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Magnetohydrodynamics turbulence: An astronomical perspective

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Abstract. Early work on magnetohydrodynamic (MHD) turbulence in the 1960s due, independently, to Iroshnikov and Kraichnan (IK) considered isotropic inertial-range spectra. Whereas laboratory experiments were not in a position to measure the spectral index, they showed that the turbulence was strongly anisotropic. Theoretical horizons correspondingly expanded in the 1980s, to accommodate both the isotropy of the IK theory and the anisotropy suggested by the experiments. Since the discovery of pulsars in 1967, many years of work on interstellar scintillation suggested that small-scale interstellar turbulence must have a hydromagnetic origin; but the IK spectrum was too flat and the ideas on anisotropic spectra too qualitative to explain the observations. In response, new theories of balanced MHD turbulence were proposed in the 1990s, which argued that the IK theory was incorrect, and made quantitative predictions of anisotropic inertial-range spectra; these theories have since found applications in many areas of astrophysics. Spacecraft measurements of solar-wind turbulence show that there is more power in Alfvén waves that travel away from the Sun than towards it. Theories of imbalanced MHD turbulence have now been proposed to address interplanetary turbulence. This very active area of research continues to be driven by astronomy.

Keywords. Magnetohydrodynamics; plasmas; turbulence; interstellar medium; solar wind.

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1. Introduction

Turbulent and magnetized plasmas are found in many environments in the Universe; some of these include the Sun, the solar wind, accretion discs, the ionized interstellar medium, molecular clouds and clusters of galaxies [1–7]. In magnetohydrodynamics (MHD), the plasma is modelled as a single component conducting fluid: currents in the plasma generate magnetic field which exert Lorentz forces on the fluid. Our focus is on the inertial-range spectra of the magnetic and velocity fields in a strongly magnetized incompressible fluid, in the limit of very large (fluid and magnetic) Reynolds numbers. Section 2 begins with the equations of incompressible MHD which govern the dynamics of the magnetic and velocity fields, and discusses the general nature of MHD turbulence. Section 3 discusses some of the early work on MHD turbulence – these include the isotropic theories of

Iroshnikov and Kraichnan (IK theory), as well as the ideas of Montgomery and collaborators on anisotropic turbulence. In §4 we discuss at length the reasons that led up to the conclusion that the IK theory is incorrect: many years of work by astronomers on interstellar turbulence, and arguments involving weak turbulence theories and resonant wave interactions. The new theories of balanced anisotropic MHD turbulence, both weak and strong, that emerged in the 1990s are described in §5. *In situ* measurements of imbalanced MHD turbulence in the solar wind are described in §6, and a theory of imbalanced, strong MHD turbulence is described in §7. We end with a few comments in §8. For more details regarding MHD turbulence, including other points of view, the reader should consult the original literature; see e.g. [8–34].

2. The general nature of MHD turbulence

The dynamics of a magnetized, incompressible plasma with uniform and constant mass density ρ, is described by the following equations that govern the behaviour of the magnetic field, $\mathbf{B}(\mathbf{x}, t)$, and the velocity field, $\mathbf{v}(\mathbf{x}, t)$:

$$\nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \cdot \mathbf{v} = 0\,; \tag{1}$$

$$\partial_t \mathbf{B} \;-\; \nabla \times (\mathbf{v} \times \mathbf{B}) = \eta \nabla^2 \mathbf{B}\,; \tag{2}$$

$$\partial_t \mathbf{v} \;+\; (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla p + \frac{(\mathbf{B} \cdot \nabla)\,\mathbf{B}}{4\pi\rho} \;+\; \nu \nabla^2 \mathbf{v} \;+\; \mathbf{f}\,, \tag{3}$$

where η is the resistivity and ν is the kinematic viscosity. $\mathbf{f}(\mathbf{x}, t)$ is the rate of velocity stirring, which is assumed to be incompressible, $\nabla \cdot \mathbf{f} = 0$. In many cases of interest to problems in turbulence, $\mathbf{f}$ is stochastic, and its statistics is assumed to be given. p is the ratio of the total pressure to the mass density, which is determined by requiring $\nabla \cdot \mathbf{v} = 0$. Equivalently, the system can also be described by the Elsasser fields [35], $\mathbf{W}^{\pm}$, defined by

$$\mathbf{W}^{+} \;=\; \mathbf{v} \;-\; \frac{\mathbf{B}}{\sqrt{4\pi\rho}}\,, \qquad \mathbf{W}^{-} \;=\; \mathbf{v} \;+\; \frac{\mathbf{B}}{\sqrt{4\pi\rho}}\,. \tag{4}$$

When ρ is uniform, we have $\nabla \cdot \mathbf{W}^{\pm} = 0$. It turns out to be very useful to describe incompressible MHD turbulence in terms of the Elsasser fields, instead of the magnetic and velocity fields. Our interest is in the inertial range of MHD turbulence, and the stirring and dissipative terms usually do not affect the physics in this range of scales; henceforth we shall drop them.

