{\bar{z}}\langle{\rm out}|a^{\rm out}_{+}(\omega,\hat{x})S\rangle+D^3_zV^z\langle a^{\rm out}_{-}(\omega,\hat{x})S|{\rm in}\rangle), \tag{76}$$

[10] The quotation marks are placed to remind us that the proposed charges do not yield a representation of the generalized BMS algebra on the radiative phase space.

[11] For supertranslations, this prescription also leads to the same Ward identities of [12].

where we used $\sqrt{\gamma}\sqrt{\gamma}\gamma^{z\bar{z}}\gamma^{z\bar{z}} = 1$. Substituting Eq. (76) in Eq. (75) we obtain the proposed identities. They take precisely the same form as the Virasoro Ward identities of [17]

$$\frac{1}{4\pi}\lim_{\omega\to 0^+}(1+\omega\partial_\omega)\int d^2z(D_{\bar{z}}^3V^{\bar{z}}\langle\text{out}|a_+^{\text{out}}(\omega,\hat{x})S|\text{in}\rangle + D_z^3V^z\langle\text{out}|a_-^{\text{out}}(\omega,\hat{x})S|\text{in}\rangle) = \sum_i J_{V_i}^{h_i}\langle\text{out}|S|\text{in}\rangle. \tag{77}$$

C. From CS theorem to generalized BMS symmetries

The purpose of this section is to show that remarkably enough, the derivation of the Virasoro Ward identities given in [17] does not make use of the CKV property of the vector fields in question, so that the identities hold for an arbitrary smooth vector field on the sphere.[12]

From Eqs. (62) and (70), the CS theorem can be written as

$$\lim_{\omega\to 0^+}(1+\omega\partial_\omega)\langle\text{out}|a_{h_s}^{\text{out}}(\omega,z,\bar{z})S|\text{in}\rangle = \sum_i J^{h_i}_{K^i_{(h_s,z,\bar{z})}}\langle\text{out}|S|\text{in}\rangle. \tag{78}$$

Let $V^A\partial_A$ be any smooth vector field on the sphere. In the following we work with $V^z\partial_z$ and $V^{\bar{z}}\partial_{\bar{z}}$ components separately. Multiplying the LHS of Eq. (78) with $h_s = -2$ by $(4\pi)^{-1}D_z^3V^z$ and integrating over $(z,\bar{z})$, we obtain the LHS of the proposed Ward identity (77) for the vector $V^z\partial_z$. The same operation on the right-hand side (RHS) of (78) is given by

$$(4\pi)^{-1}\sum_i\int d^2zD_z^3V^zJ^{h_i}_{K^i_{(-,z,\bar{z})}}\langle\text{out}|S|\text{in}\rangle = \sum_i J^{h_i}_{W_i}\langle\text{out}|S|\text{in}\rangle, \tag{79}$$

where

$$W_i \coloneqq (4\pi)^{-1}\int d^2zD_z^3V^zK^i_{(-,z,\bar{z})}. \tag{80}$$

To integrate by parts in (80), we need to specify the tensor index structure of $K^i_{(-,z,\bar{z})}$ with respect to the $(z,\bar{z})$ coordinates. This tensor structure is given by $a_-^{\text{out}}(\omega,z,\bar{z}) \sim \sigma_{\bar{z}\bar{z}}/\sqrt{\gamma}$ due to Eqs. (38) and (78). Following [17], this is captured by $\hat{\epsilon}_{\bar{z}\bar{z}} \coloneqq \sqrt{\gamma}$. We thus obtain (to avoid confusion we set $K_-^i \equiv K^i_{(-,z,\bar{z})}$)

$$\int d^2zD_z^3V^zK_-^i = \int d^2z\sqrt{\gamma}\gamma^{z\bar{z}}\gamma^{z\bar{z}}D_zD_z(D_zV^z)(\hat{\epsilon}_{\bar{z}\bar{z}}K_-^i) = -\int d^2z\sqrt{\gamma}V^zD_zD^{\bar{z}}D^{\bar{z}}(\hat{\epsilon}_{\bar{z}\bar{z}}K_-^i) = 4\pi V^{z_i}(z_i,\bar{z}_i)\partial_{z_i}, \tag{81}$$

where in the last equation we used an identity given in Eq. (6.7) of [17],

$$\gamma^{z\bar{z}}D_z^3(\hat{\epsilon}_{\bar{z}\bar{z}}K_-^i) = -4\pi\delta^{(2)}(z-z_i)\partial_{z_i}. \tag{82}$$

Using this result back in (79) we recover the RHS of the proposed Ward identity (77) for the vector $V^z\partial_z$. A similar discussion applies for $h_s = +2$ and the vector $V^{\bar{z}}\partial_{\bar{z}}$. Adding the two results one obtains the Ward identity (77) for the vector field $V^A\partial_A$.

