Chapter 2

Gravitational wave energy flux from inspiralling compact binaries in quasi-elliptical orbits: The instantaneous terms

2.1 Introduction

Inspiralling compact binaries, one of the prototype sources for laser interferometric gravitational wave (GW) detectors, are usually modelled as moving in quasi-circular orbits. This is justified since gravitational radiation reaction, under which it inspirals, circularizes the orbit towards the late stages of inspiral [138]. This late phase of inspiral and the ensuing merger phase offers promises for the gravitational wave interferometric detectors. The recently discovered double pulsar system [139, 28] has an eccentricity as low as 0.088 consistent with the circular orbit assumption for the late inspiral and pre-merger phase, believed to be reasonable enough for most of the binary systems made of neutron stars or black holes.

The theoretical modelling of the binary's phase evolution to a very high precision is called the phasing formula. This is the basic theoretical ingredient used in the construction of search templates for GW using matched filtering [140]. The two key inputs required for the construction of templates for binaries moving in quasi-circular orbits in the adiabatic approximation are the orbital energy and the GW luminosity (energy flux). These are computed using a cocktail of approximation schemes in general relativity. The schemes include the multipole decomposition, the post-Minkowskian expansion of the gravitational field or an expansion in Newton's constant G, the post-Newtonian expansion or an expansion in v/c, and the far-zone expansion or an expansion in $1/R$, where R is the distance from the source

(See [7] for a recent review).

Though the garden variety binary sources of GW are those moving in quasi-circular orbits, there is an increased recent interest in inspiralling binaries moving in ***quasi-eccentric*** orbits. Astrophysical scenarios currently exist which lead to binaries with nonzero eccentricity in the gravitational wave detector bandwidth, both terrestrial and space-based. For instance, inner binaries of hierarchical triplets undergoing Kozai oscillations [118] could not only merge due to gravitational radiation reaction [119] but a good fraction ($\sim$ 30%) of them will have eccentricity greater than about 0.1 as they enter the sensitivity band of advanced ground based interferometers. [120]. Almost all the above systems possess eccentricities well below 0.2 at 40 Hz and below 0.02 at 200 Hz. The population of stellar mass binaries in globular clusters is expected to have a thermal distribution of eccentricities [121]. In a study on the growth of intermediate BHs [122] in globular clusters it was found that the binaries have eccentricities between 0.1 and 0.2 in the LISA bandwidth. Though, supermassive black hole binaries are powerful GW sources for LISA, it is not yet conclusive if they would be in quasi-circular or quasi-eccentric orbits [141]. If a Kozai mechanism is at work, these supermassive black hole binaries could be in highly eccentric orbits and merge within the Hubble time [123]. Sources of the kind discussed above provide the prime motivation to investigate higher post-Newtonian order modelling for quasi-eccentric binaries.

The inspiral and subsequent coalescence of two compact objects results in the emission of gravitational waves (GWs). The GW luminosity from a system of two point masses in elliptic motion was first discussed by Peters and Mathews [138, 112]. The 1PN and 1.5PN accurate fluxes was provided in [142, 143, 144, 145, 146] and used to study the associated evolution of orbital elements using the 1PN 'quasi-Keplerian' representation of the binary's orbit obtained by Damour and Deruelle [147]. Gopakumar and Iyer further extended these results to 2PN order [46, 115] using the generalized quasi-Keplerian representation developed by Damour, Schafer and Wex [148, 149, 150]. The results for the energy flux and waveform presented in [46] was in perfect agreement with those obtained by Will and Wiseman using a different formalism [151]. Recently, Damour, Gopakumar and Iyer [47] discussed an analytic method for constructing high accuracy templates for the GW signals from the inspiral phase of compact binaries moving on quasi-elliptical orbits. They used an improved "method of variation of constants" to combine the three time scales involved in the elliptical orbit case, namely, orbital period, periastron precession and radiation reaction time scales, without making the usual approximation of treating the radiative time scale as an adiabatic process.

The generation problem for gravitational waves at any post-Newtonian order requires the solution to two independent problems within the given theory of gravitation. The first relates to the equation of motion of the binary and the second to the far zone fluxes of energy, angular momentum and linear momentum. The latter requires the computation of the relativistic mass and current multipole moments to appropriate post-Newtonian orders. The 3PN equations of motion (EOM) required to handle gravitational wave phasing turned out to be technically more involved due to the issues related to the ambiguities of self-field regularisation using Riesz or Hadamard regularisations [94, 152, 87] than the 2.5PN EOM required to analyse the timing problem of binary pulsars like 1913+16. Only by a deeper understanding of the origin of these ambiguities and the use of a regularisation scheme like dimensional regularisation, that respects the gauge symmetries of general relativity has the problem been uniquely resolved [87, 102] and provided the value of the ambiguity parameter ω_s or equivalently λ. We thus have in hand the requisite 3PN EOM for compact binaries moving in general orbits. The computation of GW luminosity 3PN or $(v/c)^6$ beyond the leading Einstein quadrupole formula crucially requires the computation of the 3PN accurate mass quadrupole moment. However, it was proved that three and only three ambiguity parameters (ξ, κ and ζ) exist in this case when using (extended) Hadamard regularisation [95, 96]. More recently, by use of dimensional regularisation and comparison to the Hadamard results, the three ambiguity parameters in the mass quadrupole has been determined [109] and some checks performed [153, 154]. These works thus provide the fully determined 3PN accurate mass quadrupole for general orbits, the other important ingredient to compute the 3PN accurate energy and angular momentum fluxes for inspiralling compact binaries moving in general non-circular orbits. The 3.5PN phasing of inspiralling compact binaries moving in quasi-circular orbits is now complete and available for use in GW data analysis [108, 154]. Unlike at earlier post-Newtonian orders, the 3PN contribution to energy flux come not only from the 'instantaneous' terms discussed in this chapter but also include 'hereditary' contributions arising from the ***tail*** **of** ***tails*** and ***tail-square*** terms. ***A*** semi-analytical scheme is proposed and discussed in detail in the next chapter to evaluate these history dependent contributions.

In this chapter, for binaries moving in elliptical orbits, we compute ***all*** the instantaneous contributions to the 3PN accurate GW energy flux. The orbital average of this flux is obtained using the recently constructed 3PN quasi-Keplerian parametrization of the binary's orbital motion by Memmesheimer, Gopakumar and Schafer [155]. Supplementing these, by contributions from the hereditary terms computed in the next chapter we will obtain the complete expressions for the far-zone energy flux from inspiralling compact binaries moving in eccentric orbits. The expressions will represent gravitational waves from a binary evolving negligibly under gravitational radiation reaction, including precisely upto 3PN order, the effects of eccentricity and periastron precession during epochs of inspiral when the orbital parameters are essentially constant over a few orbital revolutions. It also represents the first step towards the discussion of the ***quasi-elliptical*** case: the evolution of the binary in an elliptical orbit under gravitational radiation reaction.

The present work extends the circular orbit results at 2.5PN [156] and 3PN [95] to the elliptical orbit case. (Contrary to the previous orders, we encounter both instantaneous and hereditary terms at the 2.5PN and 3PN orders). Further, it extends earlier works on instantaneous contributions for binaries moving in elliptical orbits at 1PN [143, 144] and 2PN [46] to 3PN order. The next chapter similarly extends hereditary contributions at 1.5PN by [145, 146] to 2.5PN order and 3PN. The 3PN hereditary contributions comprise the ***tail(tail)*** and ***tail***^{2} and are extensions of [157, 158] for circular orbits to the elliptical case.

In Sec 2.2 we begin with the structure of the far-zone flux of energy, use expressions relating the radiative moments to the source moments and decompose the energy flux expression into its instantaneous parts and hereditary parts. Sec. 2.3 lists all the requisite input multipole moments in standard harmonic coordinates for binaries moving in general (non-circular) orbits. Sec. 2.4 discuss the computation of the instantaneous terms in the energy flux for standard harmonic coordinates. Sec. 2.5 recasts the flux in modified harmonic coordinates (without logs) and ADM coordinates. Secs 2.6.2 summarises the 3PN quasi-Keplerian representation required to average the flux expression over an orbit. Sec. 2.7 and Sec. 2.8 exhibits the orbital average of the energy flux in modified harmonic coordinates and ADM coordinates respectively. Sec. 2.10 exhibits the instantantaneous contributions to the energy flux in terms of gauge invariant variables.

2.2 The far-zone flux of energy

In this section, we discuss the computation of 3PN accurate energy flux for binaries moving in general (non-circular) orbits. Starting from the expression for the far zone flux in terms of the radiative multipole moments and using the relations connecting the radiative multipole moments to the source moments, we rewrite the resultant structure of the gravitational wave (GW) energy flux. It consists of the instantaneous terms investigated here, which are functions of the retarded time and hereditary terms, evaluated in the next chapter, which depends on the dynamics of the system in its entire past.

2.2.1 Far zone flux in terms of the radiative multipole moments.

Following Thorne [54], the expression for the 3PN accurate far zone energy flux in terms of symmetric trace-free (STF) radiative multipole moments read as

$$\begin{aligned}\left(\frac{d\mathcal{E}}{dt}\right)_{\text{far-zone}} &= \frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij}\right.\\ &+\frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk}+\frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm}+\frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right]\\ &\left.+\frac{1}{c^6}\left[\frac{1}{594000}U^{(1)}_{ijkmn}U^{(1)}_{ijkmn}+\frac{4}{14175}V^{(1)}_{ijkm}V^{(1)}_{ijkm}\right]+O(8)\right\}. \qquad (2.1)\end{aligned}$$

In the above U_L and V_L (with L $=$ $ijk...$ a multi-index composed of l indices) are the mass and current type radiative multipole moments respectively and $U_L^{(l)}$ and $V_L^{(l)}$ denote their l^{th} time derivatives. The moments are functions of retarded time $T_R \equiv \mathrm{T} - \frac{R}{c}$ in radiative coordinates. ε_{ipq} is the usual Levi-Civita symbol such that $\varepsilon_{123} = +1$. The shorthand $O(n)$ indicates that the post-Newtonian remainder is of order of $O(c^{-n})$.

Using the MPM formalism, the radiative moments in Eq. (2.1) can be re-expressed in terms of the source moments to an accuracy sufficient for the computation of the energy flux. For the energy flux to be complete up to 3PN approximation, one must compute the mass type radiative quadrupole U_{ij} to 3PN accuracy, mass octupole U_{ijk} and current quadrupole V_{ij} to 2PN accuracy, mass hexadecupole U_{ijkm} and current octupole V_{ijk} to 1PN accuracy and finally U_{ijkmn} and V_{ijkm} to Newtonian accuracy.

2.2.2 Radiative moments in terms of source moments

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L are given below. For the mass type moments we have [76, 156, 157, 158]

$$\begin{aligned}U_{ij}(U) &= I^{(2)}_{ij}(U)+\frac{2GM}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{2}\right]I^{(4)}_{ij}(U-\tau)\\ &+\frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty}d\tau I^{(3)}_{a<i}(U-\tau)I^{(3)}_{j>a}(U-\tau)\right.\\ &\quad+\frac{1}{7}I^{(5)}_{a<i}I_{j>a}-\frac{5}{7}I^{(4)}_{a<i}I^{(1)}_{j>a}-\frac{2}{7}I^{(3)}_{a<i}I^{(2)}_{j>a}+\frac{1}{3}\varepsilon_{ab<i}I^{(4)}_{j>a}J_b\\ &\quad\left.+4\left[W^{(2)}I_{ij}-W^{(1)}I^{(1)}_{ij}\right]\right\}\\ &+2\left(\frac{GM}{c^3}\right)^2\int_0^{+\infty}d\tau I^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right]\\ &+O(7), \qquad (2.2a)\end{aligned}$$

$$
\begin{aligned}
U_{ijk}(U) &= I^{(3)}_{ijk}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right] I^{(5)}_{ijk}(U-\tau) \\
&\quad + O(5)\,, && (2.2b) \\
U_{ijkm}(U) &= I^{(4)}_{ijkm}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30}\right] I^{(6)}_{ijkm}(U-\tau)\right. \\
&\quad + \frac{2}{5}\int_0^{+\infty} d\tau I^{(3)}_{<ij}(U-\tau) I^{(3)}_{km>}(U-\tau) \\
&\quad \left. - \frac{21}{5} I^{(5)}_{<ij} I_{km>} - \frac{63}{5} I^{(4)}_{<ij} I^{(1)}_{km>} - \frac{102}{5} I^{(3)}_{<ij} I^{(2)}_{km>}\right\} + O(4)\,, && (2.2c)
\end{aligned}
$$

where the bracket <> denotes STF projection. In the above formulas, M is the total ADM mass of the binary system. The I_L's and J_L's are the mass and current-type source moments, and $I_L^{(p)}$, $J_L^{(p)}$ denote their p-th time derivatives. W is the monopole corresponding to the set of gauge moments W_L.

For the current-type moments, on the other hand, we find

$$
\begin{aligned}
V_{ij}(U) &= J^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6}\right] J^{(4)}_{ij}(U-\tau) \\
&\quad + O(5)\,, && (2.3a) \\
V_{ijk}(U) &= J^{(3)}_{ijk}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{5}{3}\right] J^{(5)}_{ijk}(U-\tau)\right. \\
&\quad \left. + \frac{1}{10}\varepsilon_{ab<i} I^{(5)}_{j\underline{a}} I_{k>b} - \frac{1}{2}\varepsilon_{ab<i} I^{(4)}_{j\underline{a}} I^{(1)}_{k>b} - 2J_{<i} I^{(4)}_{jk>}\right\} \\
&\quad + O(4)\,. && (2.3b)
\end{aligned}
$$

[The underlined index a means that it should be excluded from the STF projection]. For all the other moments required in the computation we need only the leading order accuracy, so that

$$
\begin{aligned}
U_L(U) &= I^{(l)}_L(U) + O(3)\,, && (2.4a) \\
V_L(U) &= J^{(l)}_L(U) + O(3)\,. && (2.4b)
\end{aligned}
$$

The constant scaling the logarithm has been chosen to be r_0 to match with the choice made in the computation of tails-of-tails in [158]. It is a freely specifiable constant, entering the relation between the retarded time $U = T - R/c$ in radiative coordinates and the corresponding time $t - \rho/c$ in harmonic coordinates (where ρ is the distance of the source in harmonic coordinates). More precisely we have

$$
U = t - \frac{\rho}{c} - \frac{2GM}{c^3}\ln\left(\frac{\rho}{c\, r_0}\right) \qquad (2.5)
$$

From Eqs (2.2)-(2.3), it is clear that the radiative moments have two distinct contributions. One part which is a function only of the retarded time, $U = T - \frac{R}{c}$, and referred to as the 'instantaneous terms' forms the subject matter of this chapter. The second part on the other hand depends on the dynamics of the system in its entire past [76] and is referred to as hereditary contributions. Equally important but requiring a different treatment, it is dealt with in the next chapter as mentioned earlier.

2.2.3 Far zone flux in terms of source multipole moments: Instantaneous terms

Discussions in the earlier section on the structure of radiative multipole moments in terms of source moments allow us to write down explicitly the different kinds of contributions to the far zone energy flux up to 3PN. We have,

$$\left(\frac{d\mathcal{E}}{dt}\right) = \left(\frac{d\mathcal{E}}{dt}\right)_{\rm inst} + \left(\frac{d\mathcal{E}}{dt}\right)_{\rm hered} . \tag{2.6}$$

where the instantaneous contribution of interest here is explictly written as

$$\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)_{\rm inst} = & \frac{G}{c^5}\left\{\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij}\right.\\
&+\frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk}+\frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}I^{(5)}_{ijkm}I^{(5)}_{ijkm}+\frac{1}{84}J^{(4)}_{ijk}J^{(4)}_{ijk}\right]\\
&+\frac{8G}{5c^5}\left\{I^{(3)}_{ij}\left[I_{ij}W^{(5)}+2I^{(1)}_{ij}W^{(4)}-2I^{(3)}_{ij}W^{(2)}-I^{(4)}_{ij}W^{(1)}\right]\right\}\\
&+\frac{2G}{5c^5}I^{(3)}_{ij}\left\{-\frac{4}{7}I^{(5)}_{ai}I^{(1)}_{aj}-I^{(4)}_{ai}I^{(2)}_{aj}-\frac{4}{7}I^{(3)}_{ai}I^{(3)}_{aj}+\frac{1}{7}I^{(6)}_{ai}I_{aj}+\right.\\
&\left.\frac{1}{3}\epsilon_{abi}\left(I^{(4)}_{aj}J^{(1)}_{b}+I^{(5)}_{aj}J_{b}\right)\right\}\\
&\left.+\frac{1}{c^6}\left[\frac{1}{594000}I^{(6)}_{ijkmn}I^{(6)}_{ijkmn}+\frac{4}{14175}J^{(5)}_{ijkm}J^{(5)}_{ijkm}\right]+O(8)\right\}.
\end{aligned} \tag{2.7}$$

Since the hereditary contributions will be discussed completely in the next chapter, we refrain from giving more details about them here.

2.3 Input multipole moments in standard harmonic coordinates (with logs)

We provide, in this section, the requisite multipole moments needed for the computation of the instantaneous part of the 3PN accurate energy flux in the standard harmonic coordinate system containing log terms as used in other earlier works including [159]. These are generalisations to general non-circular orbits of expressions available in [95] for inspiralling compact binaries moving in circular orbits and explicitly computed using retarded potentials by implementing the details described in [96]. Though algebraically long and involved, the procedure is fairly algorithmic as explained in [95, 96]. Thus we skip those details here and list the final expression for these relevant source multipoles. The mass quadrupole I_{ij} is already available in [96], where the procedure used for its computation is outlined in detail. We list it here for the sake of completeness and choice of notation.

$$
\begin{aligned}
I_{ij} \;=\;& \nu m\left\{\left[\mathcal{A}-\frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r^2}\dot{r}\right]x_{\langle i}x_{j\rangle}+\mathcal{B}\frac{r^2}{c^2}v_{\langle i}v_{j\rangle}\right.\\
&\left.+2\left[\mathcal{C}\frac{r\dot{r}}{c^2}+\frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r}\right]x_{\langle i}v_{j\rangle}\right\}, \qquad (2.8\text{a})
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A} \;=\;& 1+\frac{1}{c^2}\left[v^2\left(\frac{29}{42}-\frac{29}{14}\nu\right)+\frac{Gm}{r}\left(-\frac{5}{7}+\frac{8}{7}\nu\right)\right]\\
&+\frac{1}{c^4}\left[\frac{Gm}{r}v^2\left(\frac{2021}{756}-\frac{5947}{756}\nu-\frac{4883}{756}\nu^2\right)\right.\\
&\quad+\frac{G^2m^2}{r^2}\left(-\frac{355}{252}-\frac{953}{126}\nu+\frac{337}{252}\nu^2\right)\\
&\quad+v^4\left(\frac{253}{504}-\frac{1835}{504}\nu+\frac{3545}{504}\nu^2\right)\\
&\quad\left.+\frac{Gm}{r}\dot{r}^2\left(-\frac{131}{756}+\frac{907}{756}\nu-\frac{1273}{756}\nu^2\right)\right]\\
&+\frac{1}{c^6}\left[v^6\left(\frac{4561}{11088}-\frac{7993}{1584}\nu+\frac{117067}{5544}\nu^2-\frac{328663}{11088}\nu^3\right)\right.\\
&\quad+v^4\frac{Gm}{r}\left(\frac{307}{77}-\frac{94475}{4158}\nu+\frac{218411}{8316}\nu^2+\frac{299857}{8316}\nu^3\right)\\
&\quad+\frac{G^3m^3}{r^3}\left(\frac{6285233}{207900}+\frac{15502}{385}\nu-\frac{3632}{693}\nu^2+\frac{13289}{8316}\nu^3\right.\\
&\qquad\left.-\frac{428}{105}\ln\left(\frac{r}{r_0}\right)-\frac{44}{3}\nu\ln\left(\frac{r}{r_0'}\right)\right)\\
&\quad+\frac{G^2m^2}{r^2}\dot{r}^2\left(-\frac{8539}{20790}+\frac{52153}{4158}\nu-\frac{4652}{231}\nu^2-\frac{54121}{5544}\nu^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{Gm}{r}\dot{r}^4\left(\frac{2}{99}-\frac{1745}{2772}\nu+\frac{16319}{5544}\nu^2-\frac{311}{99}\nu^3\right)\\
&+\frac{G^2m^2}{r^2}v^2\left(\frac{187183}{83160}-\frac{605419}{16632}\nu+\frac{434909}{16632}\nu^2-\frac{37369}{2772}\nu^3\right)\\
&+\frac{Gm}{r}v^2\dot{r}^2\left(-\frac{757}{5544}+\frac{5545}{8316}\nu-\frac{98311}{16632}\nu^2+\frac{153407}{8316}\nu^3\right)\Bigg], \qquad (2.8b)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B} = {} & \frac{11}{21}-\frac{11}{7}\nu\\
&+\frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{106}{27}-\frac{335}{189}\nu-\frac{985}{189}\nu^2\right)\right.\\
&\qquad\left.+v^2\left(\frac{41}{126}-\frac{337}{126}\nu+\frac{733}{126}\nu^2\right)+\dot{r}^2\left(\frac{5}{63}-\frac{25}{63}\nu+\frac{25}{63}\nu^2\right)\right]\\
&+\frac{1}{c^4}\left[v^4\left(\frac{1369}{5544}-\frac{19351}{5544}\nu+\frac{45421}{2772}\nu^2-\frac{139999}{5544}\nu^3\right)\right.\\
&\qquad+\frac{G^2m^2}{r^2}\left(-\frac{40716}{1925}-\frac{10762}{2079}\nu+\frac{62576}{2079}\nu^2-\frac{24314}{2079}\nu^3\right.\\
&\qquad\qquad\left.+\frac{428}{105}\ln\left(\frac{r}{r_0}\right)\right)\\
&\qquad+\frac{Gm}{r}\dot{r}^2\left(\frac{79}{77}-\frac{5807}{1386}\nu+\frac{515}{1386}\nu^2+\frac{8245}{693}\nu^3\right)\\
&\qquad+\frac{Gm}{r}v^2\left(\frac{587}{154}-\frac{67933}{4158}\nu+\frac{25660}{2079}\nu^2+\frac{129781}{4158}\nu^3\right)\\
&\qquad\left.+v^2\dot{r}^2\left(\frac{115}{1386}-\frac{1135}{1386}\nu+\frac{1795}{693}\nu^2-\frac{3445}{1386}\nu^3\right)\right], \qquad (2.8c)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} = {} & -\frac{2}{7}+\frac{6}{7}\nu\\
&+\frac{1}{c^2}\left[v^2\left(-\frac{13}{63}+\frac{101}{63}\nu-\frac{209}{63}\nu^2\right)\right.\\
&\qquad\left.+\frac{Gm}{r}\left(-\frac{155}{108}+\frac{4057}{756}\nu+\frac{209}{108}\nu^2\right)\right]\\
&+\frac{1}{c^4}\left[\frac{Gm}{r}v^2\left(-\frac{2839}{1386}+\frac{237893}{16632}\nu-\frac{188063}{8316}\nu^2-\frac{58565}{4158}\nu^3\right)\right.\\
&\qquad+\frac{G^2m^2}{r^2}\left(-\frac{12587}{41580}+\frac{406333}{16632}\nu-\frac{2713}{396}\nu^2+\frac{4441}{2772}\nu^3\right)\\
&\qquad+v^4\left(-\frac{457}{2772}+\frac{6103}{2772}\nu-\frac{13693}{1386}\nu^2+\frac{40687}{2772}\nu^3\right)\\
&\qquad\left.+\frac{Gm}{r}\dot{r}^2\left(\frac{305}{5544}+\frac{3233}{5544}\nu-\frac{8611}{5544}\nu^2-\frac{895}{154}\nu^3\right)\right]. \qquad (2.8d)
\end{aligned}
$$

In the above equation r_0 is an arbitrary scale that is introduced in the general MPM formalism and which then appears in the definition of the source multipole moments. r_0', related to other scales r_1' and r_2' by $m \ln r_0' = m_1 \ln r_1' + m_2 \ln r_2'$ is specific to the application of the formalism to point particle systems and comes from regularizing self-field effects in the standard harmonic coordinates. By *definition* of the ambiguity parameters these scales are taken to be the ***same*** as the two scales that appear in the final expression of the 3PN equations of motion in harmonic coordinates computed in [85, 152]. r_1', r_2' and hence r_0' are 'unphysical' in the sense that they can be arbitrarily moved by a coordinate transformation of the 'bulk' metric outside the particles or more appropriately when considering the renormalisation which follows the regularization by relevant shifts of the particles' world lines [154].