Equations (1)–(3) admit the uniformly magnetized static solution, $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{B}_0 = B_0\hat{z}$. We take this to be the unperturbed state of the plasma. In this state, the Elsasser fields, $\mathbf{W}_0^{\pm} = \mp V_{\rm A}\hat{z}$, where $V_{\rm A} = B_0/\sqrt{4\pi\rho}$ is the Alfvén speed. When this state is perturbed, we write,

$$\mathbf{W}^{+} \;=\; -V_{\rm A}\hat{z} \;+\; \mathbf{w}^{+}\,, \qquad \mathbf{W}^{-} \;=\; V_{\rm A}\hat{z} \;+\; \mathbf{w}^{-}, \tag{5}$$

where the perturbations, $\mathbf{w}^{\pm}$, are not necessarily small. The perturbations obey the equations

$$(\partial_t + V_A \partial_z)\,\mathbf{w}^+ + (\mathbf{w}^- \cdot \nabla)\mathbf{w}^+ = -\nabla p\,,$$
$$(\partial_t - V_A \partial_z)\,\mathbf{w}^- + (\mathbf{w}^+ \cdot \nabla)\mathbf{w}^- = -\nabla p\,, \tag{6}$$

where p is determined by requiring that $\nabla \cdot \mathbf{w}^{\pm} = 0$. The linearized equations are

$$(\partial_t + V_A \partial_z)\,\mathbf{w}^+ = -\nabla p\,, \qquad (\partial_t - V_A \partial_z)\,\mathbf{w}^- = -\nabla p\,. \tag{7}$$

If we consider solutions that vanish at infinity, $\nabla^2 p = 0$ implies that $p =$ constant for the linearized problem, and we have linear waves with

$$\mathbf{w}^+ = \mathbf{F}^+(x, y, z - V_A t)\,, \qquad \mathbf{w}^- = \mathbf{F}^-(x, y, z + V_A t), \tag{8}$$

where $\mathbf{F}^{\pm}$ are arbitrary vector functions with $\nabla \cdot \mathbf{F}^{\pm} = 0$. Wave propagation is along the magnetic field lines: $\mathbf{w}^+$ translates in the positive z-direction with speed V_A and $\mathbf{w}^-$ translates in the negative z-direction with speed V_A. It is useful to express this in **k**-space by assigning frequencies, $\omega^{\pm}(\mathbf{k})$ to the $\mathbf{w}^{\pm}$ waves with dispersion relations,

$$\omega^+(\mathbf{k}) = V_A k_z\,, \qquad \omega^-(\mathbf{k}) = -V_A k_z. \tag{9}$$

Being transverse waves, we must have $\mathbf{k} \cdot \tilde{\mathbf{F}}^{\pm}(\mathbf{k}) = 0$, where $\tilde{\mathbf{F}}^{\pm}$ is the Fourier transform of $\mathbf{F}^{\pm}$. In other words, for any given **k**, the waves have two independent degrees of polarization, constrained to lie in the plane perpendicular to **k**. The component perpendicular to $\hat{z}$ is called the Alfvén wave, and the other orthogonal component is called the Slow (magnetosonic) wave.

Equations (6) have a remarkable property: when either $\mathbf{w}^+$ or $\mathbf{w}^-$ is equal to $\mathbf{0}$ at the initial time, the nonlinear terms, $(\mathbf{w}^- \cdot \nabla)\mathbf{w}^+$ and $(\mathbf{w}^+ \cdot \nabla)\mathbf{w}^-$, vanish for all time. Therefore there are exact, nonlinear solutions of the form

$$\mathbf{w}^{\pm} = \mathbf{F}^{\pm}(x, y, z \mp V_A t)\,, \qquad \mathbf{w}^{\mp} = \mathbf{0}. \tag{10}$$

In other words, wavepackets that travel in one direction (i.e. either in the positive or negative z-direction) do not interact nonlinearly among themselves. Nonlinear interactions occur only between oppositely directed wavepackets. From this fact, Kraichnan [9] came to an important conclusion:

(i) Incompressible MHD turbulence in a strongly magnetized plasma can be described as being the result of (nonlinear) interactions between oppositely directed wavepackets.

By manipulating eqs (6) we can prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\, \frac{|\mathbf{w}^{\pm}|^2}{2} = 0. \tag{11}$$

This result is true even in the presence of nonlinearity, and is equivalent to the conservation of total energy and cross-helicity. Our second important conclusion now follows from eq. (11):

(ii) Wavepacket collisions conserve ±ve energies separately, and collisions can only redistribute the respective energies in **k**-space. This is true regardless of how imbalanced the situation is. For instance, if stirring puts much more energy into +ve waves than into −ve waves, the +ve waves cannot transfer any part of their energy to −ve waves. They can only scatter the −ve waves and redistribute them (and vice versa) in **k**-space.

