

Chapter 2

Quantum diffusion of a particle on a lattice in the presence of coupling to the environment modelled phenomenologically by a set of Lindblad operators

2.1 Abstract

*In this Chapter, we consider the quantum motion of a particle moving on a 1D lattice with site-diagonal dynamical disorder in the presence of a time-harmonic drive and a static uniform bias. Decoherence caused by the on-site dynamical disorder is treated through the introduction of a set of Hermitian Lindblad operators chosen so as to project on to the lattice sites. The resulting Liouville equation of motion for the reduced density matrix of the particle is solved analytically for several physical quantities of interest, i.e, the time-dependent mean and the mean-squared displacements as a function of the bias and the drive parameters. An interesting new result obtained by us is the nonlinear enhancement of the **diffusion coefficient** with increasing drive am-*

plitude, and its variation with the frequency detuning relative to the inter-site energy gap tunable by the static bias. Our expressions correctly specialize to the known exact results in the limit of zero drive and zero bias. A motivating physical realization of the above model is the Stark-Wannier ladder of states localized in the potential wells of a heterostructure superlattice subjected simultaneously to a strong electrostatic field (bias) and an electromagnetic excitation (drive). Some open questions of equivalence of the incoherence caused by the stochastic on-site modulation and the decoherence due to the Hermitian Lindblads, as also the question of dissipation are **briefly** discussed. It is also argued out that despite the heating **effects** of the Lindblad operators[19, 16] towards infinite temperature,, the band-width limited nature of the one-band Hamiltonian makes the use of the Lindblads physically meaningful

2.2 Model hamiltonian and the reduced density matrix: evolution under Hermitian Lindblad operators

2.2.1 Zero bias and zero drive

We begin by considering first the simplest case of quantum motion of a particle moving on a dynamically disordered 1D lattice (Figure 2.1) under a tight-binding one-band Hamiltonian

$$H^0 = -(V/2) \sum_l (|l\rangle\langle l+1| + |l+1\rangle\langle l|) \quad (2.1)$$

where $(-V)$ is the nearest-neighbour transfer matrix element, and the sum is over the N sites with N taken to be infinite. The effect of dynamical disorder, namely the incoherence, will be introduced through a set of Hermitian Lindblad operators, $L_l = \sqrt{\gamma} |l\rangle\langle l|$ that project on to the lattice-site orbitals, $|l\rangle$. The Lindblads represent coupling of the particle -- the dynamical degree of freedom of interest here, with the numerous environmental degrees of freedom, e.g., the thermal phonons. The

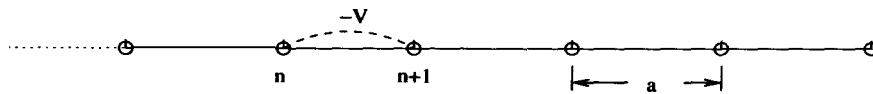


Figure 2.1: The 1D model of a dynamically disordered tight binding chain.

reduced density matrix ρ for the particle then obeys the evolution master equation (written in the operator form)[?]:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H^0, \rho] - \sum_l [L_l, [L_l, H^0]], \quad (2.2)$$

where the first term on the RHS gives the unitary evolution, while the second term gives the non-unitary (incoherent) evolution causing the initially pure density matrix ($\rho = \rho^2$ at $t = 0$) to become mixed ($\rho \neq \rho^2$ for $t > 0$). As is now well known, such a Lindbladian preserves the defining properties of the density matrix, namely, its positivity, Hermiticity, trace-class property (conservation of probability), and the gaussianity of the stochastic process. More explicitly, in the terms of its matrix elements, we have

$$\frac{\partial \rho_{mn}}{\partial t} = -\frac{iV}{2\hbar}[\rho_{m,n+1} + \rho_{m,n-1} - \rho_{m+1,n} - \rho_{m-1,n}] - \gamma \rho_{mn}[1 - \delta_{mn}]. \quad (2.3)$$

with the initial condition,

$$\rho_{mn}(t=0) = \delta_{m0}\delta_{n0}. \quad (2.4)$$

In the Fourier space, with $\beta = -\frac{V}{\hbar}$.

