Nonperturbative quasilinear approach to the shear dynamo problem

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We study large-scale dynamo action due to turbulence in the presence of a linear shear flow. Our treatment is quasilinear and equivalent to the standard "first-order smoothing approximation." However it is non perturbative in the shear strength. We first derive an integrodifferential equation for the evolution of the mean magnetic field, by systematic use of the shearing coordinate transformation and the Galilean invariance of the linear shear flow. We show that, for nonhelical turbulence, the time evolution of the cross-shear components of the mean field do not depend on any other components excepting themselves; this is valid for any Galilean-invariant velocity field, independent of its dynamics. Hence, to all orders in the shear parameter, there is no shear-current-type effect for non helical turbulence in a linear shear flow in quasilinear theory in the limit of zero resistivity. We then develop a systematic approximation of the integro-differential equation for the case when the mean magnetic field varies slowly compared to the turbulence correlation time. For nonhelical turbulence, the resulting partial differential equations can again be solved by making a shearing coordinate transformation in Fourier space. The resulting solutions are in the form of shearing waves, labeled by the wave number in the sheared coordinates. These shearing waves can grow at early and intermediate times but are expected to decay in the long time limit.

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I. INTRODUCTION

The origin of large scale magnetic fields in astrophysical systems from stars to galaxies is an issue of considerable interest. The standard paradigm involves dynamo amplification of seed magnetic fields due to turbulent flows which have helicity combined with shear. Shear flows and turbulence are ubiquitous in astrophysical systems although the turbulence in general may not be helical. However, the presence of shear by itself may open new pathways to the operation of large-scale dynamos, even if the turbulence lacks a coherent helicity [1–5]. The evidence for such large-scale dynamo action under the combined action of non helical turbulence and background shear flow comes mainly from several direct numerical simulations [1,2]. How such a dynamo works is not yet clear. One possibility is the shear-current effect [4], in which extra components of the mean electromotive force (EMF) arise due to shear, which couple components of the mean magnetic field parallel and perpendicular to the shear flow. However, there is no convergence yet on whether the sign of the relevant coupling term is such as to obtain a dynamo; some analytic calculations [6,7] and numerical experiments [1] find that the sign of the shear-current term is unfavorable for dynamo action.

In an earlier paper [8] (Paper I), we had outlined briefly a quasilinear theory of dynamo action in a linear shear flow of an incompressible fluid, which has random velocity fluctuations due either to freely decaying turbulence or generated through external forcing. Our analysis did not put any restrictions on the strength of the shear, unlike earlier analytic work, which treated shear as a small perturbation. We arrived at an integrodifferential equation for the evolution of the mean magnetic field and argued that the shear-current assisted dynamo is essentially absent in quasilinear theory in the limit of zero resistivity. In the present paper, we give detailed derivations of the main results of Paper I. We also extend our work further by deriving differential equations for the mean field, in the limit when the correlation time of the turbulence is much smaller than the time scale over which the mean field varies. This allows us to solve for the mean-field evolution in terms of the velocity correlation functions. We can draw some general conclusions on the shear dynamo independent of the exact velocity dynamics. In particular, we note that the shear dynamo can lead to transient growth of large-scale fields in the form of shearing waves, but these waves ultimately decay, even in the absence of microscopic diffusion.

In Sec. II, we formulate the shear dynamo problem. Our theory is "local" in character: in the laboratory frame we consider a background shear flow whose velocity is unidirectional (along the X_2 axis) and varies linearly in an orthogonal direction (the X_1 axis). Section III outlines a quasilinear theory of the shear dynamo. Systematic use of the shearing transformation allows us to develop a theory that is nonperturbative in the strength of the background shear. However, we ignore the complications associated with nonlinear interactions, hence, magnetohydrodynamic (MHD) turbulence and the small-scale dynamo; so our theory is quasilinear in nature, equivalent to the "first order smoothing approximation" (FOSA) [9,10]. The linear shear flow has a basic symmetry relating to measurements made by a special subset of all observers, who may be called comoving observers. This symmetry is the invariance of the equations with respect to a

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group of transformations that is a subgroup of the full Galilean group. It may be referred to as "shear-restricted Galilean invariance," or Galilean invariance (GI). It should be noted that the laboratory frame and its set of co-moving observers need not be inertial frames; in fact one of the main applications of GI is to the *shearing sheet*, which is a rotating frame. We introduce and explore the consequences of GI velocity fluctuations in Sec. IV. Such velocity fluctuations are not only compatible with the underlying symmetry of the problem, but they are expected to arise naturally. This has profound consequences for dynamo action, because the transport coefficients that define the mean EMF become spatially homogeneous in spite of the shear flow. The derivation of an integro-differential equation for the mean magnetic field is given in Sec. V. We discuss a number of ways of approximating this equation in Sec. VI, for slowly varying mean fields, all of which lead to the same set of partial differential equations for the mean-field. The mean field dynamics is further studied in Sec. VII, and VIII presents a discussion of the main results and the conclusions.

II. SHEAR DYNAMO PROBLEM

Let $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ be the unit vectors of a Cartesian coordinate system in the laboratory frame, $\boldsymbol{X}=(X_1,X_2,X_3)$ the position vector, and τ the time. The fluid velocity is given by $(-2AX_1\boldsymbol{e}_2+\boldsymbol{v})$, where A is the shear parameter (Oort's first constant) and $\boldsymbol{v}(\boldsymbol{X},\tau)$ is a randomly fluctuating velocity field. The total magnetic field, $\boldsymbol{B}'(\boldsymbol{X},\tau)$, obeys the induction equation,

$$\left(\frac{\partial}{\partial\tau}-2AX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}'+2AB_1'\boldsymbol{e}_2=\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}')+\eta\boldsymbol{\nabla}^2\boldsymbol{B}'. \tag{1}$$

The *shear dynamo problem* may be stated as follows: given some statistics of velocity fluctuations, what can be said about the magnetic field? More specific questions may be posed: does the combined action of the background shear and random velocities lead to the growth of a large-scale component of the magnetic field (i.e., a *turbulent dynamo*)? In particular, is there turbulent dynamo action when the velocity fluctuations possess mirror symmetry (i.e., when the velocity fluctuations are *nonhelical*)?

A common approach to the problem is through the theory of *mean-field electrodynamics*. Here, the action of zero-mean velocity fluctuations ($\langle\boldsymbol{v}\rangle=\boldsymbol{0}$) on some seed magnetic field is assumed to produce a total magnetic field with a well-defined *mean-field* ($\boldsymbol{B}$) and a *fluctuating-field* ($\boldsymbol{b}$):

$$\boldsymbol{B}'=\boldsymbol{B}+\boldsymbol{b},\quad \langle\boldsymbol{B}'\rangle=\boldsymbol{B},\quad \langle\boldsymbol{b}\rangle=\boldsymbol{0}, \tag{2}$$

where $\langle\ \rangle$ denotes ensemble averaging in the sense of Reynolds. Applying Reynolds averaging to the induction Eq. (1), we obtain the following equations governing the dynamics of the mean and fluctuating magnetic fields:

$$\left(\frac{\partial}{\partial\tau}-2AX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}+2AB_1\boldsymbol{e}_2=\boldsymbol{\nabla}\times\mathcal{E}+\eta\boldsymbol{\nabla}^2\boldsymbol{B}, \tag{3}$$

$$\left(\frac{\partial}{\partial\tau}-2AX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}+2Ab_1\boldsymbol{e}_2 =\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})+\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{b}-\mathcal{E})+\eta\boldsymbol{\nabla}^2\boldsymbol{b}, \tag{4}$$

where $\mathcal{E}=\langle\boldsymbol{v}\times\boldsymbol{b}\rangle$ is the mean EMF. The first step toward solving the problem is to calculate $\mathcal{E}$ and obtain a closed equation for the mean-field, $\boldsymbol{B}(\boldsymbol{X},\tau)$. In the general case, it is necessary to specify the dynamics of $\boldsymbol{v}$, which could be influenced by Lorentz forces due to both $\boldsymbol{B}$ and $\boldsymbol{b}$.

III. QUASILINEAR THEORY

To calculate the mean EMF we make some simplifying assumptions. We first make the *quasilinear* approximation in solving Eq. (4) for $\boldsymbol{b}$ by dropping terms that are quadratic in the fluctuations. Note that the dynamics of $\boldsymbol{v}$ is not prescribed; it does not imply absence of velocity dynamics. For instance, the fluid can be acted upon by Lorentz forces due to the magnetic field, Coriolis force as in the case of the *shearing sheet* or buoyancy in a convective flow. In this paper, we will not specify any particular dynamics for the velocity field. We also drop the resistive term in the interests of simplicity of presentation. Setting $\eta=0$ may seem like a drastic step, but we would like to assure the reader that the theory can be reworked without this limitation and that our main conclusions carry through, even for $\eta\neq 0$. In particular, we recover the results of this paper in the limit $\eta\to 0$. We note that the limit $\eta\to 0$ is also compatible with the physical situation in which the correlation times are small compared to the eddy turn-over time scale; so our theory is applicable when the FOSA is valid. The fluctuating velocity field is assumed be incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$). This restriction is not crucial and may be lifted without much difficulty.