Our discussion above assumed that a strong mean magnetic field is present. Is this not a limitation? After all, when we discuss the inertial range of homogeneous, isotropic hydrodynamic turbulence, we do not need to consider a large-scale velocity field. Kraichnan [9] realized that, in MHD turbulence, it is necessary to introduce a mean magnetic field, when the magnetic energy in the sub-inertial (stirring) scales exceeds the energy in the inertial range; this is sometimes referred to as the Alfvén effect. The reason for this difference in approach between hydrodynamic turbulence and MHD turbulence is the following. In the case of hydrodynamic turbulence, the velocity field of the large-scale eddies will 'sweep' eddies of smaller sizes, and this effect gives rise to observational consequences for the velocity time correlation function measured by a probe fixed at some location in the fluid. However, sweeping cannot affect the dynamics of nonlinear interactions, because a large-scale velocity field can be effectively transformed away by an appropriate Galilean boost. On the other hand, a large-scale magnetic field cannot be similarly transformed away; one way to physically see this is to note that such a field enables (Alfvén and Slow) waves that travel both up and down the field lines, and these cannot be wished away by any Galilean boost.

3. Early work on MHD turbulence

3.1 *Iroshnikov–Kraichnan (IK) theory*

It is very useful to begin with an account of a theory of incompressible MHD turbulence in a strongly magnetized plasma that was proposed in the 1960s, independently by Iroshnikov and Kraichnan [8,9]. This phenomenological model assumes isotropy of the inertial-range power spectra for the Elsasser variables, a feature which runs contrary to more modern views. However, it proves instructive to begin with the IK theory, and later see why it fails.

Assume statistically steady, isotropic excitation of ±ve waves, both with root mean squared (r.m.s.) amplitudes $w_L \ll V_A$, correlated on spatial scale L (which can be referred to as the stirring scale or outer scale). Nonlinear interactions between oppositely directed wavepackets give rise to ±ve wavepackets on smaller spatial scales. So let us consider the nature of collisions between wavepackets belonging to the inertial range, $\lambda \ll L$. Our goal is to determine how the r.m.s. amplitude on scale λ, denoted by w_λ, scales with λ.

We shall assume that the most effective collisions occur between wavepackets of similar sizes, an assumption that is also referred to as locality of interactions. From eqs (6) it follows that, in a collision between a +ve packet and a −ve packet, the r.m.s. amplitude of either packet is perturbed by an amount,

$$\delta w_\lambda \;\sim\; \frac{w_\lambda^2}{V_A}. \tag{12}$$

We imagine that a +ve wavepacket of size λ goes on to suffer a number of collisions with −ve wavepackets of similar scale, and vice versa. Successive perturbations add with random phases, so that in n collisions the r.m.s. amplitude of the perturbation will increase to $n^{1/2}(w_\lambda^2/V_A)$. Therefore, the number of collisions for perturbations to build up to order unity is

$$N_\lambda \sim \left(\frac{V_A}{w_\lambda}\right)^2. \quad (13)$$

The cascade time is the time taken for collisions to build upto order unity. Since each collision takes time $\sim\lambda/V_A$, the cascade time

$$t_\lambda \sim N_\lambda \frac{\lambda}{V_A}. \quad (14)$$

We now use Kolmogorov's hypothesis that the flux of energy cascading through the inertial range is independent of the particular spatial scale belonging to the inertial range, i.e., we assume that the energy flux, $\varepsilon \sim (w_\lambda^2/t_\lambda)$, is independent of λ. This implies that in the inertial range,

$$w_\lambda \sim w_L \left(\frac{\lambda}{L}\right)^{1/4}. \quad (15)$$

Thus we have determined the λ-dependence of the r.m.s. amplitude in the inertial range. The same result can also be expressed in **k**-space in terms of the (isotropic) energy spectrum, $E(k)k^3 \sim w_\lambda^2$, where $k \sim 1/\lambda$:

$$E(k) \sim \frac{w_L^2}{L^{1/2}\,k^{7/2}}. \quad (16)$$

It is important to note that the IK spectrum of eq. (16) is flatter than the Kolmogorov (power) spectrum of velocity fluctuations in purely hydrodynamic turbulence, $E_{\mathrm{Kol}}(k) \propto k^{-11/3}$.