D. From Ward identity to soft theorem

The CS theorem can be recovered as the Ward identity associated with the vector fields (69).[13] For the case of an outgoing negative helicity soft graviton with direction $(z_s,\bar{z}_s)$, we choose V^A in (77) by

$$V^A\partial_A = K_{(-,z_s,\bar{z}_s)} = (\bar{z}-\bar{z}_s)^{-1}(z-z_s)^2\partial_z. \tag{83}$$

One can verify that

$$D_z^3K^z_{(-,z_s,\bar{z}_s)} = 4\pi\delta^2(z-z_s). \tag{84}$$

Using (84) in (77) we recover CS theorem (78) for $h_s = -2$. Similar discussion applies for a positive helicity soft graviton.

E. Comparison with supertranslation case

We now note the following subtlety regarding this equivalence. Recall that Weinberg's soft graviton theorem is equivalent to the Ward identities associated with the supertranslation symmetries [12]. As supertranslations are parametrized by a single function, it is rather surprising that the associated Ward identities can give rise to the soft graviton theorem for both positive and negative helicity soft particles. That this is possible is due to a so-called global constraint that underlies the definition of CK spaces. On future null infinity, it is given by

$$[D_z^2C_{\bar{z}\bar{z}} - D_{\bar{z}}^2C_{zz}]_{\mathcal{I}^+_\pm} = 0. \tag{85}$$

It can be rewritten in terms of the zero mode of the news tensor as

[12] In fact, due to their singular nature, it is not clear to us how the derivation works for local CKVs.

[13] As in the case of supertranslations, this derivation requires a choice of a nonsmooth (in the present case C^1) vector field. It is understood that this is due to the use of sharp momentum eigenstates.

$$D_z^2 N_{\bar{z}\bar{z}}^{\mathrm{out}}(\omega = 0, \hat{x}) = D_{\bar{z}}^2 N_{zz}^{\mathrm{out}}(\omega = 0, \hat{x}). \tag{86}$$

This constraint ensures that the operator insertions due to positive and negative helicity soft gravitons are equivalent to each other. (For more details we refer the reader to [11].) This is consistent with the remarkable structure of Weinberg's soft term, which does not depend on the angular momenta of the external particles.

However, this constraint does not imply that the operator insertions associated with "subleading" soft positive helicity gravitons (i.e., when the leading order pole is projected out from the insertion) are equivalent to those of negative helicity gravitons. This is consistent with the fact that this subleading theorem is equivalent to the Ward identity associated with vector fields on a sphere that are parametrized by two independent functions. This is in turn reflected in the structure of the subleading CS soft term that depends on the angular momenta of the scattering particles.

VI. OUTLOOK

Motivated by the desire to understand the subleading soft graviton theorem as arising from Ward identities associated with asymptotic symmetries, we considered a *distinct* generalization of the BMS group than the one proposed in [14]. We showed that $\mathbf{G}$, which is essentially obtained by dropping a single condition from the definition of the BMS group (namely, that the vector fields defined on the conformal sphere be CKVs), is a semidirect product of supertranslations and diffeomorphisms of the conformal sphere, $\mathbf{G} = \mathrm{ST} \rtimes \mathrm{Diff}(S^2)$. We argued that $\mathbf{G}$ acts as a symmetry group on the space of all asymptotically flat geometries that are in a suitable neighborhood of Minkowski space-time.