The quasilinear approximation is equivalent to neglecting the effects of magnetohydrodynamic turbulence and small-scale dynamo action for the determination of $\mathcal{E}$. With these assumptions, the equation for $\boldsymbol{b}$ we will solve is

$$\left(\frac{\partial}{\partial\tau}-2AX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}+2Ab_1\boldsymbol{e}_2=\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) =(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B}. \tag{5}$$

A. Shearing coordinate transformation

Equation (5) is inhomogeneous in the coordinate X_1. It is convenient to exchange spatial inhomogeneity for temporal inhomogeneity, so we get rid of the $(X_1\partial/\partial X_2)$ term through a shearing transformation to space-time variables,

$$x_1=X_1,\quad x_2=X_2+2A\tau X_1,\quad x_3=X_3,\quad t=\tau. \tag{6}$$

Then partial derivatives transform as

$$\frac{\partial}{\partial X_1}=\frac{\partial}{\partial x_1}+2At\frac{\partial}{\partial x_2},$$

$$\frac{\partial}{\partial X_2}=\frac{\partial}{\partial x_2},\quad \frac{\partial}{\partial X_3}=\frac{\partial}{\partial x_3},\quad \frac{\partial}{\partial\tau}=\frac{\partial}{\partial t}+2Ax_1\frac{\partial}{\partial x_2}. \tag{7}$$

We also define variables, which are component-wise equal to the old variables,

$$\boldsymbol{H}(\boldsymbol{x},t) = \boldsymbol{B}(\boldsymbol{X},\tau), \quad \boldsymbol{h}(\boldsymbol{x},t) = \boldsymbol{b}(\boldsymbol{X},\tau), \quad \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{X},\tau) \tag{8}$$

It is important to note that, just like the old variables, the variables are expanded in the *fixed Cartesian basis of the laboratory frame*. For example, $\boldsymbol{H}=H_1\boldsymbol{e}_1+H_2\boldsymbol{e}_2+H_3\boldsymbol{e}_3$, where $H_i(\boldsymbol{x},t)=B_i(\boldsymbol{X},\tau)$, and similarly for the other variables. In the new variables, Eq. (5) becomes,

$$\frac{\partial \boldsymbol{h}}{\partial t} + 2Ah_1\boldsymbol{e}_2 = \left(\boldsymbol{H}\cdot\frac{\partial}{\partial \boldsymbol{x}} + 2AtH_1\frac{\partial}{\partial x_2}\right)\boldsymbol{u} - \left(\boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{x}} + 2Atu_1\frac{\partial}{\partial x_2}\right)\boldsymbol{H}. \tag{9}$$

Equation (9) for $\boldsymbol{h}(\boldsymbol{x},t)$ does not contain spatial derivatives of $\boldsymbol{h}$, so it can be integrated directly. We are interested in the particular solution, which vanishes at $t=0$. The solutions for $h_1(\boldsymbol{x},t)$ and $h_3(\boldsymbol{x},t)$ are

$$h_1 = \int_0^t dt' u'_{1l}[H'_l + 2At'\delta_{l2}H'_1] - \int_0^t dt'[u'_l + 2At'\delta_{l2}u'_1]H'_{1l}, \tag{10}$$

$$h_3 = \int_0^t dt' u'_{3l}[H'_l + 2At'\delta_{l2}H'_1] - \int_0^t dt'[u'_l + 2At'\delta_{l2}u'_1]H'_{3l}, \tag{11}$$

where primes denote evaluation at space-time point $(\boldsymbol{x},t')$. We have also used notation $u_{ml}=(\partial u_m/\partial x_l)$ and $H_{ml}=(\partial H_m/\partial x_l)$.

The equation for $h_2(\boldsymbol{x},t)$ involves $h_1(\boldsymbol{x},t)$; the solution is

$$h_2 = \int_0^t dt' u'_{2l}[H'_l + 2At'\delta_{l2}H'_1] - \int_0^t dt'[u'_l + 2At'\delta_{l2}u'_1]H'_{2l} - 2A\int_0^t dt' h'_1. \tag{12}$$

We need to evaluate the integral

$$\int_0^t dt' h'_1 = \int_0^t dt' \int_0^{t'} dt'' u''_{1l}[H''_l + 2At''\delta_{l2}H''_1] - \int_0^t dt' \int_0^{t'} dt''[u''_l + 2At''\delta_{l2}u''_1]H''_{1l}, \tag{13}$$

where the double-primes denote evaluation at space-time point $(\boldsymbol{x},t'')$. We now note that, for any function $f(\boldsymbol{x},t)$, the double-time integral

$$\int_0^t dt' \int_0^{t'} dt'' f(\boldsymbol{x},t'') = \int_0^t dt'' f(\boldsymbol{x},t'') \int_{t''}^t dt' = \int_0^t dt''(t-t'')f(\boldsymbol{x},t'') = \int_0^t dt'(t-t')f(\boldsymbol{x},t')$$

reduces to a single-time integral, where in the last equality we have merely replaced the dummy integration variable t'' by t'. Then

$$\int_0^t dt' h'_1 = \int_0^t dt'(t-t')u'_{1l}[H'_l + 2At'\delta_{l2}H'_1] - \int_0^t dt'(t-t')[u'_l + 2At'\delta_{l2}u'_1]H'_{1l} \tag{14}$$

can be used in Eq. (12) to get an explicit solution for $h_2(\boldsymbol{x},t)$. Combining Eqs. (10)–(12) we can write $\boldsymbol{h}(\boldsymbol{x},t)$ in component form as

$$h_m(\boldsymbol{x},t) = \int_0^t dt'[u'_{ml} - 2A(t-t')\delta_{m2}u'_{1l}][H'_l + 2At'\delta_{l2}H'_1] - \int_0^t dt'[u'_l + 2At'\delta_{l2}u'_1][H'_{ml} - 2A(t-t')\delta_{m2}H'_{1l}]. \tag{15}$$

B. Mean EMF

The expression in Eq. (15) for $\boldsymbol{h}$ should be substituted in $\mathcal{E}=\langle \boldsymbol{v}\times\boldsymbol{b}\rangle=\langle \boldsymbol{u}\times\boldsymbol{h}\rangle$. Following standard procedure, we allow $\langle\,\rangle$ to act only on the velocity variables but not the mean field; symbolically, it is assumed that $\langle \boldsymbol{uuH}\rangle=\langle \boldsymbol{uu}\rangle\boldsymbol{H}$. Interchanging the dummy indices (l,m) in the last term of Eq. (15), the mean EMF is given in component form as

$$\begin{aligned}\mathcal{E}_i(\boldsymbol{x},t) &= \epsilon_{ijm}\langle u_j h_m\rangle \\ &= \int_0^t dt'[\hat{\alpha}_{il}(\boldsymbol{x},t,t') - 2A(t-t')\hat{\beta}_{il}(\boldsymbol{x},t,t')] \\ &\quad \times[H'_l + 2At'\delta_{l2}H'_1] - \int_0^t dt'[\hat{\eta}_{iml}(\boldsymbol{x},t,t') \\ &\quad + 2At'\delta_{m2}\hat{\eta}_{i1l}(\boldsymbol{x},t,t')][H'_{lm} - 2A(t-t')\delta_{l2}H'_{1m}],\end{aligned} \tag{16}$$

where the *transport coefficients*, $(\hat{\alpha},\hat{\beta},\hat{\eta})$, are defined in terms of the $\boldsymbol{uu}$ velocity correlators by

$$\begin{aligned}\hat{\alpha}_{il}(\boldsymbol{x},t,t') &= \epsilon_{ijm}\langle u_j(\boldsymbol{x},t)u_{ml}(\boldsymbol{x},t')\rangle, \\ \hat{\beta}_{il}(\boldsymbol{x},t,t') &= \epsilon_{ij2}\langle u_j(\boldsymbol{x},t)u_{1l}(\boldsymbol{x},t')\rangle, \\ \hat{\eta}_{iml}(\boldsymbol{x},t,t') &= \epsilon_{ijl}\langle u_j(\boldsymbol{x},t)u_m(\boldsymbol{x},t')\rangle.\end{aligned} \tag{17}$$

To obtain more specific expressions for the transport coefficients, we need to provide information on the $\boldsymbol{uu}$ velocity correlators. However, it is physically more transparent to consider velocity statistics in terms of $\boldsymbol{vv}$ velocity correlators, because this is referred to the laboratory frame instead of the sheared coordinates. By definition [Eq. (8)],

$$u_m(\boldsymbol{x},t) = v_m(\boldsymbol{X}(\boldsymbol{x},t),t), \tag{18}$$

where

$$X_1 = x_1, \quad X_2 = x_2 - 2Atx_1, \quad X_3 = x_3, \quad \tau = t \tag{19}$$

are the inverse of the shearing transformation given in Eq. (6). Using

$$\frac{\partial}{\partial x_l} = \frac{\partial}{\partial X_l} - 2A\tau\delta_{l1}\frac{\partial}{\partial X_2}, \tag{20}$$

the velocity gradient u_{ml} can be written as

$$u_{ml} = \left(\frac{\partial}{\partial X_l} - 2A\tau\delta_{l1}\frac{\partial}{\partial X_2}\right)v_m = v_{ml} - 2A\tau\delta_{l1}v_{m2}, \tag{21}$$

where $v_{ml}=(\partial v_m/\partial X_l)$. Then the transport coefficients are given in terms of the $\boldsymbol{vv}$ velocity correlators by

$$\hat{\alpha}_{il}(\boldsymbol{x},t,t') = \epsilon_{ijm}[\langle v_j(\boldsymbol{X},t)v_{ml}(\boldsymbol{X}',t')\rangle - 2At'\delta_{l1}\langle v_j(\boldsymbol{X},t)v_{m2}(\boldsymbol{X}',t')\rangle],$$

$$\hat{\beta}_{il}(\boldsymbol{x},t,t') = \epsilon_{ij2}[\langle v_j(\boldsymbol{X},t)v_{1l}(\boldsymbol{X}',t')\rangle - 2At'\delta_{l1}\langle v_j(\boldsymbol{X},t)v_{12}(\boldsymbol{X}',t')\rangle],$$

$$\hat{\eta}_{iml}(\boldsymbol{x},t,t') = \epsilon_{ijl}\langle v_j(\boldsymbol{X},t)v_m(\boldsymbol{X}',t')\rangle, \tag{22}$$

where $\boldsymbol{X}$ and $\boldsymbol{X}'$ are shorthand for

$$\boldsymbol{X} = (x_1, x_2 - 2Atx_1, x_3), \quad \boldsymbol{X}' = (x_1, x_2 - 2At'x_1, x_3). \tag{23}$$

Equation (16), together with Eq. (17) or (22), gives the mean EMF in general form. $\boldsymbol{X}$ and $\boldsymbol{X}'$ can be thought of as the coordinates of the origin, at times t and t' respectively, of an observer *comoving* with the background shear flow. Therefore, the transport coefficients depend only on the velocity correlators measured by such an observer at the origin of her coordinate system. This fact will have profound consequences for dynamo action, when we consider G-invariant velocity correlators in the next section. Before discussing the Galilean invariance of the linear shear flow, we derive the form of the mean EMF for a special case, when the velocity field is "delta correlated in time."