3.2 *The work of Montgomery and his collaborators*

Laboratory experiments on plasma confinement by strong magnetic fields showed that plasma turbulence was highly anisotropic. Montgomery and Turner [36] suggested that the strong magnetic field would suppress field-aligned gradients of magnetic and velocity perturbations. So the observed anisotropic turbulence would consist of essentially perpendicular field fluctuations. Montgomery [37] also suggested that the anisotropic part could be described by the Strauss equations, which use the idea of critical balance. A lesser fraction of the turbulent energy was assumed to exist as a nearly isotropic spectrum of the IK form [37a]. Shebalin *et al* [10] argued that the anisotropic turbulence resulted from the nature of three-wave resonant interactions.

The ideas put forth by Montgomery and collaborators have physical relevance. However, we no longer believe that the anisotropic turbulence coexists with a more nearly isotropic spectrum of the IK form. In the following sections we shall see that the full physics is more complicated and more interesting:

1. The IK theory is incorrect [38].
2. Correcting the IK theory leads to a theory of balanced weak MHD turbulence [13,14], based on the three-wave resonant interactions, identified by [10,38a]. However, this theory has a limited inertial range, because nonlinear interactions strengthen.
3. When nonlinear interactions achieve strengths of order unity, the critical balance condition, suggested in [37], holds. The balanced strong MHD turbulence has an anisotropic Kolmogorov spectrum [11].

4. Why the IK theory is incorrect

Deriving the spectrum of eq. (16) is not enough. The IK theory assumes that individual collisions between wavepackets are weak. In other words, it was assumed that $N_\lambda \gg 1$ and so we need to verify if this is true. Using eqs (13) and (15), we can estimate that $N_\lambda \propto \lambda^{-1/2}$. Therefore N_λ increases as λ decreases; in other words, N_λ increases as we go deeper into the inertial range. So, it looks as if the phenomenological theory of the IK cascade is self-consistent. For many years, it seemed reasonable to suppose that the inertial-range spectrum of MHD turbulence was indeed given by eq. (16). Laboratory experiments on plasmas are very difficult to perform, and so experimental verification was not forthcoming. Also computers were not powerful enough for numerical simulations to check the predictions of the IK theory. This is where many years of astronomical observations forced theorists to look deeper into the problem.

4.1 *Enter astronomy: Interstellar scintillation and scattering*

Soon after the discovery of pulsars in 1967 [39], it was realized that the erratic fluctuations of the amplitudes of radiopulses were due to the effect of ionized interstellar medium on the propagation of radiopulses [40,41]: electron density variations in the interstellar medium cause refractive index variations, which alter the phase of a propagating radiowave. Interference and diffraction result in amplitude fluctuations which are detected as intensity scintillations, because of relative motion between the source, medium and observer. There are other related effects, such as the narrow frequency structure, pulse and angular broadening. Interstellar scintillation affects not only pulsars, but also other compact galactic and extragalactic radiosources. The scattering is generically strong. By this we mean that the fluctuations that contribute dominantly to diffractive scattering have length scales ($\sim 10^8$–10^9 cm) that are much smaller than the Fresnel length ($\sim 10^{11}$ cm). Long time variability in pulsars and extragalactic radiosources was explained by Rickett *et al* [42] as refractive scintillation on scales $\sim 10^{13}$–10^{14} cm; here the scattering is modulated weakly by refractive effects due to electron density fluctuations on the scale of the scattering disc.

These observations seemed to imply that the electron density fluctuations in the interstellar medium was close to a Kolmogorov spectrum spanning at least six decades in wavenumber. In other words, the three-dimensional power spectrum of electron density fluctuations is consistent with the form, $P(k) \propto k^{-\beta}$, where $\beta \simeq 11/3$, for k^{-1} ranging over scales 10^8–10^{14} cm [42a]. The site of these electron density fluctuations is thought to be the warm ionized medium (WIM) with mean electron density $\sim 10^{-1}\,\mathrm{cm}^{-3}$, and temperature $\sim 10^4$ K. The most common hypothesis seeks a turbulent origin.

4.2 *Why turbulence in the WIM is likely to be hydromagnetic*

The most natural model for WIM turbulence might appear to be the hydrodynamic turbulence, wherein the electron density fluctuations are passively mixed from large to small scales, just like atmospheric turbulence mixes density fluctuations – this will naturally lead to a Kolmogorov spectrum of density fluctuations, just like for density fluctuations in the Earth's atmosphere. The problem with this proposal is the following: the mean free path of electron (for Coulomb scattering) in the warm ionized medium $\ell_{\mathrm{mfp}} \sim 10^{13}$ cm. Let us assume that energy is put into the medium on some large-scale L with r.m.s. velocity amplitudes v_L. If this energy undergoes a Kolmogorov cascade, then the (inner) scale on which this hydrodynamic turbulence dissipates is

$$\ell_{\mathrm{in}} \sim \left(\frac{c_{\mathrm{s}}}{v_L}\right)^{3/4} L^{1/4} \ell_{\mathrm{mfp}}^{3/4}, \tag{17}$$

where $c_{\mathrm{s}} \sim 10\,\mathrm{km\,s^{-1}}$ is the speed of sound and $\ell_{\mathrm{mfp}} \sim 10^{13}$ cm is the mean free path (due to Coulomb scattering) in the WIM. It is reasonable to suppose that $v_L \sim c_{\mathrm{s}}$. So even if L is as small as 10^{14} cm – in fact it is likely to be larger than a parsec – the inner scale cannot be smaller than 10^{13} cm. Therefore, mixing by hydrodynamic turbulence cannot account for electron density fluctuations on scales $\sim 10^8$ cm, which are probed by diffractive interstellar scintillation.