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{m,n} \rho_{m,n} e^{-imk_1} e^{ink_2} &= \frac{i\beta}{2} \left[\sum_{m,n} \rho_{m,n+1} e^{-imk_1} e^{i(n+1)k_2} e^{-ik_2} + \dots \right] \\ &- \gamma \sum_{m,n} \rho_{m,n} e^{-imk_1} e^{ink_2} + \gamma \sum_{m,n} \delta_{m,n} e^{-imk_1} e^{ink_2}, \end{aligned} \quad (2.5)$$

with

$$\tilde{\rho}(k_1, k_2, t) = \sum_{m,n} \rho_{m,n} e^{-imk_1} e^{ink_2}, \quad \delta_{m,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)q} dq. \quad (2.6)$$

We get

$$\frac{\partial}{\partial t} \tilde{\rho}(k_1, k_2, t) = [i\beta(\cos k_2 - \cos k_1) - \gamma] \tilde{\rho}(k_1, k_2, t) + \frac{\gamma}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(k_1 - q, k_2 - q, t) dq. \quad (2.7)$$

Defining center of mass and relative wave-vectors as $p = (k_1 + k_2)/2$, $u = k_1 - k_2$ and writing $\tilde{\rho}(k_1, k_2, t) \equiv \rho(p, u, t)$, we have

$$\frac{\partial}{\partial t} \rho(p, u, t) = [2i\beta \sin p \sin(u/2) - \gamma] \rho(p, u, t) + \frac{\gamma}{2\pi} \int_{-\pi}^{\pi} \rho(p - q, u, t) dq. \quad (2.8)$$

We further define the reduced density matrix by

$$\chi(u, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(p - q, u, t) dq = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(q, u, t) dq. \quad (2.9)$$

As the dimensions of β and γ are $(time)^{-1}$, we define the dimensionless parameters as $\tau = t\beta$ and $\Gamma = \frac{\gamma}{\beta}$, and defining

$$\phi(p, u) = 2i \sin p \sin(u/2) - \Gamma. \quad (2.10)$$

The evolution equation becomes

$$\frac{\partial}{\partial \tau} \rho(p, u, \tau) = \phi(p, u) \rho(p, u, \tau) + \Gamma \chi(u, \tau). \quad (2.11)$$

We take the time(scaled) laplace transform $\tilde{\rho}(p, u, s) = \int_0^{\infty} e^{-s\tau} \rho(p, u, \tau) d\tau$ of the above equation(Eq. 2.11), and get

$$s\tilde{\rho}(p, u, s) - \rho(p, u, t=0) = \phi(p, u)\tilde{\rho}(p, u, s) + \Gamma\tilde{\chi}(u, s). \quad (2.12)$$

Now we want to calculate the value of $\rho(p, u, t=0)$. We know that

$$\begin{aligned} \rho_{m,n}(t=0) &= \langle C_m^*(t=0) C_n(t=0) \rangle = \delta_{m,0} \delta_{n,0} \\ \tilde{\rho}(k_1, k_2, t=0) &= \sum_{m,n} \rho_{m,n}(t=0) e^{-imk_1} e^{ink_2} = \sum_{m,n} \delta_{m,0} \delta_{n,0} e^{-imk_1} e^{ink_2} \\ \rho(p, u, t=0) &= \sum_{m,n} \delta_{m,0} \delta_{n,0} e^{-im(p+u/2)} e^{in(p-u/2)} = 1. \end{aligned} \quad (2.13)$$

With this we get

$$\tilde{\rho}(p, u, s) = \frac{1 + \Gamma\tilde{\chi}(u, s)}{s - \phi(p, u)}. \quad (2.14)$$

Summing the above equation i.e., Eq.(2.14) over p , we have

$$\begin{aligned} \sum_p \tilde{\rho}(p, u, s) &= \sum_p \frac{1 + \Gamma\tilde{\chi}(u, s)}{s - \phi(p, u)}, \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(p, u, s) dp &= [1 + \Gamma\tilde{\chi}(u, s)] \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dp}{s - \phi(p, u)}. \end{aligned} \quad (2.15)$$

By re-arrangements we get

$$\tilde{\chi}(u, s) = \frac{I}{1 - I\Gamma} , \quad I = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dp}{s - \phi(p, u)}. \quad (2.16)$$

Now, we want to find the mean and the mean-squared displacement of the quantum particle. For this we note

$$\begin{aligned} \tilde{\chi}(u, s) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(q, u, s) dq = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n} \tilde{\rho}_{m,n}(s) e^{-im(p+u/2)} e^{in(p-u/2)} dp \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n} \tilde{\rho}_{m,n}(s) e^{-i(m+n)\frac{u}{2}} e^{i(n-m)p} dp = \sum_{m,n} \delta_{m,n} \tilde{\rho}_{m,n}(s) e^{-i(m+n)\frac{u}{2}} = \sum_n \tilde{\rho}_{n,n}(s) e^{-inu} \\ \delta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)p} dp. \end{aligned} \quad (2.17)$$

Now, the mean displacement in the s-domain is given by $\sum_n n \tilde{\rho}_{n,n}(s)$, for

$$\frac{\partial \tilde{\chi}(u, s)}{\partial u} = -i \sum_n n \tilde{\rho}_{n,n}(s) e^{-inu}.$$

Thus, we have

$$\langle \tilde{x}(s) \rangle = i \left[\frac{\partial \tilde{\chi}(u, s)}{\partial u} \right]_{u=0}. \quad (2.18)$$