Associated with vector fields that generate $\mathrm{Diff}(S^2) = \mathbf{G}/\mathrm{ST}$ we proposed a definition of the flux in the radiative phase space of Ashtekar that was motivated by the definition of corresponding flux for the Virasoro symmetries. The reason we have not been able to derive this flux expression from first principles (as one can do for any vector field belonging to extended BMS) can be most easily understood as follows.[14]

In the case of Virasoro symmetries, the derivation of flux in the radiative phase space is based upon the action of extended BMS vector fields on the free data quantified by C_{AB} [17]. C_{AB} is the free (radiative) data in the sense that it is unconstrained and that all the other dynamical metric components in the neighborhood of null infinity are determined from Einstein's equations using C_{AB}. However, what constitutes the free data is "frame dependent" in the sense that it depends on the chosen "kinematical," leading order metric at null infinity. As an extended BMS group preserves the leading order metric at $\mathcal{I}^+$, it maps a given radiative data into a different radiative data.

[14]For pedagogy we restrict our attention to future null infinity.

Because of the fact that $\mathbf{G}$ changes the leading order structure of the metric components, we have been unable to derive the action of its proposed flux from first principles. However, we think that its appeal lies in the fact that the related Ward identities are equivalent to the subleading soft graviton theorem.

Yet another unresolved issue with H_V (as is also the case for a new class of asymptotic symmetries proposed for massless QED [34]) is that the fluxes associated with $\mathbf{G}$ do not form a closed algebra. It is conceivable that this is due to the fact that the radiative phase space of Ashtekar is based upon the existence of a fixed kinematical structure (namely, the conformal metric on the sphere and the null vector field n^a), which is in turn tied to the existence of a fixed space-time metric at leading order in r. This expectation is borne out by the fact that in the case of the massless scalar field where the radiative phase space does not refer to the entire conformal metric but only the volume form, these symmetries do indeed form a closed algebra.[15]

In light of what is said above, there appear certain natural directions in which a systematic derivation of the fluxes associated with $\mathbf{G}$ (such that they form a closed algebra) could be obtained, namely by weakening the dependence of radiative phase space on the universal structure. Detailed implementation of this idea is currently under investigation.

In summary, our proposal for $\mathbf{G}$ as a group of asymptotic symmetries for low energy gravitational scattering processes is at best a tentative one. However, because of its relationship with the subleading soft theorem, we believe that further investigation of the abovementioned issues is warranted.

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APPENDIX A: ZERO MODE SUBTLETIES OF POISSON BRACKETS

Since the subtleties we want to discuss arise from the dependence in u, in this appendix we suppress the angular components and take σ to be a scalar function on the real line parametrized by u. More precisely, we consider the

[15]Note that if this expectation turned out to be true, then both the issues mentioned above are two sides of the same coin.

phase space Γ of scalar functions on the real line with falloffs $\sigma = \sigma^{\pm} + O(u^{-\epsilon})$ as $u \to \pm\infty$ and symplectic form

$$\Omega(\sigma_1, \sigma_2) = \int (\sigma_1 \dot{\sigma}_2 - \dot{\sigma}_1 \sigma_2) du = 2 \int \sigma_1 \dot{\sigma}_2 du - [\sigma_1 \sigma_2], \tag{A1}$$

where the square brackets denote the difference in evaluation at $u = \pm\infty$.

Consider a phase space function of the form

$$F(\sigma) \coloneqq \int F(u) \sigma(u) du, \tag{A2}$$

for some smearing function $F(u)$. To find the corresponding HVF $f \coloneqq X_F$, we need to solve the equation

$$F(\sigma) = \Omega(f, \sigma) \tag{A3}$$

for $f \in \Gamma$. From (A1) it follows that we should have $F = -2\dot{f}$ and $[f\sigma] = 0 \, \forall \, \sigma$. The condition involving the boundary term can be satisfied only if $f^+ = f^- = 0$. The two conditions can be summarized by

$$\text{(i)}\ F \sim 1/|u|^{1+\epsilon} \quad \text{as } u \to \pm\infty, \text{(ii)}\ \textstyle\int_{-\infty}^{\infty} F du = 0. \tag{A4}$$