The interstellar medium is known to have a large-scale magnetic field of strength $\sim 3\,\mu$G. This implies that the Alfvén speed in the WIM $V_{\mathrm{A}} \sim 10\,\mathrm{km\,s^{-1}}$; since $V_{\mathrm{A}} \sim c_{\mathrm{s}}$, the magnetic field is dynamically non-negligible. Drawing upon the work in [37], Higdon [44] suggested in 1984 that the electron density fluctuations could be caused by 'two-dimensional isobaric entropy variations with oppositely directed gradients in temperature and density projected transverse to the local approximately uniform magnetic field'. The presence of the strong mean magnetic field would suppress the transport coefficients, and dissipation would occur on very small length scales. If the dissipation occurs through some collisionless damping, inner scale is likely to be of order the proton Larmor radius, $r_{\mathrm{L}} \sim 3 \times 10^7$ cm. Then there would be a turbulent spectrum reaching down to very small scales, allowing density fluctuations to be mixed down to the scales probed by diffractive scintillation.

The ideas on anisotropic turbulence [36,37] were not quantitative enough to derive an inertial-range spectrum. On the other hand, the IK theory predicts a spectrum that is flatter than the observed Kolmogorov value; so we seem to have resolved one problem (i.e. hydrodynamic versus MHD turbulence), only to stumble into another!

4.3 *The problem with the IK theory*

When $N_\lambda \gg 1$, nonlinear interactions between oppositely directed wavepackets are weak, and we should be able to calculate better than we have done above. There is a well-developed theory, called weak turbulence, that describes weak nonlinear interactions between nearly linear waves – see [45]. In this approach, perturbation theory is used to derive kinetic equations in **k**-space describing wave–wave interactions. Then the inertial-range energy spectra emerge as stationary solutions carrying a flux of energy to large k. The lowest order of perturbation theory, corresponding to what are called three-wave interactions in weak turbulence, is what is needed to make the IK cascade rigorous. Here, the rate of change of energy (or wave action) at a given **k** is due to either (i) the coalescence of two other waves with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ or (ii) decay of **k** into two waves with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$. For any given **k**, all $\mathbf{k}_1$ and $\mathbf{k}_2$ contribute, so long as they satisfy certain resonance conditions that depend on the dispersion relations satisfied by the linear waves. In the case of incompressible MHD, these three-wave resonance conditions are somewhat special, because of the existence of the exact, nonlinear solutions given in eq. (10). Since there are nonlinear interactions only between oppositely directed wavepackets, it means that $\mathbf{k}_1$ and $\mathbf{k}_2$ must belong to oppositely directed wavepackets. Then the three-wave resonance conditions may be written as

$$\mathbf{k}_1 + \mathbf{k}_2 \;=\; \mathbf{k}_3\,, \quad \omega_1^{\pm} + \omega_2^{\mp} \;=\; \omega_k^{\pm} \quad \text{or} \quad \omega_k^{\mp}, \tag{18}$$

where $\omega_1^{\pm} = \pm V_{\rm A} k_{1z}$, $\omega_2^{\mp} = \mp V_{\rm A} k_{2z}$, $\omega_k^{\pm} = \pm V_{\rm A} k_z$ and $\omega_k^{\mp} = \mp V_{\rm A} k_z$. Note the resemblance to 'momentum–energy' relations in quasiparticle interactions in quantum field theory or condensed matter physics. These resonance conditions have a remarkable property, noted by Shebalin *et al* [10], that one of the k_{1z} or k_{2z} must be zero. This fact may be readily seen by considering the z-component of the wavevector resonance condition, $k_{1z} + k_{2z} = k_z$, together with the frequency resonance condition. If, for instance, we choose $\mathbf{k}_1$ and **k** to be $+$ waves and $\mathbf{k}_2$ to be a $-$ wave, then the frequency resonance condition is, $k_{1z} - k_{2z} = k_z$. In this case, it is clear that $k_{2z} = 0$, implying that $k_{1z} = k_z$. Hence waves with values of k_z not present initially cannot be created by nonlinear interactions during wavepacket collisions. In other words, there is no cascade of energy to smaller spatial scales in the z-direction, and hence the turbulence must be anisotropic.