Similarly, the mean-squared displacement is given by

$$\langle \tilde{x}^2(s) \rangle = \sum_n n^2 \tilde{\rho}_{n,n}(s) = - \left[\frac{\partial^2 \tilde{\chi}(u, s)}{\partial u^2} \right]_{u=0}. \quad (2.19)$$

Now differentiating Eq.(2.16), w. r. t. (u) and finding the derivatives of the integral **I** at $u = 0$, and using Eq.(2.18) and Eq.(2.19), we obtain

$$\begin{aligned} \langle \tilde{x}(s) \rangle &= 0 , \\ \langle x(t) \rangle &= 0. \end{aligned} \quad (2.20)$$

$$\begin{aligned} \langle \tilde{x}^2(s) \rangle &= \frac{1}{s^2(s + \Gamma)} , \\ \langle x^2(\tau) \rangle &= -\frac{1}{\Gamma^2} + \frac{1}{\Gamma} \tau + \frac{1}{\Gamma^2} e^{-\Gamma \tau}. \end{aligned} \quad (2.21)$$

Recalling, $\tau = t\beta$ and $\Gamma = \gamma/\beta$, where $\beta = V/\hbar$, $\gamma = 2q_0/\hbar^2$, we obtain

$$\langle x^2(t) \rangle = -\frac{\beta^2}{\gamma^2} + \frac{\beta^2}{\gamma} t + \frac{\beta^2}{\gamma^2} e^{-\gamma t} , \quad (2.22)$$

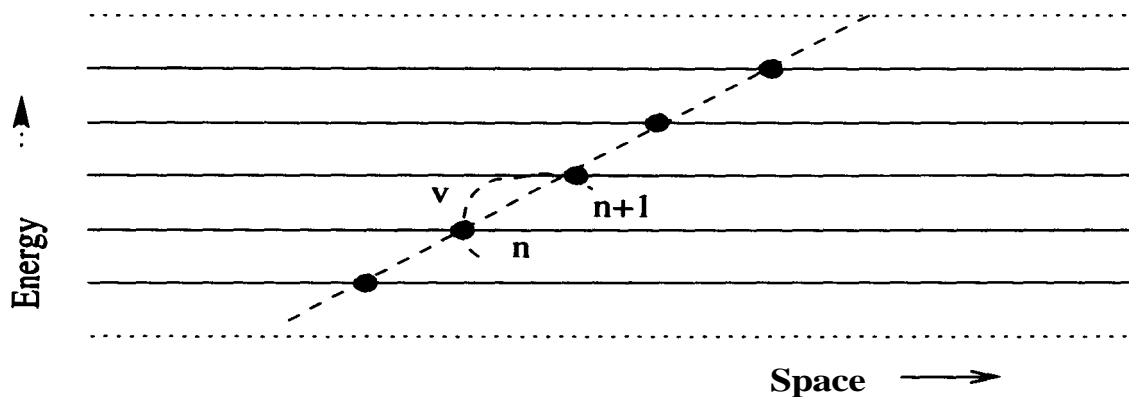


Figure 2.2: The 1-D model of a dynamically disordered tight binding lattice with bias.

which reduces to the classical case in the large time ($t \gg 1/\gamma$) limit with $\langle x^2(t) \rangle \sim (\beta^2/\gamma)t$ giving the diffusion coefficient $D = \beta^2/2\gamma$. In the small time limit it goes **ballistically** as t^2 as expected, while the mean displacement $\langle x(t) \rangle$ is zero. The same results were obtained by solving the Schroedinger equation with a time-dependent random potential (Gaussian White Noise, GWN)[1, 2] by using the Novikov theorem[21].

This shows the equivalence of time evolution of the quantum particle obtained by solving the Schroedinger wave equation with a time-dependent GWN potential and the evolution by Lindbladian master equation for a tight-binding lattice Hamiltonian.

2.2.2 Non-zero bias and zero drive

Next, we consider the more interesting case where the model lattice hamiltonian has a systematic bias in that there is a constant energy mismatch between the successive site energies. The system hamiltonian in this case is,

$$H^\alpha = -\frac{V}{2} \sum_l [|l\rangle\langle l+1| + |l+1\rangle\langle l|] + \sum_l \alpha l [|l\rangle\langle l|], \quad (2.23)$$

with a as the site-energy level spacing. With this hamiltonian, the Lindbladian master equation(Eq.2.2) for the time evolution of the density matrix is now,

$$\frac{\partial \rho_{mn}}{\partial t} = -\frac{iV}{2\hbar} [\rho_{m,n+1} + \rho_{m,n-1} - \rho_{m+1,n} - \rho_{m-1,n}] - \gamma \rho_{mn} [1 - \delta_{mn}] - \frac{i\alpha}{\hbar} [m - n] \rho_{mn}. \quad (2.24)$$

$$\frac{V}{\hbar} \equiv \frac{1}{\text{time}}, \quad \frac{a}{\hbar} \equiv \frac{1}{\text{time}}, \quad \gamma \equiv \frac{1}{\text{time}}.$$