C. Delta-correlated-in-time velocity correlator

Although somewhat artificial, it is not uncommon to study dynamo action due to velocity fields whose correlation times are supposed so small that the two-point correlator taken between space-time points $(\boldsymbol{R},\tau)$ and $(\boldsymbol{R}',\tau')$ is assumed to be

$$\langle v_i(\boldsymbol{R},\tau)v_j(\boldsymbol{R}',\tau')\rangle = \delta(\tau-\tau')T_{ij}(\boldsymbol{R},\boldsymbol{R}',\tau). \tag{24}$$

Incompressibility implies that

$$\frac{\partial T_{ij}}{\partial R_i} = 0, \quad \frac{\partial T_{ij}}{\partial R_j'} = 0. \tag{25}$$

We define

$$T_{ijl}(\boldsymbol{R},\tau) = \left(\frac{\partial T_{ij}}{\partial R_l'}\right)_{\boldsymbol{R}'=\boldsymbol{R}}. \tag{26}$$

The delta-function ensures that $\boldsymbol{X}$ and $\boldsymbol{X}'$ defined in Eq. (23) are equal to each other. Then the velocity correlators

$$\langle v_i(\boldsymbol{X},t)v_j(\boldsymbol{X}',t')\rangle = \delta(t-t')T_{ij}(\boldsymbol{X},\boldsymbol{X},t),$$

$$\langle v_i(\boldsymbol{X},t)v_{jl}(\boldsymbol{X}',t')\rangle = \delta(t-t')T_{ijl}(\boldsymbol{X},t). \tag{27}$$

Substitute Eq. (27) in Eq. (22) for the transport coefficients,

$$\hat{\alpha}_{il}(\boldsymbol{x},t,t') = \delta(t-t')\epsilon_{ijm}[T_{jml} - 2At\delta_{l1}T_{jm2}],$$

$$\hat{\beta}_{il}(\boldsymbol{x},t,t') = \delta(t-t')\epsilon_{ij2}[T_{j1l} - 2At\delta_{l1}T_{j12}],$$

$$\hat{\eta}_{iml}(\boldsymbol{x},t,t') = \delta(t-t')\epsilon_{ijl}T_{jm}, \tag{28}$$

and use these expressions in Eq. (16). The delta-function ensures that the integrals over time can all be performed explicitly, so the mean EMF is

$$\mathcal{E}_i = \epsilon_{ijm}[T_{jml} - 2At\delta_{l1}T_{jm2}][H_l + 2At\delta_{l2}H_1] - \epsilon_{ijl}[T_{jm} + 2At\delta_{m2}T_{j1}]H_{lm}. \tag{29}$$

It is useful to write the EMF in terms of the original variables and laboratory frame coordinates. To this end we transform

$$H_{lm} = \left(\frac{\partial}{\partial X_m} - 2A\tau\delta_{m1}\frac{\partial}{\partial X_2}\right)B_l = B_{lm} - 2A\tau\delta_{m1}B_{l2}, \tag{30}$$

where $B_{lm}=(\partial B_l/\partial X_m)$. Then the explicit dependence of $\mathcal{E}_i$ on the shear parameter A cancels out, and the mean EMF assumes the simple form,

$$\mathcal{E}_i = \epsilon_{ijm}T_{jml}B_l - \epsilon_{ijl}T_{jm}B_{lm}, \tag{31}$$

which is identical to the familiar expression in the absence of background shear. Therefore we conclude that, to obtain nontrivial effects due to the shear flow, it is necessary to consider velocity correlators with nonzero correlation times. Henceforth, we shall consider the general case of finite velocity correlation times.

IV. GALILEAN INVARIANCE

The linear shear flow has a basic symmetry relating to measurements made by a special subset of all observers. We define a co-moving observer as one whose velocity with respect to the laboratory frame is equal to the velocity of the background shear flow, and whose Cartesian coordinate axes are aligned with those of the laboratory frame. A co-moving observer can be labeled by the coordinates, $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$ with respect to the laboratory frame, of her origin at time $\tau=0$. Different labels identify different co-moving observers and vice versa. As the labels run over all possible values, they exhaust the set of all co-moving observers. The origin of the coordinate axes of a co-moving observer translates with uniform velocity; its position with respect to the origin of the laboratory frame is given by

$$\boldsymbol{X}_c(\tau) = (\xi_1, \xi_2 - 2A\tau\xi_1, \xi_3). \tag{32}$$

An event with space-time coordinates $(\boldsymbol{X},\tau)$ in the laboratory frame has space-time coordinates $(\widetilde{\boldsymbol{X}},\widetilde{\tau})$ with respect to the co-moving observer, given by

$$\tilde{\boldsymbol{X}} = \boldsymbol{X} - \boldsymbol{X}_c(\tau), \quad \tilde{\tau} = \tau - \tau_0, \tag{33}$$

where the arbitrary constant τ_0 allows for translation in time as well.

Let $[\tilde{\boldsymbol{B}}'(\tilde{\boldsymbol{X}},\tilde{\tau}),\tilde{\boldsymbol{B}}(\tilde{\boldsymbol{X}},\tilde{\tau}),\tilde{\boldsymbol{b}}(\tilde{\boldsymbol{X}},\tilde{\tau}),\tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}},\tilde{\tau})]$ denote the total, the mean, the fluctuating magnetic fields and the fluctuating velocity field, respectively, as measured by the co-moving observer. These are all equal to the respective quantities measured in the laboratory frame,

$$[\tilde{\boldsymbol{B}}'(\tilde{\boldsymbol{X}},\tilde{\tau}),\tilde{\boldsymbol{B}}(\tilde{\boldsymbol{X}},\tilde{\tau}),\tilde{\boldsymbol{b}}(\tilde{\boldsymbol{X}},\tilde{\tau}),\tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}},\tilde{\tau})] = [\boldsymbol{B}'(\boldsymbol{X},\tau),\boldsymbol{B}(\boldsymbol{X},\tau),\boldsymbol{b}(\boldsymbol{X},\tau),\boldsymbol{v}(\boldsymbol{X},\tau)]. \tag{34}$$

That this must be true may be understood as follows. Magnetic fields are invariant under nonrelativistic boosts, so the total, mean and fluctuating magnetic fields must be the same in both frames. To see that the fluctuating velocity fields must be the same, we note that the total fluid velocity measured by the co-moving observer is, by definition, equal to $[-2A\tilde{X}_1\boldsymbol{e}_2+\tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}},\tilde{\tau})]$. This must be equal to the difference between the velocity in the laboratory frame, $[-2AX_1\boldsymbol{e}_2+\boldsymbol{v}(\boldsymbol{X},\tau)]$, and $(-2A\xi_1\boldsymbol{e}_2)$, which is the velocity of the co-moving observer with respect to the laboratory frame. Using $\tilde{X}=X-\xi_1$, we see that $\tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}},\tilde{\tau})=\boldsymbol{v}(\boldsymbol{X},\tau)$.

The *Galilean coordinate transformation* given in Eq. (33) implies that partial derivatives are related through

$$\frac{\partial}{\partial \boldsymbol{X}} = \frac{\partial}{\partial \tilde{\boldsymbol{X}}}, \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tilde{\tau}} + 2A\xi_1\frac{\partial}{\partial \tilde{X}_2}. \tag{35}$$

Note that the combination $(\partial/\partial\tau-2AX_1\partial/\partial X_2)=(\partial/\partial\tilde{\tau}-2A\tilde{X}_1\partial/\partial\tilde{X}_2)$ is invariant in form. The other partial derivatives occurring in Eqs. (1), (3), and (4) are spatial derivatives which, by the second of Eqs. (35), are the same in both frames. Therefore, Eqs. (1), (3), and (4) are invariant under the simultaneous transformations given in Eqs. (33) and (34). We note that this symmetry property is actually invariance under a subset of the full ten-parameter Galilean group, parametrized by the five quantities $(\xi_1,\xi_2,\xi_3,\tau_0,A)$; for brevity we will refer to this restricted symmetry as GI.

There is a fundamental difference between the coordinate transformations associated with Galilean invariance [Eq. (33)] and the shearing transformation [Eq. (6)]. The former relates different co-moving observers, whereas the latter describes a time-dependent distortion of the coordinates axes of one observer. Comparing Eq. (35) with Eq. (7), we note that the relationship between old and new variables is homogeneous for the Galilean transformation, whereas it is inhomogeneous for the shearing transformation.

It is important to note that the laboratory frame and its set of co-moving observers need not be inertial frames. Indeed, one of the main applications of our theory is to the *shearing sheet* which is a rotating frame providing a local description of a differentially rotating disk; in addition to other forces, the velocity field is affected by the Coriolis force. The only requirement is that the magnetic field satisfies the induction Eq. (1).

A. Galilean-invariant velocity correlators

Naturally occurring velocity fields are Galilean-invariant, and this has a strong impact on the velocity statistics. We consider the n-point velocity correlator measured by the observer in the laboratory frame. Let this observer correlate v_{j_1} at space-time location $(\boldsymbol{R}_1,\tau_1)$, with v_{j_2} at space-time location $(\boldsymbol{R}_2,\tau_2)$, and so on upto v_{j_n} at space-time location $(\boldsymbol{R}_n,\tau_n)$. Now consider a co-moving observer, the position vector of whose origin is given by $\boldsymbol{X}_c(\tau)$ of Eq. (32). An identical experiment performed by this observer must yield the same results, the measurements now made at the space-time points denoted by $(\boldsymbol{R}_1+\boldsymbol{X}_c(\tau_1),\tau_1);(\boldsymbol{R}_2+\boldsymbol{X}_c(\tau_2),\tau_2);\ldots;(\boldsymbol{R}_n+\boldsymbol{X}_c(\tau_n),\tau_n)$. If the velocity statistics is GI, the n-point velocity correlator must satisfy the condition

$$\langle v_{j_1}(\boldsymbol{R}_1,\tau_1)\cdots v_{j_n}(\boldsymbol{R}_n,\tau_n)\rangle = \langle v_{j_1}(\boldsymbol{R}_1+\boldsymbol{X}_c(\tau_1),\tau_1)\cdots v_{j_n}(\boldsymbol{R}_n+\boldsymbol{X}_c(\tau_n),\tau_n)\rangle, \tag{36}$$

for all $(\boldsymbol{R}_1,\ldots\boldsymbol{R}_n;\tau_1,\ldots\tau_n;\boldsymbol{\xi})$. In quasilinear theory we require only the two-point velocity correlators, for which

$$\langle v_i(\boldsymbol{R},\tau)v_j(\boldsymbol{R}',\tau')\rangle = \langle v_i[\boldsymbol{R}+\boldsymbol{X}_c(\tau),\tau]v_j[\boldsymbol{R}'+\boldsymbol{X}_c(\tau'),\tau']\rangle \tag{37}$$

for all $(\boldsymbol{R},\boldsymbol{R}',\tau,\tau',\boldsymbol{\xi})$. We also need to work out the correlation between velocities and their gradients,

$$\begin{aligned}\langle v_i(\boldsymbol{R},\tau)v_{jl}(\boldsymbol{R}',\tau')\rangle &= \frac{\partial}{\partial R'_l}\langle v_i(\boldsymbol{R},\tau)v_j(\boldsymbol{R}',\tau')\rangle \\ &= \frac{\partial}{\partial R'_l}\langle v_i(\boldsymbol{R}+\boldsymbol{X}_c(\tau),\tau)v_j(\boldsymbol{R}'+\boldsymbol{X}_c(\tau'),\tau')\rangle \\ &= \langle v_i(\boldsymbol{R}+\boldsymbol{X}_c(\tau),\tau)v_{jl}(\boldsymbol{R}'+\boldsymbol{X}_c(\tau'),\tau')\rangle.\end{aligned} \tag{38}$$