The 2PN mass octupole for general orbits is the next of the non-trivial moments required for what follows. It is given by:

$$\begin{aligned}
I_{ijk} \;=\;& \nu m \sqrt{1-4\nu}\ \mathrm{STF}_{ijk}\left\{ x_{ijk}\left[-1+\frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{5}{6}-\frac{13\nu}{6}\right)+v^2\left(-\frac{5}{6}+\frac{19}{6}\nu\right)\right]\right.\right.\\
&+\frac{1}{c^4}\left[v^4\left(-\frac{257}{440}+\frac{7319}{1320}\nu-\frac{5501}{440}\nu^2\right)+\frac{G^2m^2}{r^2}\left(\frac{47}{33}+\frac{1591}{132}\nu-\frac{235}{66}\nu^2\right)\right.\\
&\left.\left.+\frac{Gm}{r}\dot{r}^2\left(\frac{247}{1320}-\frac{531}{440}\nu+\frac{1347}{440}\nu^2\right)+\frac{Gm}{r}v^2\left(-\frac{3853}{1320}+\frac{14257}{1320}\nu+\frac{17371}{1320}\nu^2\right)\right]\right]\\
&+x_{ij}v_k\frac{r\dot{r}}{c^2}\left[1-2\nu+\frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{2461}{660}-\frac{8689}{660}\nu-\frac{1389}{220}\nu^2\right)\right.\right.\\
&\left.\left.+v^2\left(\frac{13}{22}-\frac{107}{22}\nu+\frac{102}{11}\nu^2\right)\right]\right]+x_iv_{jk}\frac{r^2}{c^2}\left[-1+2\nu+\frac{1}{c^2}\left[v^2\left(-\frac{61}{110}+\frac{519}{110}\nu-\frac{504}{55}\nu^2\right)\right.\right.\\
&\left.\left.+\dot{r}^2\left(\frac{1}{11}-\frac{4}{11}\nu+\frac{3}{11}\nu^2\right)+\frac{Gm}{r}\left(-\frac{1949}{330}-\frac{62}{165}\nu+\frac{483}{55}\nu^2\right)\right]\right]\\
&\left.+v_{ijk}\frac{\dot{r}r^3}{c^4}\left(-\frac{13}{55}+\frac{52}{55}\nu-\frac{39}{55}\nu^2\right)\right\}+O(6)\,. \qquad (2.9)
\end{aligned}$$

The other mass-type moments needed in this work read as

$$\begin{aligned}
I_{ijkl} \;=\;& \nu m\,\mathrm{STF}_{ijkl}\left\{x_{ijkl}\left[1-3\nu+\frac{1}{c^2}\left[\left(\frac{103}{110}-\frac{147}{22}\nu+\frac{279}{22}\nu^2\right)v^2\right.\right.\right.\\
&\left.\left.-\left(\frac{10}{11}-\frac{61}{11}\nu+\frac{105}{11}\nu^2\right)\frac{Gm}{r}\right]\right]-\frac{72}{55}v_ix_{jkl}\frac{r\dot{r}}{c^2}(1-5\nu+5\nu^2)\\
&\left.+\frac{78}{55}v_{ij}x_{kl}\frac{r^2}{c^2}(1-5\nu+5\nu^2)\right\}, \qquad (2.10)
\end{aligned}$$

$$\begin{aligned}
I_{ijklm} &= -\nu m\sqrt{1-4\nu}(1-2\nu)\mathrm{STF}_{ijklm}\left\{x_{ijklm}\right\}, && (2.11)\\
I_{ijklmn} &= \nu m(1-5\nu+5\nu^2)\mathrm{STF}_{ijklmn}\left\{x_{ijklmn}\right\}. && (2.12)
\end{aligned}$$

In the above and what follows, $x_{ijk\ldots} = x_ix_jx_k\ldots$ and $v_{ijk\ldots} = v_iv_jv_k\ldots$, $\nu = m_1m_2/m^2$ and STF_L denotes that the terms inside the bracket are symmetric and trace-free in the indices listed.

W, the monopole corresponding to the set of gauge moments W_L is given by:

$$W = \frac{1}{3}\,\nu\, m\, \mathbf{x}.\mathbf{v} + O(2). \tag{2.13}$$

The other new input needed is the current quadrupole to 2PN accuracy which reads as:

$$\begin{aligned}
J_{ij} \;=\;& m\,\nu\,\sqrt{1-4\nu}\;\mathrm{STF}_{ij}\left\{\varepsilon_{abi}x_{ja}v_b\left[-1+\frac{1}{c^2}\left[\frac{G\,m}{r}\left(-\frac{27}{14}-\frac{15}{7}\,\nu\right)+v^2\left(-\frac{13}{28}+\frac{17}{7}\,\nu\right)\right]\right.\right.\\
&+\frac{1}{c^4}\left[v^4\left(-\frac{29}{84}+\frac{11}{3}\,\nu-\frac{505}{56}\,\nu^2\right)+\frac{G^2\,m^2}{r^2}\left(\frac{43}{252}+\frac{1543}{126}\,\nu-\frac{293}{84}\,\nu^2\right)\right.\\
&\left.\left.+\frac{Gm}{r}\dot{r}^2\left(\frac{5}{252}+\frac{241}{252}\,\nu+\frac{335}{84}\,\nu^2\right)+\frac{Gm}{r}v^2\left(-\frac{671}{252}+\frac{1297}{126}\,\nu+\frac{121}{12}\,\nu^2\right)\right]\right]\\
&+\varepsilon_{abi}x^a v^{jb}\,\frac{r\dot{r}}{c^2}\left[-\frac{5}{28}+\frac{5}{14}\,\nu+\frac{1}{c^2}\left[v^2\left(-\frac{25}{168}+\frac{25}{24}\,\nu-\frac{25}{14}\,\nu^2\right)\right.\right.\\
&\left.\left.\left.+\frac{G\,m}{r}\left(-\frac{103}{63}-\frac{337}{126}\,\nu+\frac{173}{84}\,\nu^2\right)\right]\right]\right\}+O(6)\,.
\end{aligned} \tag{2.14}$$

The remaining current moments required are given by

$$\begin{aligned}
J_{ijk} \;=\;& \nu m\,\mathrm{STF}_{ijk}\epsilon_{kab}\left\{x_{aij}v_b\left[1-3\nu\right.\right.\\
&\left.+\frac{1}{c^2}\left[\left(\frac{41}{90}-\frac{77}{18}\nu+\frac{185}{18}\nu^2\right)v^2+\left(\frac{14}{9}-\frac{16}{9}\nu-\frac{86}{9}\nu^2\right)\frac{Gm}{r}\right]\right]\\
&\left.+\frac{7}{45}x^a v^{ijb}\frac{r^2}{c^2}(1-5\nu+5\nu^2)\;+\frac{2}{9}x_{ai}v_{bj}\frac{r\,\dot{r}}{c^2}(1-5\nu+5\nu^2)\right\},
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
J_{ijkl} \;&=\; -\nu\, m\,\sqrt{1-4\nu}(1-2\nu)\mathrm{STF}_{ijkl}\left\{\epsilon_{lab}\,x_{aijk}v_b\right\}, && (2.16)\\
J_{ijklm} \;&=\; \nu\, m\,(1-5\nu+5\nu^2)\,\mathrm{STF}_{ijklm}\left\{\epsilon_{mab}\,x_{aijkl}v_b\right\}. && (2.17)
\end{aligned}$$

The computation of the fluxes involves the time derivatives of the source moments. 3PN accurate fluxes require the 3PN equations of motion for inspiralling compact binaries which is now complete [88, 160, 88, 102]. For the present work, where the multipole moments are computed in standard harmonic coordinates and reduced to the centre of mass (CM) coordinates, we require the 3PN accurate equation of motion (acceleration) in the CM coordinates in the standard harmonic gauge. This was computed in [159] and given by:

$$a^i = \frac{dv^i}{dt} = -\frac{m}{r^2}\left[(1+\mathcal{A}_E)\,n^i+\mathcal{B}_E\,v^i\right]+O\left(\frac{1}{c^7}\right)\,, \tag{2.18}$$

and where the coefficients $\mathcal{A}_E$ and $\mathcal{B}_E$ are:

$$
\begin{aligned}
\mathcal{A}_E \;=\;& \frac{1}{c^2}\left\{-\frac{3\,\dot{r}^2\,\nu}{2} + v^2 + 3\,\nu\,v^2 - \frac{m}{r}\,(4+2\,\nu)\right\} \\
+\;& \frac{1}{c^4}\left\{\frac{15\,\dot{r}^4\,\nu}{8} - \frac{45\,\dot{r}^4\,\nu^2}{8} - \frac{9\,\dot{r}^2\,\nu\,v^2}{2} + 6\,\dot{r}^2\,\nu^2\,v^2 + 3\,\nu\,v^4 - 4\,\nu^2\,v^4\right. \\
&\quad + \frac{m}{r}\left(-2\,\dot{r}^2 - 25\,\dot{r}^2\,\nu - 2\,\dot{r}^2\,\nu^2 - \frac{13\,\nu\,v^2}{2} + 2\,\nu^2\,v^2\right) \\
&\quad \left. + \frac{m^2}{r^2}\left(9 + \frac{87\,\nu}{4}\right)\right\} \\
+\;& \frac{1}{c^5}\left\{-\frac{24\,\dot{r}\,\nu\,v^2}{5}\frac{m}{r} - \frac{136\,\dot{r}\,\nu\,m^2}{15\,r^2}\right\} \\
+\;& \frac{1}{c^6}\left\{-\frac{35\,\dot{r}^6\,\nu}{16} + \frac{175\,\dot{r}^6\,\nu^2}{16} - \frac{175\,\dot{r}^6\,\nu^3}{16} + \frac{15\,\dot{r}^4\,\nu\,v^2}{2}\right. \\
&\quad - \frac{135\,\dot{r}^4\,\nu^2\,v^2}{4} + \frac{255\,\dot{r}^4\,\nu^3\,v^2}{8} - \frac{15\,\dot{r}^2\,\nu\,v^4}{2} + \frac{237\,\dot{r}^2\,\nu^2\,v^4}{8} \\
&\quad - \frac{45\,\dot{r}^2\,\nu^3\,v^4}{2} + \frac{11\,\nu\,v^6}{4} - \frac{49\,\nu^2\,v^6}{4} + 13\,\nu^3\,v^6 \\
&\quad + \frac{m}{r}\left(79\,\dot{r}^4\,\nu - \frac{69\,\dot{r}^4\,\nu^2}{2} - 30\,\dot{r}^4\,\nu^3 - 121\,\dot{r}^2\,\nu\,v^2 + 16\,\dot{r}^2\,\nu^2\,v^2\right. \\
&\qquad \left. + 20\,\dot{r}^2\,\nu^3\,v^2 + \frac{75\,\nu\,v^4}{4} + 8\,\nu^2\,v^4 - 10\,\nu^3\,v^4\right) \\
&\quad + \frac{m^2}{r^2}\left(\dot{r}^2 + \frac{32573\,\dot{r}^2\,\nu}{168} + \frac{11\,\dot{r}^2\,\nu^2}{8} - 7\,\dot{r}^2\,\nu^3 + \frac{615\,\dot{r}^2\,\nu\,\pi^2}{64} - \frac{26987\,\nu\,v^2}{840}\right. \\
&\qquad \left. + \nu^3\,v^2 - \frac{123\,\nu\,\pi^2\,v^2}{64} - 110\,\dot{r}^2\,\nu\,\ln\left(\frac{r}{r_0'}\right) + 22\,\nu\,v^2\,\ln\left(\frac{r}{r_0'}\right)\right) \\
&\quad \left. + \frac{m^3}{r^3}\left(-16 - \frac{41911\,\nu}{420} + \frac{44\,\lambda\,\nu}{3} - \frac{71\,\nu^2}{2} + \frac{41\,\nu\,\pi^2}{16}\right)\right\}\,, \qquad (2.19a)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_E \;=\;& \frac{1}{c^2}\left\{-4\,\dot{r} + 2\,\dot{r}\,\nu\right\} \\
+\;& \frac{1}{c^4}\left\{\frac{9\,\dot{r}^3\,\nu}{2} + 3\,\dot{r}^3\,\nu^2 - \frac{15\,\dot{r}\,\nu\,v^2}{2} - 2\,\dot{r}\,\nu^2\,v^2\right. \\
&\quad \left. + \frac{m}{r}\left(2\,\dot{r} + \frac{41\,\dot{r}\,\nu}{2} + 4\,\dot{r}\,\nu^2\right)\right\} \\
+\;& \frac{1}{c^5}\left\{\frac{8\,\nu\,v^2}{5}\frac{m}{r} + \frac{24\,\nu\,m^2}{5\,r^2}\right\} \\
+\;& \frac{1}{c^6}\left\{-\frac{45\,\dot{r}^5\,\nu}{8} + 15\,\dot{r}^5\,\nu^2 + \frac{15\,\dot{r}^5\,\nu^3}{4} + 12\,\dot{r}^3\,\nu\,v^2\right. \\
&\quad - \frac{111\,\dot{r}^3\,\nu^2\,v^2}{4} - 12\,\dot{r}^3\,\nu^3\,v^2 - \frac{65\,\dot{r}\,\nu\,v^4}{8} + 19\,\dot{r}\,\nu^2\,v^4 + 6\,\dot{r}\,\nu^3\,v^4
\end{aligned}
$$

$$
+\frac{m}{r}\left(\frac{329\,\dot{r}^3\,\nu}{6}+\frac{59\,\dot{r}^3\,\nu^2}{2}+18\,\dot{r}^3\,\nu^3-15\,\dot{r}\,\nu\,v^2-27\,\dot{r}\,\nu^2\,v^2-10\,\dot{r}\,\nu^3\,v^2\right)
$$

$$
+\frac{m^2}{r^2}\left(-4\,\dot{r}-\frac{18169\,\dot{r}\,\nu}{840}+25\,\dot{r}\,\nu^2+8\,\dot{r}\,\nu^3-\frac{123\,\dot{r}\,\nu\,\pi^2}{32}\right.
$$

$$
\left.\left.+44\,\dot{r}\,\nu\,\ln\left(\frac{r}{r'_0}\right)\right)\right\}\,. \tag{2.19b}
$$

Recall that λ is no longer arbitrary but now uniquely determined and given by $\lambda = -3080/1987$.

2.4 3PN energy flux in standard harmonic coordinates

Using the multipole moments in Eqs. (2.8) - (2.17), one computes the required time derivatives as required in Eq. (2.7). Though lengthy it is straightforward to compute the different parts constituting the instantaneous terms in the energy flux at 3PN order. We have,

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right) = & \left[\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{0PN}}+\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{1PN}}+\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{2PN}}\right.\\
& \left.+\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{2.5PN}}+\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{3PN}}\right]_{\mathrm{inst}}+\left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{her}}+\mathcal{O}(7)\,,
\end{aligned} \tag{2.20}
$$

$$
\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{0PN}} = \frac{32}{5}\frac{G^3 m^4 \nu^2}{c^5 r^4}\left\{v^2-\frac{11}{12}\dot{r}^2\right\}, \tag{2.21a}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{1PN}} = & \frac{32}{5}\frac{G^3 m^4 \nu^2}{c^7 r^4}\left\{v^4\left(\frac{785}{336}-\frac{71}{28}\nu\right)+\dot{r}^2 v^2\left(-\frac{1487}{168}+\frac{58}{7}\nu\right)\right.\\
& +\frac{G m}{r}v^2\left(-\frac{170}{21}+\frac{10}{21}\nu\right)+\dot{r}^4\left(\frac{687}{112}-\frac{155}{28}\nu\right)\\
& \left.+\frac{G m}{r}\dot{r}^2\left(\frac{367}{42}-\frac{5}{14}\nu\right)+\frac{G^2 m^2}{r^2}\left(\frac{1}{21}-\frac{4}{21}\nu\right)\right\},
\end{aligned} \tag{2.21b}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{2PN}} = & \frac{32}{5}\frac{G^3 m^4 \nu^2}{c^9 r^4}\left\{v^6\left(\frac{47}{14}-\frac{5497}{504}\nu+\frac{2215}{252}\nu^2\right)\right.\\
& +\dot{r}^2 v^4\left(-\frac{573}{56}+\frac{1713}{28}\nu-\frac{1573}{42}\nu^2\right)\\
& +\frac{G m}{r}v^4\left(-\frac{247}{14}+\frac{5237}{252}\nu-\frac{199}{36}\nu^2\right)\\
& +\dot{r}^4 v^2\left(\frac{1009}{84}-\frac{5069}{56}\nu+\frac{631}{14}\nu^2\right)\\
& +\frac{G m}{r}\dot{r}^2 v^2\left(\frac{4987}{84}-\frac{8513}{84}\nu+\frac{2165}{84}\nu^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^2 m^2}{r^2} v^2 \left(\frac{281473}{9072} + \frac{2273}{252}\nu + \frac{13}{27}\nu^2\right) \\
&+\dot{r}^6 \left(-\frac{2501}{504} + \frac{10117}{252}\nu - \frac{2101}{126}\nu^2\right) \\
&+\frac{G m}{r}\dot{r}^4 \left(-\frac{5585}{126} + \frac{60971}{756}\nu - \frac{7145}{378}\nu^2\right) \\
&+\frac{G^2 m^2}{r^2}\dot{r}^2 \left(-\frac{106319}{3024} - \frac{1633}{504}\nu - \frac{16}{9}\nu^2\right) \\
&+\frac{G^3 m^3}{r^3} \left(-\frac{253}{378} + \frac{19}{7}\nu - \frac{4}{27}\nu^2\right)\Bigg\}\,,
\end{aligned}
\tag{2.21c}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{2.5\mathrm{PN}} = {}& \frac{32}{5}\frac{G^3 m^4 \nu^2}{c^{10} r^4}\Bigg\{\dot{r}\,\nu\Bigg(-\frac{12349}{210}\frac{G m}{r}v^4 + \frac{4524}{35}\frac{G m}{r}v^2\dot{r}^2 - \frac{2753}{126}\frac{G^2 m^2}{r^2}v^2 \\
&-\frac{985}{14}\frac{G m}{r}\dot{r}^4 + \frac{13981}{630}\frac{G^2 m^2}{r^2}\dot{r}^2 - \frac{1}{315}\frac{G^3 m^3}{r^3}\Bigg)\Bigg\}\,,
\end{aligned}
\tag{2.21d}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{3\mathrm{PN}} = {}& \frac{32}{5}\frac{G^3 m^4 \nu^2}{c^{11} r^4}\Bigg\{v^8\left(\frac{80315}{14784} - \frac{694427}{22176}\nu + \frac{604085}{11088}\nu^2 - \frac{16985}{462}\nu^3\right) \\
&+\dot{r}^2 v^6 \left(-\frac{31499}{1008} + \frac{1119913}{5544}\nu - \frac{44701}{132}\nu^2 + \frac{38725}{231}\nu^3\right) \\
&+\frac{G m}{r} v^6 \left(-\frac{61669}{3696} + \frac{95321}{1008}\nu - \frac{955013}{11088}\nu^2 + \frac{47255}{1386}\nu^3\right) \\
&+\dot{r}^4 v^4 \left(\frac{204349}{2464} - \frac{3522149}{7392}\nu + \frac{2354753}{3696}\nu^2 - \frac{109447}{462}\nu^3\right) \\
&+\frac{G m}{r}\dot{r}^2 v^4 \left(\frac{136695}{1232} - \frac{202693}{336}\nu + \frac{744377}{1232}\nu^2 - \frac{931099}{5544}\nu^3\right) \\
&+\frac{G^2 m^2}{r^2} v^4 \Bigg(\frac{598614941}{2494800} - \frac{856}{35}\ln\left(\frac{r}{r_0}\right) \\
&+\left[\frac{39896}{2079} - \frac{369}{64}\pi^2\right]\nu + \frac{1300907}{33264}\nu^2 - \frac{161783}{24948}\nu^3\Bigg) \\
&+\dot{r}^6 v^2 \left(-\frac{1005979}{11088} + \frac{2589599}{5544}\nu - \frac{1322141}{2772}\nu^2 + \frac{90455}{693}\nu^3\right) \\
&+\frac{G m}{r}\dot{r}^4 v^2 \left(-\frac{715157}{3696} + \frac{35158037}{33264}\nu - \frac{3672143}{3696}\nu^2 + \frac{871025}{4158}\nu^3\right) \\
&+\frac{G^2 m^2}{r^2}\dot{r}^2 v^2 \Bigg(-\frac{35629009}{37800} + \frac{3424}{35}\ln\left(\frac{r}{r_0}\right) \\
&+\left[-\frac{150739}{1232} + \frac{861}{32}\pi^2\right]\nu - \frac{453247}{1848}\nu^2 + \frac{496081}{8316}\nu^3\Bigg) \\
&+\frac{G^3 m^3}{r^3} v^2 \Bigg(-\frac{24608492}{155925} + \frac{856}{105}\ln\left(\frac{r}{r_0}\right) \\
&+\left[-\frac{6356291}{22680} + \frac{44}{3}\ln\left(\frac{r}{r'_0}\right) + \frac{451}{64}\pi^2\right]\nu + \frac{3725}{462}\nu^2 - \frac{841}{2268}\nu^3\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\dot{r}^8\left(\frac{1507925}{44352}-\frac{20365}{126}\nu+\frac{687305}{5544}\nu^2-\frac{32755}{1386}\nu^3\right)\\
&+\frac{Gm}{r}\dot{r}^6\left(\frac{5476951}{55440}-\frac{671765}{1232}\nu+\frac{5205019}{11088}\nu^2-\frac{860477}{11088}\nu^3\right)\\
&+\frac{G^2m^2}{r^2}\dot{r}^4\left(\frac{115627817}{166320}-\frac{214}{3}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.+\left[\frac{42671}{792}-\frac{697}{32}\pi^2\right]\nu+\frac{1099355}{4752}\nu^2-\frac{825331}{16632}\nu^3\right)\\
&+\frac{G^3m^3}{r^3}\dot{r}^2\left(\frac{3202601}{23100}-\frac{1712}{315}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.+\left[\frac{6220199}{22680}-\frac{88}{9}\ln\left(\frac{r}{r_0'}\right)-\frac{1763}{192}\pi^2\right]\nu+\frac{57577}{1848}\nu^2-\frac{43018}{6237}\nu^3\right)\\
&+\frac{G^4m^4}{r^4}\left(\frac{37571}{8316}-\frac{14962}{891}\nu-\frac{3019}{594}\nu^2-\frac{866}{6237}\nu^3\right)\Bigg\}. \qquad (2.21e)
\end{aligned}
$$

The new results of this chapter are the instantaneous terms at 2.5PN and 3PN. Up to 2PN order, all the terms match with those in [138, 143, 46]. As one may notice, the 2.5PN terms in the above equation are all proportional to $\dot{r}$ and hence are zero for the circular orbit case in agreement with the results of [156]. The $\dot{r}$ dependence of these terms is also a crucial aspect when we discuss the orbital average of the 2.5PN terms in Sec. 2.7. The 3PN terms provide the generalizations of the circular orbit results at 2.5PN and 3PN in Refs. [156] and [95] (respectively). As expected, the two constants r_0 and r_0' present in the expression of the mass quadrupole moment appear in the final expression for the flux. The dependence of the instantaneous terms on the scale r_0 should exactly cancel a similar contribution coming from the tail terms as we will see in the next chapter generalising the situation for circular orbits. The dependence on r_0' is more involved. For circular orbits, it was shown [95] that this 'unphysical or gauge' dependence disappears when the total flux is expressed in terms of the gauge invariant parameter x related to the asymptotic gravitational wave frequency. We shall examine later the analog of this cancellation in the general orbit case. We also exhibit alternative representations of the energy flux for elliptical orbits, one of which is in terms of gauge invariant variables related to those suggested by [155].

2.5 3PN Energy Flux in alternative coordinates

The first prominent application of the present computation is the evolution of the orbital elements under gravitational wave radiation reaction to order 3PN beyond the leading quadrupolar radiation reaction. This requires one to average over an orbit, the instantaneous expressions for the energy flux obtained in Sec. 2.4. Averaging over an orbit is most conveniently accomplished by the use of the generalized quasi-Keplerian representation for the orbit. Recently, such a representation has been constructed to 3PN accuracy in both harmonic and ADM coordinates by Mernrnesheimer, Gopakumar and Schafer [155].

The standard harmonic coordinates used up till now though useful for analytical algebraic checks, contain gauge-dependent logarithm terms that are not very convenient in numerical calculations. More seriously, in the presense of the log terms a simple generalised quasi-Keplerian representation is not possible impeding the averaging of the flux over an orbit. Ref. [155] thus only provides the generalised quasi-Keplerian representation at 3PN for only the modified harmonic coordinates (without the log terms). Consequently it would be useful to have the expresssion for the energy flux in a modified harmonic coordinate system like the one explicitly used in [161]. This would require us to re-express the instantaneous expressions for the energy flux Eq. (2.21) in standard harmonic coordinates (with logs) as used in [159] in terms of corresponding variables in the modified harmonic coordinates without logs as used in [161]. We first provide this in this section.

Many related numerical relativity studies are in ADM-type coordinates and hence for future applications we transform the energy flux in modified harmonic coordinates suitably and provide explicitly an expression for the energy flux in ADM coordinates.