Now, we define $\delta = \frac{\alpha}{V}$, $\Gamma = \frac{\hbar\gamma}{V}$, $\tau = \frac{tV}{\hbar}$, as dimensionless parameters. So we have

$$\frac{\partial \rho_{mn}}{\partial \tau} = -\frac{i}{2}[\rho_{m,n+1} + \rho_{m,n-1} - \rho_{m+1,n} - \rho_{m-1,n}] - i\delta[m-n]\rho_{mn} - \Gamma\rho_{mn}[1 - \delta_{mn}]. \quad (2.25)$$

Writing the above equation in the Fourier space, by defining

$$\tilde{\rho}(k_1, k_2, t) = \sum_{m,n} \rho_{mn}(t) e^{-imk_1} e^{ink_2}, \quad (2.26)$$

we get

$$\begin{aligned} \frac{\partial}{\partial \tau} \sum_{m,n} \rho_{mn}(\tau) e^{-imk_1} e^{ink_2} &= \frac{i}{2} \left[\sum_{m,n} \rho_{m+1,n}(\tau) e^{-imk_1} e^{ink_2} + \sum_{m,n} \rho_{m-1,n}(\tau) e^{-imk_1} e^{ink_2} - \dots \right] \\ &- i\delta \left[\sum_{m,n} (\tau)m\rho_{mn}(\tau) e^{-imk_1} e^{ink_2} - \sum_{mn} n\rho_{mn}(\tau) e^{-imk_1} e^{ink_2} \right] \\ &- \Gamma \sum_{m,n} \rho_{mn}(\tau) e^{-imk_1} e^{ink_2} + \Gamma \sum_{m,n} \delta_{mn} \rho_{mn}(\tau) e^{-imk_1} e^{ink_2}. \end{aligned} \quad (2.27)$$

Using the representation $\delta_{mn} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)q} dq$ in the last term of the above equation,

we obtain

$$\begin{aligned} \frac{\partial \tilde{\rho}(k_1, k_2, \tau)}{\partial \tau} &= \frac{i}{2} \left[\sum_{m+1,n} \rho_{m+1,n}(\tau) e^{-i(m+1)k_1} e^{ink_2} e^{ik_1} + \sum_{m-1,n} \rho_{m-1,n}(\tau) e^{-i(m-1)k_1} e^{ink_2} e^{-ik_1} \right] \\ &- \frac{i}{2} \left[\sum_{m,n-1} \rho_{m,n-1}(\tau) e^{-imk_1} e^{i(n-1)k_2} e^{ik_2} + \sum_{m,n+1} \rho_{m,n+1}(\tau) e^{-imk_1} e^{i(n+1)k_2} e^{-ik_2} \right] \\ &- i\delta \left[i \frac{\partial}{\partial k_1} \sum_{m,n} \rho_{mn}(\tau) e^{-imk_1} e^{ink_2} + i \frac{\partial}{\partial k_2} \sum_{m,n} \rho_{mn}(\tau) e^{-imk_1} e^{ink_2} \right] \\ &- \Gamma \tilde{\rho}(k_1, k_2, \tau) + \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(k_1 - q, k_2 - q, \tau) dq, \text{ or} \end{aligned} \quad (2.28)$$

$$\begin{aligned} \frac{\partial \tilde{\rho}(k_1, k_2, \tau)}{\partial \tau} &= \frac{i}{2} [e^{ik_1} + e^{-ik_1} - e^{ik_2} - e^{-ik_2}] \tilde{\rho}(k_1, k_2, \tau) \\ &+ \delta \left[\frac{\partial}{\partial k_1} + \frac{\partial}{\partial k_2} \right] \tilde{\rho}(k_1, k_2, \tau) - \Gamma \tilde{\rho}(k_1, k_2, \tau) + \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(k_1 - q, k_2 - q, \tau) dq, \end{aligned} \quad (2.29)$$

$$\begin{aligned} \frac{\partial \tilde{\rho}(k_1, k_2, \tau)}{\partial \tau} &= - \left[\Gamma + 2i \sin\left(\frac{k_1 + k_2}{2}\right) \sin\left(\frac{k_1 - k_2}{2}\right) \right] \tilde{\rho}(k_1, k_2, \tau) \\ &+ \delta \left[\frac{\partial}{\partial k_1} + \frac{\partial}{\partial k_2} \right] \tilde{\rho}(k_1, k_2, \tau) + \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(k_1 - q, k_2 - q, \tau) dq. \end{aligned} \quad (2.30)$$