If we now set

$$\boldsymbol{R}=\boldsymbol{R}'=\boldsymbol{0}, \quad \tau=t, \quad \tau'=t', \quad (\xi_1,\xi_2,\xi_3)=(x_1,x_2,x_3) \tag{39}$$

we will have

$$\boldsymbol{X}_c(\tau)=(x_1,x_2-2Atx_1,x_3), \quad \boldsymbol{X}_c(\tau')=(x_1,x_2-2At'x_1,x_3). \tag{40}$$

Comparing Eq. (40) with Eq. (23), we see that $\boldsymbol{X}_c(\tau)$ and $\boldsymbol{X}_c(\tau')$ are equal to $\boldsymbol{X}$ and $\boldsymbol{X}'$, which are quantities that enter as arguments in the velocity correlators of Eqs. (22) defining the transport coefficients. Hence, reading Eqs. (37) and (38) from right to left, the velocity correlators,

$$\langle v_i(\boldsymbol{X},t)v_j(\boldsymbol{X}',t')\rangle = \langle v_i(\boldsymbol{0},t)v_j(\boldsymbol{0},t')\rangle = R_{ij}(t,t'),$$

$$\langle v_i(\boldsymbol{X},t)v_{jl}(\boldsymbol{X}',t')\rangle = \langle v_i(\boldsymbol{0},t)v_{jl}(\boldsymbol{0},t')\rangle = S_{ijl}(t,t'), \tag{41}$$

are independent of space, and are given by the functions, $R_{ij}(t,t')$ and $S_{ijl}(t,t')$. Symmetry and incompressiblity imply that $R_{ij}(t,t')=R_{ji}(t',t)$ and $S_{ijj}(t,t')=0$. Note that the turbulence will, in general, be affected by the background shear and the velocity correlators will not be isotropic. In particular, $R_{ij}(t,t')$ will not be proportional to the unit tensor, δ_{ij}.

B. Galilean-invariant mean EMF

The transport coefficients are completely determined by the form of the velocity correlator. Using Eqs. (41) in Eqs. (22), we can see that the GI transport coefficients,

$$\hat{\alpha}_{il}(t,t') = \epsilon_{ijm}[S_{jml}(t,t') - 2At'\,\delta_{l1}S_{jm2}(t,t')],$$

$$\hat{\beta}_{il}(t,t') = \epsilon_{ij2}[S_{j1l}(t,t') - 2At'\,\delta_{l1}S_{j12}(t,t')],$$

$$\hat{\eta}_{iml}(t,t') = \epsilon_{ijl}R_{jm}(t,t'), \qquad (42)$$

are independent of space. Galilean invariance is the fundamental reason that the velocity correlators, hence the transport coefficients, are independent of space. The derivation given above is purely mathematical, relying on the basic freedom of choice of parameters made in Eq. (39), but we can also understand the results more physically. $\boldsymbol{X}$ and $\boldsymbol{X}'$, as given by Eq. (23), can be thought of as the location of the origin of a co-moving observer at times t and t', respectively. Thus when the observer correlates velocities at $\boldsymbol{X}=\boldsymbol{X}_c(t)$ and $\boldsymbol{X}'=\boldsymbol{X}_c(t')$, it will be the same as correlating the velocities at her origin, but at different times. Then GI implies that the velocity correlators must be equal to those measured by *any* co-moving observer at her origin at times t and t'. In particular, this must be true for the observer in the laboratory frame, which explains Eqs. (41), consequently Eqs. (42).

We can derive an expression for the G-invariant mean EMF by using Eqs. (42) for the transport coefficients in Eq. (16). The integrands can be simplified as follows:

$$\hat{\alpha}_{il}(t,t')[H'_l + 2At'\,\delta_{l2}H'_1] = \epsilon_{ijm}[S_{jml}(t,t') - 2At'\,\delta_{l1}S_{jm2}(t,t')] \times [H'_l + 2At'\,\delta_{l2}H'_1] = \epsilon_{ijm}S_{jml}(t,t')H'_l,$$

$$\hat{\beta}_{il}(t,t')[H'_l + 2At'\,\delta_{l2}H'_1] = \epsilon_{ij2}[S_{j1l}(t,t') - 2At'\,\delta_{l1}S_{j12}(t,t')] \times [H'_l + 2At'\,\delta_{l2}H'_1] = \epsilon_{ij2}S_{j1l}(t,t')H'_l,$$

$$[\hat{\eta}_{iml}(t,t') + 2At'\,\delta_{m2}\hat{\eta}_{i1l}(t,t')]H'_{lm} = \epsilon_{ijl}[R_{jm}(t,t') + 2At'\,\delta_{m2}R_{j1}(t,t')]H'_{lm},$$

$$[\hat{\eta}_{im2}(t,t') + 2At'\,\delta_{m2}\hat{\eta}_{i12}(t,t')]H'_{1m} = \epsilon_{ij2}\delta_{l1}[R_{jm}(t,t') + 2At'\,\delta_{m2}R_{j1}(t,t')]H'_{lm}.$$

Define

$$C_{jml}(t,t') = S_{jml}(t,t') - 2A(t-t')\,\delta_{m2}S_{j1l}(t,t'),$$

$$D_{jm}(t,t') = R_{jm}(t,t') + 2At'\,\delta_{m2}R_{j1}(t,t'). \qquad (43)$$

The mean EMF can now be written compactly as

$$\mathcal{E}_i(\boldsymbol{x},t) = \epsilon_{ijm}\int_0^t dt'\,C_{jml}(t,t')H'_l - \int_0^t dt'[\epsilon_{ijl} - 2A(t-t')\,\delta_{l1}\epsilon_{ij2}] \times D_{jm}(t,t')H'_{lm}, \qquad (44)$$

where the $\boldsymbol{x}$ dependence of $\mathcal{E}$ comes about only through the mean field, $\boldsymbol{H}(\boldsymbol{x},t)$, and its spatial gradients, because the G-invariant transport coefficients are independent of $\boldsymbol{x}$.

V. MEAN-FIELD INDUCTION EQUATION

Applying the shearing transformation given in Eqs. (6) and (7) to the mean field Eq. (3), we see that the mean field, $\boldsymbol{H}(\boldsymbol{x},t)$, obeys

$$\frac{\partial H_i}{\partial t} + 2A\,\delta_{i2}H_1 = (\boldsymbol{\nabla}\times\mathcal{E})_i + \eta\boldsymbol{\nabla}^2 H_i, \qquad (45)$$

where

$$(\boldsymbol{\nabla})_p \equiv \frac{\partial}{\partial X_p} = \frac{\partial}{\partial x_p} + 2At\delta_{p1}\frac{\partial}{\partial x_2}. \qquad (46)$$

It may be verified that Eq. (45) preserves the condition $\boldsymbol{\nabla}\cdot\boldsymbol{H}=0$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{H} \equiv \frac{\partial H_p}{\partial X_p} = H_{pp} + 2AtH_{12} = 0. \qquad (47)$$

We now use Eqs. (44) and (46) to evaluate $\boldsymbol{\nabla}\times\mathcal{E}$.

$$(\boldsymbol{\nabla}\times\mathcal{E})_i = \epsilon_{ipq}\frac{\partial\mathcal{E}_q}{\partial X_p} = \epsilon_{ipq}\left(\frac{\partial}{\partial x_p} + 2At\delta_{p1}\frac{\partial}{\partial x_2}\right)\mathcal{E}_q = \epsilon_{ipq}\epsilon_{qjm}\int_0^t dt'\,C_{jml}(t,t')[H'_{lp} + 2At\delta_{p1}H'_{l2}] - \int_0^t dt'\,D_{jm}(t,t')[\epsilon_{ipq}\epsilon_{qjl} - 2A(t-t')\,\delta_{l1}\epsilon_{ipq}\epsilon_{qj2}] \times [H'_{lmp} + 2At\delta_{p1}H'_{lm2}].$$

Expanding $\epsilon_{ipq}\epsilon_{qjm}=(\delta_{ij}\delta_{mp} - \delta_{im}\delta_{jp})$, the contribution from the C term is

$$(\boldsymbol{\nabla}\times\mathcal{E})_i^C = \int_0^t dt'[C_{ipl} - C_{pil}][H'_{lp} + 2At\delta_{p1}H'_{l2}]. \qquad (48)$$

Evaluating the D term is a bit more involved. Again, we begin by expanding $\epsilon_{ipq}\epsilon_{qjl}=(\delta_{ij}\delta_{lp} - \delta_{il}\delta_{jp})$. Then we get

$$(\boldsymbol{\nabla}\times\mathcal{E})_i^D = \int_0^t dt'\,D_{pm}\{H'_{ipm} + 2At\delta_{p1}H'_{i2m} - 2A(t-t')\,\delta_{i2}[H'_{1pm} + 2At\delta_{p1}H'_{12m}]\} - \int_0^t dt'\,D_{im}[H'_{ppm} + 2At'H'_{12m}]. \qquad (49)$$