For economy of presentation, we avoid rewriting the long expressions for the far-zone flux in these alternative coordinates but present them in terms of the flux in standard harmonic coordinates and an additional ***'correction term'.***

2.5.1 Energy Flux in modified harmonic coordinates without logs

In order to obtain the expression for the energy flux in modified harmonic coordinates, we need to consider the effect of the 3PN coordinate transformation that removes such log terms as discussed in [86]. Following [102] these logs can be equivalently removed by the following shift formula on the particle world-lines:

$$\xi_1^i = \frac{22}{3}\frac{m_1^2 m_2}{c^6 r^2}\, n^i \ln\left(\frac{r}{r_1'}\right), \tag{2.22a}$$

$$\xi_2^i = -\frac{22}{3}\frac{m_1 m_2^2}{c^6 r^2}\, n^i \ln\left(\frac{r}{r_2'}\right) \tag{2.22b}$$

Under the shift by ξ, the acceleration of the first particle is shifted by

$$\delta_\xi a_1^i = \ddot{\xi}_1^i - \left(\xi_1^j - \xi_2^j\right)\partial_j a_1^i. \tag{2.23}$$

Consequently the relative acceleration is shifted by

$$\delta_\xi a^i = -\frac{m^3\nu}{r^4}\left\{\left[\left(110\dot r^2 - 22v^2\right)n^i - 44\dot r v^i\right]\ln\left(\frac{r}{r_0'}\right) + \left(-\frac{176}{3}\dot r^2 + \frac{22}{3}v^2 - \frac{22}{3}\frac{m}{r}\right)n^i + \frac{44}{3}\dot r v^i\right\} \tag{2.24}$$

Adding the above shift to the expression for the relative acceleration in standard harmonic coordinates in [159] yields the expression for the acceleration in modified harmonic coordinates:

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2} = \frac{m}{r^2}\left[(-1+A)\mathbf{n} + B\mathbf{v}\right] , \tag{2.25}$$

where A and B represent post-Newtonian terms. In this appropriate harmonic gauge, writing $A = A_1 + A_2 + \dots$ and $B = B_1 + B_2 + \dots$, the expressions for A and B read [161]:

$$A_1 = 2(2+\nu)\frac{m}{r} - (1+3\nu)v^2 + \frac{3}{2}\nu\dot r^2 , \tag{2.26a}$$

$$\begin{aligned} A_2 = {}& -\frac{3}{4}(12+29\nu)\left(\frac{m}{r}\right)^2 - \nu(3-4\nu)v^4 - \frac{15}{8}\nu(1-3\nu)\dot r^4 + \frac{1}{2}\nu(13-4\nu)\frac{m}{r}v^2 \\ & + (2+25\nu+2\nu^2)\frac{m}{r}\dot r^2 + \frac{3}{2}\nu(3-4\nu)v^2\dot r^2 , \end{aligned} \tag{2.26b}$$

$$A_{5/2} = \frac{8}{5}\nu\frac{m}{r}\dot r\left(\frac{17}{3}\frac{m}{r} + 3v^2\right) , \tag{2.26c}$$

$$\begin{aligned} A_3 = {}& \left[16 + \left(\frac{1399}{12} - \frac{41}{16}\pi^2\right)\nu + \frac{71}{2}\nu^2\right]\left(\frac{m}{r}\right)^3 + \nu\left[\frac{20827}{840} + \frac{123}{64}\pi^2 - \nu^2\right]\left(\frac{m}{r}\right)^2 v^2 \\ & - \left[1 + \left(\frac{22717}{168} + \frac{615}{64}\pi^2\right)\nu + \frac{11}{8}\nu^2 - 7\nu^3\right]\left(\frac{m}{r}\right)^2\dot r^2 \\ & - \frac{1}{4}\nu(11 - 49\nu + 52\nu^2)v^6 + \frac{35}{16}\nu(1 - 5\nu + 5\nu^2)\dot r^6 - \frac{1}{4}\nu\left(75 + 32\nu - 40\nu^2\right)\frac{m}{r}v^4 \\ & - \frac{1}{2}\nu\left(158 - 69\nu - 60\nu^2\right)\frac{m}{r}\dot r^4 + \nu\left(121 - 16\nu - 20\nu^2\right)\frac{m}{r}v^2\dot r^2 \\ & + \frac{3}{8}\nu\left(20 - 79\nu + 60\nu^2\right)v^4\dot r^2 - \frac{15}{8}\nu\left(4 - 18\nu + 17\nu^2\right)v^2\dot r^4 , \end{aligned} \tag{2.26d}$$

$$B_1 = 2(2-\nu)\dot r , \tag{2.27a}$$

$$B_2 = -\frac{1}{2}\dot r\left[(4 + 41\nu + 8\nu^2)\frac{m}{r} - \nu(15+4\nu)v^2 + 3\nu(3+2\nu)\dot r^2\right] , \tag{2.27b}$$

$$B_{5/2} = -\frac{8}{5}\nu\frac{m}{r}\left(3\frac{m}{r} + v^2\right) , \tag{2.27c}$$

$$\begin{aligned} B_3 = {}& \dot r\left\{\left[4 + \left(\frac{5849}{840} + \frac{123}{32}\pi^2\right)\nu - 25\nu^2 - 8\nu^3\right]\left(\frac{m}{r}\right)^2 + \frac{1}{8}\nu\left(65 - 152\nu - 48\nu^2\right)v^4 \right. \\ & + \frac{15}{8}\nu\left(3 - 8\nu - 2\nu^2\right)\dot r^4 + \nu\left(15 + 27\nu + 10\nu^2\right)\frac{m}{r}v^2 \end{aligned}$$

$$-\frac{1}{6}\nu\left(329+177\nu+108\nu^2\right)\frac{m}{r}\dot{r}^2-\frac{3}{4}\nu\left(16-37\nu-16\nu^2\right)v^2\dot{r}^2\Bigg\}\,. \tag{2.27d}$$

As expected, the log dependence in the above transformation exactly cancels the log dependence of the acceleration in the standard harmonic coordinates 2.8a. Some 3PN coefficients in the EOM are also modified and the final result agrees with that displayed in [161].

The only other modification vis a vis the calculation of the energy flux in standard harmonic coordinates is the part related to the mass quadrupole which must be computed to 3PN accuracy.

Under the above shift formula the mass quadruple I_{ij} is shifted by

$$\delta_\xi I_{ij}=STF_{ij}\left(\frac{44}{3}\frac{m^4\,\nu^2}{r^3}\ln\left(\frac{r}{r_0'}\right)x_{ij}\right), \tag{2.28}$$

which exactly cancels the $\ln r_0'$ dependence of the mass quadrupole in standard harmonic coordinates. Thus in the modified harmonic gauge the $\ln r_0'$ dependence of the mass quadrupole also vanishes as expected. The rest of the expression of the mass quadrupole remains exactly the same as in the standard harmonic coordinates Eq. (2.8a) and not re-written here.

Taking into account these modified contributions to the 3PN energy flux from the mass quadrupole and adding it to the other multipole contributions that remain unchanged, one finally obtains the energy flux in the modified harmonic coordinates without logs.

$$\begin{aligned}\left(\dot{\mathcal{E}}\right)_{\rm MHar} &= \left(\dot{\mathcal{E}}\right)_{\rm SHar\to Mhar}+\frac{G^3\,m^4\,\nu^2}{c^{11}\,r^4}\left\{\frac{G^2\,m^2\,\nu}{r^2}\left[\frac{704}{5}\,v^4-\frac{17248}{15}\,\dot{r}^2\,v^2+\frac{9856}{9}\,\dot{r}^4\right.\right.\\ &\left.\left.-\frac{704}{5}\frac{Gm}{r}v^2+\frac{704}{3}\frac{Gm}{r}\dot{r}^2+\left(-\frac{1408}{15}\frac{Gm}{r}v_2+\frac{2816}{45}\frac{Gm}{r}\dot{r}^2\right)\log\left(\frac{r}{r_0'}\right)\right]\right\}\end{aligned} \tag{2.29}$$

To avoid any confusion arising from the above very compact notation, let us remind the reader that the expression for the energy flux in the modified harmonic coordinates would involve variables solely in the modified harmonic cordinates. Thus the u, $\dot{r}$ and r would be the variables in the modified harmonic coordinates. The symbol $\left(\dot{\mathcal{E}}\right)_{\rm SHar\to Mhar}$ thus schematically represents the expression given in Eq. (2.21) in standard harmonic coordinates but where u^2, $\dot{r}^2$ and m/r are the modified harmonic variables $v^2_{\rm MHar}$, $\dot{r}^2_{\rm Mhar}$ and $m/r_{\rm Mhar}$. To avoid making the notation too heavy we refrain from putting a subscript MHar on v, $\dot{r}$ or r. Since the transformation starts at 3PN there is no need to put any such label on the *difference* terms and this is often useful to remember.

2.5.2 Energy Flux in ADM coordinates

To transform the energy flux to from standard harmonic to ADM coordinates we require the 'contact' transformations connecting the standard harmonic coordinates (with log terms) and ADM coordinates. They are given by [159]:

$$
\begin{aligned}
\mathbf{r}_{\mathrm{SHar}} &= \mathbf{r}_{\mathrm{ADM}} + \left\{ \frac{Gm}{c^4}\left[v^2\left(\frac{5}{8}\nu\right) + \dot{r}^2\left(-\frac{1}{8}\nu\right) + \frac{Gm}{r}\left(3\nu + \frac{1}{4}\right)\right] \right. \\
&\quad + \frac{Gm\nu}{c^6}\left[v^4\left(\frac{1}{2} - \frac{11}{8}\nu\right) + \dot{r}^2 v^2\left(-\frac{5}{16} + \frac{15}{16}\nu\right) + \dot{r}^4\left(\frac{1}{16} - \frac{5}{16}\nu\right) \right. \\
&\quad + \frac{Gm}{r}v^2\left(\frac{451}{48} + \frac{3}{8}\nu\right) + \frac{Gm}{r}\dot{r}^2\left(-\frac{161}{48} + \frac{5}{2}\nu\right) \\
&\quad \left.\left. + \frac{G^2m^2}{r^2}\left(-\frac{2773}{280} + \frac{22}{3}\ln\left(\frac{r}{r_0'}\right) - \frac{21}{32}\pi^2\right)\right]\right\} \mathbf{n} \\
&\quad + \left\{ \frac{Gm}{c^4}\dot{r}\left(-\frac{9}{4}\nu\right) \right. \\
&\quad \left. + \frac{Gm\nu}{c^6}\dot{r}\left[v^2\left(-\frac{17}{8} + \frac{21}{4}\nu\right) + \dot{r}^2\left(\frac{5}{12} - \frac{29}{24}\nu\right) + \frac{Gm}{r}\left(-\frac{43}{3} - 5\nu\right)\right]\right\} \mathbf{v}, \qquad (2.30)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_{\mathrm{SHar}} &= \mathbf{v}_{\mathrm{ADM}} + \left\{ \frac{Gm}{c^4 r}\left[v^2\left(-\frac{13}{8}\nu\right) + \dot{r}^2\left(\frac{17}{8}\nu\right) + \frac{Gm}{r}\left(\frac{1}{4} + \frac{21}{4}\nu\right)\right] \right. \\
&\quad + \frac{Gm\nu}{c^6 r}\left[v^4\left(-\frac{13}{8} + \frac{31}{8}\nu\right) + \dot{r}^2 v^2\left(\frac{49}{16} - \frac{127}{16}\nu\right) \right. \\
&\quad + \dot{r}^4\left(-\frac{19}{16} + \frac{53}{16}\nu\right) + \frac{Gm}{r}v^2\left(-\frac{9}{16} - \frac{25}{8}\nu\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{165}{16} + \frac{45}{4}\nu\right) \\
&\quad \left.\left. + \frac{G^2m^2}{r^2}\left(-\frac{3839}{840} + \frac{22}{3}\ln\left(\frac{r}{r_0'}\right) - \frac{21}{32}\pi^2 + \frac{1}{2}\nu\right)\right]\right\} \mathbf{v} \\
&\quad + \left\{ \frac{Gm}{c^4 r}\dot{r}\left[v^2\left(-\frac{7}{8}\nu\right) + \dot{r}^2\left(\frac{3}{8}\nu\right) + \frac{Gm}{r}\left(-\frac{1}{2} - \frac{19}{4}\nu\right)\right] \right. \\
&\quad + \frac{Gm\nu}{c^6 r}\dot{r}\left[v^4\left(-\frac{9}{8} + \frac{13}{4}\nu\right) + \dot{r}^2 v^2\left(\frac{19}{16} - \frac{65}{16}\nu\right) + \dot{r}^4\left(-\frac{5}{16} + \frac{25}{16}\nu\right) \right. \\
&\quad + \frac{Gm}{r}v^2\left(-\frac{37}{2} + \frac{31}{8}\nu\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{99}{8} - \frac{259}{24}\nu\right) \\
&\quad \left.\left. + \frac{G^2m^2}{r^2}\left(\frac{28807}{840} - 22\ln\left(\frac{r}{r_0'}\right) + \frac{63}{32}\pi^2 - \frac{13}{4}\nu\right)\right]\right\} \mathbf{n}, \qquad (2.31a)
\end{aligned}
$$

$$
\begin{aligned}
r_{\mathrm{SHar}} &= r_{\mathrm{ADM}} + \frac{Gm}{c^4}\left\{ v^2\left(\frac{5}{8}\nu\right) + \dot{r}^2\left(-\frac{19}{8}\nu\right) + \frac{Gm}{r}\left(\frac{1}{4} + 3\nu\right)\right\} \\
&\quad + \frac{Gm\nu}{c^6}\left\{ v^4\left(\frac{1}{2} - \frac{11}{8}\nu\right) + \dot{r}^2 v^2\left(-\frac{39}{16} + \frac{99}{16}\nu\right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\dot{r}^4\left(\frac{23}{48}-\frac{73}{48}\nu\right)+\frac{Gm}{r}v^2\left(\frac{451}{48}+\frac{3}{8}\nu\right)\\
&+\frac{Gm}{r}\dot{r}^2\left(-\frac{283}{16}-\frac{5}{2}\nu\right)+\frac{G^2m^2}{r^2}\left(-\frac{2773}{280}+\frac{22}{3}\ln\left(\frac{r}{r_0'}\right)-\frac{21}{32}\pi^2\right)\Bigg\}, \qquad (2.31b)
\end{aligned}
$$

$$
\begin{aligned}
v^2{}_{\mathrm{SHar}} \;=\;& v^2_{\mathrm{ADM}}+\frac{Gm}{c^4 r}\left\{v^4\left(-\frac{13}{4}\nu\right)+\dot{r}^2v^2\left(\frac{5}{2}\nu\right)+\dot{r}^4\left(\frac{3}{4}\nu\right)\right.\\
&\left.+\frac{Gm}{r}v^2\left(\frac{1}{2}+\frac{21}{2}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(-1-\frac{19}{2}\nu\right)\right\}\\
&+\frac{Gm\nu}{c^6 r}\left\{v^6\left(-\frac{13}{4}+\frac{31}{4}\nu\right)+\dot{r}^2v^4\left(\frac{31}{8}-\frac{75}{8}\nu\right)+\dot{r}^4v^2\left(-\frac{3}{2}\nu\right)+\dot{r}^6\left(-\frac{5}{8}+\frac{25}{8}\nu\right)\right.\\
&+\frac{Gm}{r}v^4\left(-\frac{9}{8}-\frac{25}{4}\nu\right)+\frac{Gm}{r}\dot{r}^2v^2\left(-\frac{131}{8}+\frac{121}{4}\nu\right)+\frac{Gm}{r}\dot{r}^4\left(\frac{99}{4}-\frac{259}{12}\nu\right)\\
&+\frac{G^2m^2}{r^2}v^2\left(-\frac{3839}{420}+\frac{44}{3}\ln\left(\frac{r}{r_0'}\right)-\frac{21}{16}\pi^2+\nu\right)\\
&\left.+\frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{28807}{420}-44\ln\left(\frac{r}{r_0'}\right)+\frac{63}{16}\pi^2-\frac{13}{2}\nu\right)\right\}, \qquad (2.31c)
\end{aligned}
$$

$$
\begin{aligned}
\dot{r}^2_{\mathrm{SHar}} \;=\;& \dot{r}^2_{\mathrm{ADM}}+\frac{Gm}{c^4 r}\dot{r}^2\left\{v^2\left(-\frac{19}{2}\nu\right)+\dot{r}^2\left(\frac{19}{2}\nu\right)+\frac{Gm}{r}\left(-\frac{1}{2}+\nu\right)\right\}\\
&+\frac{Gm\nu}{c^6 r}\left\{v^4\left(-\frac{39}{4}+\frac{99}{4}\nu\right)+\dot{r}^2v^2\left(\frac{163}{12}-\frac{443}{12}\nu\right)+\dot{r}^4\left(-\frac{23}{6}+\frac{73}{6}\nu\right)\right.\\
&+\frac{Gm}{r}v^2\left(-\frac{1603}{24}-\frac{17}{2}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(\frac{1777}{24}+\frac{131}{12}\nu\right)\\
&\left.+\frac{G^2m^2}{r^2}\left(\frac{6242}{105}-\frac{88}{3}\ln\left(\frac{r}{r_0'}\right)+\frac{21}{8}\pi^2-\frac{11}{2}\nu\right)\right\}. \qquad (2.31d)
\end{aligned}
$$

The above equations provide the 3PN generalization of Eq. (4.6) of [46]. They also incorporate the corrected transformation between ADM and harmonic co-ordinates at 2PN, as given in [47]. Using these equations, one can transform the energy flux in standard harmonic coordinates, Eq. (2.21), into an expression for the energy flux in ADM coordinates. In a notation similar to that introduced above. we obtain:

$$
\begin{aligned}
\left(\dot{\mathcal{E}}\right)_{\mathrm{ADM}} \;=\;& \left(\dot{\mathcal{E}}\right)_{\mathrm{SHar}\to\mathrm{ADM}}-\frac{G^4m^5\nu^2}{c^9 r^5}\left\{v^4\left(\frac{184}{5}\nu\right)-\dot{r}^2v^2\left(\frac{736}{5}\nu\right)\right.\\
&\left.+\frac{Gm}{r}v^2\left(\frac{16}{5}+\frac{48}{5}\nu\right)+\dot{r}^4\left(\frac{320}{3}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(-\frac{12}{5}-\frac{56}{15}\nu\right)\right\}\\
&-\frac{G^4m^5\nu^2}{c^{11}r^5}\left\{v^6\left(\frac{5886}{35}\nu-\frac{1616}{7}\nu^2\right)+\dot{r}^2v^4\left(-\frac{129866}{105}\nu+\frac{21598}{15}\nu^2\right)\right.\\
&+\frac{Gm}{r}v^4\left(-\frac{37582}{105}\nu+\frac{7528}{35}\nu^2\right)+\dot{r}^4v^2\left(\frac{689434}{315}\nu-\frac{714608}{315}\nu^2\right)\\
&+\frac{Gm}{r}\dot{r}^2v^2\left(-\frac{936}{35}+\frac{201251}{105}\nu-\frac{14086}{21}\nu^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^2m^2}{r^2}v^2\left(-\frac{272}{7}+\left[-\frac{21296}{75}-\frac{42}{5}\pi^2\right]\nu+\frac{96}{35}\nu^2\right)\\
&+\dot{r}^6\left(-\frac{116138}{105}\nu+\frac{110986}{105}\nu^2\right)\\
&+\frac{Gm}{r}\dot{r}^4\left(\frac{328}{15}-\frac{181019}{105}\nu+\frac{143924}{315}\nu^2\right)\\
&+\frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{1612}{35}+\left[\frac{316264}{1575}+\frac{28}{5}\pi^2\right]\nu+\frac{828}{35}\nu^2\right)\\
&+\frac{G^3m^3}{r^3}\left(\frac{16}{35}+\frac{128}{35}\nu-\frac{768}{35}\nu^2\right)\\
&+\left(\frac{G^2m^2}{r^2}v^2\left(\frac{1408}{15}\nu\right)-\frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{2816}{45}\nu\right)\right)\ln\left(\frac{r}{r_0'}\right)\Bigg\}, \qquad (2.32)
\end{aligned}
$$

where $(\dot{\mathcal{E}})_{\rm SHar\to ADM}$ denotes the expression for the flux in standard harmonic coordinates Eq. (2.21) but where v^2, $\dot{r}^2$ and m/r are the ADM variables v_A^2, $\dot{r}_A^2$ and m/r_A. A close examination of the two terms on the RHS of $\delta\dot{\mathcal{E}}$ Eq. (2.32) reveals that the terms with $\ln r_0'$ of $(\dot{\mathcal{E}})_{\rm SHar\to ADM}$ exactly cancel with those of $\delta\dot{\mathcal{E}}$ and the final flux in ADM coordinate is free of $\ln r_0'$. This is consistent with the general understanding that $\ln r_0'$ is a feature of the harmonic coordinates and that the ADM coordinates will *not* contain this. The cancellation of the $\ln r_0'$ terms provides a useful internal check on the long algebra involved in these computations.

Unfortunately, for the further computations in this chapter that allows one to extract a more gauge-invariant description of the inspiral the above description is inadequate for reasons explained earlier. The modified harmonic coordinate (without logs) is better suited for these purposes. In order to perform as many independent checks on the long and involved algebra, we have found it expeditious to use two different harmonic coordinate systems; one containing the (gauge dependent) log terms *á **la*** [159] and another harmonic coordinate system without log terms as in [86, 161]. We conclude the section with an expression for the difference between the energy flux in the modified harmonic coordinate and ADM coordinates. As before this requires the relevant contact transformations given by:

$$
\begin{aligned}
\mathbf{r}_{\rm MHar} \;=\; \mathbf{r}_{\rm ADM}&+\Bigg\{\frac{Gm}{c^4}\left[v^2\left(\frac{5}{8}\nu\right)+\dot{r}^2\left(-\frac{1}{8}\nu\right)+\frac{Gm}{r}\left(\frac{1}{4}+3\nu\right)\right]\\
&+\frac{Gm}{c^6}\nu\Bigg[v^4\left(\frac{1}{2}-\frac{11}{8}\nu\right)+\dot{r}^2v^2\left(-\frac{5}{16}+\frac{15}{16}\nu\right)+\dot{r}^4\left(\frac{1}{16}-\frac{5}{16}\nu\right)\\
&+\frac{Gm}{r}v^2\left(\frac{451}{48}+\frac{3}{8}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(-\frac{161}{48}+\frac{5}{2}\nu\right)+\frac{G^2m^2}{r^2}\left(-\frac{2773}{280}-\frac{21}{32}\pi^2\right)\Bigg]\Bigg\}\,\mathbf{n}\\
&+\Bigg\{\frac{Gm}{c^4}\,\dot{r}\left(-\frac{9}{4}\nu\right)\\
&+\frac{Gm\,\nu}{c^6}\,\dot{r}\left[v^2\left(-\frac{17}{8}+\frac{21}{4}\nu\right)+\dot{r}^2\left(\frac{5}{12}-\frac{29}{24}\nu\right)+\frac{Gm}{r}\left(-\frac{43}{3}-5\nu\right)\right]\Bigg\}\,\mathbf{v}, \qquad (2.33)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_{\rm MHar} &= \mathbf{v}_{\rm ADM} + \left\{ \frac{Gm}{c^4 r}\left[v^2\left(-\frac{13}{8}\nu\right) + \dot r^2\left(\frac{17}{8}\nu\right) + \frac{Gm}{r}\left(\frac{1}{4}+\frac{21}{4}\nu\right)\right]\right.\\
&\quad + \frac{Gm\nu}{c^6 r}\left[v^4\left(-\frac{13}{8}+\frac{31}{8}\nu\right) + \dot r^2 v^2\left(\frac{49}{16}-\frac{127}{16}\nu\right) + \dot r^4\left(-\frac{19}{16}+\frac{53}{16}\nu\right)\right.\\
&\quad \left.\left. + \frac{Gm}{r}v^2\left(-\frac{9}{16}-\frac{25}{8}\nu\right) + \frac{Gm}{r}\dot r^2\left(\frac{165}{16}+\frac{45}{4}\nu\right) + \frac{G^2m^2}{r^2}\left(-\frac{3839}{840}-\frac{21}{32}\pi^2+\frac{1}{2}\nu\right)\right]\right\}\mathbf{v}\\
&\quad + \left\{ \frac{Gm}{c^4 r}\dot r\left[v^2\left(-\frac{7}{8}\nu\right) + \dot r^2\left(\frac{3}{8}\nu\right) + \frac{Gm}{r}\left(-\frac{1}{2}-\frac{19}{4}\nu\right)\right]\right.\\
&\quad + \frac{Gm\nu}{c^6 r}\dot r\left[v^4\left(-\frac{9}{8}+\frac{13}{4}\nu\right) + \dot r^2 v^2\left(\frac{19}{16}-\frac{65}{16}\nu\right) + \dot r^4\left(-\frac{5}{16}+\frac{25}{16}\nu\right)\right.\\
&\quad + \frac{Gm}{r}v^2\left(-\frac{37}{2}+\frac{31}{8}\nu\right) + \frac{Gm}{r}\dot r^2\left(\frac{99}{8}-\frac{259}{24}\nu\right)\\
&\quad \left.\left. + \frac{G^2m^2}{r^2}\left(\frac{7549}{280}+\frac{63}{32}\pi^2-\frac{13}{4}\nu\right)\right]\right\}\mathbf{n}, \qquad (2.34a)\\
r_{\rm MHar} &= r_{\rm ADM} + \frac{Gm}{c^4}\left\{ v^2\left(\frac{5}{8}\nu\right) + \dot r^2\left(-\frac{19}{8}\nu\right) + \frac{Gm}{r}\left(\frac{1}{4}+3\nu\right)\right\}\\
&\quad + \frac{Gm\nu}{c^6}\left\{ v^4\left(\frac{1}{2}-\frac{11}{8}\nu\right) + \dot r^2 v^2\left(-\frac{39}{16}+\frac{99}{16}\nu\right) + \dot r^4\left(\frac{23}{48}-\frac{73}{48}\nu\right)\right.\\
&\quad \left. + \frac{Gm}{r}v^2\left(\frac{451}{48}+\frac{3}{8}\nu\right) + \frac{Gm}{r}\dot r^2\left(-\frac{283}{16}-\frac{5}{2}\nu\right) + \frac{G^2m^2}{r^2}\left(-\frac{2773}{280}-\frac{21}{32}\pi^2\right)\right\}, \qquad (2.34b)\\
v^2_{\rm MHar} &= v^2_{\rm ADM} + \frac{Gm}{c^4 r}\left\{ v^4\left(-\frac{13}{4}\nu\right) + \dot r^2 v^2\left(\frac{5}{2}\nu\right) + \dot r^4\left(\frac{3}{4}\nu\right) + \frac{Gm}{r}v^2\left(\frac{1}{2}+\frac{21}{2}\nu\right)\right.\\
&\quad \left. + \frac{Gm}{r}\dot r^2\left(-1-\frac{19}{2}\nu\right)\right\}\\
&\quad + \frac{Gm\nu}{c^6 r}\left\{ v^6\left(-\frac{13}{4}+\frac{31}{4}\nu\right) + \dot r^2 v^4\left(\frac{31}{8}-\frac{75}{8}\nu\right) + \dot r^4 v^2\left(-\frac{3}{2}\nu\right) + \dot r^6\left(-\frac{5}{8} + 25\right.\right.\\
&\quad + \frac{Gm}{r}v^4\left(-\frac{9}{8}-\frac{25}{4}\nu\right) + \frac{Gm}{r}\dot r^2 v^2\left(-\frac{131}{8}+\frac{121}{4}\nu\right) + \frac{Gm}{r}\dot r^4\left(\frac{99}{4}-\frac{259}{12}\nu\right)\\
&\quad \left. + \frac{G^2m^2}{r^2}v^2\left(-\frac{3839}{420}-\frac{21}{16}\pi^2+\nu\right) + \frac{G^2m^2}{r^2}\dot r^2\left(\frac{7549}{140}+\frac{63}{16}\pi^2-\frac{13}{2}\nu\right)\right\}, \qquad (2.34c)\\
\dot r^2_{\rm MHar} &= \dot r^2_{\rm ADM} + \frac{Gm}{c^4 r}\dot r^2\left\{ v^2\left(-\frac{19}{2}\nu\right) + \dot r^2\left(\frac{19}{2}\nu\right) + \frac{Gm}{r}\left(-\frac{1}{2}+\nu\right)\right\} +\\
&\quad \frac{Gm\nu}{c^6 r}\dot r^2\left\{ v^4\left(-\frac{39}{4}+\frac{99}{4}\nu\right) + \dot r^2 v^2\left(\frac{163}{12}-\frac{443}{12}\nu\right) + \dot r^4\left(-\frac{23}{6}+\frac{73}{6}\nu\right)\right.\\
&\quad + \frac{Gm}{r}v^2\left(\frac{1603}{24} \quad \frac{17}{2}\nu\right) + \frac{Gm}{r}\dot r^2\left(\frac{1777}{24}+\frac{131}{12}\nu\right)\\
&\quad \left. + \frac{G^2m^2}{r^2}\left(\frac{4702}{105}+\frac{21}{8}\pi^2-\frac{11}{2}\nu\right)\right\}. \qquad (2.34d)
\end{aligned}
$$

Employing the above equations, one can transform the energy flux in modified harmonic coordinates, Eq. (2.21), into an expression for the energy flux in **ADM** coordinates. We obtain,

$$
\begin{aligned}
\left(\dot{\mathcal{E}}\right)_{\rm ADM} &= \left(\dot{\mathcal{E}}\right)_{\rm MHar\to ADM} - \frac{G^4m^5\nu^2}{c^9r^5}\left\{v^4\left(\frac{184}{5}\nu\right) - \dot{r}^2v^2\left(\frac{736}{5}\nu\right)\right. \\
&+\frac{Gm}{r}v^2\left(\frac{16}{5}+\frac{48}{5}\nu\right) + \dot{r}^4\left(\frac{320}{3}\nu\right) + \left.\frac{Gm}{r}\dot{r}^2\left(-\frac{12}{5}-\frac{56}{15}\nu\right)\right\} \\
&-\frac{G^4m^5\nu^2}{c^{11}r^5}\left\{v^6\left(\frac{5886}{35}\nu - \frac{1616}{7}\nu^2\right) + \dot{r}^2v^4\left(-\frac{129866}{105}\nu + \frac{21598}{15}\nu^2\right)\right. \\
&+\frac{Gm}{r}v^4\left(-\frac{22798}{105}\nu + \frac{7528}{35}\nu^2\right) + \dot{r}^4v^2\left(\frac{689434}{315}\nu - \frac{714608}{315}\nu^2\right) \\
&+\frac{Gm}{r}\dot{r}^2v^2\left(-\frac{936}{35} + \frac{16103}{21}\nu - \frac{14086}{21}\nu^2\right) \\
&+\frac{G^2m^2}{r^2}v^2\left(-\frac{272}{7} + \left[-\frac{31856}{75} - \frac{42}{5}\pi^2\right]\nu + \frac{96}{35}\nu^2\right) \\
&+\dot{r}^6\left(-\frac{116138}{105}\nu + \frac{110986}{105}\nu^2\right) \\
&+\frac{Gm}{r}\dot{r}^4\left(\frac{328}{15} - \frac{198097}{315}\nu + \frac{143924}{315}\nu^2\right) \\
&+\frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{1612}{35} + \left[\frac{685864}{1575} + \frac{28}{5}\pi^2\right]\nu + \frac{828}{35}\nu^2\right) \\
&+\left.\frac{G^3m^3}{r^3}\left(\frac{16}{35} + \frac{128}{35}\nu - \frac{768}{35}\nu^2\right)\right\}. \qquad (2.35)
\end{aligned}
$$

Once again, for the last time we reiterate that $\left(\dot{\mathcal{E}}\right)_{\rm MHar\to ADM}$ denotes the energy flux expression in the modified harmonic coordinates Eq. (2.29) where $v^2_{\rm MHar}$, $\dot{r}^2_{\rm MHar}$ and $m/r_{\rm MHar}$ are replaced by the ADM variables $v^2_{\rm ADM}$, $\dot{r}^2_{\rm ADM}$ and $m/r_{\rm ADM}$.