Inserting the center-of-mass and the relative wave-vectors as $\mathbf{P} = \frac{k_1+k_2}{2}$ and $u = k_1 - k_2$ respectively, and writing $\tilde{\rho}(k_1, k_2, \tau)$ as $\rho(p, u, \tau)$, the above equation transforms to

$$\left[\frac{\partial}{\partial \tau} - \delta \frac{\partial}{\partial p} \right] \rho(p, u, \tau) = -[\Gamma + 2i \sin(u/2) \sin(p)] \rho(p, u, \tau) + \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} \rho(q, u, \tau) dq. \quad (2.31)$$

As before, we take the time Laplace transform of the above equation,

$$\tilde{\chi}(u, s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \int_0^{\infty} \rho(p, u, \tau) e^{-s\tau} d\tau$$

giving

$$s\tilde{\rho}(p, u, s) - \rho(p, u, 0) = \delta \frac{\partial}{\partial p} \tilde{\rho}(p, u, s) - [\Gamma + 2i \sin(u/2) \sin(p)] \tilde{\rho}(p, u, s) + \Gamma \tilde{\chi}(u, s). \quad (2.32)$$

Using the initial condition $\rho(p, u, 0) = 1$, we get

$$-\delta \frac{\partial}{\partial p} \tilde{\rho}(p, u, s) = -[s + \Gamma + 2i \sin(u/2) \sin(p)] \tilde{\rho}(p, u, s) + 1 + \Gamma \tilde{\chi}(u, s). \quad (2.33)$$

Now writing $\nu = \frac{s+\Gamma}{\delta}$, $f_u = \frac{2}{\delta} \sin \frac{u}{2}$, and $-[1 + \Gamma \tilde{\chi}(u, s)]/2 = Q(u, s)$, we obtain

$$\frac{\partial}{\partial p} \tilde{\rho}(p, u, s) - [\nu + i f_u \sin(p)] \tilde{\rho}(p, u, s) = Q(u, s). \quad (2.34)$$

After solving the above first order P.D.E. in p , we obtain

$$\tilde{\rho}(p, u, s) = Q(u, s) e^{\nu p - i f_u \cos p} \int e^{-\nu p + i f_u \cos p} dp + C(u, s) e^{\nu p - i f_u \cos p}. \quad (2.35)$$

The integration 'constant' $C(u, s)$ is calculated by setting $\mathbf{P} = 0$, in the above equation (Eq.(2.35)) as

$$C(u, s) = \tilde{\rho}(p=0, u, s) e^{i f_u} - Q(u, s) \left[\int e^{-\nu p + i f_u \cos p} dp \right]_{p=0}. \quad (2.36)$$

The transformed density matrix in the s -domain at $p = 0$ is given as

$$\tilde{\rho}(0, u, s) = \sum_{m,n} \tilde{\rho}_{mn}(s) e^{-i(m+n)u/2}. \quad (2.37)$$

In the long-time limit, due to decoherence by the dissipative medium, the contribution of the off-diagonal elements to $\tilde{\rho}_{mn}(s)$ becomes very small, and only the diagonal elements ($m = n$) will contribute to the time evolution of the density matrix. Under this

condition, we get a closed solution of Eq.(2.35). By putting $m = n$ in Eq.(2.37), we now get

$$\tilde{\rho}(0, u, s) = \sum_m \tilde{\rho}_{mm}(s) e^{-imu}. \quad (2.38)$$

Now, from Eq.(2.17) we have

$$\tilde{\chi}(u, s) = \tilde{\rho}(0, u, s) = \sum_m \tilde{\rho}_{mm}(s) e^{-imu}, \quad (2.39)$$

and thus from Eq.(2.36) and Eq.(2.39), we get

$$C(u, s) = \tilde{\chi}(u, s) e^{if_u} - Q(u, s) \left[\int e^{\phi(p, u)} dp \right]_{p=0}, \quad (2.40)$$

where we have defined $\phi(p, u) = -\nu p \dagger$ if, cosp . Next, we sum Eq.(2.35) over p and get

$$\sum_p \tilde{\rho}(p, u, s) = Q(u, s) \sum_p e^{-\phi(p, u)} \int e^{\phi(p, u)} dp + C(u, s) \sum_p e^{-\phi(p, u)}, \text{ or} \quad (2.41)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(p, u, s) dp = Q(u, s) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\phi(p, u)} \int e^{\phi(p, u)} dp dp + C(u, s) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\phi(p, u)} dp. \quad (2.42)$$

Now it is convenient to define the integrals

$$\begin{aligned} \tilde{\chi}(u, s) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(p, u, s) dp, \quad I = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\phi(p, u)} \left[\int e^{\phi(p, u)} dp \right] dp \\ I_2 &= \int e^{\phi(p, u)} dp, \quad I_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\phi(p, u)} dp, \quad I_{2p} = \left[\int e^{\phi(p, u)} dp \right]_{p=0}. \end{aligned} \quad (2.43)$$