Only for F satisfying (i) and (ii) does (A2) admit a HVF, in which case it is given by

$$X_F = f(u) = -\frac{1}{2} \int_{-\infty}^{u} F(u') du'. \tag{A5}$$

The PB between a pair of functions F and G satisfying (i) and (ii) can then be written as

$$\{F, G\} = \Omega(X_F, X_G) \tag{A6}$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} du du' G(u) F(u') \theta(u - u') \tag{A7}$$

$$= -\frac{1}{4} \int_{-\infty}^{\infty} du du' G(u) F(u') \mathrm{sign}(u - u'), \tag{A8}$$

where $\theta(u - u')$ is the step function. It is clear that there is no way to extract PBs for $\{\sigma(u), \sigma(u')\}$ (there is not even a unique expression). Let us nevertheless use the form (A8) to set

$$“\{\sigma(u), \sigma(u')\} = -\frac{1}{4} \mathrm{sign}(u - u'),” \tag{A9}$$

and see how we get a contradiction. Equation (A9) is the analogue of Eq. (2.12) of [12] (our conventions for PBs differ by a sign with those used in [12]). An example of the contradiction found in [12] is as follows. Consider the phase space function

$$H(\sigma) \coloneqq [\sigma] = \sigma^+ - \sigma^- = \Omega(1, \sigma). \tag{A10}$$

It admits a HVF given by $X_H = 1$. Its action on $\sigma(u)$ is simply given by $X_H(\sigma(u)) = 1$. Since $\sigma(u)$ does not admit a HVF (not even in a distributional sense), we cannot interpret this action in terms of Poisson brackets. If we nevertheless do so, we find

$$“\{\sigma(u), [\sigma]\}” = X_H(\sigma(u)) = 1. \tag{A11}$$

But using (A9) we get

$$\begin{aligned} “\{\sigma(u), [\sigma]\} &= \{\sigma(u), \sigma^+\} - \{\sigma(u), \sigma^-\} \\ &= \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2},” \end{aligned} \tag{A12}$$

and hence the contradiction.

APPENDIX B: DERIVATION OF ACTION OF BMS ON RADIATIVE PHASE SPACE

In [4] it is shown that given a derivative D_a compatible with (q_{ab}, n^a) and a new frame $(q'_{ab}, n'^a) = (\omega^2 q_{ab}, \omega^{-1} n^a)$, there exists a natural derivative D'_a compatible with the new frame given by Eq. (4.5) of [4],

$$D'_a k_b = D_a k_b - 2\omega^{-1} k_{(a} \partial_{b)} \omega + \omega^{-1} q_{ab} \omega^c k_c, \tag{B1}$$

where ω^c is any vector satisfying $\omega^c q_{bc} = D_b \omega$. The corresponding map $[D_a] \to [D'_a]$ between equivalence classes of derivatives provides the isomorphism between the phase spaces associated with the two frames.

Under the action of a general BMS vector field ξ^a, the “transformed derivative” $D'_a \approx D_a + \delta_\xi D_a$ is compatible with the frame $(q'_{ab}, n'^a) \approx ((1 + 2\alpha) q_{ab}, (1 - \alpha) n^a)$. To obtain the BMS action on the original phase space, we use the aforementioned isomorphism to map D'_a to a derivative compatible (q_{ab}, n^a). The resulting derivative, D''_a, is obtained by performing the substitutions $D'_a \to D''_a$, $D_a \to D'_a$, and $\omega \to 1 - \alpha$ in (B1),

$$\begin{aligned} D''_a k_b - D_a k_b = (\delta_\xi D_a) k_b + 2(1 + \alpha) k_{(a} \partial_{b)} \alpha \\ - (1 + \alpha)(2 + \alpha) q_{ab} q^{cd} l_c \partial_d \alpha. \end{aligned} \tag{B2}$$

Choosing k_b such that $n^b k_b = 1$ and taking the trace-free part, Eq. (B2) gives us the desired action,

$$(X_\xi)_{ab} = ([\mathcal{L}_\xi, D_a] k_b + 2k_{(a} \partial_{b)} \alpha)^{\mathrm{TF}}, \tag{B3}$$

where the contribution of the last term in Eq. (B2) is zero as it is pure trace and we have dropped $O(\alpha^2)$ terms. Equation (B3) precisely matches with the Hamiltonian vector field associated with a BMS vector field ξ^a as given in Eq. (4.14) of [4].