5. Anisotropic MHD turbulence

It turns out that the different polarizations of the Alfvén waves and Slow waves play a very important role in the turbulent cascade. Viewed physically, we may think of the $(\mathbf{w}^{\pm}\cdot\nabla)\mathbf{w}^{\mp}$ terms as being responsible for the nonlinear cascade [45a]. During collisions between two oppositely directed wavepackets, we may think of $(\mathbf{w}^{\pm}\cdot\nabla)$ as coming from the wavepacket that offers a perturbation, and the $\mathbf{w}^{\mp}$ wavepacket as the one that suffers the perturbation. The polarization of an Alfvén wave is such that its $\mathbf{w}^{\pm}$ is perpendicular to $\hat{z}$ and, for $\lambda \ll L$, the polarization of a Slow wave is almost parallel to $\hat{z}$. Then we estimate that $(\mathbf{w}^{\pm}\cdot\nabla) \sim (w_\lambda/\lambda)$ when the perturbing wavepacket is an Alfvén wave, and $(\mathbf{w}^{\pm}\cdot\nabla) \sim (w_\lambda/L)$ when the perturbing wavepacket is a Slow wave. Clearly an Alfvén wave is a much stronger perturber than a Slow wave. Thus, it comes as no surprise that Slow waves are mostly scattered by Alfvén waves, but do not significantly perturb Alfvén waves; the cascade of Slow waves is driven by the Alfvén waves. Hence in our estimates below we ignore Slow waves, and restrict our attention to Alfvén waves.

5.1 *Balanced weak MHD turbulence*

The cascade is due to the resonant three-wave interactions discussed above (see [13,14,19, 21]). Consider again the steady, balanced and isotropic excitation of ±ve Alfvén waves, with amplitudes $w_L \ll V_A$ and scale L, which initiates a weak cascade to smaller spatial scales. As we have discussed above, there is no transfer of energy to smaller spatial scales in the direction parallel to the mean magnetic field. No parallel cascade implies that wavepackets of transverse scale $\lambda \ll L$ have parallel scales L. A collision between a +ve Alfvén wavepacket and a −ve Alfvén wavepacket now takes time $\sim L/V_A$, because the wavepackets have parallel scales $\sim L$. In one collision a wavepacket is perturbed by the amount

$$\delta w_\lambda \sim \frac{L w_\lambda^2}{\lambda V_A} < w_\lambda. \tag{19}$$

The perturbations add with random phases. The number of collisions for perturbations to build up to order unity is

$$N_\lambda \sim \left(\frac{\lambda V_A}{L w_\lambda}\right)^2. \tag{20}$$

The cascade time

$$t_\lambda \sim N_\lambda \frac{L}{V_A}. \tag{21}$$

Kolmogorov's hypothesis of the λ-independence of the energy flux

$$\varepsilon \sim \frac{w_\lambda^2}{t_\lambda} \tag{22}$$

implies that in the inertial range, the r.m.s. amplitudes are given by

$$w_\lambda \sim w_L \left(\frac{\lambda}{L}\right)^{1/2}. \tag{23}$$

The energy spectrum, $E(\mathbf{k})$, is highly anisotropic in **k**-space, and now depends on both k_z and $k_\perp$. The precise dependence of E on k_z is not so interesting, because there is no parallel cascade; it suffices to know that E is largely confined to the region $|k_z| < L^{-1}$. However, the dependence of E on $k_\perp$ is of great interest, because this has been established by the cascade in the transverse directions, described above. Accounting for this anisotropy, we can estimate $E(k_\perp, k_z)\, k_\perp^2\, L^{-1} \sim w_\lambda^2$, where $k_\perp \sim \lambda^{-1}$. Now, using eq. (23), we have

$$E(k_\perp, k_z) \sim \frac{w_L^2}{k_\perp^3}, \qquad \text{for} \quad |k_z| < L^{-1}. \tag{24}$$

Being a theory based on weak turbulence, the turbulent cascade can be described rigorously using kinetic equations for energy transfer (see [19,21]).

Having obtained the inertial-range energy spectrum of the balanced weak cascade, we need to perform checks of self-consistency regarding the assumed weakness of the cascade. Using eqs (20) and (23), we estimate that

$$N_\lambda \sim \left(\frac{V_A}{w_L}\right)^2 \frac{\lambda}{L}. \tag{25}$$

We must first verify that the cascade initiated at the stirring scale L is weak to begin with, which implies that we must have $N_L \gg 1$. From eq. (25), we see that this can be satisfied if $w_L \ll V_{\rm A}$. In other words, isotropic, balanced stirring initiates a weak cascade if the r.m.s. amplitudes at the stirring scale is much less than the Alfvén speed. A more geometric way of stating this is that a weak cascade is initiated if stirring bends field lines (of the mean magnetic field) only by small angles.