With these definitions of the integrals, and using Eq.(2.36), we get

$$\tilde{\chi}(u, s) = \frac{I_1 I_{2p} - I}{\delta \dagger \Gamma I - \Gamma I_1 I_{2p} - \delta I_1 e^{if_u}}. \quad (2.44)$$

We know from (Eq.(2.17) and Eq.(2.18)) that $\langle \tilde{x}(s) \rangle = i \left[\frac{\partial \tilde{\chi}(u, s)}{\partial u} \right]_{u=0}$. Accordingly, we differentiate Equation (2.44) w. r. t. (u) and set $u = 0$. Finally, after some algebra, we obtain

$$\langle \tilde{x}(s) \rangle = \delta \left[\frac{s + \Gamma}{s^2 [(s + \Gamma)^2 + \delta^2]} \right] \frac{z}{z - 1}, \quad z = \frac{\sinh \pi \nu}{\pi \nu} \quad (2.45)$$

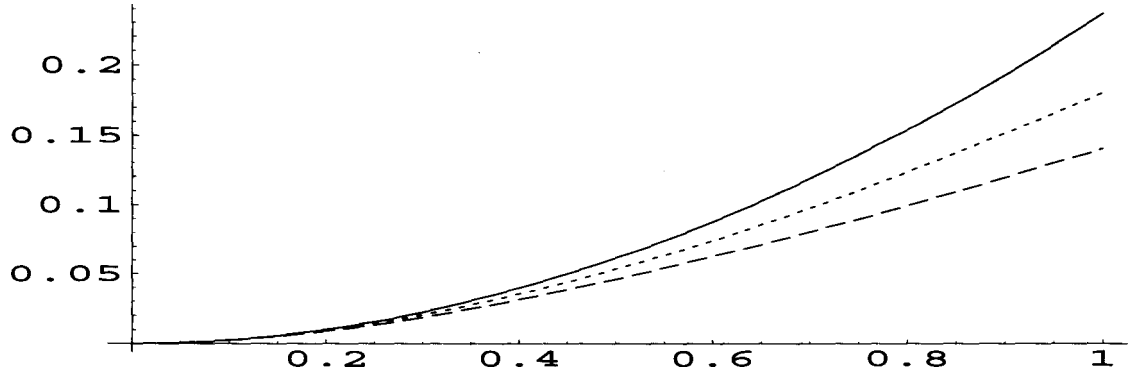


Figure 2.3: The plot of mean displacement $x(\tau)$ (vertical-axis) vs. scaled time $\tau = t/t_0$, $t_0 = \hbar/V$, (horizontal axis). The solid line is for $\Gamma = t_0\gamma = 0.1$, dotted ($\Gamma = 1$), and the dashed line ($\Gamma = 2$), with constant $\delta = \alpha/V = 0.5$.

The above equation (Eq.(2.45)) was numerically inverted to time domain, and the mean displacement vs. scaled time is plotted in Fig.(2.3).

2.2.2.1 Small δ - limit:

We can set δ positive, without loss of generality. Small δ limit means that energy level separation $\alpha \ll$ transfer matrix element V . Under the limit $\delta \rightarrow 0$, $\nu = \frac{s+\Gamma}{\delta} \rightarrow \infty$ thus

$$\frac{z}{z-1} = \frac{\sinh \pi\nu}{\sinh \pi\nu - \pi\nu} \simeq \frac{e^{\pi\nu}/2}{e^{\pi\nu}/2 - \pi\nu} \simeq \frac{e^{\pi\nu}/2}{e^{\pi\nu}/2} \simeq 1. \quad (2.46)$$

Thus, taking the inverse laplace transform of Eq.(2.45), we obtain

$$\langle x(\tau) \rangle \simeq \delta \left[\frac{\Gamma}{\Gamma^2 + \delta^2} \tau + \frac{\delta^2 - \Gamma^2}{(\Gamma^2 + \delta^2)^2} \{1 - e^{-\Gamma\tau} \cos \delta\tau\} - \frac{2\delta\Gamma}{(\delta^2 + \Gamma^2)^2} e^{-\Gamma\tau} \sin \delta\tau \right]. \quad (2.47)$$

One further approximation can be performed for $\delta \ll \Gamma$

$$\langle x(\tau) \rangle \simeq \delta \left[-\frac{1}{\Gamma^2} + \frac{1}{\Gamma} \tau + \frac{1}{\Gamma^2} e^{-\Gamma\tau} \cos \delta\tau \right]. \quad (2.48)$$