APPENDIX C: SYMPLECTIC ACTION OF GENERALIZED BMS GENERATORS

The proof of (51) is essentially the same as that for BMS generators. The difference with the BMS case is that $\mathcal{L}_V q_{AB}$ contains trace-free components, which we denote by t_{AB},

$$\mathcal{L}_V q_{AB} = 2\alpha q_{AB} + t_{AB}, \quad \text{where } q^{AB} t_{AB} = 0. \tag{C1}$$

Similarly,

$$\mathcal{L}_V q^{AB} = -2\alpha q^{AB} - t^{AB}, \tag{C2}$$

with $t^{AB} = t_{CD} q^{AC} q^{BD}$. That these "non-CKV" terms do not spoil (51) will follow from the fact that they will always appear contracted with two other trace-free tensors and a metric, yielding a vanishing result. For instance, if $\sigma^1_{AB}, \sigma^2_{AB}$, and t^{AB} are symmetric and trace-free, then

$$q^{AC} t^{BD} \sigma^1_{AB} \sigma^2_{CD} = 0, \tag{C3}$$

as can be seen by writing the expression in $(z, \bar{z})$ components.

The expression (50) for X^{hard}_V is

$$(X^{\text{hard}}_V)_{AB} = (\mathcal{L}_V \sigma_{AB})^{\text{TF}} + \alpha u \dot{\sigma}_{AB} - \alpha \sigma_{AB}, \tag{C4}$$

where TF denotes the trace-free part with respect to q_{AB}. The evaluation of (51) involves three terms associated with each of the terms in (C4),

$$\Omega(X^{\text{hard}}_V(\sigma_1), \sigma_2) = \Omega((\mathcal{L}_V \sigma_1)^{\text{TF}}, \sigma_2) + \Omega(\alpha u \dot{\sigma}_1, \sigma_2) - \Omega(\alpha \sigma_1, \sigma_2). \tag{C5}$$

The first contribution to (51) is

$$\begin{aligned}
&\Omega((\mathcal{L}_V \sigma_1)^{\text{TF}}, \sigma_2) - 1 \leftrightarrow 2 \\
&= \int du d^2V q^{AC} q^{BD} (\mathcal{L}_V \sigma^1_{AB} \dot{\sigma}^2_{BC} - \mathcal{L}_V \dot{\sigma}^1_{AB} \sigma^2_{CD}) - 1 \leftrightarrow 2 \\
&= \int du d^2V q^{AC} q^{BD} (\mathcal{L}_V (\sigma^1_{AB} \dot{\sigma}^2_{BC}) - \mathcal{L}_V (\dot{\sigma}^1_{AB} \sigma^2_{CD})) \\
&= 2 \int du d^2V \alpha q^{AC} q^{BD} (\sigma^1_{AB} \dot{\sigma}^2_{BC} - \dot{\sigma}^1_{AB} \sigma^2_{CD}),
\end{aligned} \tag{C6}$$

where we used $q^{AC} q^{BD} (\mathcal{L}_V \sigma^1_{AB})^{\text{TF}} \dot{\sigma}^2_{BC} = q^{AC} q^{BD} \mathcal{L}_V \sigma^1_{AB} \dot{\sigma}^2_{BC}$, $\mathcal{L}_V (d^2V) = 2\alpha d^2V$, and Eqs. (C2) and (C3). The second contribution to (51) is

$$\begin{aligned}
&\Omega(\alpha u \dot{\sigma}_1, \sigma_2) - 1 \leftrightarrow 2 \\
&= \int du d^2V q^{AC} q^{BD} (\alpha u \dot{\sigma}^1_{AB} \dot{\sigma}^2_{BC} - \alpha \partial_u (u \dot{\sigma}^1_{AB}) \sigma^2_{CD}) - 1 \leftrightarrow 2 \\
&= \int du d^2V q^{AC} q^{BD} (\alpha (u \dot{\sigma}^1_{AB}) \dot{\sigma}^2_{CD}) - 1 \leftrightarrow 2 = 0,
\end{aligned} \tag{C7}$$

where we used $\lim_{u \to \pm\infty} u \dot{\sigma}_{AB} = 0$ so that no boundary contribution arises from the integration by parts in u. Finally, it is easy to see that the last contribution to (51) exactly cancels (C6).

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