The second integral vanishes because the factor in [ ] multiplying D_{im} is zero: to see this, differentiate the divergence-free condition of Eq. (47) with respect to x_m. Gathering together Eqs. (48) and (49), we have

$$(\nabla\times\mathcal{E})_i = \int_0^t dt'[C_{iml}-C_{mil}][H'_{lm}+2At\delta_{m1}H'_{l2}] + \int_0^t dt' D_{jm}\{H'_{ijm}+2At\delta_{j1}H'_{i2m} - 2A(t-t')\delta_{i2}[H'_{1jm}+2At\delta_{j1}H'_{12m}]\}. \quad (50)$$

Thus, the mean field $\boldsymbol{H}(\boldsymbol{x},t)$ satisfies the mean-field induction equation,

$$\frac{\partial H_i}{\partial t} + 2A\delta_{i2}H_1 = \eta\nabla^2 H_i + \int_0^t dt'[C_{iml}-C_{mil}] \times[H'_{lm}+2At\delta_{m1}H'_{l2}] + \int_0^t dt' D_{jm}\{H'_{ijm}+2At\delta_{j1}H'_{i2m} - 2A(t-t')\delta_{i2}[H'_{1jm}+2At\delta_{j1}H'_{12m}]\}. \quad (51)$$

Equation (51) gives a closed set of integro-differential equations governing the dynamics of the mean field, $\boldsymbol{H}(\boldsymbol{x},t)$, valid for arbitrary values of A. Some of its important properties are as follows.

(1) Only the part of $C_{iml}(t,t')$ that is antisymmetric in the indices (i,m) contributes.

(2) The $D_{jm}(t,t')$ terms are such that $(\nabla\times\mathcal{E})_i$ involves only H_i for $i=1$ and $i=3$, whereas $(\nabla\times\mathcal{E})_2$ depends on both H_2 and H_1. This means that the mean-field induction Eq. (51) determining the time evolution of $H_1(\boldsymbol{x},t)$ and $H_3(\boldsymbol{x},t)$ are closed, whereas the equation for $H_2(\boldsymbol{x},t)$ involves both $H_2(\boldsymbol{x},t)$ and $H_1(\boldsymbol{x},t)$. Thus, $H_1(\boldsymbol{x},t)$ [or $H_3(\boldsymbol{x},t)$] can be computed by using only the initial data $H_1(\boldsymbol{x},0)$ [or $H_3(\boldsymbol{x},0)$]. The equation for H_2 involves both H_2 and H_1, and can then be solved. The implications for the original field, $\boldsymbol{B}(\boldsymbol{X},\tau)$, can be read off, because it is equal to $\boldsymbol{H}(\boldsymbol{x},t)$ componentwise [i.e., $B_i(\boldsymbol{X},\tau)=H_i(\boldsymbol{x},t)$]. Thus, the $D_{jm}(t,t')$ terms do not couple either B_1 or B_3 with any other components, excepting themselves. In demonstrating this, we have not assumed that either the shear is small, or that $\boldsymbol{H}(\boldsymbol{x},t)$ is such a slow function of time that it can be pulled out the time integral in Eq. (50).

(3) When the turbulence is non helical, $C_{iml}(t,t')=0$, but $D_{jm}(t,t')\neq 0$. In this case, there is no shear-current type effect, in quasilinear theory in the limit of zero resistivity. This result should be compared with earlier work discussed in [4,6,7], where there is explicit coupling of B_2 and B_1 in the evolution equation for B_1. A generalization of Eq. (51) to the case of nonzero resistivity has been worked out in [11]. It is interesting to note that the corresponding generalization of C_{iml} that appears in this case need not vanish for non helical turbulence. However, it is expected to vanish in the formal limit of zero resistivity, consistent with our result given above.

VI. INDUCTION EQUATION FOR A SLOWLY VARYING MEAN-FIELD

A. Mean EMF

The mean EMF given in Eq. (44) is a *functional* of H_l and H_{lm}. When the mean-field is slowly varying compared to velocity correlation times, we expect to be able to approximate $\mathcal{E}$ as a *function* of H_l and H_{lm}. In this case, the mean-field induction equation would reduce to a set of coupled partial differential equations, instead of the more formidable set of coupled integro-differential equations given by Eqs. (45) and (50). Sheared coordinates are useful—perhaps indispensable—for calculations, but physical interpretation is simplest in the laboratory frame. Hence, we derive an expression for the mean EMF in terms the original variables B_l and B_{lm}. The result may be stated simply as

$$\mathcal{E}_i = \alpha_{il}(\tau)B_l(\boldsymbol{X},\tau) - \eta_{iml}(\tau)\frac{\partial B_l}{\partial X_m},$$

$$\alpha_{il}(\tau) = \epsilon_{ijm}\int_0^\tau d\tau'[C_{jml}(\tau,\tau') + 2A(\tau-\tau')\delta_{l1}C_{jm2}(\tau,\tau')],$$

$$\eta_{iml}(\tau) = \epsilon_{ijl}\int_0^\tau d\tau'[R_{jm}(\tau,\tau') - 2A(\tau-\tau')\delta_{m2}R_{j1}(\tau,\tau')], \quad (52)$$

which is derived below by two different methods.

1. Method I: use of a perturbative solution for $\boldsymbol{H}(\boldsymbol{x},t')$

Consider the mean-field Eq. (45) when $\mathcal{E}$ can be considered small. We introduce an ordering parameter $\varepsilon \ll 1$ and consider $\mathcal{E}$ to be $O(\varepsilon)$. Then a perturbative solution of Eq. (45) in the $\eta\to 0$ limit is

$$H_l(\boldsymbol{x},t') = H_l(\boldsymbol{x},t) + 2A(t-t')\delta_{l2}H_1(\boldsymbol{x},t) + O(\varepsilon). \quad (53)$$

We can also consider perturbative solutions with nonzero η, but using them in Eq. (44) for $\mathcal{E}$ would not be correct, because Eq. (44) was derived in the limit $\eta\to 0$. We now use Eq. (53) in Eq. (44),

$$\mathcal{E}_i(\boldsymbol{x},t) = H_l\epsilon_{ijm}\int_0^t dt' C_{jml}(t,t') + 2AH_1\epsilon_{ijm} \times\int_0^t dt'(t-t')C_{jm2}(t,t') - H_{lm}\epsilon_{ijl} \times\int_0^t dt' D_{jm}(t,t') + O(\varepsilon^2). \quad (54)$$

Transform to the original field variables, using $H_l=B_l$ and $H_{lm}=B_{lm}-2At\delta_{m1}B_{l2}$, which is given in Eq. (30). The C terms remain unaltered and can be seen to combine to equal $\alpha_{il}B_l$. Work out the D term using the expression for D_{jm} given in Eq. (43),

$$H_{lm}\int_0^t dt' D_{jm} = [B_{lm} - 2At\delta_{m1}B_{l2}]\int_0^t dt'[R_{jm} + 2At'\delta_{m2}R_{j1}]$$

$$= B_{lm}\int_0^t dt' R_{jm} - 2AB_{l2}\int_0^t dt'(t-t')R_{j1}.$$

Using the above result, and ignoring $O(\varepsilon^2)$ terms in Eq. (54), we obtain the result stated in Eq. (52).

2. Method II: Taylor expansion of $\boldsymbol{B}(\boldsymbol{X}',\tau'=t')$

This is the standard approach, although not as short as the one given above. We express $\boldsymbol{H}(\boldsymbol{x},t')=\boldsymbol{B}(\boldsymbol{X}',\tau'=t')$ and Taylor expand $\boldsymbol{B}$ inside the integral in Eq. (44). As in Eq. (23),

$$\boldsymbol{X} = (x_1, x_2 - 2Atx_1, x_3), \quad \boldsymbol{X}' = (x_1, x_2 - 2At'x_1, x_3).$$

Writing $\boldsymbol{X}'=\boldsymbol{X}+2A(t-t')x_1\boldsymbol{e}_2$, we Taylor expand:

$$H_l' \equiv H_l(\boldsymbol{x},t') = B_l(\boldsymbol{X}',t')$$
$$= B_l(\boldsymbol{X}+2A(t-t')x_1\boldsymbol{e}_2,t')$$
$$= B_l(\boldsymbol{X},t) + 2A(t-t')x_1B_{l2} - (t-t')\frac{\partial B_l}{\partial t} + \cdots.$$

We now use the mean-field induction Eq. (3) to evaluate $(\partial B_l/\partial t)$. As earlier we drop the contributions from $(\boldsymbol{\nabla}\times\mathcal{E})$ and the η term and get

$$\frac{\partial B_l}{\partial t} = 2Ax_1B_{l2} - 2A\delta_{l2}B_1 + \cdots. \quad (55)$$

Then

$$H_l' = B_l(\boldsymbol{X},t) + 2A(t-t')x_1B_{l2} - (t-t')[2Ax_1B_{l2} - 2A\delta_{l2}B_1] + \cdots = B_l + 2A(t-t')\delta_{l2}B_1 + \cdots. \quad (56)$$

Note that the inhomogeneous terms proportional to x_1 mutually cancel. It is clear, on physical grounds that they must, because the mean EMF given by Eq. (44) is GI, and any valid approximation of a GI expression must preserve this symmetry. In particular, this implies that transport coefficients cannot depend on x_1. We now use Eq. (56) inside the time integrals of Eq. (44). $B_l=B_l(\boldsymbol{X},t)$ is a function of $(\boldsymbol{x},t)$ and can be pulled out of the integral over t'. Work out the C and D terms separately,

$$\mathcal{E}_i^C = \epsilon_{ijm}\int_0^t dt' C_{jml}(t,t')H_l'$$

$$= \epsilon_{ijm}\int_0^t dt' C_{jml}[B_l + 2A(t-t')\delta_{l2}B_1]$$

$$= \epsilon_{ijm}\int_0^t dt'[C_{jml}B_l + 2A(t-t')C_{jm2}B_1] = \alpha_{il}B_l.$$