Equation (25) implies more interesting consequences: the transverse cascade strengthens as λ decreases, because N_λ decreases when λ decreases. Therefore the assumption of the weakness of interactions must break down when $N_\lambda \sim 1$, which happens when $\lambda \sim \lambda_* \sim L(w_L/V_{\rm A})^2$. Therefore, the inertial-range spectrum of eq. (24) is valid only for transverse scales larger than λ_*.

5.2 *Balanced strong MHD turbulence*

When $N_\lambda \sim 1$, the assumption of the weakness of the cascade is no longer even approximately valid. The turbulent cascade turns strong, and was first described by Goldreich and Sridhar [11]. They conjectured that N_λ remains of order unity for smaller values of λ all the way down to the dissipation scale. This is equivalent to assuming that the cascade time remains of order the wave period, a condition that may be referred to as critical balance. The balanced weak cascade described above turns into a balanced strong cascade for $\lambda < \lambda_*$. Instead of following the transition from weak to strong, we prefer to describe the inertial range of the balanced strong cascade when it is initiated at the stirring scale itself because there is less clutter in the description.

Consider steady, balanced and isotropic excitation of $\pm$ve waves, with amplitudes $w_L \sim V_{\rm A}$ and scale L, which initiates a strong cascade to smaller spatial scales. Wavepackets of transverse scale $\lambda < L$ can now possess parallel scales that are smaller than L. This is because the resonance conditions are not in force when nonlinear interactions are strong. So, in addition to a transverse cascade, a parallel cascade can also occur. Let eddies of transverse scale λ possess parallel scale Λ_λ, which is as yet an unknown function of the transverse scale λ. Critical balance implies that

$$t_\lambda \sim \frac{\lambda}{w_\lambda} \sim \frac{\Lambda_\lambda}{V_{\rm A}}. \qquad (26)$$

As before, we invoke Kolmogorov's hypothesis of the λ-independence of the energy flux

$$\varepsilon \sim \frac{w_\lambda^2}{t_\lambda} \qquad (27)$$

which implies that in the inertial range

$$w_\lambda \sim V_A \left(\frac{\lambda}{L}\right)^{1/3}, \quad \Lambda_\lambda \sim L^{1/3}\lambda^{2/3}. \qquad (28)$$

The energy spectrum is again anisotropic and may be estimated as $E(k_\perp, k_z)\, k_\perp^2\, (\Lambda_\lambda)^{-1} \sim w_\lambda^2$, where $k_\perp \sim \lambda^{-1}$. Now, using eq. (28), we have

$$E(k_\perp, k_z) \sim \frac{w_L^2}{L^{1/3}k_\perp^{10/3}}, \qquad \text{for} \quad |k_z| < \frac{k_\perp^{2/3}}{L^{1/3}}. \qquad (29)$$

Summary of the balanced strong cascade:

1. The cascade is critically balanced in that the cascade time is of order of the wave period throughout the inertial range.
2. The r.m.s. amplitudes have a scaling with the transverse scale which is of the Kolmogorov form.
3. The turbulent cascade occurs in the transverse as well as parallel directions. However, the cascade occurs predominantly in the transverse directions, because the parallel correlation lengths scale as the 2/3rd power of the transverse correlation lengths.

It has sometimes been argued that there is a 'dynamical alignment' of velocity and magnetic fields, resulting in spectra that are flatter than Kolmogorov [25,46]. However, as discussed below, this would not be in agreement with solar wind data.

6. Interplanetary turbulence

The balanced strong cascade described in the previous section has found application in many areas of astronomy. However, only laboratory experiments (which are difficult to perform) and the solar wind offer the possibility of direct measurements of MHD turbulence. The solar wind is a result of the expansion of the hot solar corona, flowing at several hundred km s^{-1}. It is also magnetized, with an Alfvén speed near Earth of a few tens of km s^{-1}. Since the 1960s spacecrafts have travelled outside the Earth's magnetosphere into the space. Interplanetary turbulence has been studied, using many years of *in situ* measurements of the fluctuations of magnetic and velocity fields in the solar wind [47]. Belcher and Davis [48] first established that MHD scale fluctuations in the solar wind are predominantly Alfvénic; i.e., magnetic and velocity were either correlated or anticorrelated. In other words, they showed that the fluctuations were composed predominantly of one kind of Elsasser field, w^+ or w^-, but not both at any given time. Moreover, the Alfvén waves are mostly propagating away from the Sun – whether these are w^+ or w^- depends on the polarity of the local mean magnetic field. Thus, MHD turbulence in the solar wind can be expected to be highly imbalanced.