This is odd in δ as it should be. Long time behaviour is

$$\langle x(\tau) \rangle \sim \frac{\delta}{\Gamma} \tau. \quad (2.49)$$

Small time behaviour is

$$\langle x(\tau) \rangle \sim -\frac{\delta}{2} \tau^2. \quad (2.50)$$

2.2.2.2 Small s - analysis, long time behaviour:

For small s we mean $s \ll \Gamma$ and $\sinh \pi \nu \simeq \sinh(\frac{\pi \Gamma}{\delta}) + s \frac{\pi}{\delta} \cosh(\frac{\pi \Gamma}{\delta})$, we obtain

$$\langle x(\tau) \rangle \simeq \frac{\delta \Gamma}{\Gamma^2 + \delta^2} \left[\frac{\sinh(\frac{\pi \Gamma}{\delta})}{\sinh(\frac{\pi \Gamma}{\delta}) - \frac{\pi \Gamma}{\delta}} \right] \quad (2.51)$$

For $\delta \ll \Gamma$, the Eq.(51) gives $\langle x(\tau) \rangle \sim \frac{\delta}{\Gamma} \tau$ as expected.

It is readily seen that the mean displacement is zero when δ is zero, i.e., when all the lattice sites have same energy (as in case A with no bias). Also, the expression for $\langle x(\tau) \rangle$ is an odd function of δ , as it should be.

The expression for mean-squared displacement ($x^2(s)$) in the s -domain is obtained by doubly differentiating Eq.(2.44). We obtain

$$\begin{aligned} \langle x^2(s) \rangle = & - \left[\frac{\partial^2 \bar{\chi}(u, s)}{\partial u^2} \right]_{u=0} = \frac{I''_u - I''_{3u}}{\delta + \Gamma(I_u - I_{3u}) - \delta I_{1u}} + 2 \frac{[I'_{3u} - I'_u][\Gamma(I'_u - I'_{3u}) - \delta I'_{1u} - i]}{[\delta + \Gamma(I_u - I_{3u}) - \delta I_{1u}]^2} \\ & + \frac{[I_{3u} - I_u][\Gamma(I''_u - I''_{3u}) - \delta I''_{1u} - 2iI_{1u} + I_{1u}/\delta]}{[\delta + \Gamma(I_u - I_{3u}) - \delta I_{1u}]^2} - \frac{2[I_{3u} - I_u][\Gamma(I'_u - I'_{3u}) - \delta I'_{1u} - iI_{1u}]^2}{[\delta + \Gamma(I_u - I_{3u}) - \delta I_{1u}]^3} \quad (2.52) \end{aligned}$$

The values of various integrals are given in the Appendix(2A).

2.2.3 Non-zero bias and non-zero drive

Next we consider one more physically realizable case in which the lattice is present in an external electromagnetic drive (semiconductor heterostructure superlattice with Stark-Wannier ladder of states present in a tunable laser field). Our model is described by a tight-binding one-band hamiltonian

$$H^\omega = -E_0 \cos \omega t \sum_l [|l\rangle \langle l+1| + |l+1\rangle \langle l|] + \sum_l \alpha l |l\rangle \langle l|, \quad (2.53)$$

with the evolution master equation

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H^\omega, \rho] + \frac{\Gamma}{2} \sum_l \{ 2L_l \rho L_l - \rho L_l L_l - L_l L_l \rho \} \quad (2.54)$$

where $E_0 \cos \omega t$ is the time dependent drive of amplitude E_0 and circular frequency ω , appearing as the nearest-neighbour transfer matrix element. It may be noted, that in

the limit $w = 0$, this simulates the usual transfer matrix element $-V$. In terms of the matrix elements

$$\frac{\partial \rho_{mn}}{\partial t} = \frac{iE_0}{\hbar} [\rho_{m+1,n} + \rho_{m-1,n} - \rho_{m,n-1} - \rho_{m,n+1}] - i\frac{\alpha}{\hbar} (m\rho_{mn} - n\rho_{mn}) - \gamma[1 - \delta_{mn}]\rho_{mn} \quad (2.55)$$

The quantities $\frac{E_0}{\hbar}$, $\frac{\alpha}{\hbar}$, and γ have a dimension of $time^{-1}$. So, we define the scaled quantities $t_0 = \frac{\hbar}{E_0}$, $\delta = \frac{\alpha}{E_0}$, $\Gamma = t_0\gamma$, and $\tau = \frac{t}{t_0}$. With this we have

$$\frac{\partial \rho_{mn}}{\partial \tau} = i \cos \omega \tau [\rho_{m+1,n} + \rho_{m-1,n} - \rho_{m,n-1} - \rho_{m,n+1}] - i\delta (m\rho_{mn} - n\rho_{mn}) - \Gamma[1 - \delta_{mn}]\rho_{mn} \quad (2.56)$$