To calculate the D terms, we note that $H_{lm}'=(\partial H_l/\partial x_m)$. Since the integrals over t' is performed at constant $\boldsymbol{x}$, $(\partial/\partial x_m)$ can be pulled out of the integral,

$$\mathcal{E}_i^D = -\frac{\partial}{\partial x_m}\int_0^t dt'[\epsilon_{ijl} - 2A(t-t')\delta_{l1}\epsilon_{ij2}]D_{jm}(t,t')H_l'.$$

Work out

$$[\epsilon_{ijl} - 2A(t-t')\delta_{l1}\epsilon_{ij2}]H_l'$$
$$= [\epsilon_{ijl} - 2A(t-t')\delta_{l1}\epsilon_{ij2}][B_l + 2A(t-t')\delta_{l2}B_1]$$
$$= \epsilon_{ijl}B_l(\boldsymbol{X},t).$$

Then

$$\mathcal{E}_i^D = -\epsilon_{ijl}\frac{\partial B_l}{\partial x_m}\int_0^t dt' D_{jm}(t,t').$$

The quantity

$$\frac{\partial B_l}{\partial x_m} = \left(\frac{\partial}{\partial X_m} - 2At\delta_{m1}\frac{\partial}{\partial X_2}\right)B_l = B_{lm} - 2At\delta_{m1}B_{l2}$$

can be regarded as a function of $(\boldsymbol{X},t)$ [or equivalently $(\boldsymbol{x},t)$], and we are free to take it *inside* the t' integral. When this is done and the expression for D_{jm} given in Eq. (43) is used, we have

$$\mathcal{E}_i^D = -\epsilon_{ijl}\int_0^t dt'[B_{lm} - 2At\delta_{m1}B_{l2}][R_{jm} + 2At'\delta_{m2}R_{j1}]$$

$$= -\epsilon_{ijl}B_{lm}\int_0^t dt'[R_{jm} - 2A(t-t')\delta_{m2}R_{j1}]$$

$$= -\eta_{iml}B_{lm}. \quad (57)$$

B. Calculation of $\boldsymbol{\nabla}\times\mathcal{E}$

We need to calculate $\boldsymbol{\nabla}\times\mathcal{E}$ for the mean EMF of Eq. (52). Work out the α and η terms separately.

$$(\boldsymbol{\nabla}\times\mathcal{E})_i^\alpha = \epsilon_{ipq}\alpha_{il}B_{lp}$$

$$= B_{lp}\epsilon_{ipq}\epsilon_{qjm}\int_0^\tau d\tau'[C_{jml} + 2A(\tau-\tau')\delta_{l1}C_{jm2}]$$

$$= B_{lm}\int_0^\tau d\tau'[C_{iml} + 2A(\tau-\tau')\delta_{l1}C_{im2}]$$

$$- B_{lj}\int_0^\tau d\tau'[C_{jil} + 2A(\tau-\tau')\delta_{l1}C_{ji2}]$$

$$= B_{lm}\int_0^\tau d\tau'\{C_{iml} - C_{mil} + 2A(\tau-\tau')\delta_{l1} \times[C_{im2} - C_{mi2}]\}. \quad (58)$$

Note that only the part of C_{iml} that is antisymmetric in the indices (i,m) contributes.

$$(\nabla \times \mathcal{E})_i^\eta = -\epsilon_{ipq}\eta_{qml}B_{lpm} = B_{lpm}\epsilon_{ipq}\epsilon_{qjl}\int_0^\tau d\tau'[R_{jm} - 2A(\tau-\tau')\delta_{m2}R_{j1}] = B_{ijm}\int_0^\tau d\tau'[R_{jm} - 2A(\tau-\tau')\delta_{m2}R_{j1}], \tag{59}$$

where we have used $B_{ll} \equiv \nabla \cdot \boldsymbol{B} = 0$. We note that Eqs. (58) and (59) can also be derived directly from the expression for $\nabla \times \mathcal{E}$, given in Eq. (50). This is an interesting exercise as it allows us to formulate an alternate criteria on when the integral equation for $\boldsymbol{B}$ can be approximated by differential equations. We examine such an approximation further below.

C. Approximating the integral equation directly

It is convenient to work with the Fourier transform of $\boldsymbol{H}(\boldsymbol{x},t)$,

$$\tilde{\boldsymbol{H}}(\boldsymbol{k},t) = \int d^3x \boldsymbol{H}(\boldsymbol{x},t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x}). \tag{60}$$

We also define the vector $\boldsymbol{K}(\boldsymbol{k},t) = (k_1 + 2Atk_2, k_2, k_3)$ and $K^2 = |\boldsymbol{K}|^2 = (k_1+2Atk_2)^2 + k_2^2 + k_3^2$: note that $\boldsymbol{K}\cdot\boldsymbol{X} = \boldsymbol{k}\cdot\boldsymbol{x}$. The magnetic field in the original variables, $\boldsymbol{B}(\boldsymbol{X},\tau)$, can be recovered by using the shearing transformation, Eq. (6), to write $(\boldsymbol{x},t)$ in terms of the laboratory frame coordinates $(\boldsymbol{X},\tau)$,

$$\boldsymbol{B}(\boldsymbol{X},\tau) = \boldsymbol{H}(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\tilde{\boldsymbol{H}}(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{\boldsymbol{H}}(\boldsymbol{k},\tau)\exp(i\boldsymbol{K}(\boldsymbol{k},\tau)\cdot\boldsymbol{X}). \tag{61}$$

From Eq. (51), the Fourier transformed induction equation becomes

$$\frac{\partial \tilde{H}_i}{\partial t} + 2A\delta_{i2}\tilde{H}_1 = -\eta K^2\tilde{H}_i + i\int_0^t dt'[C_{iml} - C_{mil}] \times [\tilde{H}_l' k_m + 2At\delta_{m1}\tilde{H}_l' k_2] - \int_0^t dt' D_{jm}\{\tilde{H}_i' k_j k_m + 2At\delta_{j1}\tilde{H}_i' k_2 k_m\} + \int_0^t dt' D_{jm}\{2A(t-t')\delta_{i2}[\tilde{H}_1' k_j k_m + 2At\delta_{j1}\tilde{H}_1' k_2 k_m]\}. \tag{62}$$

Let us again simplify the integrals corresponding to the C term, say T^C and D term, say T^D, separately. Using the definition of $\boldsymbol{K}(\boldsymbol{k},t)$, the C term simplifies to

$$T_i^C = iK_m(\boldsymbol{k},t)\int_0^t dt'[C_{iml} - C_{mil}]\tilde{H}_l'. \tag{63}$$

We now assume that the mean field is slowly varying compared to the correlation time τ_c of the turbulence and Taylor expand $\tilde{H}_l(\boldsymbol{k},t')$ about t (this assumption can later be checked for its self consistency). We get

$$\tilde{H}_l(\boldsymbol{k},t') = \tilde{H}_l(\boldsymbol{k},t) - (t-t')\frac{\partial \tilde{H}_l}{\partial t} + \cdots = [\tilde{H}_l(\boldsymbol{k},t) + 2A(t-t')\delta_{l2}\tilde{H}_1] - (t-t') \times \left[\frac{\partial \tilde{H}_l}{\partial t} + 2A\delta_{l2}\tilde{H}_1\right] + \cdots, \tag{64}$$

where in the second line we have added and subtracted a term $2A(t-t')\delta_{l2}\tilde{H}_1$. Substituting this expansion in Eq. (63), the C term becomes

$$T_i^C = iK_m(\boldsymbol{k},t)\tilde{H}_l\int_0^t dt'\{C_{iml} - C_{mil} + 2A(t-t')\delta_{l1} \times [C_{im2} - C_{mi2}]\} - iK_m(\boldsymbol{k},t)\left[\frac{\partial \tilde{H}_l}{\partial t} + 2A\delta_{l2}\tilde{H}_1\right] \times \int_0^t dt'(t-t')[C_{iml} - C_{mil}]. \tag{65}$$

Now consider the D terms. Again using the definition of $\boldsymbol{K}(\boldsymbol{k},t)$ and $D_{jm} = R_{jm} + 2At'\delta_{m2}R_{j1}$, we can simplify this to

$$T_i^D = -K_jK_m\int_0^t dt'[\tilde{H}_i' - 2A(t-t')\delta_{i2}\tilde{H}_1'] \times [R_{jm} - 2A(t-t')\delta_{m2}R_{j1}]. \tag{66}$$

Again assume that the mean field is slowly varying compared to the correlation time τ_c of the turbulence and Taylor expand $\tilde{H}_l(\boldsymbol{k},t')$ about t. To first order in $(t-t')$, we have

$$[\tilde{H}_i' - 2A(t-t')\delta_{i2}\tilde{H}_1'] = \tilde{H}_i - (t-t')\left[\frac{\partial \tilde{H}_i}{\partial t} + 2A\delta_{i2}\tilde{H}_1\right] + \cdots.$$

Substituting this expansion in Eq. (66) gives

$$T_i^D = -K_jK_m\tilde{H}_i\int_0^t dt'[R_{jm} - 2A(t-t')\delta_{m2}R_{j1}] + K_jK_m\left[\frac{\partial \tilde{H}_i}{\partial t} + 2A\delta_{i2}\tilde{H}_1\right]\int_0^t dt'(t-t') \times [R_{jm} - 2A(t-t')\delta_{m2}R_{j1}]. \tag{67}$$

The expressions for T_i^C and T_i^D given in Eqs. (65) and (67) can be simplified. In both equations, the second terms are proportional to the LHS of the induction (62). As before we ignore microscopic diffusion and write

$$\frac{\partial \tilde{H}_i}{\partial t} + 2A\delta_{i2}\tilde{H}_1 \simeq T_i^C + T_i^D.$$

Then Eqs. (65) and (67) can be written as

$$T_i^C = iK_m(\boldsymbol{k},t)\widetilde{H}_l\int_0^t dt'\{C_{iml} - C_{mil} + 2A(t-t')\delta_{l1} \times[C_{im2} - C_{mi2}]\} - iK_m(\boldsymbol{k},t)[T_l^C + T_l^D] \times\int_0^t dt'(t-t')[C_{iml} - C_{mil}],$$

$$T_i^D = - K_jK_m\widetilde{H}_i\int_0^t dt'[R_{jm} - 2A(t-t')\delta_{m2}R_{j1}] + K_jK_m[T_i^C + T_i^D]\int_0^t dt'(t-t') \times[R_{jm} - 2A(t-t')\delta_{m2}R_{j1}]. \quad (68)$$

When these equations are added together, they result in a set of three-coupled linear equations for the unknown quantities $[T_1^C+T_1^D]$, $[T_2^C+T_2^D]$, and $[T_3^C+T_3^D]$. It is straightforward to solve this system of equations, but the solutions assume a form, which is needlessly complicated for our purposes. We are interested in the limit of short velocity correlations times, τ_c. In this case both T_i^C and T_i^D are well approximated by their respective first terms,

$$T_i^C = iK_m(\boldsymbol{k},t)\widetilde{H}_l\int_0^t dt'\{C_{iml} - C_{mil} + 2A(t-t')\delta_{l1} \times[C_{im2} - C_{mi2}]\},$$

$$T_i^D = - K_jK_m\widetilde{H}_i\int_0^t dt'[R_{jm} - 2A(t-t')\delta_{m2}R_{j1}]. \quad (69)$$

These are exactly the Fourier transforms of Eq. (58) for $(\boldsymbol{\nabla}\times\mathcal{E})_i^{\alpha}$, and Eq. (59) for $(\boldsymbol{\nabla}\times\mathcal{E})_i^{\eta}$.