Before we discuss the calculation of the orbital average of the energy flux in Sec. 2.7, we recall briefly the useful Keplerian representation of planetary motion. We then summarize the generalized quasi-Keplerian representation at 3PN order in the ADM coordinates recently obtained in [155] which, as mentioned earlier, is another essential input for the computations to follow.

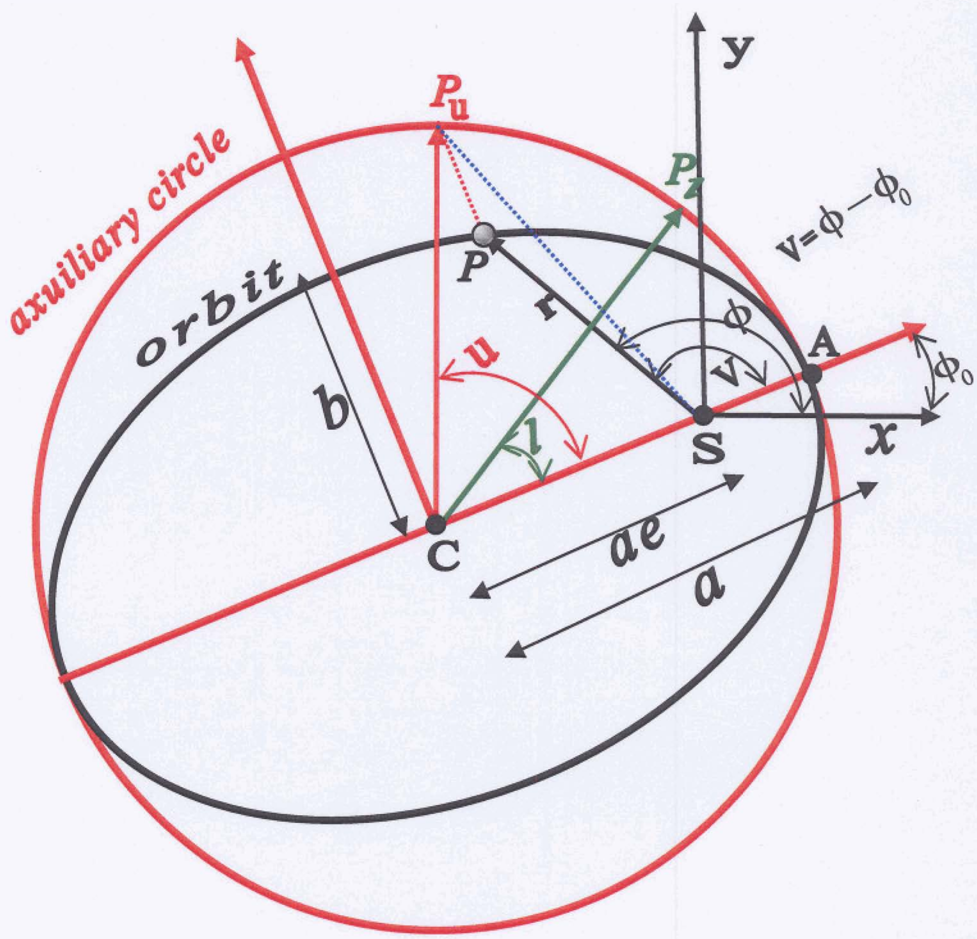

Figure 2.1: **This figure illustrates an object "P" moving** in **elliptical orbit around the focus S.** a, b **and** e **are the** semi-major axis, **semi-minor axis and** eccentricity **respectively.** r **is the position vector** which **makes angle** V **with the** x **axes. The angle** V called **the** "true anomaly". C **is the center of the orbit and** the **circle.** u **and** l are the eccentric anomaly **and mean** anomaly **respectively.** At $u = 0$, $l = V = 0$ and $\phi = \phi_0$. When $u = \pi$ then $l = V = \pi$ and $\phi = \pi + \phi_0$. **Further** Area(SP_uA)/Area(SPA)= $(\pi a^2/2)/(\pi a b/2) \equiv$ a/b **and also that** Area(CP_lA) **equals** Area(SP_uA).

2.6 The generalised quasi-Keplerian representation

2.6.1 The Keplerian representation

The Keplerian parametrisation of a particle moving in a general orbit with $0 \leq e \leq 1$) is given by:

$$
\begin{aligned}
r &= a(1 - e\cos u)\,, && (2.36a)\\
l \equiv n(t - t_0) &= u - e\sin u, && (2.36b)\\
(\phi - \phi_0) &= V\,, && (2.36c)\\
\text{where, } V &= 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{u}{2}\right]. && (2.36d)
\end{aligned}
$$

In the above, the three angles V, u and l (measured rom the perhelion) are called the true anomaly, the eccentric anomaly and the mean anomaly respectively. The orbit has semi-major (minor) axis $a(b)$, eccentricity e and mean motion n. The geometrical meaning of the various anomalies (or angles) emerges from the following geometrical construction. Construct an auxiliary circle for the orbit with a diameter equal to the orbit's semimajor axis a as in the Fig. [2.6.1] where C is the center of auxiliary circle and ellipse, S one of the focii of the ellipse, P the position of the orbiting body, A the perihelion, P_u the projection of the orbiting body on the auxiliary circle, (therefore $a/b \equiv$ Area(SP_uA)/Area(SPA)) and finally P_l, a point on the circle such that Area(CP_lA)$\equiv$ Area(SP_uA).

2.6.2 3PN generalised quasi-Keplerian representation: Summary

The Keplerian representation discussed in the previous section was for the leading Newtonian order. The quasi-Keplerian representation at 1PN was introduced by Damour and Deruelle [147] to discuss the problem of binary pulsar timing. At this order, relativistic periastron precession first appears and complicates the simpler Newtonian picture. This elegant formulation will play a crucial role in the our computation of the hereditary terms in the next chapter and will be discussed in more detail there. The 2PN extension of this work in the ADM coordinates was next given by Damour, Schafer and Wex [148, 149, 150] and is referred to as generalized QK representation. In a more recent work, the 3PN parametrization of the orbital motion of the binary was constructed by Memmeshiemer, Gopakumar and Schafer [155] in both ADM and modified harmonic coordinates. In ADM-type coordinates Eq. (19) of Ref. [155] provides the 3PN parametrization which reads as

$$
\begin{aligned}
r &= a, (1 - e_r \cos u)\,, && (2.37a)\\
l \equiv n(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(V - u)
\end{aligned}
$$

$$
\begin{aligned}
& + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right) \sin V + \frac{i_{6t}}{c^6} \sin 2V + \frac{h_{6t}}{c^6} \sin 3V , && (2.37b) \\
\frac{2\pi}{\Phi}(\phi - \phi_0) &= V + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right) \sin 2V + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right) \sin 3V \\
& + \frac{i_{6\phi}}{c^6} \sin 4V + \frac{h_{6\phi}}{c^6} \sin 5V , && (2.37c) \\
\text{where, } V &= 2 \arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right]. && (2.37d)
\end{aligned}
$$

V is the *3PN* generalisation of the Keplerian true anomaly. In the above a_r, e_r, l, u, n, e_t, e_ϕ and $2\pi/\Phi$ are some *3PN* accurate semi-major axis, radial eccentricity, mean anomaly, eccentric anomaly, mean motion, 'time' eccentricity, angular eccentricity and angle of advance of periastron per orbital revolution respectively. Notice that the equations contain three kinds of 'eccentricities' e,, e, and e_ϕ labelled after the coordinates t, r, and ϕ respectively, that as we shall see differ from each other starting at the 1PN order. The notation $\Phi/2\pi \equiv K = 1 + k$ is used in the next chapter and mentioned here for completeness.

In ref. ([155]) the explicit dependence of the orbital elements and all the coefficients in Eq. *2.37* above has been obtained as a *PN* series in terms of the *3PN* conserved orbital energy and angular momentum. They form the basis for our computation of the average energy **flux** at *3PN* order and in ADM coordinates they are given by

$$
\begin{aligned}
a_r &= \frac{1}{(-2E)}\Bigg\{1 + \frac{(-2E)}{4c^2}(-7+\nu) + \frac{(-2E)^2}{16c^4}\Big[(1 + 10\nu + \nu^2) \\
& + \frac{1}{(-2Eh^2)}(-68 + 44\nu)\Big] + \frac{(-2E)^3}{192c^6}\Big[3 - 9\nu - 6\nu^2 \\
& + 3\nu^3 + (-2Eh^2)\left(864 + \left(-3\pi^2 - 2212\right)\nu + 432\nu^2\right) \\
& + \frac{}{(-2Eh^2)^2}\left(-6432 + (13488 - 240n^2)\nu - 768\nu^2\right)\Big]\Bigg\}, && (2.38a) \\
{e_r}^2 &= 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\Big\{24 - 4\nu + 5(-3+\nu)(-2Eh^2)\Big\} \\
& + \frac{(-2E)^2}{8c^4}\Big\{52 + 2\nu + 2\nu^2 - \left(80 - 55\nu + 4\nu^2\right)(-2Eh^2) \\
& - \frac{8}{(-2Eh^2)}(-17 + 11\nu)\Big\} + \frac{(-2E)^3}{192c^6}\Big\{-768 - 6\nu\pi^2 \\
& - 344\nu - 216\nu^2 + 3(-2Eh^2)\Big(-1488 + 1556\nu - 319\nu^2 \\
& + 4\nu^3\Big) - \frac{4}{(-2Eh^2)}\left(588 - 8212\nu + 177\nu\pi^2 + 480\nu^2\right) \\
& + \frac{192}{(-2Eh^2)^2}\left(134 - 281\nu + 5\nu\pi^2 + 16\nu^2\right)\Big\}, && (2.38b)
\end{aligned}
$$

$$
\begin{aligned}
n &= (-2E)^{3/2}\Bigg\{1 + \frac{(-2E)}{8c^2}(-15+\nu) + \frac{(-2E)^2}{128c^4}\Bigg[555 + 30\nu \\
&\quad +11\nu^2 + \frac{192}{\sqrt{(-2Eh^2)}}(-5+2\nu)\Bigg] + \frac{(-2E)^3}{3072c^6}\Bigg[-29385 \\
&\quad -4995\nu - 315\nu^2 + 135\nu^3 - \frac{16}{(-2Eh^2)^{3/2}}\Big(10080 + 123\nu\pi^2 \\
&\quad -13952\nu + 1440\nu^2\Big) + \frac{5760}{\sqrt{(-2Eh^2)}}\left(17 - 9\nu + 2\nu^2\right)\Bigg]\Bigg\}, \qquad (2.38c) \\
e_t^2 &= 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\Big\{-8 + 8\nu - (-17+7\nu)(-2Eh^2)\Big\} \\
&\quad + \frac{(-2E)^2}{8c^4}\Big\{8 + 4\nu + 20\nu^2 - (-2Eh^2)(112 - 47\nu + 16\nu^2) \\
&\quad -24\sqrt{(-2Eh^2)}(-5+2\nu) + \frac{4}{(-2Eh^2)}(17 - 11\nu) \\
&\quad - \frac{24}{\sqrt{(-2Eh^2)}}(5 - 2\nu)\Big\} \\
&\quad + \frac{(-2E)^3}{192c^6}\Big\{24(-2+5\nu)\left(-23 + 10\nu + 4\nu^2\right) - 15\Big(-528 \\
&\quad +200\nu - 77\nu^2 + 24\nu^3\Big)(-2Eh^2) - 72(265 - 193\nu \\
&\quad +46\nu^2)\sqrt{(-2Eh^2)} - \frac{2}{(-2Eh^2)}\Big(6732 + 117\nu\pi^2 - 12508\nu \\
&\quad +2004\nu^2\Big) + \frac{2}{\sqrt{(-2Eh^2)}}\Big(16380 - 19964\nu + 123\nu\pi^2 \\
&\quad +3240\nu^2\Big) - \frac{2}{(-2Eh^2)^{3/2}}\Big(10080 + 123\nu\pi^2 - 13952\nu \\
&\quad +1440\nu^2\Big) + \frac{96}{(-2Eh^2)^2}\Big(134 - 281\nu + 5\nu\pi^2 + 16\nu^2\Big)\Big\}, \qquad (2.38d) \\
g_{4t} &= \frac{3(-2E)^2}{2}\left\{\frac{5-2\nu}{\sqrt{(-2Eh^2)}}\right\}, \qquad (2.38e) \\
g_{6t} &= \frac{(-2E)^3}{192}\Big\{\frac{1}{(-2Eh^2)^{3/2}}\Big(10080 + 123\nu\pi^2 - 13952\nu \\
&\quad +1440\nu^2\Big) + \frac{1}{\sqrt{(-2Eh^2)}}\left(-3420 + 1980\nu - 648\nu^2\right)\Big\}, \qquad (2.38f) \\
f_{4t} &= -\frac{1}{8}\frac{(-2E)^2}{\sqrt{(-2Eh^2)}}\Big\{(4+\nu)\nu\sqrt{(1+2Eh^2)}\Big\}, \qquad (2.38g) \\
f_{6t} &= \frac{(-2E)^3}{192}\Big\{\frac{1}{(-2Eh^2)^{3/2}}\frac{1}{\sqrt{1+2Eh^2}}\Big(1728 - 4148\nu + 3\nu\pi^2 \\
&\quad +600\nu^2 + 33\nu^3\Big) + 3\frac{\sqrt{(-2Eh^2)}}{\sqrt{(1+2Eh^2)}}\nu\left(-64 - 4\nu + 23\nu^2\right)
\end{aligned}
$$

$$+\frac{1}{\sqrt{(-2\,E\,h^2)(1+2\,E\,h^2)}}(-1728+4232\,\nu-3\,\nu n^2$$

$$-627\,\nu^2-105\,\nu^3)\Big)\Big\}\,, \tag{2.38h}$$

$$i_{6t} = \frac{(-2\,E)^3}{32}\,\nu\left\{\frac{(1+2\,E\,h^2)}{(-2\,E\,h^2)^{3/2}}\,(23+12\,\nu+6\,\nu^2)\right\}, \tag{2.38i}$$

$$h_{6t} = \frac{13\,(-2\,E)^3}{192}\,\nu^3\left(\frac{1+2\,E\,h^2}{-2\,E\,h^2}\right)^{3/2}, \tag{2.38j}$$

$$\Phi = 2\,\pi\left\{1+\frac{3}{c^2h^2}+\frac{(-2\,E)^2}{4\,c^4}\left[\frac{3}{(-2\,E\,h^2)}\,(-5+2\,\nu)\right.\right.$$

$$\left.+\frac{15}{(-2E\,h^2)^2}\,(7-2\,\nu)\right]+\frac{(-2\,E)^3}{128\,c^6}\left[\frac{24}{(-2\,E\,h^2)}(5-5\nu\right.$$

$$+4\nu^2)-(-2E\,h^2)^2\Big(10080-13952\,\nu+123\,\nu\,\pi^2+1440\,\nu^2\Big)$$

$$\left.\left.+\frac{5}{(-2E\,h^2)^3}(7392-8000\,\nu+123\,\nu\,\pi^2+336\,\nu^2)\right)\right]\Bigg\}, \tag{2.38k}$$

$$f_{4\phi} = \frac{(-2E)^2}{8}\,\frac{(1+2\,E\,h^2)}{(-2\,E\,h^2)^2}\,\nu\,(1-3\,\nu)\,, \tag{2.38l}$$

$$f_{6\phi} = \frac{(-2\,E)^3}{256}\left\{\frac{4\,\nu}{(-2\,E\,h^2)}\Big(-11-40\,\nu+24\nu^2\Big)\right.$$

$$+\frac{1}{(-2\,E\,h^2)^2}\Big(-256+1192\,\nu-49\,\nu\,\pi^2+336\,\nu^2-80\,\nu^3\Big)$$

$$\left.+\frac{1}{(-2\,E\,h^2)^3}\Big(256+49\,\nu\,\pi^2-1076\,\nu-384\,\nu^2-40\,\nu^3\Big)\right\}, \tag{2.38m}$$

$$g_{4\phi} = -\frac{3(-2\,E)^2}{32}\,\frac{\nu^2}{(-2\,E\,h^2)^2}(1+2\,E\,h^2)^{3/2}\,, \tag{2.38n}$$

$$g_{6\phi} = \frac{(-2\,E)^3}{768}\,\sqrt{(1+2\,E\,h^2)}\left\{-\frac{3}{(-2\,E\,h^2)}\,\nu^2\,(9-26\,\nu)\right.$$

$$-\frac{1}{(-2\,E\,h^2)^2}\,\nu\Big(220+3\,\pi^2+312\,\nu+150\,\nu^2\Big)$$

$$\left.+\frac{1}{(-2\,E\,h^2)^3}\,\nu\Big(220+3\,\pi^2+96\,\nu+45\,\nu^2\Big)\right\}, \tag{2.38o}$$

$$i_{6\phi} = \frac{(-2\,E)^3}{128}\,\frac{(1+2\,E\,h^2)^2}{(-2\,E\,h^2)^3}\,\nu\Big(5+28\,\nu+10\,\nu^2\Big)\,, \tag{2.38p}$$

$$h_{6\phi} = \frac{5\,(-2\,E)^3}{256}\,\frac{\nu^3}{(-2\,E\,h^2)^3}\,(1+2\,E\,h^2)^{5/2}\,, \tag{2.38q}$$

$$e_\phi{}^2 = 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\Big\{24+(-15+\nu)\,(-2\,E\,h^2)\Big\}$$

$$+\frac{(-2\,E)^2}{16\,c^4}\Big\{-32+176\,\nu+18\,\nu^2-(-2\,E\,h^2)(160-30\,\nu$$

$$+3\,\nu^2)+\frac{1}{(-2\,E\,h^2)}\,\Big(408-232\,\nu-15\,\nu^2\Big)$$

$$+\frac{(-2E)^3}{384c^6}\Big\{-16032+2764\nu+3\nu\pi^2+4536\nu^2+234\nu^3$$
$$-36\Big(248-80\nu+13\nu^2+\nu^3\Big)(-2Eh^2)-\frac{6}{(-2Eh^2)}\Big(2456$$
$$-26860\nu+581\nu\pi^2+2689\nu^2+10\nu^3\Big)+\frac{3}{(-2Eh^2)^2}\Big(27776$$
$$-65436\nu+1325\nu\pi^2+3440\nu^2-70\nu^3\Big)\Big\}. \tag{2.38r}$$

Note that the PN expansion for k begins at 1PN. The three eccentricities e_r, e_t and e_ϕ, which differ from each other at PN orders, are related by

$$\begin{aligned}
e_t &= e_r\Big\{1+\frac{(-2E)}{2c^2}(-8+3\nu)+\frac{(-2E)^2}{8c^4}\frac{1}{(-2Eh^2)}\Big[-34+22\nu \\
&+(-60+24\nu)\sqrt{(-2Eh^2)}+\Big(72-33\nu+12\nu^2\Big)(-2Eh^2)\Big] \\
&+\frac{(-2E)^3}{192c^6}\frac{1}{(-2Eh^2)^2}\Big[-6432+13488\nu-240\nu\pi^2 \\
&-768\nu^2+\Big(-10080+13952\nu-123\nu\pi^2 \\
&-1440\nu^2\Big)\sqrt{(-2Eh^2)}+(2700-4420\nu-3\nu\pi^2 \\
&+1092\nu^2)(-2Eh^2)+(9180-6444\nu+1512\nu^2)(-2Eh^2)^{3/2} \\
&+\Big(-3840+1284\nu-672\nu^2+240\nu^3\Big)(-2Eh^2)^2\Big]\Big\}, \qquad (2.39a) \\
e_\phi &= e_r\Big\{1+\frac{(-2E)}{2c^2}\nu+\frac{(-2E)^2}{32c^4}\frac{1}{(-2Eh^2)}\Big[136-56\nu-15\nu^2 \\
&+\nu(20+11\nu)(-2Eh^2)\Big]+\frac{(-2E)^3}{768c^6}\frac{1}{(-2Eh^2)^2}\Big[31872 \\
&-88404\nu+2055\nu\pi^2+4176\nu^2-210\nu^3+\Big(2256 \\
&+10228\nu-15\nu\pi^2-2406\nu^2-450\nu^3\Big)(-2Eh^2)+6\nu(136 \\
&+34\nu+31\nu^2)(-2Eh^2)^2\Big]\Big\}. \qquad (2.39b)
\end{aligned}$$