Besides the transient coronal mass ejections, there are two classes of solar wind, the fast ($\sim$750 km s^{-1}) and the slow ($\sim$250–500 km s^{-1}). The fast wind flows out of coronal holes which have open magnetic field geometries, whereas the slow wind appears to originate near the boundaries of the coronal holes. The fast wind is more steady than the slow wind, and turbulence is best studied in the fast wind at high solar latitudes near solar minimum. The Ulysses spacecraft has explored the solar wind at high latitudes, from October 1990 to June 2009, and a lot of what we know about MHD turbulence in the solar wind comes from the analysis of data accumulated over many years. Solar wind speeds are much greater than either the orbital speeds of spacecrafts or the speed of MHD waves, and a spacecraft time series is approximately a straight line cut which is like a snapshot of the solar wind plasma. By Taylor's hypothesis, we can relate the measurement frequencies, f, to a wavenumber: $k_0 = 2\pi f/V$, where V is the solar wind speed. Let $P_{ij}^{\text{red}}(f)$ be the (reduced) temporal spectrum of correlation between the ith and jth components of magnetic/velocity fluctuation. This is related to the three-dimensional spectrum in the plasma frame, $P_{ij}(\mathbf{k})$, by,

$$P_{ij}^{\rm red}(f) \;=\; \int {\rm d}^3k\ \delta(k_0 - \hat{\mathbf{V}}\cdot\mathbf{k})\,P_{ij}(\mathbf{k})\,. \tag{30}$$

If the turbulence is isotropic, it is possible to recover $P_{ij}(\mathbf{k})$ from a measurement of $P_{ij}^{\rm red}(f)$. However, when the turbulence is anisotropic – as is expected of MHD turbulence – it is necessary to use additional assumptions.

Wicks *et al* [49] have recently measured the power and spectral index anisotropy of the magnetic field fluctuations in the fast solar wind. Their measurements span scales larger than the outer scale down to the ion Larmor radius, thus covering the entire inertial range of turbulence. These are consistent with the anisotropic Kolmogorov spectrum predicted by Goldreich and Sridhar [11], described above. However, interplanetary turbulence is highly imbalanced, whereas the theory of [11] applies only to balanced, strong MHD turbulence. Now I describe an appropriate generalization to the imbalanced case.

7. Imbalanced strong MHD turbulence

When MHD turbulence is imbalanced, the energy flux of the +ve waves is different from that of the −ve waves; the cross-helicity of the turbulence is non-zero. This is clearly the more general and generic case of MHD turbulence, and the account given below is based on [27,49a]. Consider the steady, imbalanced and isotropic excitation of ±ve waves on some large scale L, with unequal Elsasser energy fluxes, ε^+ and ε^-. An anisotropic turbulent cascade carries energy to small scales. As before, let $\lambda \ll L$ be a transverse scale in the inertial range. The task is to determine the r.m.s. Elsasser amplitudes, $w_\lambda^\pm$ and the corresponding parallel correlation lengths, $\Lambda_\lambda^\pm$. The analysis is complicated. So we only give a summary here and refer the reader to [27] for details:

1. In common with the balanced strong cascade, the energy spectra of both Elsasser waves are of the anisotropic Kolmogorov form
$$w_\lambda^\pm \;\sim\; \frac{(\varepsilon^\pm)^{2/3}}{(\varepsilon^\mp)^{1/3}}\,\lambda^{1/3}\,. \tag{31}$$
Therefore, the ratio of the Elsasser amplitudes is independent of scale, and is equal to the ratio of the corresponding energy fluxes. Thus we can infer turbulent flux ratios from the amplitude ratios, providing insight into the origin of the turbulence.
2. The parallel correlation lengths of ±ve waves are equal to each other on all scales, and proportional to the 2/3rd power of the transverse correlation length.
$$\Lambda_\lambda^\pm \;\equiv\; \Lambda_\lambda \;\sim\; \frac{(\varepsilon^-)^{1/3}}{(\varepsilon^+)^{2/3}}\,V_{\rm A}\lambda^{2/3}\,. \tag{32}$$
3. The equality of cascade time and waveperiod (critical balance) that characterizes the strong balanced cascade does not apply to the Elsasser field with the larger amplitude. Instead, the more general criterion that always applies to both Elsasser fields is that the cascade time is equal to the correlation time of the straining imposed by oppositely-directed waves.
4. In the limit that the energy fluxes are equal, the turbulence corresponds to the balanced strong cascade.

8. Comments

There are other theories of imbalanced cascades [29–31,33,34] which differ from the account given in the previous section. It is intriguing that the most recent work by Wicks *et al* [50] presents results on measurements of turbulence in the solar wind by the WIND spacecraft, which do not appear to be fully explained by any of the theories of incompressible imbalanced MHD turbulence! The solar wind is probably the best laboratory we have, to explore MHD turbulence.

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