We now state the conditions under which the approximations given in Eqs. (69) are valid. Let us define the quantities α_0 and $\eta_{\rm turb}$ as typical values of the time integrals of the velocity correlators, S_{jml} and R_{jm}, respectively (for homogeneous and isotropic turbulence, α_0 is of order the magnitude of the usual α effect, and $\eta_{\rm turb}$ would be comparable to the magnitude of the usual turbulent diffusion coefficient). For any wave number, K, we can define time scales, $t_\alpha=(K\alpha_0)^{-1}$ and $t_\eta=(K^2\eta_{\rm turb})^{-1}$, associated with α_0 and $\eta_{\rm turb}$. When τ_c is small enough such that

$$\tau_c \ll t_\alpha, t_\eta, \quad A\tau_c \ll 1,$$

$$A\tau_c^2 \ll t_\alpha, t_\eta, \quad A\tau_c \geq 1, \quad (70)$$

then both T_i^C and T_i^D are well approximated by their respective first terms, as given in Eqs. (69). The time scales, $t_\alpha=(K\alpha_0)^{-1}$ and $t_\eta=(K^2\eta_{\rm turb})^{-1}$, depend on the spatial scale, K^{-1}, which is a time-dependent quantity for $k_2\neq 0$; at late times, $K\sim|2Atk_2|$ and this makes the quantities t_α and t_η decreasing functions of time. With this fact taken into account, the inequalities given in Eq. (70) translate into upper limits on the time over which the expressions in Eq. (69) serve as good approximations to T_i^C and T_i^D.

D. Mean-field induction equation

We gather together here the results obtained in this section. When the mean-field is slowly varying, it satisfies the following partial differential equation:

$$\left(\frac{\partial}{\partial\tau} - 2AX_1\frac{\partial}{\partial X_2}\right)B_i + 2A\delta_{i2}B_1 = \widetilde{\alpha}_{imj}(\tau)\frac{\partial B_j}{\partial X_m} + \widetilde{\eta}_{jm}(\tau)\frac{\partial^2 B_i}{\partial X_j\,\partial X_m} + \eta\boldsymbol{\nabla}^2B_i, \quad (71)$$

where

$$\widetilde{\alpha}_{imj}(\tau) = \int_0^\tau d\tau'\{C_{imj} - C_{mij} + 2A(\tau-\tau')\delta_{j1}[C_{im2} - C_{mi2}]\},$$

$$\widetilde{\eta}_{jm}(\tau) = \frac{1}{2}\int_0^\tau d\tau'\{R_{jm} + R_{mj} - 2A(\tau-\tau')[\delta_{m2}R_{j1} + \delta_{j2}R_{m1}]\}. \quad (72)$$

In the above integrals $C_{imj}=C_{imj}(\tau,\tau')$, $R_{jm}=R_{jm}(\tau,\tau')$, etc. Some comments are as follows.

(1) Note that $\widetilde{\alpha}_{imj}$ is antisymmetric in the indices (i,m), whereas $\widetilde{\eta}_{jm}$ is symmetric in the indices (j,m).

(2) $\widetilde{\eta}_{jm}$ terms do not lead to coupling of any component of $\boldsymbol{B}$ with any other component.

VII. MEAN-FIELD DYNAMICS FOR NONHELICAL VELOCITY STATISTICS

When the velocity fluctuations are nonhelical, $S_{imj}(\tau,\tau')=0$, so that both $C_{imj}(\tau,\tau')$ and $\widetilde{\alpha}_{mj}(\tau)$ vanish (in specific models of the velocity dynamics we find that the generated velocity fluctuations are indeed nonhelical, if the forcing is nonhelical even in the presence of shear). Then the evolution of the mean field (over times when the inequalities of Eqs. (70) are satisfied) is determined by

$$\left(\frac{\partial}{\partial\tau} - 2AX_1\frac{\partial}{\partial X_2}\right)B_i + 2A\delta_{i2}B_1 = \widetilde{\eta}_{jm}(\tau)\frac{\partial^2 B_i}{\partial X_j\,\partial X_m} + \eta\boldsymbol{\nabla}^2B_i. \quad (73)$$

Note that $\widetilde{\eta}_{jm}$ depends on the nature of the stirring and will, in general, be a function of time; this will be the case, say, for decaying turbulence. However, for statistically stationary stirring, $\widetilde{\eta}_{jm}$ will become time independent, after an initial transient evolution.

Equation (73) is inhomogeneous in the spatial coordinates so, as before, we find it convenient to work with the new variable, $\boldsymbol{H}(\boldsymbol{x},t)$, and transform Eq. (73) to the shearing coordinates $(\boldsymbol{x},t)$,

$$\frac{\partial H_i}{\partial t} + 2A\delta_{i2}H_1 = \widetilde{\eta}_{jm}(\tau)\frac{\partial^2 H_i}{\partial X_j\,\partial X_m} + \eta\boldsymbol{\nabla}^2H_i, \quad (74)$$

where (see Eq. (46))

$$\frac{\partial}{\partial X_p} = \frac{\partial}{\partial x_p} + 2At\delta_{p1}\frac{\partial}{\partial x_2}; \quad \boldsymbol{\nabla}^2 = \left(\frac{\partial}{\partial x_p} + 2At\delta_{p1}\frac{\partial}{\partial x_2}\right)^2. \tag{75}$$

Equation (74) is homogeneous in $\boldsymbol{x}$ but not in t, so we take a spatial Fourier transform defined earlier in Eq. (60). Then $\tilde{\boldsymbol{H}}(\boldsymbol{k},t)$ satisfies

$$\frac{\partial \tilde{H}_i}{\partial t} + 2A\delta_{i2}\tilde{H}_1 = -[\tilde{\eta}_{jm}(t)K_jK_m + \eta K^2]\tilde{H}_i, \tag{76}$$

where the vector $\boldsymbol{K}(\boldsymbol{k},t)=(k_1+2Atk_2,k_2,k_3)$ and $K^2=|\boldsymbol{K}|^2=(k_1+2Atk_2)^2+k_2^2+k_3^2$, as before. It may be verified that this equation preserves the Fourier version of the divergence condition of Eq. (47), namely $\boldsymbol{K}\cdot\tilde{\boldsymbol{H}}(\boldsymbol{k},t)=0$. The solution is

$$\tilde{H}_1(\boldsymbol{k},t) = \tilde{H}_1(\boldsymbol{k},0)\mathcal{G}(\boldsymbol{k},t),$$

$$\tilde{H}_2(\boldsymbol{k},t) = [\tilde{H}_2(\boldsymbol{k},0) - 2At\tilde{H}_1(\boldsymbol{k},0)]\mathcal{G}(\boldsymbol{k},t),$$

$$\tilde{H}_3(\boldsymbol{k},t) = \tilde{H}_3(\boldsymbol{k},0)\mathcal{G}(\boldsymbol{k},t), \tag{77}$$

where $\tilde{\boldsymbol{H}}(\boldsymbol{k},0)$ are given initial conditions satisfying $\boldsymbol{k}\cdot\tilde{\boldsymbol{H}}(\boldsymbol{k},0)=0$, ensuring that $\boldsymbol{K}\cdot\tilde{\boldsymbol{H}}(\boldsymbol{k},t)=0$. The Green's function, $\mathcal{G}(\boldsymbol{k},t)$, is zero for $t<0$ and is defined for $t\geq 0$ by

$$\mathcal{G}(\boldsymbol{k},t) = \exp\left[-\int_0^t ds(\tilde{\eta}_{jm}(s)K_jK_m + \eta K^2)\right]. \tag{78}$$

In the integrand, $K_j=k_j+2As\delta_{j1}k_2$ should be regarded as a function of $\boldsymbol{k}$ and s, and the s integral performed at fixed $\boldsymbol{k}$. Then $\mathcal{G}(\boldsymbol{k},t)$ can be written as the product of a *microscopic* Green's function, $\mathcal{G}_\eta(\boldsymbol{k},t)$, and a *turbulent* Green's function, $\mathcal{G}_t(\boldsymbol{k},t)$,

$$\mathcal{G}(\boldsymbol{k},t) = \mathcal{G}_\eta(\boldsymbol{k},t)\cdot\mathcal{G}_t(\boldsymbol{k},t),$$

$$\mathcal{G}_\eta(\boldsymbol{k},t) = \exp\left[-\eta\left(k^2t + 2Ak_1k_2t^2 + \frac{4}{3}A^2k_2^2t^3\right)\right],$$

$$\mathcal{G}_t(\boldsymbol{k},t) = \exp[-Q_{jm}(t)k_jk_m], \tag{79}$$

where the time-dependent symmetric matrix $Q_{jm}(t)$ is given by

$$Q_{jm}(t) = \int_0^t ds\{\tilde{\eta}_{jm}(s) + 2As[\delta_{j2}\tilde{\eta}_{1m}(s) + \delta_{m2}\tilde{\eta}_{j1}(s)] + 4A^2\delta_{j2}\delta_{m2}s^2\tilde{\eta}_{11}(s)\} \tag{80}$$

in terms of time integrals of $\tilde{\eta}_{jm}(\tau)$, which are assumed to be known functions depending on the velocity correlators, $R_{jm}(\tau,\tau')$, as given in Eq. (72).