The presense of log terms in the standard harmonic coordinates obstructs the construction of a generalised quasi-Keplerian representation which crucially exploits the fact that at order 3PN the radial equation is a fourth order ***polynomial*** in $1/r$. [155] thus constructs the GQK representation for the modified harmonic coordinates. The representation in modified harmonic coordinates is of the same form as above but the corresponding equations for the orbital elements are given by [155]

$$
\begin{aligned}
\mathrm{a,} \;=\; & \frac{1}{(-2E)}\Bigg\{1+\frac{(-2E)}{4c^2}(-7+\nu)+\frac{(-2E)^2}{16c^4}\Bigg[1+\nu^2 \\
& +\frac{16}{(-2\mathrm{Eh}^2)}(-4+7\,\mathrm{v})]+\frac{(-2E)^3}{6720\,c^6}\Bigg[105-105\,\mathrm{v} \\
& +105\,\mathrm{v}^3+\frac{}{(-2\mathrm{Eh}^2)}(26880+4305\,\mathrm{n}^2\mathrm{v}-215408\,\mathrm{v} \\
& +47040\,\mathrm{v}^2)-\frac{}{(-2Eh^2)^2}(53760-176024\,\mathrm{v}+4305\,\pi^2\nu \\
& +15120\,\nu^2)\Bigg)\Bigg]\Bigg\},
\end{aligned}
\tag{2.40a}
$$

$$
\begin{aligned}
e_r^{\,2} \;=\; & 1+2Eh^2+\frac{(-2E)}{4c^2}\Big\{24-4\,\mathrm{v}+5\,(-3+\mathrm{v})(-2Eh^2)\Big\} \\
& +\frac{(-2E)^2}{8c^4}\Big\{60+148\,\nu+2\,\nu^2-(-2Eh^2)\Big(80-45\,\mathrm{v}+4\,\nu^2\Big) \\
& +\frac{}{(-2Eh^2)}(4-7\,\mathrm{v})\Big\}+\frac{(-2E)^3}{6720\,c^6}\Big\{-3360+181264\,\nu \\
& +8610\mathrm{n}^2\mathrm{v}-67200\mathrm{v}^2+105(-2Eh^2)\Big(-1488+1120\,\nu \\
& -195\,\nu^2+4\,\nu^3\Big)-\frac{80}{(-2Eh^2)}\Big(1008-21130\,\nu+861\,\pi^2\nu \\
& +2268\,\nu^2\Big)+\frac{16}{(-2Eh^2)^2}\Big(53760-176024\,\nu+4305\,\pi^2\nu \\
& +15120\,\nu^2\Big)\Big\},
\end{aligned}
\tag{2.40b}
$$

$$
\begin{aligned}
n \;=\; & (-2E)^{3/2}\Bigg\{1+\frac{(-2E)}{8\,c^2}(-15+\nu)+\frac{(-2E)^2}{128\,c^4}\Bigg[555+30\,\nu \\
& +11\,\nu^2+\frac{192}{\sqrt{(-2Eh^2)}}(-5+2\,\nu)\Bigg]+\frac{(-2E)^3}{3072\,c^6}\Bigg[-29385 \\
& -4995\,\nu-315\,\nu^2+135\,\nu^3+\frac{5760}{\sqrt{(-2Eh^2)}}(17-9\,\nu+2\,\nu^2) \\
& -\frac{16}{(-2Eh^2)^{3/2}}\Big(10080-13952\,\nu+123\,\pi^2\nu+1440\,\nu^2\Big)\Bigg]\Bigg\},
\end{aligned}
\tag{2.40c}
$$

$$
\begin{aligned}
e_t^{\,2} \;=\; & 1+2Eh^2+\frac{(-2E)}{4\,c^2}\Big\{-8+8\,\nu-(-2Eh^2)(-17+7\,\nu)\Big\} \\
& +\frac{(-2E)^2}{8\,c^4}\Big\{12+72\,\nu+20\,\nu^2-24\,\sqrt{(-2Eh^2)}\,(-5+2\,\nu) \\
& -(-2Eh^2)(112-47\,\nu+16\,\nu^2)-\frac{16}{(-2Eh^2)}(-4+7\,\nu) \\
& +\frac{24}{\sqrt{(-2Eh^2)}}(-5+2\,\nu)\Big\}+\frac{(-2E)^3}{6720\,c^6}\Big\{23520-464800\,\nu \\
& +179760\,\nu^2+16800\,\nu^3-2520\,\sqrt{(-2Eh^2)}(265-193\,\nu
\end{aligned}
$$

$$
\begin{aligned}
&+46\nu^2) - 525(-2Eh^2)\Big(-528 + 200\nu - 77\nu^2 + 24\nu^3\Big) \\
&-\frac{6}{(-2Eh^2)}(73920 - 260272\nu + 4305\pi^2\nu + 61040\nu^2\Big) \\
&+\frac{70}{\sqrt{(-2Eh^2)}}(16380 - 19964\nu + 123\pi^2\nu + 3240\nu^2\Big) \\
&+\frac{}{(-2Eh^2)^2}(53760 - 176024\nu + 4305\pi^2\nu + 15120\nu^2\Big) \\
&-\frac{70}{(-2Eh^2)^{3/2}}(10080 - 13952\nu + 123\pi^2\nu + 1440\nu^2\Big)\Big\}, \qquad (2.40d)
\end{aligned}
$$

$$
g_{4t} = -\frac{3(-2E)^2}{2}\cdot\frac{1}{\qquad}(-5 + 2\nu)\Big\}, \qquad (2.40e)
$$

$$
\begin{aligned}
g_{6t} = {} & \frac{(-2E)^3}{192}\Big\{ \qquad \Big(10080 - 13952\nu + 123\pi^2\nu \\
&+1440\nu^2) + \frac{}{\sqrt{(-2Eh^2)}}(-3420 + 1980\nu - 648\nu^2)\Big\}, \qquad (2.40f)
\end{aligned}
$$

$$
f_{4t} = -\frac{(-2E)^2}{8}\Big\{\frac{\sqrt{1+2Eh^2}}{\sqrt{(-2Eh^2)}}\nu\,(-15 + \nu)\}, \qquad (2.40g)
$$

$$
\begin{aligned}
f_{6t} = {} & \frac{(-2E)^3}{2240}\Big\{\frac{1}{(-2Eh^2)^{3/2}\sqrt{1+2Eh^2}}(22400 + 43651\nu \\
&-1435\pi^2\nu - 20965\nu^2 + 385\nu^3\Big) \\
&+\frac{1}{\sqrt{(-2Eh^2)}\sqrt{1+2Eh^2}}\Big(-22400 - 49321\nu \\
&+27300\nu^2 + 1435\pi^2\nu - 1225\nu^3\Big) \\
&+\frac{35\sqrt{(-2Eh^2)}}{\sqrt{1+2Eh^2}}\nu\,(297 - 175\nu + 23\nu^2)\Big\}, \qquad (2.40h)
\end{aligned}
$$

$$
i_{6t} = \frac{(-2E)^3}{16}\Big\{\frac{1+2Eh^2}{(-2Eh^2)^{3/2}}\nu\Big(116 - 49\nu + 3\nu^2\Big)\Big\}, \qquad (2.40i)
$$

$$
h_{6t} = \frac{(-2E)^3}{192}\Big\{\Big(\frac{1+2Eh^2}{(-2Eh^2)}\Big)^{3/2}\nu\,\Big(23 - 73\nu + 13\nu^2\Big)\Big\}, \qquad (2.40j)
$$

$$
\begin{aligned}
\Phi = {} & 2\pi\Big\{1 + \frac{3}{h^2c^2} + \frac{(-2E)^2}{4c^4}\Big[\frac{3}{(-2Eh^2)}(-5 + 2\nu) \\
&-\frac{15}{(-2Eh^2)^2}(-7 + 2\nu)\Big] + \frac{(-2E)^3}{128c^6}\Big[\frac{5}{(-2Eh^2)^3}\Big(7392 \\
&-8000\nu + 336\nu^2 + 123\pi^2\nu\Big) + \frac{24}{(-2Eh^2)}(5 - 5\nu + 4\nu^2) \\
&-\frac{1}{(-2Eh^2)^2}\Big(10080 - 13952\nu + 123\pi^2\nu + 1440\nu^2\Big)\Big]\Big\}, \qquad (2.40k)
\end{aligned}
$$

$$
\begin{aligned}
f_{4\phi} &= \frac{(-2E)^2}{8}\left\{\frac{1+2Eh^2}{(-2Eh^2)^2}\left(1+19\,\nu-3\,\nu^2\right)\right\}, &(2.40l)\\
f_{6\phi} &= \frac{(-2E)^3}{26880}\left\{\frac{1}{(-2Eh^2)^3}\left(67200+994704\,\nu-30135\,\pi^2\nu\right.\right.\\
&\left.-335160\,\nu^2-4200\,\nu^3\right)+\frac{1}{(-2Eh^2)^2}\left(-60480-991904\,\nu\right.\\
&\left.+30135\,\pi^2\nu+428400\,\nu^2-8400\,\nu^3\right)+\frac{1}{(-2Eh^2)}\left(840\right.\\
&\left.\left.+141680\,\nu-99960\,\nu^2+10080\,\nu^3\right)\right\}, &(2.40m)\\
g_{4\phi} &= -\frac{(-2E)^2}{32}\left\{\frac{\left(1+2Eh^2\right)^{3/2}}{(-2Eh^2)^2}\,\nu\,(-1+3\,\nu)\right\}, &(2.40n)\\
g_{6\phi} &= \frac{(-2E)^3}{8960}\,\frac{1}{\sqrt{1+2Eh^2}}\left\{-35\,\nu\,(14-49\,\nu+26\,\nu^2)\right.\\
&-\frac{1}{(-2Eh^2)}\,\nu\left(-36196+1435\,\pi^2+29225\,\nu-2660\,\nu^2\right)\\
&+\frac{1}{(-2Eh^2)^2}\,\nu\left(-71867+2870\,\pi^2+56035\,\nu-2275\,\nu^2\right)\\
&\left.-\frac{1}{(-2Eh^2)^3}\,\nu\left(-36161+1435\,\pi^2+28525\,\nu-525\,\nu^2\right)\right\}, &(2.40o)\\
i_{6\phi} &= \frac{(-2E)^3}{192}\left\{\frac{(1+2Eh^2)^2}{(-2Eh^2)^3}\,\nu\,(82-57\,\nu+15\,\nu^2)\right\}, &(2.40p)\\
h_{6\phi} &= \frac{(-2E)^3}{256}\left\{\frac{(1+2Eh^2)^{5/2}}{(-2Eh^2)^3}\,\nu\,(1-5\,\nu+5\,\nu^2)\right\}, &(2.40q)\\
{e_\phi}^2 &= 1+2Eh^2+\frac{(-2E)}{4\,c^2}\left\{24+(-2Eh^2)(-15+\nu)\right\}\\
&+\frac{(-2E)^2}{16c^4}\left\{-40+34\,\nu+18\,\nu^2-(-2Eh^2)(160\right.\\
&\left.-31\,\nu+3\,\nu^2)-\frac{1}{(-2Eh^2)}(-416+91\,\nu+15\,\nu^2)\right\}\\
&+\frac{(-2E)^3}{13440\,c^6}\left\{-584640-17482\,\nu-4305\,\pi^2\nu-7350\,\nu^2\right.\\
&+8190\,\nu^3-420(-2Eh^2)\left(744-248\,\nu+31\,\nu^2+3\,\nu^3\right)\\
&-\frac{14}{(-2Eh^2)}\left(36960-341012\,\nu+4305\,\pi^2\nu-225\,\nu^2\right.\\
&\left.+150\,\nu^3\right)-\frac{1}{(-2Eh^2)^2}\left(-2956800+5627206\,\nu\right.\\
&\left.\left.-81795\,\pi^2\nu+14490\,\nu^2+7350\,\nu^3\right)\right\}. &(2.40r)
\end{aligned}
$$

In (modified) harmonic coordinates too, there are PN accurate relations connecting the three eccentricities e_r, e_t and e_ϕ. These relations read

$$
\begin{aligned}
e_t &= e_r\Big\{1 + \frac{(-2E)}{2c^2}(3\nu - 8) + \frac{(-2E)^2}{4c^4}\frac{1}{(-2Eh^2)}\Big[-16 + 28\nu \\
&\quad +(-30 + 12\nu)\sqrt{(-2Eh^2)} + (36 - 19\nu + 6\nu^2)(-2Eh^2)\Big] \\
&\quad +\frac{(-2E)^3}{6720c^6}\frac{1}{h^4(-2E)^2}\Big[-215040 + 704096\nu - 17220\pi^2\nu \\
&\quad -60480\nu^2 + 35\Big(-10080 + 13952\nu - 123\pi^2\nu \\
&\quad -1440\nu^2\Big)\sqrt{(-2Eh^2)} + \Big(87360 - 354848\nu + 4305\pi^2\nu \\
&\quad +105840\nu^2\Big)(-2Eh^2) + \Big(-134400 + 54600\nu - 28560\nu^2 \\
&\quad +8400\nu^3\Big)(-2Eh^2)^2 + (321300 - 225540\nu \\
&\quad +52920\nu^2)(-2Eh^2)^{3/2}\Big]\Big\}, \qquad (2.41a) \\
e_\phi &= e_r\Big\{1 + \frac{(-2E)}{2c^2}\nu + \frac{(-2E)^2}{32c^4}\frac{1}{(-2Eh^2)}\Big[160 + 357\nu - 15\nu^2 \\
&\quad +(-\nu + 11\nu^2)(-2Eh^2)\Big] + \frac{(-2E)^3}{8960c^6}\frac{1}{(-2Eh^2)^2}\Big[412160 \\
&\quad +1854\nu - 18655\pi^2\nu - 166110\nu^2 - 2450\nu^3 + \Big(24640 \\
&\quad -182730\nu + 7175\pi^2\nu + 156520\nu^2 - 5250\nu^3\Big)(-2Eh^2) \\
&\quad +70\nu(-1 - \nu + 31\nu^2)(-2Eh^2)^2\Big]\Big\}. \qquad (2.41b)
\end{aligned}
$$

We conclude with a recall of an important point related to the use of gauge invariant variables in the elliptical orbit case as stressed by [155]. Damour and Schafer [148] showed that the functional form of n and Φ as functions of gauge invariant variables like E and h is the same in different coordinate systems (gauges). From the explicit expressions in [155] for n and Φ in the ADM and modified harmonic coordinates the gauge invariance of these two parameters is explicit. This prompted [155] to suggest the use of variables $x_{\rm MGS} = (Gmn/c^3)^{2/3}$ and $k' = (\Phi - 2\pi)/6\pi$ as gauge invariant variables in the general orbit case. In the present chapter we propose a variant of the former: $x = (Gmn\Phi/2\pi c^3)^{2/3} = (GmnKc^3)^{2/3} = (Gmn(1+k)c^3)^{2/3}$. The choice we propose is the obvious generalisation of gauge invariant variable x in the circular orbit case and thus facilitates the straightforward reading out of the circular orbit limit.

2.7 Orbital average of the energy flux in modified harmonic coordinates

2.7.1 The instantaneous terms

To average the energy flux over an orbit we will require the use of a 3PN quasi-Keplerian representation in harmonic coordinates. Consequently, the averaging over an orbit is only possible in the modified harmonic coordinates without the log terms and discussed next.

The computation of the orbital average involves evaluation of the the integral,

$$\langle\dot{\mathcal{E}}\rangle = \frac{1}{P}\int_0^P \dot{\mathcal{E}}(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{dl}{du}\,\dot{\mathcal{E}}(u)du\,. \tag{2.42}$$

Using the GQK representation of the orbit discussed in the earlier section, we can transform the expression for the energy flux $\dot{\mathcal{E}}$ (r, $\dot{r}^2, v^2$) or more exactly $(dl/du\times\dot{\mathcal{E}})(r,\dot{r}^2,v^2)$ to $(dl/du\times \dot{\mathcal{E}})(x,$ e,, $u)$.[1] Recall that in the expression of the flux at the 3PN order there are log terms too and hence it is convenient to rewrite the expression as

$$\frac{l}{du}\,\dot{\mathcal{E}} = \sum_{N=3}\left\{\alpha_N(e_t)\frac{1}{(1-e_t\cos u)^N} + \beta_N(e_t)\frac{\sin u}{(1-e_t\cos u)^N} + \gamma_N(e_t)\frac{\ln(1-\text{e}, \cos\text{u})}{(1-e_t\cos u)^N}\right\}, \tag{2.43}$$

where the non-vanishing β_N's and γ_N's read as

$$\beta_6(e_t) = x^{15/2}\nu^3\left\{-\frac{40576}{1575}e_t\right\}, \tag{2.44a}$$

$$\beta_7(e_t) = x^{15/2}\nu^3\left\{\frac{36544}{525}e_t - \frac{1728}{175}e_t^3\right\}, \tag{2.44b}$$

$$\beta_8(e_t) = x^{15/2}\nu^3\left\{-\frac{9856}{45}e_t + \frac{9856}{45}e_t^3\right\}, \tag{2.44c}$$

$$\beta_9(e_t) = x^{15/2}\nu^3\left\{-\frac{11712}{35}e_t + \frac{23424}{35}e_t^3 - \frac{11712}{35}e_t^5\right\}, \tag{2.44d}$$

$$\gamma_5(x,e_t) = \frac{6848}{525}\nu^2x^8, \tag{2.45a}$$

$$\gamma_6(x,e_t) = -\frac{109568}{1575}\nu^2x^8, \tag{2.45b}$$

$$\gamma_7(x,e_t) = \nu^2x^8\left\{\frac{27392}{315} - \frac{150656}{525}\left(1-e_t^2\right)\right\}, \tag{2.45c}$$

[1]Ref. [46] uses Gm/a_r and e_r while [47] employs Gmn/c^3 and e,. We propose the use of e, and $x = (Gmn\Phi/2\pi c^3)^{2/3}$ for reasons outlined in the previous section 2.6.2.

$$\gamma_8(x,e_t) = \frac{27392}{45}\nu^2 x^8\left(1-e_t^2\right), \tag{2.45d}$$

$$\gamma_9(x,e_t) = -\frac{6848}{15}\nu^2 x^8\left(1-e_t^2\right)^2 \tag{2.45e}$$

Finally, the α_N's are given by:

$$\begin{aligned}
\alpha_3 = {} & \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\left\{-\frac{1}{12}\left(1-e_t^2\right)^{7/2}\right. \\
& +x\left\{\left(1-e_t^2\right)^{5/2}\left(\frac{13}{21}+\frac{3}{14}e_t^2\right)+\frac{5}{252}\left(1-e_t^2\right)^{7/2}\nu\right\} \\
& +x^2\left\{\left(1-e_t^2\right)^{3/2}\left(\frac{163}{252}+\frac{107}{72}e_t^2-\frac{65}{252}e_t^4\right.\right. \\
& \left.+\left(-\frac{293}{144}-\frac{313}{168}e_t^2+\frac{149}{1008}e_t^4\right)\nu\right) \\
& \left.+\left(1-e_t^2\right)^3\left(\frac{5}{12}-\frac{1}{6}\nu\right)+\frac{5}{756}\left(1-e_t^2\right)^{7/2}\nu^2\right\} \\
& +x^3\left\{\frac{5}{6}+\frac{35}{3}e_t^2-\frac{40}{3}e_t^4+\frac{5}{6}e_t^6\right. \\
& +\left(-\frac{85}{27}+\frac{53}{108}e_t^2+\frac{187}{54}e_t^4-\frac{29}{36}e_t^6\right)\nu \\
& +\sqrt{1-e_t^2}\left(-\frac{1447}{154}+\frac{14459}{2772}e_t^2-\frac{5395}{2772}e_t^4+\frac{1}{66}e_t^6\right. \\
& +\left(\frac{80347}{33264}+\frac{53377}{4752}e_t^2+\frac{1987}{1188}e_t^4-\frac{89}{11088}e_t^6\right)\nu^2 \\
& +\nu\left(-\frac{238345}{9504}+\frac{4681}{2464}e_t^4-\frac{26729}{66528}e_t^6+\frac{205}{192}\pi^2\right. \\
& \left.\left.+e_t^2\left(-\frac{542761}{22176}+\frac{205}{768}\pi^2\right)\right)\right) \\
& +\left(1-e_t^2\right)^2\left(\left(\frac{5}{18}+\frac{11}{36}e_t^2\right)\nu^2+\frac{41}{1152}\nu\pi^2\right) \\
& \left.\left.-\frac{5}{37422}\left(1-e_t^2\right)^{7/2}\nu^3\right\}\right\},
\end{aligned} \tag{2.46a}$$

$$\begin{aligned}
\alpha_4 = {} & \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\left\{\frac{1}{6}\left(1-e_t^2\right)^{7/2}\right. \\
& +x\left\{\left(1-e_t^2\right)^{5/2}\left(-\frac{103}{42}+\frac{11}{14}e_t^2\right)-\frac{11}{126}\left(1-e_t^2\right)^{7/2}\nu\right\} \\
& +x^2\left\{\left(1-e_t^2\right)^{3/2}\left(\frac{2489}{252}-\frac{3043}{252}e_t^2-\frac{391}{252}e_t^4\right.\right. \\
& \left.+\left(\frac{2089}{378}+\frac{1405}{378}e_t^2-\frac{659}{378}e_t^4\right)\nu\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(1-e_t^2\right)^3\left(-\frac{95}{24}+\frac{19}{12}\nu\right)+\frac{41}{216}\left(1-e_t^2\right)^{7/2}\nu^2\Bigg\}\\
&+x^3\left\{\frac{1025}{42}-\frac{6955}{84}e_t^2+\frac{1415}{21}e_t^4-\frac{755}{84}e_t^6\right.\\
&+\left(\frac{27337}{1512}-\frac{3625}{189}e_t^2-\frac{8891}{1512}e_t^4+\frac{1759}{252}e_t^6\right)\nu\\
&+\sqrt{1-e_t^2}\left(\frac{9287}{770}+\frac{93938}{3465}e_t^2-\frac{190241}{6930}e_t^4+\frac{1741}{3465}e_t^6\right.\\
&+\left(-\frac{775267}{99792}-\frac{303179}{11088}e_t^2+\frac{165871}{33264}e_t^4-\frac{49865}{99792}e_t^6\right)\nu^2\\
&+\nu\left(\frac{1301467}{55440}+\frac{138139}{33264}e_t^6-\frac{533}{192}\pi^2\right.\\
&+e_t^2\left(\frac{1676425}{33264}-\frac{41}{384}\pi^2\right)\\
&\left.\left.+e_t^4\left(\frac{3007979}{166320}+\frac{41}{192}\pi^2\right)\right)\right)\\
&+\left(1-e_t^2\right)^2\left(\left(-\frac{194}{63}-\frac{1241}{504}e_t^2\right)\nu^2-\frac{779}{2304}\nu\pi^2\right)\\
&\left.+\frac{46769}{299376}\left(1-e_t^2\right)^{7/2}\nu^3\right\}\Bigg\}, \qquad (2.46\text{b})
\end{aligned}
$$

$$
\begin{aligned}
\alpha_5 \;=\; &\frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\left\{\frac{11}{12}\left(1-e_t^2\right)^{9/2}\right.\\
&+x\left(1-e_t^2\right)^{7/2}\left(-\frac{199}{28}-\frac{19}{12}e_t^2+\left(\frac{317}{252}+\frac{163}{252}e_t^2\right)\nu\right)\\
&+x^2\left\{\left(1-e_t^2\right)^2\left(-\frac{40}{3}+\frac{215}{12}e_t^2-\frac{55}{12}e_t^4\right.\right.\\
&\left.+\left(\frac{16}{3}-\frac{43}{6}e_t^2+\frac{11}{6}e_t^4\right)\nu\right)\\
&+\left(1-e_t^2\right)^{5/2}\left(\frac{211}{4536}-\frac{20107}{1134}e_t^2+\frac{169}{84}e_t^4\right.\\
&\left.+\left(\frac{22481}{6048}-\frac{17375}{6048}e_t^2-\frac{161}{144}e_t^4\right)\nu\right)\\
&\left.+\left(1-e_t^2\right)^{7/2}\left(\frac{1747}{6048}-\frac{5}{27}e_t^2\right)\nu^2\right\}\\
&+x^3\left\{\left(1-e_t^2\right)\left(\frac{535}{84}-\frac{2245}{21}e_t^2+\frac{9215}{84}e_t^4-\frac{55}{6}e_t^6\right.\right.\\
&+\left(-\frac{37}{126}-\frac{631}{42}e_t^2+\frac{523}{28}e_t^4-\frac{121}{36}e_t^6\right)\nu^2\\
&+\nu\left(\frac{49925}{756}+\frac{319}{36}e_t^6-\frac{41}{36}\pi^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+e_t^4\left(-\frac{20681}{378}-\frac{451}{1152}\pi^2\right)\\
&+e_t^2\left(-\frac{7631}{378}+\frac{1763}{1152}\pi^2\right)\Bigg)\Bigg)\\
&+\left(1-e_t^2\right)^{3/2}\Bigg(\frac{2840351}{113400}+\frac{202321}{7425}e_t^2-\frac{5995927}{178200}e_t^4-\frac{571}{462}e_t^6\\
&+\left(\frac{4444501}{399168}-\frac{5110477}{199584}e_t^2+\frac{544699}{57024}e_t^4+\frac{22357}{33264}e_t^6\right)\nu^2\\
&+\nu\Bigg(\frac{432910831}{1496880}+\frac{773}{672}e_t^6-\frac{3731}{384}\pi^2\\
&+e_t^4\left(-\frac{859295}{21384}-\frac{41}{64}\pi^2\right)\\
&+e_t^2\left(\frac{119057287}{598752}+\frac{1927}{768}\pi^2\right)\Bigg)\Bigg)\\
&+\left(1-e_t^2\right)^{7/2}\left(-\frac{214}{105}\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right)+\left(\frac{52739}{1197504}+\frac{8377}{74844}e_t^2\right)\nu^3\right)\Bigg\}\Bigg\}, \qquad (2.46c)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_6 \;=\;& \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{\\
&x\left(1-e_t^2\right)^{9/2}\left(-\frac{19}{21}+\frac{10}{7}\nu\right)\\
&+x^2\Bigg\{\left(1-e_t^2\right)^{5/2}\left(\frac{3527}{6048}+\frac{2519}{1512}e_t^2-\frac{13603}{6048}e_t^4\right)\nu^2\\
&+\left(\frac{275}{8}-\frac{275}{8}e_t^2\right)\left(1-e_t^2\right)^3\\
&+\left(1-e_t^2\right)^{7/2}\left(\frac{26993}{567}-\frac{811}{252}e_t^2+\left(-\frac{67987}{2016}-\frac{122093}{6048}e_t^2\right)\nu\right)\\
&-\frac{55}{4}\left(1-e_t^2\right)^4\nu\Bigg\}\\
&+x^3\Bigg\{\left(1-e_t^2\right)^2\Bigg(-\frac{9935}{56}+180e_t^2-\frac{145}{56}e_t^4\\
&+\left(\frac{355}{21}-\frac{955}{168}e_t^2-\frac{1885}{168}e_t^4\right)\nu^2\\
&+\nu\Bigg(-\frac{73699}{504}+\frac{211}{42}e_t^4+\frac{2255}{768}\pi^2\\
&+e_t^2\left(\frac{71167}{504}-\frac{2255}{768}\pi^2\right)\Bigg)\Bigg)\\
&+\left(1-e_t^2\right)^{5/2}\Bigg(-\frac{5542769}{19800}+\frac{38278951}{623700}e_t^2-\frac{271079}{27720}e_t^4\\
&+\left(-\frac{1743347}{133056}+\frac{14730817}{199584}e_t^2+\frac{2076271}{399168}e_t^4\right)\nu^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{119953}{44352}+\frac{904027}{199584}e_t^2-\frac{728477}{399168}e_t^4\right)\nu^3\\
&+\nu\left(-\frac{6782819}{68040}+\frac{4267325}{66528}e_t^4+\frac{11767}{768}\pi^2\right.\\
&\left.\left.\left.+e_t^2\left(-\frac{532918877}{2993760}-\frac{779}{768}\pi^2\right)\right)\right)\\
&\left.\left.+\frac{3424}{315}\left(1-e_t^2\right)^{7/2}\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right)\right\}\right\}, \qquad (2.46\text{d})
\end{aligned}
$$

$$
\begin{aligned}
\alpha_7 \;=\;& \frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\left\{\vphantom{\frac{1}{1}}\right.\\
&x\left(1-e_t^2\right)^{11/2}\left(\frac{687}{112}-\frac{155}{28}\nu\right)\\
&+x^2\left\{\left(1-e_t^2\right)^{7/2}\left(-\frac{404113}{3024}+\frac{395419}{3024}e_t^2+\frac{23}{8}e_t^4\right.\right.\\
&+\left(\frac{59663}{864}-\frac{236669}{6048}e_t^2-\frac{5027}{168}e_t^4\right)\nu\\
&\left.\left.+\left(-\frac{49099}{6048}+\frac{19147}{6048}e_t^2+\frac{104}{21}e_t^4\right)\nu^2\right)\right\}\\
&+x^3\left\{\left(1-e_t^2\right)^3\left(-\frac{12585}{56}+\frac{12585}{56}e_t^2\right)\right.\\
&+\left(1-e_t^2\right)^{7/2}\left(\frac{415399769}{277200}-\frac{188784109}{277200}e_t^2\right.\\
&+\frac{25253}{1232}e_t^4-\left(-\frac{9844}{315}+\frac{4708}{105}e_t^2\right)\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right)\\
&+\left(-\frac{16992433}{199584}+\frac{1829777}{18144}e_t^2+\frac{30019}{1008}e_t^4\right)\nu^2\\
&+\left(-\frac{49187}{22176}-\frac{1409411}{199584}e_t^2+\frac{1843}{1386}e_t^4\right)\nu^3\\
&+\nu\left(\frac{85884049}{285120}-\frac{74927}{672}e_t^4-\frac{4223}{192}\pi^2\right.\\
&\left.\left.\left.+e_t^2\left(-\frac{179175593}{399168}+\frac{533}{32}\pi^2\right)\right)\right)\\
&\left.\left.+\left(1-e_t^2\right)^4\left(\frac{1281}{4}\nu-\frac{645}{7}\nu^2\right)\right\}\right\}, \qquad (2.46\text{e})
\end{aligned}
$$

$$
\begin{aligned}
\alpha_8 &= \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{ \\
&x^2\left(1-e_t^2\right)^{11/2}\left(\frac{12113}{252}+\frac{139117}{6048}\nu-\frac{36371}{6048}\nu^2\right) \\
&+x^3\Bigg\{\left(1-e_t^2\right)^{9/2}\Bigg(-\frac{148891823}{83160}+\frac{5022439}{27720}e_t^2-\frac{856}{9}\ln\left(\frac{c^2 r_0}{Gm}x\right) \\
&+\left(\frac{12030505}{28512}+\frac{24965915}{199584}e_t^2\right)\nu^2 \\
&+\left(-\frac{1695697}{66528}+\frac{1914239}{199584}e_t^2\right)\nu^3 \\
&+\nu\left(-\frac{3159669019}{1995840}-\frac{4628329}{12096}e_t^2+\frac{10045}{256}\pi^2\right)\Bigg) \\
&+\left(1-e_t^2\right)^5\left(\frac{10305}{32}-\frac{6711}{16}\nu+\frac{465}{4}\nu^2\right)\Bigg\}\Bigg\}, \qquad (2.46f) \\
\alpha_9 &= \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{ \\
&x^2\left(1-e_t^2\right)^{13/2}\left(\frac{2501}{504}-\frac{10117}{252}\nu+\frac{2101}{126}\nu^2\right) \\
&+x^3\Bigg\{\left(1-e_t^2\right)^{11/2}\Bigg(\frac{7559609}{9504}-\frac{305995}{5544}e_t^2+\frac{214}{3}\ln\left(\frac{c^2 r_0}{Gm}x\right) \\
&+\left(-\frac{169279}{384}-\frac{303437}{2079}e_t^2\right)\nu^2 \\
&+\left(\frac{10892993}{133056}+\frac{12107}{4158}e_t^2\right)\nu^3 \\
&+\nu\left(\frac{315967481}{266112}+\frac{563993}{2079}e_t^2-\frac{697}{32}\pi^2\right)\Bigg)\Bigg\}\Bigg\}, \qquad (2.46g) \\
\alpha_{10} &= \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{x^3\left(1-e_t^2\right)^{13/2}\Bigg(-\frac{6094651}{55440}+\frac{19107917}{88704}\nu \\
&-\frac{7252559}{88704}\nu^2-\frac{1349011}{44352}\nu^3\Bigg)\Bigg\}, \qquad (2.46h) \\
\alpha_{11} &= \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{x^3\left(1-e_t^2\right)^{15/2}\Bigg(\frac{1507925}{44352}-\frac{20365}{126}\nu \\
&+\frac{687305}{5544}\nu^2-\frac{32755}{1386}\nu^3\Bigg)\Bigg\}. \qquad (2.46i)
\end{aligned}
$$