The solution in the original variables, $\boldsymbol{B}(\boldsymbol{X},\tau)$, can be recovered by using the shearing transformation, Eq. (6), to write $(\boldsymbol{x},t)$ in terms of the laboratory frame coordinates $(\boldsymbol{X},\tau)$ [see Eq. (61)],

$$\boldsymbol{B}(\boldsymbol{X},\tau) = \boldsymbol{H}(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\tilde{\boldsymbol{H}}(\boldsymbol{k},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{\boldsymbol{H}}(\boldsymbol{k},\tau)\exp(i\boldsymbol{K}(\boldsymbol{k},\tau)\cdot\boldsymbol{X}). \tag{81}$$

Equivalently, the solution is given in component form as

$$B_1(\boldsymbol{X},\tau) = \int \frac{d^3k}{(2\pi)^3}\tilde{B}_1(\boldsymbol{k},0)\mathcal{G}(\boldsymbol{k},\tau)\exp[i\boldsymbol{K}(\boldsymbol{k},\tau)\cdot\boldsymbol{X}],$$

$$B_2(\boldsymbol{X},\tau) = \int \frac{d^3k}{(2\pi)^3}[\tilde{B}_2(\boldsymbol{k},0) - 2A\tau\tilde{B}_1(\boldsymbol{k},0)]\mathcal{G}(\boldsymbol{k},\tau)\exp[i\boldsymbol{K}(\boldsymbol{k},\tau)\cdot\boldsymbol{X}],$$

$$B_3(\boldsymbol{X},\tau) = \int \frac{d^3k}{(2\pi)^3}\tilde{B}_3(\boldsymbol{k},0)\mathcal{G}(\boldsymbol{k},\tau)\exp[i\boldsymbol{K}(\boldsymbol{k},\tau)\cdot\boldsymbol{X}], \tag{82}$$

where we have written the initial condition, $\tilde{\boldsymbol{H}}(\boldsymbol{k},0)=\tilde{\boldsymbol{B}}(\boldsymbol{k},0)$, with $\boldsymbol{k}\cdot\tilde{\boldsymbol{B}}(\boldsymbol{k},0)=0$.

Some comments are as follows.

(1) The above solution for $\boldsymbol{B}(\boldsymbol{X},\tau)$ is a linear superposition of *shearing waves* of the form $\exp(i\boldsymbol{K}(\boldsymbol{k},\tau)\cdot\boldsymbol{X})=\exp[i(k_1+2A\tau k_2)X_1+ik_2X_2+ik_3X_3]$, indexed by the triplet of numbers (k_1,k_2,k_3).

(2) Whether the waves grow or decay depends on the time dependence of the Green's function, $\mathcal{G}(\boldsymbol{k},\tau)=\mathcal{G}_\eta(\boldsymbol{k},\tau)\cdot\mathcal{G}_t(\boldsymbol{k},\tau)$. The first term, $\mathcal{G}_\eta$, is known explicitly and describes the ultimately decay of the shearing waves (on the long resistive time scale), although these could be transiently amplified. The second term, $\mathcal{G}_t$, depends on the properties of the time-dependent symmetric matrix $Q_{jm}(\tau)$. Shearing waves can grow if $Q_{jm}(\tau)$ has at least one negative eigenvalue of large enough magnitude. To translate this requirement into an explicit statement on dynamo action requires developing a dynamical theory of the velocity correlators, $R_{jm}(\tau,\tau')$, because $Q_{jm}(\tau)$ depends on time integrals over $R_{jm}(\tau,\tau')$.

In specific cases it is possible that the velocity dynamics is such that $\tilde{\eta}_{jm}(\tau)$ becomes independent of τ, in the long time limit (this is generic when steady forcing competes with dissipation). Taking the zero of time to be after this stationary state has been reached, we can do the s integrals in Eq. (80) explicitly and write

$$Q_{jm}(t)k_jk_m = t(\tilde{\eta}_{jm}k_jk_m) + 2At^2(\tilde{\eta}_{1m}k_mk_2) + \frac{4}{3}A^2t^3(\tilde{\eta}_{11}k_2^2). \tag{83}$$

We can now make further statements on the dynamo growth using Eq. (83). Note that the linear shear of the form that we have adopted is likely to lead to a nonzero $\tilde{\eta}_{12}$, but is not expected to couple the X_3 component with other components, and thus we expect $\tilde{\eta}_{13}=\tilde{\eta}_{23}=0$. Then

$$-Q_{jm}k_jk_m = -t[\tilde{\eta}_{11}k_1^2 + \tilde{\eta}_{22}k_2^2 + 2\tilde{\eta}_{12}k_1k_2 + \tilde{\eta}_{33}k_3^2)] - 2At^2[\tilde{\eta}_{11}k_1k_2 + \tilde{\eta}_{12}k_2^2] - \frac{4}{3}At^3\tilde{\eta}_{11}k_2^2. \quad (84)$$

The term linear in t will dominate at early times while the term proportional to t^3 will dominate eventually. Thus, at early times we need one of the eigenvalues of the matrix

$$\begin{pmatrix} \tilde{\eta}_{11} & \tilde{\eta}_{12} & 0 \\ \tilde{\eta}_{12} & \tilde{\eta}_{22} & 0 \\ 0 & 0 & \tilde{\eta}_{33} \end{pmatrix}$$

to be negative for dynamo growth. These eigenvalues are

$$\lambda_{\pm} = \frac{(\tilde{\eta}_{11} + \tilde{\eta}_{22})}{2} \pm \frac{|\tilde{\eta}_{11} - \tilde{\eta}_{22}|}{2}\left[1 + 4\frac{\tilde{\eta}_{12}^2}{(\tilde{\eta}_{11} - \tilde{\eta}_{22})^2}\right]^{1/2},$$

$$\lambda_3 = \tilde{\eta}_{33}. \quad (85)$$

Nonzero values of $\tilde{\eta}_{12}$ or negative values of the diagonal elements of the turbulent diffusion tensor favor growth at early times. Preliminary work on simple models of velocity dynamics that we are exploring suggests that $\tilde{\eta}_{22}$ can become negative but $\tilde{\eta}_{11}$ and $\tilde{\eta}_{33}$ remain positive; this happens because the turbulence is strongly affected by the background shear and the velocity correlators are not isotropic. Thus, a nonzero k_2 seems to be required for growth initially.

At intermediate times, when the t^2 term dominates we can always choose shearing waves with an appropriate sign and magnitude of k_1k_2 such that $2At^2(\tilde{\eta}_{11}k_1k_2+\tilde{\eta}_{12}k_2^2)$ is negative, and there is growth of the mean field. On the other hand, all shearing waves with nonzero k_2 will eventually decay, in the long time limit $t\to\infty$, if $\tilde{\eta}_{11}>0$, as then the t^3 term is negative definite. Thus it seems likely that the shear dynamo can have shearing wave solutions, which grow for some time if they have nonzero X_2 dependence, but which will eventually decay. As already emphasized above, one needs to develop a dynamical theory of the velocity correlators, for deriving more explicit results on dynamo action, due to nonhelical turbulence and shear. It is, in general, not an easy task to make analytical progress on a dynamical theory. However, in the limit of low fluid Reynolds numbers, a perturbative analysis is possible and the velocity correlators can be computed explicitly. Such an analysis has been undertaken by Singh and Sridhar, and preliminary results for nonhelical forcing indicate that the turbulent diffusion coefficient $\tilde{\eta}_{22}$ can indeed become negative. Also our conclusions are based on the differential equation approximation, which is valid for a finite period and thus we need to solve the integral equation for the mean field evolution directly, to firm up the above results.

VIII. CONCLUSIONS

We have studied here large-scale dynamo action due to turbulence in the presence of a linear shear flow. Systematic use of the shearing coordinate transformation and the Galilean invariance of a linear shear flow allows us to develop a quasilinear theory of the shear dynamo which, we emphasize, is non perturbative in the shear parameter. The result is an integro-differential equation for the evolution of the mean magnetic field. We showed using this equation that for nonhelical turbulence, the time evolution of the cross-shear components of the mean field do not depend on any other components excepting themselves. This implies that there is essentially no shear-current type effect in quasilinear theory in the limit of zero resistivity. Our result is valid for any Galilean-invariant velocity field, independent of its dynamics.

We then derived differential equations for the mean-field evolution, by developing a systematic approximation to the integro-differential equation, assuming the mean field varies on time scales much longer than the correlation times of the turbulence. For nonhelical velocity correlators, these equations can be solved in terms of shearing waves. These waves can grow transiently at early and intermediate times. However it is likely that they will eventually decay at asymptotically late times. More explicit statements about the behavior of the shearing wave solutions requires developing a dynamical theory of velocity correlators in shear flows. It is also important to directly solve the integral equation for the mean field as the differential equation approximation is valid only for a limited period.

Growth of large-scale magnetic fields in the presence of shear and nonhelical turbulence has been reported in some direct numerical simulations [1,2]. Whether we can understand these numerical results through our quasilinear theory depends on the existence (or otherwise) of growing solutions to the integral Eq. (51) for the mean field. This in turn relies on the form of the velocity correlators, which will be strongly affected by shear and highly anisotropic; hence it is difficult to guess their tensorial forms *a priori*, and it is necessary to develop a dynamical theory of velocity correlators. We cannot discount the possibility that effects we have ignored may also play a role. Perhaps the initial growth of the shearing wave in the mean field, for large enough shear, is sustained by an effect which breaks one of our assumptions. One possibility is that helicity fluxes arising due to shear, turbulence and an inhomogeneous mean magnetic field [10,12] induce a nonlinear alpha effect when the Lorentz forces become strong. Another is the possible presence of an incoherent alpha-shear dynamo [1,13] in these simulations. A third possibility is that if even transient growth makes nonaxisymmetric mean fields strong enough, they themselves might drive motions which could lead to sustained dynamo action; this seems remniscent of some of the subcritical dynamos discussed by [14]. Clearly further studies of various aspects of the shear dynamo, particularly incorporating velocity dynamics can only be more fruitful.

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