It is worth noting that the β_N's correspond to all the 2.5PN terms while the γ_N represent the log terms at order 3PN. Recall here the useful formulas

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{sinu}}{(1-\mathrm{e}\cos u)^N}\mathrm{du} = 0\,, \tag{2.47}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{(1-e\cos u)^N} = \frac{1}{(1-e^2)^{N/2}}P_{(N-1)}\left(\frac{1}{\sqrt{1-e^2}}\right), \tag{2.48}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\ln(1-e\cos u)}{(1-\mathrm{e}\cos u)^N}\mathrm{du} = \frac{(-1)^{(N-1)}}{(\mathrm{N}-1}\frac{d^{(N-1)}\mathcal{Y}(e,y)}{dy^{(N-1)}}\bigg|_{y=1}\,. \tag{2.49}$$

where,

$$\begin{aligned}\mathcal{Y}(e,y) &= \frac{1}{\sqrt{y^2-e^2}}\left\{\ln\left[\frac{1}{2}\left(1+\sqrt{1-e^2}\right)\right]-2\ln\left[y+\sqrt{y^2-e^2}\right]\right.\\ &\quad\left.+2\ln\left[y-1+\sqrt{1-e^2}+\sqrt{y^2-e^2}\right]\right\}\end{aligned} \tag{2.50}$$

Implementing all the above integrations, the expression for the energy flux can be averaged over an orbit in the modified harmonic coordinates to order 3PN extending the results of [46] at 2PN (in ADM coordinates). We have,

$$\begin{aligned}<\dot{\mathcal{E}}>_{\mathrm{MHar}} &= \frac{32\nu^2x^5}{5}\frac{1}{\left(1-e_t^2\right)^{7/2}}\Big(<\dot{\mathcal{E}}_{\mathrm{N}}>_{\mathrm{MHar}}+x<\dot{\mathcal{E}}_{\mathrm{1PN}}>_{\mathrm{MHar}}\\ &\quad+x^2<\dot{\mathcal{E}}_{\mathrm{2PN}}>_{\mathrm{MHar}}+x^3<\dot{\mathcal{E}}_{\mathrm{3PN}}>_{\mathrm{MHar}}\Big)\,.\end{aligned} \tag{2.51}$$

$$<\dot{\mathcal{E}}_N>_{\mathrm{Mhar}} = 1+e_t^2\frac{73}{24}+e_t^4\frac{37}{96}\,, \tag{2.52a}$$

$$\begin{aligned}<\dot{\mathcal{E}}_{\mathrm{1PN}}>_{\mathrm{Mhar}} &= \frac{1}{\left(1-e_t^2\right)}\\ &\quad\left\{\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)+e_t^2\left(\frac{10475}{672}-\frac{1081}{36}\nu\right)\right.\\ &\quad\left.+e_t^4\left(\frac{10043}{384}-\frac{311}{12}\nu\right)+e_t^6\left(\frac{2179}{1792}-\frac{851}{576}\nu\right)\right\},\end{aligned} \tag{2.52b}$$

$$\begin{aligned}<\dot{\mathcal{E}}_{\mathrm{2PN}}>_{\mathrm{Mhar}} &= \frac{1}{\left(1-e_t^2\right)^2}\\ &\quad\left\{-\frac{203471}{9072}+\frac{12799}{504}\nu+\frac{65}{18}\nu^2+e_t^2\left(-\frac{3807197}{18144}+\frac{116789}{2016}\nu+\frac{5935}{54}\nu^2\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+e_t^4\left(-\frac{268447}{24192}-\frac{2465027}{8064}\nu+\frac{247805}{864}\nu^2\right)\\
&+e_t^6\left(\frac{1307105}{16128}-\frac{416945}{2688}\nu+\frac{185305}{1728}\nu^2\right)\\
&+e_t^8\left(\frac{86567}{64512}-\frac{9769}{4608}\nu+\frac{21275}{6912}\nu^2\right)\\
&+\sqrt{1-e_t^2}\left[\left(\frac{35}{2}-7\nu\right)+e_t^2\left(\frac{6425}{48}-\frac{1285}{24}\nu\right)\right.\\
&\left.\left.+e_t^4\left(\frac{5065}{64}-\frac{1013}{32}\nu\right)+e_t^6\left(\frac{185}{96}-\frac{37}{48}\nu\right)\right]\right\},
\end{aligned}
\tag{2.52c}
$$

$$
\begin{aligned}
<\dot{\mathcal{E}}_{3\mathrm{PN}}>_{\mathrm{Mhar}} = &\frac{1}{\left(1-e_t^2\right)^3}\\
&\left\{\frac{1266161801}{9979200}+\left(\frac{8009293}{54432}-\frac{41}{64}\pi^2\right)\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right.\\
&+e_t^2\left(\frac{27805251167}{19958400}+\left[\frac{654126203}{272160}+\frac{4879}{1536}\pi^2\right]\nu-\frac{1179281}{3024}\nu^2-\frac{53696}{243}\nu^3\right)\\
&+e_t^4\left(\frac{670405291}{415800}+\left[\frac{763187017}{136080}-\frac{29971}{1024}\pi^2\right]\nu+\frac{142865}{192}\nu^2-\frac{10816087}{7776}\nu^3\right)\\
&+e_t^6\left(\frac{1121282527}{1478400}+\left[\frac{1147175951}{1451520}-\frac{84501}{4096}\pi^2\right]\nu+\frac{100111945}{48384}\nu^2-\frac{983251}{648}\nu^3\right)\\
&+e_t^8\left(\frac{22052148101}{141926400}+\left[-\frac{32334863}{129024}-\frac{4059}{4096}\pi^2\right]\nu+\frac{80211601}{193536}\nu^2-\frac{4586539}{15552}\nu^3\right)\\
&+e_t^{10}\left(-\frac{8977637}{11354112}+\frac{9287}{48384}\nu+\frac{8977}{55296}\nu^2-\frac{567617}{124416}\nu^3\right)\\
&+\sqrt{1-e_t^2}\left[\left(-\frac{165761}{1008}+\frac{287}{192}\pi^2\right)\nu+e_t^2\left(-\frac{14935421}{6048}+\frac{52685}{4608}\pi^2\right)\nu\right.\\
&+e_t^4\left(-\frac{31082483}{8064}+\frac{41533}{6144}\pi^2\right)\nu+e_t^6\left(-\frac{40922933}{48384}+\frac{1517}{9216}\pi^2\right)\nu\\
&\left.+e_t^8\left(-\frac{1073}{288}\nu\right)\right]+\left(\frac{1712}{105}+\frac{14552}{63}e_t^2+\frac{553297}{1260}e_t^4+\frac{187357}{1260}e_t^6\right.\\
&\left.\left.+\frac{10593}{2240}e_t^8\right)\ln\left[\left(\frac{c^2 r_0}{Gm}x\right)\frac{1+\sqrt{1-e_t^2}}{2\left(1-e_t^2\right)}\right]\right\}.
\end{aligned}
\tag{2.52d}
$$

It should be noted that in the above there is no term at 2.5PN. The 2.5PN contribution is proportional to $\dot{r}$ and vanishes after averaging since it always includes only 'odd' terms. For ease of presentation we have ***not*** put a label on e_t to indicate that it represents the eccentricity in the modified harmonic coordinates e_t^{MHar}. Since x is gauge invariant, no such label is required on it. It is important to keep track of these facts when comparing formulas in different gauges, as we will eventually do.

The circular orbit limit of the above expression is obtained by setting $e_t = 0$. One obtains,

$$
\begin{aligned}
<\dot{\mathcal{E}}>|_{\odot} \;=\; & \frac{32}{5}x^5\nu^2\left\{1+x\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)+x^2\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)+\right.\\
& x^3\left(\frac{1266161801}{9979200}-\frac{1712}{105}\ln\left(\frac{Gm}{c^2xr_0}\right)+\left[-\frac{14930989}{272160}+\frac{41}{48}\pi^2-\frac{88}{3}\theta\right]\nu\right.\\
& \left.\left.-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right)\right\}. \qquad (2.53)
\end{aligned}
$$

The above expression is in exact agreement with Eq. (12.5) of [95] after converting the $y = Gm/c^2 r_{\rm SHar}$ there to the gauge invariant variable x. It should be kept in mind that this is ***only*** the instantaneous contribution and not the complete 3PN accurate energy flux for the circular orbit case.

2.7.2 The log terms in the energy flux

As mentioned before, this chapter is concerned ***only*** with the computation of the instantaneous terms in the energy flux. The complete 3PN energy flux ***also*** include important hereditary contributions composed of tails at orders 1.5PN and 2.5PN and tails-of-tails and tails-squared terms at order 3PN. These will be investigated and computed in the next chapter. Only after computing the ***complete*** energy flux can one discuss in detail the structure of the log terms in the energy flux, the cancellation of the $\log r_0$ terms and finally the circular orbit limit of this term as a check.

2.8 Orbital average of the energy flux in ADM coordinates

Starting from the expression for the instantaneous energy flux in ADM coordinates Eq. (2.35), employing the appropriate 3PN GQK representation from [155] and following the procedure outlined in the previous section, we can average the energy flux over an orbit in the ADM coordinates. The calculation proceeds exactly as before. The β_N's and γ_N's are exactly the same. The alphas, except α_{11}, are different as expected,

$$
\begin{aligned}
\alpha_3 \;=\; & \frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\left\{-\frac{1}{12}\left(1-e_t^2\right)^{7/2}\right.\\
& +x\left\{\left(1-e_t^2\right)^{5/2}\left(\frac{13}{21}+\frac{3}{14}e_t^2\right)+\frac{5}{252}\left(1-e_t^2\right)^{7/2}\nu\right\}\\
& +x^2\left\{\left(1-e_t^2\right)^{3/2}\left(\frac{163}{252}+\frac{107}{72}e_t^2-\frac{65}{252}e_t^4\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{293}{144}-\frac{313}{168}e_t^2+\frac{149}{1008}e_t^4\right)\nu\Bigg)\\
&+\left(1-e_t^2\right)^3\left(\frac{5}{12}-\frac{1}{6}\nu\right)+\left(1-e_t^2\right)^{7/2}\frac{5}{756}\nu^2\Bigg\}\\
&+x^3\Bigg\{\frac{5}{6}+\frac{35}{3}e_t^2-\frac{40}{3}e_t^4+\frac{5}{6}e_t^6\\
&+\left(-\frac{85}{27}+\frac{53}{108}e_t^2+\frac{187}{54}e_t^4-\frac{29}{36}e_t^6\right)\nu\\
&+\sqrt{1-e_t^2}\Bigg(-\frac{1447}{154}+\frac{3326}{693}e_t^2-\frac{5395}{2772}e_t^4+\frac{1}{66}e_t^6\\
&+\Bigg(-\frac{238345}{9504}+\frac{205}{192}\pi^2+\left(-\frac{699841}{22176}+\frac{205}{768}\pi^2\right)e_t^2\\
&+\frac{4681}{2464}e_t^4-\frac{26729}{66528}e_t^6\Bigg)\nu\\
&+\left(\frac{80347}{33264}+\frac{53377}{4752}e_t^2+\frac{1987}{1188}e_t^4-\frac{89}{11088}e_t^6\right)\nu^2\Bigg)\\
&+\left(1-e_t^2\right)^2\left(\frac{41}{1152}\pi^2\nu+\left(\frac{5}{18}+\frac{11}{36}e_t^2\right)\nu^2\right)\\
&-\left(1-e_t^2\right)^{7/2}\frac{5}{37422}\nu^3\Bigg\}\Bigg\}, \qquad (2.54a)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_4 \;=\;& \frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{\frac{1}{6}\left(1-e_t^2\right)^{7/2}\\
&+x\left\{\left(1-e_t^2\right)^{5/2}\left(-\frac{103}{42}+\frac{11}{14}e_t^2\right)-\frac{11}{126}\left(1-e_t^2\right)^{7/2}\nu\right\}\\
&+x^2\Bigg\{\left(1-e_t^2\right)^{3/2}\Bigg(\frac{617}{63}-\frac{1511}{126}e_t^2-\frac{391}{252}e_t^4\\
&+\left(\frac{7411}{1512}+\frac{671}{189}e_t^2-\frac{1439}{1512}e_t^4\right)\nu\Bigg)\\
&+\left(1-e_t^2\right)^3\left(-\frac{95}{24}+\frac{19}{12}\nu\right)+\frac{41}{216}\left(1-e_t^2\right)^{7/2}\nu^2\Bigg\}\\
&+x^3\Bigg\{\frac{1025}{42}-\frac{6955}{84}e_t^2+\frac{1415}{21}e_t^4-\frac{755}{84}e_t^6\\
&+\left(\frac{27337}{1512}-\frac{3625}{189}e_t^2-\frac{8891}{1512}e_t^4+\frac{1759}{252}e_t^6\right)\nu\\
&+\sqrt{1-e_t^2}\Bigg(\frac{4946}{385}+\frac{93113}{3465}e_t^2-\frac{188261}{6930}e_t^4+\frac{1741}{3465}e_t^6\\
&+\left(-\frac{7204}{891}-\frac{125591}{5544}e_t^2+\frac{17579}{16632}e_t^4-\frac{47059}{49896}e_t^6\right)\nu^2\\
&+\nu\Bigg(\frac{4447}{154}+\frac{65803}{33264}e_t^6-\frac{491}{192}\pi^2
\end{aligned}
$$

$$
\begin{aligned}
& +e_t^2\left(\frac{1904191}{33264}-\frac{41}{384}\pi^2\right)+e_t^4\left(\frac{368867}{16632}-\frac{1}{192}\pi^2\right)\Bigg)\Bigg)\\
& +\left(1-e_t^2\right)^2\left(\left(-\frac{194}{63}-\frac{1241}{504}e_t^2\right)\nu^2-\frac{779}{2304}\nu\pi^2\right)\\
& +\frac{46769}{299376}\left(1-e_t^2\right)^{7/2}\nu^3\Bigg\}\Bigg\}, \qquad (2.54\text{b})
\end{aligned}
$$

$$
\begin{aligned}
\alpha_5 = \; & \frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\left\{\frac{11}{12}\left(1-e_t^2\right)^{9/2}\right.\\
& +x\left\{\left(1-e_t^2\right)^{7/2}\left(-\frac{199}{28}-\frac{19}{12}e_t^2+\left(\frac{317}{252}+\frac{163}{252}e_t^2\right)\nu\right)\right\}\\
& +x^2\left\{\left(1-e_t^2\right)^2\left(-\frac{40}{3}+\frac{215}{12}e_t^2-\frac{55}{12}e_t^4\right.\right.\\
& \left.+\left(\frac{16}{3}-\frac{43}{6}e_t^2+\frac{11}{6}e_t^4\right)\nu\right)\\
& +\left(1-e_t^2\right)^{5/2}\left(\frac{767}{2268}-\frac{78727}{4536}e_t^2+\frac{169}{84}e_t^4\right.\\
& \left.+\left(\frac{14863}{3024}+\frac{10981}{1512}e_t^2-\frac{161}{144}e_t^4\right)\nu\right)\\
& \left.+\left(1-e_t^2\right)^{7/2}\left(\frac{1747}{6048}-\frac{5}{27}e_t^2\right)\nu^2\right\}\\
& +x^3\left\{\left(1-e_t^2\right)\left(\frac{535}{84}-\frac{2245}{21}e_t^2+\frac{9215}{84}e_t^4-\frac{55}{6}e_t^6\right.\right.\\
& +\left(-\frac{37}{126}-\frac{631}{42}e_t^2+\frac{523}{28}e_t^4-\frac{121}{36}e_t^6\right)\nu^2\\
& +\nu\left(\frac{49925}{756}+\frac{319}{36}e_t^6-\frac{41}{36}\pi^2\right.\\
& \left.\left.+e_t^4\left(-\frac{20681}{378}-\frac{451}{1152}\pi^2\right)+e_t^2\left(-\frac{7631}{378}+\frac{1763}{1152}\pi^2\right)\right)\right)\\
& +\left(1-e_t^2\right)^{3/2}\left(\frac{1154563}{56700}+\frac{1681072}{51975}e_t^2\right.\\
& -\frac{10439041}{311850}e_t^4-\frac{571}{462}e_t^6+\left(\frac{1144315}{99792}\right.\\
& \left.-\frac{4780907}{99792}e_t^2-\frac{31915}{7128}e_t^4+\frac{22357}{33264}e_t^6\right)\nu^2\\
& +\nu\left(\frac{323491919}{1197504}+\frac{773}{672}e_t^6-\frac{245}{24}\pi^2\right.\\
& \left.\left.+e_t^4\left(-\frac{15879091}{1197504}-\frac{41}{64}\pi^2\right)+e_t^2\left(\frac{133906099}{598752}-\frac{383}{768}\pi^2\right)\right)\right)\\
& \left.\left.+\left(1-e_t^2\right)^{7/2}\left(-\frac{214}{105}\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right)+\left(\frac{52739}{1197504}+\frac{8377}{74844}e_t^2\right)\nu^3\right)\right\}\right\}, \qquad (2.54\text{c})
\end{aligned}
$$

$$
\begin{aligned}
\alpha_6 \;=\;& \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{\\
& x\left(1-e_t^2\right)^{9/2}\left(-\frac{19}{21}+\frac{10}{7}\nu\right)\\
& +x^2\Bigg\{\left(1-e_t^2\right)^{5/2}\left(\frac{3527}{6048}+\frac{2519}{1512}e_t^2-\frac{13603}{6048}e_t^4\right)\nu^2\\
& +\left(\frac{275}{8}-\frac{275}{8}e_t^2\right)\left(1-e_t^2\right)^3\\
& +\left(1-e_t^2\right)^{7/2}\Bigg(\frac{55309}{1134}-\frac{811}{252}e_t^2\\
& +\left(-\frac{10891}{504}-\frac{12269}{1512}e_t^2\right)\nu\Bigg)-\frac{55}{4}\left(1-e_t^2\right)^4\nu\Bigg\}\\
& +x^3\Bigg\{\left(1-e_t^2\right)^2\Bigg(-\frac{9935}{56}+180e_t^2-\frac{145}{56}e_t^4\\
& +\left(\frac{355}{21}-\frac{955}{168}e_t^2-\frac{1885}{168}e_t^4\right)\nu^2\Bigg)\\
& +\left(1-e_t^2\right)^{5/2}\Bigg(-\frac{39979133}{138600}+\frac{2894909}{44550}e_t^2-\frac{271079}{27720}e_t^4\\
& +\left(\frac{46421}{4158}+\frac{1811035}{49896}e_t^2+\frac{188885}{49896}e_t^4\right)\nu^2\\
& +\left(-\frac{119953}{44352}+\frac{904027}{199584}e_t^2-\frac{728477}{399168}e_t^4\right)\nu^3\\
& +\nu\Bigg(-\frac{25170443}{108864}+\frac{1405303}{33264}e_t^4+\frac{8701}{768}\pi^2\\
& +e_t^2\left(-\frac{20144681}{171072}-\frac{3341}{768}\pi^2\right)\Bigg)\Bigg)\\
& +\left(1-e_t^2\right)^3\nu\left(-\frac{73699}{504}-\frac{211}{42}e_t^2+\frac{2255}{768}\pi^2\right)\\
& +\frac{3424}{315}\left(1-e_t^2\right)^{7/2}\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right)\Bigg\}\Bigg\},
\end{aligned}
\tag{2.54d}
$$

$$
\begin{aligned}
\alpha_7 \;=\;& \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\Bigg\{\\
& x\left(1-e_t^2\right)^{11/2}\left(\frac{687}{112}-\frac{155}{28}\nu\right)\\
& +x^2\Bigg\{\left(1-e_t^2\right)^{7/2}\Bigg(-\frac{408271}{3024}+\frac{399577}{3024}e_t^2+\frac{23}{8}e_t^4\\
& +\left(\frac{15467}{216}-\frac{31513}{756}e_t^2-\frac{5027}{168}e_t^4\right)\nu
\end{aligned}
$$

$$
\begin{aligned}
& +\left(-\frac{49099}{6048}+\frac{19147}{6048}e_t^2+\frac{104}{21}e_t^4\right)\nu^2\Big)\Big\} \\
& +x^3\Big\{\left(1-e_t^2\right)^3\left(-\frac{12585}{56}+\frac{12585}{56}e_t^2\right) \\
& +\left(1-e_t^2\right)^{7/2}\Big(\frac{420179819}{277200}-\frac{93593867}{138600}e_t^2+\frac{25253}{1232}e_t^4 \\
& -\left(-\frac{9844}{315}+\frac{4708}{105}e_t^2\right)\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right) \\
& +\left(-\frac{13102921}{199584}+\frac{70247}{2592}e_t^2+\frac{30019}{1008}e_t^4\right)\nu^2 \\
& +\left(-\frac{49187}{22176}-\frac{1409411}{199584}e_t^2+\frac{1843}{1386}e_t^4\right)\nu^3 \\
& +\nu\Big(\frac{112873811}{399168}-\frac{74927}{672}e_t^4-\frac{8425}{384}\pi^2 \\
& +e_t^2\left(-\frac{138547181}{399168}+\frac{533}{32}\pi^2\right)\Big)\Big) \\
& +\left(1-e_t^2\right)^4\left(\frac{1281}{4}\nu-\frac{645}{7}\nu^2\right)\Big\}\Big\}, \qquad (2.54e)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_8 \;=\; & \frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\Big\{x^2\Big\{\left(1-e_t^2\right)^{11/2}\left(\frac{12113}{252}+\frac{11737}{1512}\nu-\frac{36371}{6048}\nu^2\right)\Big\} \\
& +x^3\Big\{\left(1-e_t^2\right)^{9/2}\Big(-\frac{18534196}{10395}+\frac{5022439}{27720}e_t^2 \\
& +\left(\frac{57056845}{199584}+\frac{2206805}{199584}e_t^2\right)\nu^2 \\
& +\left(-\frac{1695697}{66528}+\frac{1914239}{199584}e_t^2\right)\nu^3 \\
& +\nu\left(-\frac{68439443}{57024}-\frac{707185}{3024}e_t^2+\frac{11123}{256}\pi^2\right) \\
& -\frac{856}{9}\ln\left(\frac{c^2\,r_0}{G\,m}\,x\right)\Big) \\
& +\left(1-e_t^2\right)^5\left(\frac{10305}{32}-\frac{6711}{16}\nu+\frac{465}{4}\nu^2\right)\Big\}\Big\}, \qquad (2.54f)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_9 \;=\; & \frac{32}{5}\frac{\nu^2x^5}{\left(1-e_t^2\right)^{7/2}}\Big\{ \\
& x^2\left(1-e_t^2\right)^{13/2}\left(\frac{2501}{504}-\frac{10117}{252}\nu+\frac{2101}{126}\nu^2\right) \\
& +x^3\left(1-e_t^2\right)^{11/2}\Big(\frac{52101107}{66528}-\frac{305995}{5544}e_t^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{3865945}{8064}-\frac{303437}{2079}e_t^2\right)\nu^2\\
&+\left(\frac{10892993}{133056}+\frac{12107}{4158}e_t^2\right)\nu^3\\
&+\nu\left(\frac{10483967}{9504}+\frac{563993}{2079}e_t^2-\frac{697}{32}\pi^2\right)\\
&+\frac{214}{3}\ln\left(\frac{c^2 r_0}{Gm}x\right)\bigg)\bigg\}, \qquad (2.54g)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{10} &= \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\left\{x^3\left(1-e_t^2\right)^{13/2}\left(-\frac{6094651}{55440}+\frac{254537}{3168}\nu\right.\right.\\
&\left.\left.+\frac{4356379}{88704}\nu^2-\frac{1349011}{44352}\nu^3\right)\right\}, \qquad (2.54h)\\
\alpha_{11} &= \frac{32}{5}\frac{\nu^2 x^5}{\left(1-e_t^2\right)^{7/2}}\left\{x^3\left(1-e_t^2\right)^{15/2}\left(\frac{1507925}{44352}-\frac{20365}{126}\nu\right.\right.\\
&\left.\left.+\frac{687305}{5544}\nu^2-\frac{32755}{1386}\nu^3\right)\right\}. \qquad (2.54i)
\end{aligned}
$$

The final result turns out to be:

$$
\begin{aligned}
<\dot{\mathcal{E}}>_{\rm ADM} &= \frac{32\nu^2x^5}{5}\frac{1}{\left(1-e_t^2\right)^{7/2}}\left(<\dot{\mathcal{E}}_{\rm N}>_{\rm ADM}+x<\dot{\mathcal{E}}_{\rm 1PN}>_{\rm ADM}\right.\\
&\left.+x^2<\dot{\mathcal{E}}_{\rm 2PN}>_{\rm ADM}+x^3<\dot{\mathcal{E}}_{\rm 3PN}>_{\rm ADM}\right). \qquad (2.55)
\end{aligned}
$$

$$
\begin{aligned}
<\dot{\mathcal{E}}_N>_{\rm ADM} &= 1+e_t^2\frac{73}{24}+e_t^4\frac{37}{96} \qquad (2.56a)\\
<\dot{\mathcal{E}}_{\rm 1PN}>_{\rm ADM} &= \frac{1}{\left(1-e_t^2\right)}\left\{\left(\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)+e_t^2\left(\frac{10475}{672}-\frac{1081}{36}\nu\right)\right.\right.\\
&\left.\left.+e_t^4\left(\frac{10043}{384}-\frac{311}{12}\nu\right)+e_t^6\left(\frac{2179}{1792}-\frac{851}{576}\nu\right)\right)\right\} \qquad (2.56b)\\
<\dot{\mathcal{E}}_{\rm 2PN}>_{\rm ADM} &= \frac{1}{\left(1-e_t^2\right)^2}\\
&\left\{-\frac{203471}{9072}+\frac{12799}{504}\nu+\frac{65}{18}\nu^2+e_t^2\left(-\frac{3866543}{18144}+\frac{4691}{2016}\nu+\frac{5935}{54}\nu^2\right)\right.\\
&+e_t^4\left(-\frac{369751}{24192}-\frac{3039083}{8064}\nu+\frac{247805}{864}\nu^2\right)\\
&+e_t^6\left(\frac{1302443}{16128}-\frac{215077}{1344}\nu+\frac{185305}{1728}\nu^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+e_t^8\left(\frac{86567}{64512}-\frac{9769}{4608}\nu+\frac{21275}{6912}\nu^2\right)\\
&+\sqrt{1-e_t^2}\left[\frac{35}{2}-7\nu+e_t^2\left(\frac{6425}{48}-\frac{1285}{24}\nu\right)\right.\\
&\left.\left.+e_t^4\left(\frac{5065}{64}-\frac{1013}{32}\nu\right)+e_t^6\left(\frac{185}{96}-\frac{37}{48}\nu\right)\right]\right\}
\end{aligned}
\tag{2.56c}
$$

$$
\begin{aligned}
<\dot{\mathcal{E}}_{\rm 3PN}>_{\rm ADM} \;=\;& \frac{1}{\left(1-e_t^2\right)^3}\\
&\left\{\frac{1266161801}{9979200}+\left[\frac{8009293}{54432}-\frac{41}{64}\pi^2\right]\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right.\\
&+e_t^2\left(\frac{27685797767}{19958400}+\left[\frac{250838045}{108864}+\frac{31255}{1536}\pi^2\right]\nu+\frac{133487}{6048}\nu^2-\frac{53696}{243}\nu^3\right)\\
&+e_t^4\left(\frac{5135886353}{3326400}+\left[\frac{479870915}{108864}-\frac{7459}{1024}\pi^2\right]\nu+\frac{1305967}{576}\nu^2-\frac{10816087}{7776}\nu^3\right)\\
&+e_t^6\left(\frac{352339259}{492800}+\left[-\frac{4938799}{145152}-\frac{78285}{4096}\pi^2\right]\nu+\frac{34228207}{12096}\nu^2-\frac{983251}{648}\nu^3\right)\\
&+e_t^8\left(\frac{21840664301}{141926400}+\left[-\frac{36513893}{129024}-\frac{4059}{4096}\pi^2\right]\nu+\frac{86104369}{193536}\nu^2-\frac{4586539}{15552}\nu^3\right)\\
&+e_t^{10}\left(-\frac{8977637}{11354112}+\frac{9287}{48384}\nu+\frac{8977}{55296}\nu^2-\frac{567617}{124416}\nu^3\right)\\
&+\sqrt{1-e_t^2}\left[\left(-\frac{165761}{1008}+\frac{287}{192}\pi^2\right)\nu+e_t^2\left(-\frac{14935421}{6048}+\frac{52685}{4608}\pi^2\right)\nu\right.\\
&+e_t^4\left(-\frac{31082483}{8064}+\frac{41533}{6144}\pi^2\right)\nu+e_t^6\left(-\frac{40922933}{48384}+\frac{1517}{9216}\pi^2\right)\nu\\
&\left.+e_t^8\left(-\frac{1073}{288}\nu\right)\right]+\left(\frac{1712}{105}+\frac{14552}{63}e_t^2+\frac{553297}{1260}e_t^4+\frac{187357}{1260}e_t^6\right.\\
&\left.\left.+\frac{10593}{2240}e_t^8\right)\ln\left[\left(\frac{c^2\,r_0}{Gm}x\right)\frac{1+\sqrt{1-e_t^2}}{2\left(1-e_t^2\right)}\right]\right\}
\end{aligned}
\tag{2.56d}
$$

As before there is no 2.5PN term in the energy flux after averaging. The circular orbit limit as expected is in agreement with Eq. (2.53). The contributions at Newtonian and 1PN orders have the same form in the modified harmonic coordinates and ADM coordinates consistent with the fact that the two coordinates differ starting only at 2PN. The e_t in the above expression now represents $e_t^{\rm ADM}$, the time eccentricity in ADM coordinates.

A useful internal consistency check of the algebraic correctness of different representations of the energy flux, the coordinate transformations linking the various gauges, and the work of [155] on the construction of the 3PN generalised quasi-Keplerian representation is the verification of the equality of Eqs. (2.52) and (2.56) using the following transformation between the time eccenticities $e_t^{\rm MHar}$ and $e_t^{\rm ADM}$:

$$
\begin{aligned}
e_t^{\rm MHar} &= e_t^{\rm ADM}\left\{1-\frac{x^2}{\left(1-e_t^2\right)}\left(\frac{1}{4}+\frac{17}{4}\nu\right)-\right.\\
&\quad \frac{x^3}{\left(1-e_t^2\right)^2}\left(\frac{1}{2}+\frac{1}{2}e_t^2+\left(\frac{16739}{1680}-\frac{21\pi^2}{16}+\frac{249}{16}e_t^2\right)\nu\right.\\
&\quad \left.\left.-\left(\frac{83}{24}+\frac{241}{24}e_t^2\right)\nu^2\right)\right\}
\end{aligned}
\tag{2.57}
$$

This relation derives from using Eqs. (20d) and (25d) and rewriting the E and h^2 dependence in terms of x and e,. There is no ambiguity in not having a label on the e_t in the 2PN and 3PN terms above.

2.9 Test particle limit

An important check on the results of our computation is in the test particle limit where results for the energy flux in the eccentric orbit case is available (to second order in eccentricity e^2) from computations using an entirely different method based on black hole perturbation theory in a Schwarzschild background. To compare the result of our MPM computation with the result obtained in the Ref. [162], we take the test particle limit ($\nu \to 0$) of our results and expand it in powers of e_t retaining only terms up to e_t^2. Implementing this procedure, the instantaneous contribution to the energy flux in the test mass limit is given by

$$
\begin{aligned}
\dot{\mathcal{E}}^{\rm Inst.}_{\nu\to 0} &= \frac{32}{5}\nu^2 x^5\\
&\quad \left\{1-\frac{1247}{336}x-\frac{44711}{9072}x^2+x^3\left(\frac{1266161801}{9979200}+\frac{1712}{105}\ln\left[\frac{c^2 r_0}{Gm}x\right]\right)+\right.\\
&\quad \left. e_t^2\left(\frac{157}{24}-\frac{187}{168}x-\frac{84547}{756}x^2+x^3\left(\frac{22718275589}{9979200}+\frac{1712}{105}\ln\left[\frac{c^2 r_0}{Gm}x\right]\right)\right)\right\}\\
&\quad +O(e_t^4).
\end{aligned}
\tag{2.58}
$$

In the next chapter, the hereditary contributions will also be computed using MPM in the test mass limit. Together with the instantaneous term computed above it will yield the total energy flux in the test mass limit of our MPM computations that can then be compared to the result in Ref. [162] obtained by perturbation methods in the Schwarzschild background.

2.10 Gauge invariant expression for the energy flux: instantaneous terms

In the previous sections the energy flux was represented using x a gauge invariant variable and e_t which however is coordinate dependent. The variable e_t is useful in extracting the circular limit for which it has value zero. In this section we explore the possibility of rewriting the flux in terms of two gauge invariant observables defined earlier: x and k'. This can be achieved either by starting from the average energy flux in terms of variables x and e, and rewriting e_t in terms of x and k' or alternatively by working from the beginning with the expression for the flux in terms of x and k'. We have checked that they lead to the same results. The computation can be done independently both in the modified harmonic coordinate and in ADM coordinates. The end result is identical proving the gauge invariance of the energy flux and providing a gauge invariant expression of the energy flux. We have

$$\begin{aligned} <\dot{\mathcal{E}}> \;=\; & \frac{32\nu^2x^5}{5}\left(\frac{x}{k'}\right)^{-13/2}\Big(<\dot{\mathcal{E}}_{\rm N}> +x<\dot{\mathcal{E}}_{\rm 1PN}> +x^2<\dot{\mathcal{E}}_{\rm 2PN}> \\ & +x^3<\dot{\mathcal{E}}_{\rm 3PN}>\Big). \end{aligned} \tag{2.59}$$

$$<\dot{\mathcal{E}}_N> \;=\; \left(\frac{x}{k'}\right)^3\frac{425}{96}+\left(\frac{x}{k'}\right)^4\left(-\frac{61}{16}\right)+\left(\frac{x}{k'}\right)^5\frac{37}{96}, \tag{2.60a}$$

$$\begin{aligned} <\dot{\mathcal{E}}_{\rm 1PN}> \;=\; & \left\{\left(\frac{x}{k'}\right)^2\left(-\frac{289}{3}+\frac{3605}{384}\nu\right)+\left(\frac{x}{k'}\right)^3\left(\frac{1865}{24}+\frac{3775}{384}\nu\right)\right. \\ & \left.+\left(\frac{x}{k'}\right)^4\left(-\frac{5297}{336}-\frac{2725}{384}\nu\right)+\left(\frac{x}{k'}\right)^5\left(\frac{139}{112}+\frac{259}{1152}\nu\right)\right\}, \end{aligned} \tag{2.60b}$$

$$\begin{aligned} <\dot{\mathcal{E}}_{\rm 2PN}> \;=\; & \left\{\frac{x}{k'}\left(\frac{267725837}{258048}+\left[\frac{1440583}{2304}-\frac{609875}{24576}\pi^2\right]\nu+\frac{24395}{1024}\nu^2\right)\right. \\ & +\left(\frac{x}{k'}\right)^2\left(-\frac{51894953}{82944}+\left[-\frac{583921}{512}+\frac{497125}{24576}\pi^2\right]\nu+\frac{1625}{48}\nu^2\right) \\ & +\left(\frac{x}{k'}\right)^3\left(\frac{49183667}{387072}+\left[\frac{14718145}{32256}-\frac{32595}{8192}\pi^2\right]\nu+\frac{37145}{4608}\nu^2\right) \\ & +\left(\frac{x}{k'}\right)^{7/2}\left(-\frac{305}{16}+\frac{61}{8}\nu\right)+\left(\frac{x}{k'}\right)^4\left(-\frac{2145781}{64512}\right. \\ & \left.+\left[-\frac{505639}{10752}+\frac{1517}{8192}\pi^2\right]\nu-\frac{105}{16}\nu^2\right) \\ & \left.+\left(\frac{x}{k'}\right)^{9/2}\left(\frac{185}{48}-\frac{37}{24}\nu\right)+\left(\frac{x}{k'}\right)^5\left(\frac{744545}{258048}+\frac{19073}{32256}\nu+\frac{2849}{27648}\nu^2\right)\right\} \end{aligned} \tag{2.60c}$$

$$
\begin{aligned}
<\dot{\mathcal{E}}_{3\mathrm{PN}}> \;=\; & \left\{\frac{149899221067}{7741440}+\left[-\frac{186642211961}{3096576}+\frac{46739713}{32768}\pi^2\right]\nu\right. \\
& +\left[\frac{66297815}{6144}-\frac{8315825}{32768}\pi^2\right]\nu^2-\frac{415625}{12288}\nu^3+\left(\frac{x}{k'}\right)^{1/2}\left(-\frac{161249}{192}\right) \\
& +\left(\frac{x}{k'}\right)\left(-\frac{66998702987}{2073600}+\left[\frac{71509958869}{1032192}-\frac{117241181}{98304}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{24611099}{2304}+\frac{6633185}{49152}\pi^2\right]\nu^2+\frac{4346075}{12288}\nu^3\right)+\left(\frac{x}{k'}\right)^{3/2}\frac{3727559}{2880} \\
& +\left(\frac{x}{k'}\right)^2\left(\frac{4774135897}{322560}+\left[-\frac{330028752779}{13934592}+\frac{29862965}{114688}\pi^2\right]\nu\right. \\
& \left.+\left[\frac{15103071}{7168}-\frac{3258475}{294912}\pi^2\right]\nu^2-\frac{2249695}{18432}\nu^3\right) \\
& +\left(\frac{x}{k'}\right)^{5/2}\left(-\frac{928043}{5760}+\left[-\frac{1879}{1152}-\frac{2501}{1536}\pi^2\right]\nu-\frac{5605}{192}\nu^2\right) \\
& +\left(\frac{x}{k'}\right)^3\left(-\frac{2740721737}{1290240}+\left[\frac{74294239}{24576}-\frac{8351167}{688128}\pi^2\right]\nu\right. \\
& \left.+\left[\frac{6154165}{64512}-\frac{615}{1024}\pi^2\right]\nu^2+\frac{298895}{6144}\nu^3\right) \\
& +\left(\frac{x}{k'}\right)^{7/2}\left(-\frac{3913177}{37800}+\left[-\frac{351499}{12096}+\frac{1517}{4608}\pi^2\right]\nu+\frac{1153}{32}\nu^2\right) \\
& +\left(\frac{x}{k'}\right)^4\left(\frac{1758850201}{141926400}+\left[-\frac{185390651}{1032192}+\frac{68757}{229376}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{5900711}{387072}-\frac{1517}{16384}\pi^2\right]\nu^2-\frac{2568655}{331776}\nu^3\right) \\
& +\left(\frac{x}{k'}\right)^{9/2}\left(\frac{51335}{2688}-\frac{10951}{2688}\nu-\frac{481}{192}\nu^2\right) \\
& +\left(\frac{x}{k'}\right)^5\left(\frac{2635805}{405504}+\frac{891535}{3096576}\nu+\frac{4537}{27648}\nu^2+\frac{106375}{995328}\nu^3\right) \\
& +\left(\frac{161249}{192}-\frac{125939}{80}\left(\frac{x}{k'}\right)+\frac{263113}{288}\left(\frac{x}{k'}\right)^2\right. \\
& \left.\left.-\frac{168953}{1008}\left(\frac{x}{k'}\right)^3+\frac{10593}{2240}\left(\frac{x}{k'}\right)^4\right)\log\left[\left(\frac{c^2 r_0}{Gm}x\right)\frac{1+\sqrt{(x/k')}}{2(x/k')}\right]\right\}. \qquad (2.60\text{d})
\end{aligned}
$$

The circular orbit limit of the above gauge invariant representation agrees once again with that in [95]. Unlike the (x, e_t) representation where the circular orbit limit is explicit, the gauge invariant representation requires a little more algebra and care to recover the circular orbit limit. More precisely, it requires the following limit of x/k' in the circular orbit case:

$$\left(\frac{x}{k'}\right)_\odot = 1 + x\left(-\frac{9}{2} + \frac{7}{3}\nu\right) + x^2\left(-\frac{9}{4} + \left[\frac{397}{12} - \frac{41}{32}\pi^2\right]\nu + \frac{28}{9}\nu^2\right)$$
$$+x^3\left(\frac{11547}{16} + \left[-\frac{354967}{288} + \frac{41779}{1536}\pi^2\right]\nu + \left[\frac{25025}{144} - \frac{7093}{1536}\pi^2\right]\nu^2\right.$$
$$\left.+\frac{1223}{324}\nu^3\right). \tag{2.61}$$

Though the combination of x and k' used above is one choice of gauge invariant variables, it may be interesting to use the combination $x_{\rm MGS} = \xi = \left(\frac{Gmn}{c^3}\right)^{2/3}$ and k' since these variables are related to the two basic periodicities in the problem: the radial period n and the periastron precession period k. In a numerical computation one can well anticipate the convenience of such a choice. In their terms the final results are given by:

$$<\dot{\mathcal{E}}> = \frac{32\nu^2\xi^5}{5}\left(\frac{\xi}{k'}\right)^{-13/2}\left(<\dot{\mathcal{E}}_{\rm N}> + \xi<\dot{\mathcal{E}}_{\rm 1PN}> + \xi^2<\dot{\mathcal{E}}_{\rm 2PN}>\right.$$
$$\left.+\xi^3<\dot{\mathcal{E}}_{\rm 3PN}>\right). \tag{2.62}$$

$$<\dot{\mathcal{E}}_N> = \left(\frac{\xi}{k'}\right)^3\frac{425}{96} + \left(\frac{\xi}{k'}\right)^4\left(-\frac{61}{16}\right) + \left(\frac{\xi}{k'}\right)^5\frac{37}{96}, \tag{2.63a}$$

$$<\dot{\mathcal{E}}_{\rm 1PN}> = \left(\frac{\xi}{k'}\right)^2\left(-\frac{7973}{96} + \frac{3605}{384}\nu\right) + \left(\frac{\xi}{k'}\right)^3\left(\frac{2815}{48} + \frac{3775}{384}\nu\right)$$
$$+\left(\frac{\xi}{k'}\right)^4\left(-\frac{2927}{224} - \frac{2725}{384}\nu\right) + \left(\frac{\xi}{k'}\right)^5\left(\frac{139}{112} + \frac{259}{1152}\nu\right), \tag{2.63b}$$

$$<\dot{\mathcal{E}}_{\rm 2PN}> = \frac{\xi}{k'}\left(\frac{193149965}{258048} + \left[\frac{1505473}{2304} - \frac{609875}{24576}\pi^2\right]\nu + \frac{24395}{1024}\nu^2\right)$$
$$+\left(\frac{\xi}{k'}\right)^2\left(-\frac{21248873}{82944} + \left[-\frac{1676263}{1536} + \frac{497125}{24576}\pi^2\right]\nu + \frac{1625}{48}\nu^2\right)$$
$$+\left(\frac{\xi}{k'}\right)^3\left(\frac{8557235}{387072} + \left[\frac{13115845}{32256} - \frac{32595}{8192}\pi^2\right]\nu + \frac{37145}{4608}\nu^2\right)$$
$$+\left(\frac{\xi}{k'}\right)^{7/2}\left(-\frac{305}{16} + \frac{61}{8}\nu\right) + \left(\frac{\xi}{k'}\right)^4\left(-\frac{1425205}{64512} + \left[-\frac{483883}{10752}\right.\right.$$
$$\left.\left.+\frac{1517}{8192}\pi^2\right]\nu - \frac{105}{16}\nu^2\right) + \left(\frac{\xi}{k'}\right)^{9/2}\left(\frac{185}{48} - \frac{37}{24}\nu\right)$$
$$+\left(\frac{\xi}{k'}\right)^5\left(\frac{744545}{258048} + \frac{19073}{32256}\nu + \frac{2849}{27648}\nu^2\right), \tag{2.63c}$$

$$<\dot{\mathcal{E}}_{\rm 3PN}> = \frac{24856363771}{1105920} + \left[-\frac{180833781305}{3096576} + \frac{44300213}{32768}\pi^2\right]\nu$$

$$
\begin{aligned}
&+\left[\frac{66736925}{6144}-\frac{8315825}{32768}\pi^2\right]\nu^2-\frac{415625}{12288}\nu^3+\left(\frac{\xi}{k'}\right)^{1/2}\left(-\frac{161249}{192}\right)\\
&+\left(\frac{\xi}{k'}\right)\left(-\frac{56069997}{1600}+\left[\frac{65674771189}{1032192}-\frac{35766227}{32768}\pi^2\right]\nu\right.\\
&\left.+\left[-\frac{24221099}{2304}+\frac{6633185}{49152}\pi^2\right]\nu^2+\frac{4346075}{12288}\nu^3\right)+\left(\frac{\xi}{k'}\right)^{3/2}\frac{3727559}{2880}\\
&+\left(\frac{\xi}{k'}\right)^2\left(\frac{29942574607}{1935360}+\left[-\frac{286905469499}{13934592}+\frac{26668655}{114688}\pi^2\right]\nu\right.\\
&\left.+\left[\frac{139567849}{64512}-\frac{3258475}{294912}\pi^2\right]\nu^2-\frac{2249695}{18432}\nu^3\right)\\
&+\left(\frac{\xi}{k'}\right)^{5/2}\left(-\frac{1806443}{5760}+\left[\frac{68393}{1152}-\frac{2501}{1536}\pi^2\right]\nu-\frac{5605}{192}\nu^2\right)\\
&+\left(\frac{\xi}{k'}\right)^3\left(-\frac{3083727757}{1290240}+\left[\frac{448291945}{172032}-\frac{7204315}{688128}\pi^2\right]\nu\right.\\
&\left.+\left[\frac{2343925}{64512}-\frac{615}{1024}\pi^2\right]\nu^2+\frac{298895}{6144}\nu^3\right)\\
&+\left(\frac{\xi}{k'}\right)^{7/2}\left(-\frac{1228151}{18900}+\left[-\frac{537979}{12096}+\frac{1517}{4608}\pi^2\right]\nu+\frac{1153}{32}\nu^2\right)\\
&+\left(\frac{\xi}{k'}\right)^4\left(\frac{6263347451}{141926400}+\left[-\frac{19852995}{114688}+\frac{68757}{229376}\pi^2\right]\nu\right.\\
&\left.+\left[-\frac{202295}{14336}-\frac{1517}{16384}\pi^2\right]\nu^2-\frac{2568655}{331776}\nu^3\right)\\
&+\left(\frac{\xi}{k'}\right)^{9/2}\left(\frac{51335}{2688}-\frac{10951}{2688}\nu-\frac{481}{192}\nu^2\right)\\
&+\left(\frac{\xi}{k'}\right)^5\left(\frac{2635805}{405504}+\frac{891535}{3096576}\nu+\frac{4537}{27648}\nu^2+\frac{106375}{995328}\nu^3\right)\\
&+\left(\frac{161249}{192}-\frac{125939}{80}\left(\frac{\xi}{k'}\right)+\frac{263113}{288}\left(\frac{\xi}{k'}\right)^2\right.\\
&\left.-\frac{168953}{1008}\left(\frac{\xi}{k'}\right)^3+\frac{10593}{2240}\left(\frac{\xi}{k'}\right)^4\right)\log\left[\left(\frac{c^2\,r_0}{G\,m}\,\xi\right)\frac{1+\sqrt{(\xi/k')}}{2(\xi/k')}\right]. \qquad (2.63d)
\end{aligned}
$$

The circular orbit limit of this form can be derived using the limit of ξ/k' in this limit given by:

$$
\begin{aligned}
\left(\frac{\xi}{k'}\right)_{\odot} &= 1+\xi\left(-\frac{13}{2}+\frac{7}{3}\nu\right)+\xi^2\left(-\frac{41}{4}+\left[\frac{151}{4}-\frac{41}{32}\pi^2\right]\nu+\frac{28}{9}\nu^2\right)\\
&+\xi^3\left(\frac{31865}{48}+\left[-\frac{303415}{288}+\frac{33907}{1536}\pi^2\right]\nu+\left[\frac{25249}{144}-\frac{7093}{1536}\pi^2\right]\nu^2\right.\\
&\left.+\frac{1223}{324}\nu^3\right). \qquad (2.64)
\end{aligned}
$$

Once again the results are consistent with the results of [95].

In QK-orbit the expression of ξ/k' in ADM coordinate is given by

$$
\begin{aligned}
\xi_{\text{ADM}} = {} & \left(1-e_t^2\right) + \frac{\xi^2}{\left(1-e_t^2\right)}\left\{-\frac{13}{2}+\frac{7}{3}\nu \right. \\
& \left. + e_t^2\left(\frac{9}{4}-\frac{1}{6}\nu\right) + e_t^4\left(\frac{17}{4}-\frac{13}{6}\nu\right)\right\} \\
& + \frac{\xi^2}{\left(1-e_t^2\right)^2}\left\{-\frac{21}{4}+\left[\frac{143}{4}-\frac{41}{32}\pi^2\right]\nu+\frac{28}{9}\nu^2 \right. \\
& + e_t^2\left(-\frac{49}{4}+\left[-\frac{11}{4}+\frac{123}{128}\pi^2\right]\nu-\frac{19}{3}\nu^2\right) \\
& + e_t^4\left(\frac{465}{16}+\left[-\frac{281}{6}+\frac{41}{128}\pi^2\right]\nu+\frac{125}{24}\nu^2\right) \\
& + e_t^6\left(-\frac{185}{16}+\frac{83}{6}\nu-\frac{143}{72}\nu^2\right) \\
& \left. + \sqrt{1-e_t^2}\left[-5+e_t^4(10-4\nu)+2\nu+e_t^2(-5+2\nu)\right]\right\} \\
& + \frac{\xi^3}{\left(1-e_t^2\right)^3}\left\{\frac{33305}{48}+\left[-\frac{105533}{96}+\frac{11521}{512}\pi^2\right]\nu \right. \\
& + \left[\frac{25729}{144}-\frac{7093}{1536}\pi^2\right]\nu^2+\frac{1223}{324}\nu^3 \\
& + e_t^2\left(-\frac{9373}{24}+\left[\frac{9837}{16}-\frac{9667}{512}\pi^2\right]\nu \right. \\
& \left. + \left[-\frac{11533}{144}+\frac{7339}{1536}\pi^2\right]\nu^2-\frac{17705}{1296}\nu^3\right) \\
& + e_t^4\left(-\frac{6943}{32}+\left[\frac{13393}{48}-\frac{1239}{512}\pi^2\right]\nu \right. \\
& \left. + \left[-\frac{3161}{144}-\frac{779}{1536}\pi^2\right]\nu^2+\frac{2251}{432}\nu^3\right) \\
& + e_t^6\left(-\frac{21269}{192}+\left[\frac{24623}{96}-\frac{615}{512}\pi^2\right]\nu \right. \\
& \left. + \left[-\frac{4573}{48}+\frac{533}{1536}\pi^2\right]\nu^2+\frac{7081}{1296}\nu^3\right) \\
& + e_t^8\left(\frac{4691}{192}-\frac{2449}{48}\nu+\frac{671}{36}\nu^2-\frac{1021}{1296}\nu^3\right) \\
& + \sqrt{1-e_t^2}\left[-30+\left[\frac{412}{9}-\frac{41}{96}\pi^2\right]\nu-\frac{10}{3}\nu^2\right.
\end{aligned}
$$

$$+e_t^2\left(-\frac{415}{4}+\left[\frac{1021}{9}-\frac{41}{96}\pi^2\right]\nu-20\nu^2\right)$$
$$+e_t^4\left(\frac{355}{2}+\left[-\frac{1781}{9}+\frac{41}{48}\pi^2\right]\nu+29\nu^2\right)$$
$$+e_t^6\left(-\frac{175}{4}+\frac{116}{3}\nu-\frac{17}{3}\nu^2\right)\Bigg]\Bigg\}. \tag{2.65}$$

By substituting the expression of ξ/k' in Eq. (2.63) we obtain the average of the orbital energy flux in term of ξ and e_t in ADM coordinates which is given by

$$\begin{aligned}<\dot{\mathcal{E}}>_{\mathrm{ADM}} &= \frac{32\nu^2\xi^5}{5}\frac{1}{(1-e_t^2)^{7/2}}\Big(<\dot{\mathcal{E}}_{\mathrm{N}}>_{\mathrm{ADM}}+\xi<\dot{\mathcal{E}}_{\mathrm{1PN}}>_{\mathrm{ADM}}+\xi^2<\dot{\mathcal{E}}_{\mathrm{2PN}}>_{\mathrm{ADM}} \\ &\quad +\xi^3<\dot{\mathcal{E}}_{\mathrm{3PN}}>_{\mathrm{ADM}}\Big).\end{aligned} \tag{2.66}$$

$$<\dot{\mathcal{E}}_N>_{\mathrm{ADM}} = 1+e_t^2\frac{73}{24}+e_t^4\frac{37}{96}, \tag{2.67a}$$

$$\begin{aligned}<\dot{\mathcal{E}}_{\mathrm{1PN}}>_{\mathrm{ADM}} &= \frac{1}{\left(1-e_t^2\right)} \\ &\quad \left\{\frac{2113}{336}-\frac{35}{12}\nu+e_t^2\left(\frac{10305}{224}-\frac{1081}{36}\nu\right)\right. \\ &\quad \left.+e_t^4\left(\frac{3841}{128}-\frac{311}{12}\nu\right)+e_t^6\left(\frac{2179}{1792}-\frac{851}{576}\nu\right)\right\},\end{aligned} \tag{2.67b}$$

$$\begin{aligned}<\dot{\mathcal{E}}_{\mathrm{2PN}}>_{\mathrm{ADM}} &= \frac{1}{\left(1-e_t^2\right)^2} \\ &\quad \left\{\frac{299701}{9072}-\frac{16601}{504}\nu+\frac{65}{18}\nu^2\right. \\ &\quad +e_t^2\left(\frac{5817277}{18144}-\frac{908501}{2016}\nu+\frac{5935}{54}\nu^2\right) \\ &\quad +e_t^4\left(\frac{11282477}{24192}-\frac{6150947}{8064}\nu+\frac{247805}{864}\nu^2\right) \\ &\quad +e_t^8\left(\frac{86567}{64512}-\frac{9769}{4608}\nu+\frac{21275}{6912}\nu^2\right) \\ &\quad +e_t^6\left(\frac{1801955}{16128}-\frac{750385}{4032}\nu+\frac{185305}{1728}\nu^2\right) \\ &\quad +\sqrt{1-e_t^2}\left[\frac{35}{2}-7\nu+e_t^2\left(\frac{6425}{48}-\frac{1285}{24}\nu\right)\right. \\ &\quad \left.\left.+e_t^4\left(\frac{5065}{64}-\frac{1013}{32}\nu\right)+e_t^6\left(\frac{185}{96}-\frac{37}{48}\nu\right)\right]\right\},\end{aligned} \tag{2.67c}$$

$$
\begin{aligned}
< \dot{\mathcal{E}}_{3PN} >_{ADM} \;=\; & \frac{1}{\left(1-e_t^2\right)^3} \\
& \left\{\frac{62181833}{158400} + \left[-\frac{32799587}{54432} + \frac{779}{64}\pi^2\right]\nu + \frac{261337}{3024}\nu^2 - \frac{775}{324}\nu^3\right. \\
& +e_t^2\left(\frac{412054561}{105600} + \left[-\frac{607915099}{108864} + \frac{96035}{1536}\pi^2\right]\nu\right. \\
& \left.+\frac{12559915}{6048}\nu^2 - \frac{53696}{243}\nu^3\right) \\
& +e_t^4\left(\frac{1531759583}{211200} + \left[-\frac{1115502439}{108864} + \frac{22723}{3072}\pi^2\right]\nu\right. \\
& \left.+\frac{12398711}{1728}\nu^2 - \frac{10816087}{7776}\nu^3\right) \\
& +e_t^6\left(\frac{89256796753}{26611200} + \left[-\frac{727977199}{145152} - \frac{219685}{12288}\pi^2\right]\nu\right. \\
& \left.+\frac{2206527}{448}\nu^2 - \frac{983251}{648}\nu^3\right) \\
& +e_t^8\left(\frac{12105629567}{47308800} + \left[-\frac{56423453}{129024} - \frac{4059}{4096}\pi^2\right]\nu\right. \\
& \left.+\frac{103625201}{193536}\nu^2 - \frac{4586539}{15552}\nu^3\right) \\
& +e_t^{10}\left(-\frac{8977637}{11354112} + \frac{9287}{48384}\nu + \frac{8977}{55296}\nu^2 - \frac{567617}{124416}\nu^3\right) \\
& +\sqrt{1-e_t^2}\left[\frac{30556517}{151200} + \frac{455}{12}\nu^2 + \left[-\frac{284705}{1008} + \frac{287}{192}\pi^2\right]\nu\right. \\
& +e_t^2\left(\frac{251168231}{100800} + \left[-\frac{20078741}{6048} + \frac{52685}{4608}\pi^2\right]\nu + \frac{43559}{72}\nu^2\right) \\
& +e_t^4\left(\frac{1336667951}{403200} + \left[-\frac{35699627}{8064} + \frac{41533}{6144}\pi^2\right]\nu + \frac{303985}{288}\nu^2\right) \\
& +e_t^6\left(\frac{1558169203}{2419200} + \left[-\frac{42190997}{48384} + \frac{1517}{9216}\pi^2\right]\nu + \frac{73357}{288}\nu^2\right) \\
& \left.+e_t^8\left(\frac{185}{48} - \frac{1073}{288}\nu + \frac{407}{288}\nu^2\right)\right] + \left(\frac{1712}{105} + \frac{14552}{63}e_t^2 + \frac{553297}{1260}e_t^4\right. \\
& \left.\left.+\frac{187357}{1260}e_t^6 + \frac{10593}{2240}e_t^8\right)\ln\left[\left(\frac{c^2 r_0}{Gm}x\right)\frac{1+\sqrt{1-e_t^2}}{2\left(1-e_t^2\right)}\right]\right\}. \qquad (2.67d)
\end{aligned}
$$

Similarly the expression of ξ/k' and < & > in modified harmonic coordinate are given by:

$$
\begin{aligned}
\left(\frac{\xi}{k'}\right)_{\text{Mhar}} = {} & \left(1-e_t^2\right)+\frac{\xi^2}{\left(1-e_t^2\right)}\left\{-\frac{13}{2}+\frac{7}{3}\nu\right. \\
& \left.+e_t^2\left(\frac{9}{4}-\frac{1}{6}\nu\right)+e_t^4\left(\frac{17}{4}-\frac{13}{6}\nu\right)\right\} \\
& +\frac{\xi^2}{\left(1-e_t^2\right)^2}\left\{-\frac{21}{4}+\left[\frac{143}{4}-\frac{41}{32}\pi^2\right]\nu+\frac{28}{9}\nu^2\right. \\
& +e_t^2\left(-\frac{51}{4}+\left[-\frac{45}{4}+\frac{123}{128}\pi^2\right]\nu-\frac{19}{3}\nu^2\right) \\
& +e_t^4\left(\frac{473}{16}+\left[-\frac{115}{3}+\frac{41}{128}\pi^2\right]\nu+\frac{125}{24}\nu^2\right) \\
& +e_t^6\left(-\frac{185}{16}+\frac{83}{6}\nu-\frac{143}{72}\nu^2\right) \\
& \left.+\sqrt{1-e_t^2}\left[-5+2\nu+e_t^4(10-4\nu)+e_t^2(-5+2\nu)\right]\right\} \\
& +\frac{\xi^3}{\left(1-e_t^2\right)^3}\left\{\frac{33305}{48}+\left[-\frac{105533}{96}+\frac{11521}{512}\pi^2\right]\nu\right. \\
& +\left[\frac{25729}{144}-\frac{7093}{1536}\pi^2\right]\nu^2+\frac{1223}{324}\nu^3 \\
& +e_t^2\left(-\frac{1187}{3}+\left[\frac{883417}{1680}-\frac{8323}{512}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{7885}{144}+\frac{7339}{1536}\pi^2\right]\nu^2-\frac{17705}{1296}\nu^3\right) \\
& +e_t^4\left(-\frac{6743}{32}+\left[\frac{624803}{1680}-\frac{2583}{512}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{6569}{144}-\frac{779}{1536}\pi^2\right]\nu^2+\frac{2251}{432}\nu^3\right) \\
& +e_t^6\left(-\frac{21485}{192}+\left[\frac{24247}{96}-\frac{615}{512}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{1551}{16}+\frac{533}{1536}\pi^2\right]\nu^2+\frac{7081}{1296}\nu^3\right) \\
& +e_t^8\left(\frac{4691}{192}-\frac{2449}{48}\nu+\frac{671}{36}\nu^2-\frac{1021}{1296}\nu^3\right) \\
& +\sqrt{1-e_t^2}\left[-30+\left[\frac{412}{9}-\frac{41}{96}\pi^2\right]\nu-\frac{10}{3}\nu^2\right. \\
& +e_t^2\left(-\frac{415}{4}+\left[\frac{1021}{9}-\frac{41}{96}\pi^2\right]\nu-20\nu^2\right)
\end{aligned}
$$

$$+e_t^4\left(\frac{355}{2}+\left[-\frac{1781}{9}+\frac{41}{48}\pi^2\right]\nu+29\nu^2\right)$$
$$+e_t^6\left(-\frac{175}{4}+\frac{116}{3}\nu-\frac{17}{3}\nu^2\right)\Bigg]\Bigg\}. \tag{2.68}$$

By substituting the expression of ξ/k' in Eq. (2.63) we obtain the average of the orbital energy flux in term of ξ and e, in ADM coordinates which is given by

$$\begin{aligned} <\dot{\mathcal{E}}>_{\rm Mhar} &= \frac{32\nu^2\xi^5}{5}\frac{1}{(1-e_t^2)^{7/2}}\Big(<\dot{\mathcal{E}}_{\rm N}>_{\rm Mhar}+\xi<\dot{\mathcal{E}}_{\rm 1PN}>_{\rm Mhar}+\xi^2<\dot{\mathcal{E}}_{\rm 2PN}>_{\rm Mhar} \\ &\quad +\xi^3<\dot{\mathcal{E}}_{\rm 3PN}>_{\rm Mhar}\Big). \end{aligned} \tag{2.69}$$

$$<\dot{\mathcal{E}}_N>_{\rm Mhar} = 1+e_t^2\frac{73}{24}+e_t^4\frac{37}{96}, \tag{2.70a}$$

$$\begin{aligned} <\dot{\mathcal{E}}_{\rm 1PN}>_{\rm Mhar} &= \frac{1}{\left(1-e_t^2\right)} \\ &\quad \left\{\frac{2113}{336}-\frac{35}{12}\nu+e_t^2\left(\frac{10305}{224}-\frac{1081}{36}\nu\right)\right. \\ &\quad \left.+e_t^4\left(\frac{3841}{128}-\frac{311}{12}\nu\right)+e_t^6\left(\frac{2179}{1792}-\frac{851}{576}\nu\right)\right\}, \end{aligned} \tag{2.70b}$$

$$\begin{aligned} <\dot{\mathcal{E}}_{\rm 2PN}>_{\rm Mhar} &= \frac{1}{\left(1-e_t^2\right)^2} \\ &\quad \left\{\frac{299701}{9072}-\frac{16601}{504}\nu+\frac{65}{18}\nu^2\right. \\ &\quad +e_t^2\left(\frac{5876623}{18144}-\frac{796403}{2016}\nu+\frac{5935}{54}\nu^2\right) \\ &\quad +e_t^4\left(\frac{11383781}{24192}-\frac{5576891}{8064}\nu+\frac{247805}{864}\nu^2\right) \\ &\quad +e_t^6\left(\frac{1806617}{16128}-\frac{1461143}{8064}\nu+\frac{185305}{1728}\nu^2\right) \\ &\quad +e_t^8\left(\frac{86567}{64512}-\frac{9769}{4608}\nu+\frac{21275}{6912}\nu^2\right) \\ &\quad +\sqrt{1-e_t^2}\left[\frac{35}{2}-7\nu+e_t^2\left(\frac{6425}{48}-\frac{1285}{24}\nu\right)\right. \\ &\quad \left.\left.+e_t^4\left(\frac{5065}{64}-\frac{1013}{32}\nu\right)+e_t^6\left(\frac{185}{96}-\frac{37}{48}\nu\right)\right]\right\}, \end{aligned} \tag{2.70c}$$

$$
\begin{aligned}
<\dot{\mathcal{E}}_{3\mathrm{PN}}>_{\mathrm{Mhar}} \;=\;& \frac{1}{\left(1-e_t^2\right)^3} \\
& \left\{\frac{62181833}{158400}+\left[-\frac{32799587}{54432}+\frac{779}{64}\pi^2\right]\nu+\frac{261337}{3024}\nu^2-\frac{775}{324}\nu^3\right. \\
& +e_t^2\left(\frac{2926351327}{739200}+\left[-\frac{179679691}{38880}+\frac{69659}{1536}\pi^2\right]\nu\right. \\
& \left.+\frac{5033933}{3024}\nu^2-\frac{53696}{243}\nu^3\right) \\
& +e_t^4\left(\frac{520596161}{70400}+\left[-\frac{2120453191}{272160}-\frac{44813}{3072}\pi^2\right]\nu\right. \\
& \left.+\frac{9766595}{1728}\nu^2-\frac{10816087}{7776}\nu^3\right) \\
& +e_t^6\left(\frac{12938933779}{3801600}+\left[-\frac{5935795609}{1451520}-\frac{238333}{12288}\pi^2\right]\nu\right. \\
& \left.+\frac{22389337}{5376}\nu^2-\frac{983251}{648}\nu^3\right) \\
& +e_t^8\left(\frac{12176124167}{47308800}+\left[-\frac{7463489}{18432}-\frac{4059}{4096}\pi^2\right]\nu\right. \\
& \left.+\frac{97732433}{193536}\nu^2-\frac{4586539}{15552}\nu^3\right) \\
& +e_t^{10}\left(-\frac{8977637}{11354112}+\frac{9287}{48384}\nu+\frac{8977}{55296}\nu^2-\frac{567617}{124416}\nu^3\right) \\
& +\left(1-e_t^2\right)^{1/2}\left[\frac{30556517}{151200}+\left[-\frac{284705}{1008}+\frac{287}{192}\pi^2\right]\nu+\frac{455}{12}\nu^2\right. \\
& +e_t^6\left(\frac{1558169203}{2419200}+\left[-\frac{42190997}{48384}+\frac{1517}{9216}\pi^2\right]\nu+\frac{73357}{288}\nu^2\right) \\
& +e_t^2\left(\frac{251168231}{100800}+\left[-\frac{20078741}{6048}+\frac{52685}{4608}\pi^2\right]\nu+\frac{43559}{72}\nu^2\right) \\
& +e_t^4\left(\frac{1336667951}{403200}+\left[-\frac{35699627}{8064}+\frac{41533}{6144}\pi^2\right]\nu+\frac{303985}{288}\nu^2\right) \\
& \left.+e_t^8\left(\frac{185}{48}-\frac{1073}{288}\nu+\frac{407}{288}\nu^2\right)\right]+\left(\frac{1712}{105}+\frac{14552}{63}e_t^2+\frac{553297}{1260}e_t^4\right. \\
& \left.\left.+\frac{187357}{1260}e_t^6+\frac{10593}{2240}e_t^8\right)\ln\left[\left(\frac{c^2 r_0}{Gm}x\right)\frac{1+\sqrt{1-e_t^2}}{2\left(1-e_t^2\right)}\right]\right\}. \qquad (2.70\mathrm{d})
\end{aligned}
$$

The above results in Eqs. (2.67) and (2.70) can be obtained directly by choosing our variables ξ and e_t as we do in the Secs. 2.7 and 2.8.

From Eq. (2.67), one can recover the result of [115] up to 2pN after converting from e_t

to e_r. This provides another check on our elliptical orbit result, after taking into account the error there which has been corrected in [47] and also here in this chapter. For completeness we give also the 3pN result in modified harmonic coordinate.

We conclude by an expression of the energy flux in terms of two alternative gauge-invariant variables x and z, the latter by construction vanishing in the circular orbit limit and given by

$$z \equiv \left(\frac{x}{k'}\right)_{\odot} - \left(\frac{x}{k'}\right). \tag{2.71}$$

This allows one to explicitly see the circular orbit coefficients without any further algebra. Unlike x, ξ or k', which are 'exactly' gauge invariant, by construction the explicit expression of z is gauge invariant only upto a particular PN order, in this case 3PN.

$$\begin{aligned} <\dot{\mathcal{E}}> \;=\; & \frac{32\nu^2x^5}{5}\frac{1}{(1-z)^{7/2}}\Big(<\dot{\mathcal{E}}_{\rm N}> + x <\dot{\mathcal{E}}_{\rm 1PN}> + x^2 <\dot{\mathcal{E}}_{\rm 2PN}> \\ & + x^3 <\dot{\mathcal{E}}_{\rm 3PN}>\Big). \end{aligned} \tag{2.72}$$

$$<\dot{\mathcal{E}}_{\rm N}> \;=\; 1 + \frac{73}{24}z + \frac{37}{96}z^2, \tag{2.73a}$$

$$\begin{aligned} <\dot{\mathcal{E}}_{\rm 1PN}> \;=\; & \frac{1}{1-z} \\ & \left\{-\frac{1247}{336} - \frac{35}{12}\nu + z\left(-\frac{171}{14} - \frac{761}{48}\nu\right)\right. \\ & \left. + z^2\left(-\frac{3625}{384} - \frac{373}{48}\nu\right) + z^3\left(-\frac{139}{112} - \frac{259}{1152}\nu\right)\right\}, \end{aligned} \tag{2.73b}$$

$$\begin{aligned} <\dot{\mathcal{E}}_{\rm 2PN}> \;=\; & \frac{1}{(1-z)^2} \\ & \left\{\frac{93259}{9072} + \frac{6205}{504}\nu + \frac{65}{18}\nu^2\right. \\ & + z\left(-\frac{8849363}{72576} + \left[\frac{453637}{1008} - \frac{32185}{3072}\pi^2\right]\nu + \frac{13361}{288}\nu^2\right) \\ & + z^2\left(-\frac{10696723}{96768} + \left[\frac{2278795}{4032} - \frac{13735}{1024}\pi^2\right]\nu + \frac{64763}{1152}\nu^2\right) \\ & + z^3\left(\frac{13873}{1152} + \left(\frac{537181}{8064} - \frac{7585}{8192}\pi^2\right)\nu + \frac{30191}{3456}\nu^2\right) \\ & + z^4\left(\frac{744545}{258048} + \frac{19073}{32256}\nu + \frac{2849}{27648}\nu^2\right) \\ & \left. + (1-z)^{5/2}\left[-\frac{365}{24} + \frac{73}{12}\nu + z\left(-\frac{185}{48} + \frac{37}{24}\nu\right)\right]\right\}, \end{aligned} \tag{2.73c}$$

$$
\begin{aligned}
<\dot{\mathcal{E}}_{3PN}> \;=\; & \frac{1}{(1-z)^3} \\
& \left\{\frac{1403349929}{9979200}+\left[-\frac{10181615}{54432}+\frac{4961}{2304}\pi^2\right]\nu+\frac{32143}{3024}\nu^2-\frac{775}{324}\nu^3\right. \\
& +z\left(\frac{1151327326241}{79833600}+\left[-\frac{5087164487}{217728}+\frac{64859827}{129024}\pi^2\right]\nu\right. \\
& \left.+\left[\frac{46934095}{24192}-\frac{328615}{9216}\pi^2\right]\nu^2-\frac{927845}{10368}\nu^3\right) \\
& +z^2\left(\frac{1898735413}{123200}+\left[-\frac{4984509773}{217728}+\frac{35039461}{73728}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{22788071}{48384}+\frac{1350335}{36864}\pi^2\right]\nu^2-\frac{465865}{1728}\nu^3\right) \\
& +z^3\left(-\frac{9778049927}{3379200}+\left[\frac{599158247}{96768}-\frac{26642989}{172032}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{430128131}{193536}+\frac{248665}{4096}\pi^2\right]\nu^2-\frac{36483685}{248832}\nu^3\right) \\
& +z^4\left(-\frac{10996715097}{31539200}+\left[\frac{807749}{1728}-\frac{4486589}{344064}\pi^2\right]\nu\right. \\
& \left.+\left[-\frac{103128329}{774144}+\frac{98605}{32768}\pi^2\right]\nu^2-\frac{1343675}{124416}\nu^3\right) \\
& +z^5\left(-\frac{2635805}{405504}-\frac{891535}{3096576}\nu-\frac{4537}{27648}\nu^2-\frac{106375}{995328}\nu^3\right) \\
& +\sqrt{1-z}\left[-\frac{18559}{1350}-\frac{16493447}{50400}z-\frac{102893087}{201600}z^2\right. \\
& \left.-\frac{1142417}{151200}z^3+\frac{51335}{2688}z^4\right]+(1-z)^{5/2}\left[\left[\frac{513323}{3024}-\frac{2993}{2304}\pi^2\right]\nu\right. \\
& -\frac{3013}{72}\nu^2+z\left(\left[\frac{59675}{864}-\frac{1517}{4608}\pi^2\right]\nu-\frac{5503}{144}\nu^2\right) \\
& \left.+z^2\left(-\frac{10951}{2688}\nu-\frac{481}{192}\nu^2\right)\right]+\left(\frac{1712}{105}+\frac{14552}{63}z\right. \\
& \left.\left.+\frac{553297}{1260}z^2+\frac{187357}{1260}z^3+\frac{10593}{2240}z^4\right)\ln\left[\left(\frac{c^2 r_0}{Gm}x\right)\frac{1+\sqrt{1-z}}{2(1-z)}\right]\right\} \qquad (2.73\text{d})
\end{aligned}
$$

The circular orbit limit is given by $z = 0$ and reduces to the result for the instantaneous contribution in [95] rewritten in term of x.

2.11 Concluding remarks

The instantaneous contributions to the 3PN gravitational wave luminosity from the inspiral phase of a binary system of compact objects moving in an elliptical orbit is computed using the Multipolar post-Minkowskian wave generation formalism. The new inputs for this calculation include the mass octupole and current quadrupole at 2PN for general orbits and the 3PN accurate mass quadrupole. Using the 3PN quasi-Keplerian representation of elliptical orbits obtained recently the flux is averaged over the binary's orbit. The expression for the instantaneous contributions averaged over an orbit is presented in different coordinate systems: Standard harmonic coordinates (with logs), modified harmonic coordinates (without logs) and ADM coordinates. Alternative ***gauge invariant*** expressions are also provided. Supplementing the instantaneous contributions of this chapter by the important hereditary contributions arising from tails, tails-of-tails and tails squared terms calculated in the next chapter, the complete energy flux can be obtained.

For binaries moving on circular orbits the energy flux has been computed to 3.5PN order [95]. The extension of these results to eccentric orbits would be interesting. However, some uncomputed modules remain in the formalism to compute the multipole moments required for the 3.5PN generation in the general orbit case. We leave this to a future investigation.