After applying the rotating-wave approximation[18] with $\rho_{mn} = \bar{\rho}_{mn} e^{-i\delta(m-n)\tau}$ the evolution of reduced density matrix in co-ordinate space is

$$\frac{\partial \bar{\rho}_{mn}}{\partial \tau} = \frac{i}{2} \left[e^{i(\theta-\delta)\tau} [\bar{\rho}_{m+1,n} - \bar{\rho}_{m,n-1}] + e^{-i(\theta-\delta)\tau} [\bar{\rho}_{m-1,n} - \bar{\rho}_{m,n+1}] \right] - \Gamma \bar{\rho}_{mn} [1 - \delta_{mn}] \quad (2.57)$$

Here, we have defined $\theta = \omega t_0$, $\tau = t/t_0$, and $t_0 = \hbar/E_0$. By writing, $\mathbf{A} = \theta - \delta$. The \mathbf{A} is the detuning $\omega t_0 - \delta$ between drive frequency w and scaled energy level spacing δ . Noting that

$$\begin{aligned} \bar{\rho}(k_1, k_2, \tau) &= \sum_{m,n} \bar{\rho}_{mn}(\tau) e^{-imk_1} e^{ink_2} \\ \bar{\rho}_{m,n+1} &\rightarrow e^{-ik_2} \bar{\rho}(k_1, k_2, \tau), \quad \bar{\rho}_{m+1,n} \rightarrow e^{ik_1} \bar{\rho}(k_1, k_2, \tau) \\ \bar{\rho}_{m,n-1} &\rightarrow e^{ik_2} \bar{\rho}(k_1, k_2, \tau), \quad \bar{\rho}_{m-1,n} \rightarrow e^{-ik_1} \bar{\rho}(k_1, k_2, \tau), \end{aligned} \quad (2.58)$$

we obtain

$$\frac{\partial \bar{\rho}(k_1, k_2, \tau)}{\partial \tau} = (i[\cos(k_1 + \Delta\tau) - \cos(k_2 + \Delta\tau)] - \Gamma) \bar{\rho}(k_1, k_2, \tau) + \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} \bar{\rho}(k_1 - q, k_2 - q, \tau) dq. \quad (2.59)$$

Performing the co-ordinate transformations $p = (k_1 + k_2)/2$, $u = k_2 - k_1$ and defining $\bar{\rho}(k_1, k_2, \tau) \equiv \varrho(p, u, \tau)$, we have

$$\frac{\partial \varrho(p, u, \tau)}{\partial \tau} = [2i \sin(p + \Delta\tau) \sin(u/2) - \Gamma] \varrho(p, u, \tau) + \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} \varrho(p - q, u, \tau) dq. \quad (2.60)$$

By defining

$$\bar{\chi}(u, \tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varrho(p - q, u, \tau) dq = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varrho(q, u, \tau) dq, \quad (2.61)$$

$$\varphi(p, u, \tau) = - \int [2i \sin(p + \Delta\tau) \sin(u/2) - \Gamma] d\tau = \frac{2i}{\Delta} \cos(p + \Delta\tau) \sin(u/2) + \Gamma\tau, \quad (2.62)$$

the solution of the first order P.D.E (Eq.(2.60)) is

$$\varrho(p, u, \tau) = \Gamma e^{-\varphi(p, u, \tau)} \int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau + C_1(p, u) e^{-\varphi(p, u, \tau)}. \quad (2.63)$$

Summing over p, we get

$$\sum_p \varrho(p, u, \tau) = \Gamma \sum_p e^{-\varphi(p, u, \tau)} \int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau + \sum_p C_1(p, u) e^{-\varphi(p, u, \tau)}, \quad (2.64)$$

or

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dp \varrho(p, u, \tau) = \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} dp e^{-\varphi(p, u, \tau)} \int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau + \frac{1}{2\pi} \int_{-\pi}^{\pi} dp C_1(p, u) e^{-\varphi(p, u, \tau)}, \quad (2.65)$$

or

$$\bar{\chi}(u, \tau) = \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} e^{-\varphi(p, u, \tau)} I_4 dp + \frac{1}{2\pi} \int_{-\pi}^{\pi} C_1(p, u) e^{-\varphi(p, u, \tau)} dp, \quad I_4 = \int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau. \quad (2.66)$$

To calculate $C_1(p, u)$, we put $\tau = 0$ in Eq.(2.63), and use the initial condition (Eq.(4)), i.e., $\bar{\rho}_{mn}(t = 0) = \rho_{mn}(t = 0) = \delta_{m0} \delta_{n0}$. We get

$$\varrho(p, u, \tau = 0) = \sum_{m, n} \bar{\rho}_{mn}(0) e^{-im(p-u/2)} e^{in(p+u/2)} = \sum_{m, n} \delta_{m0} \delta_{n0} e^{-im(p-u/2)} e^{in(p+u/2)} = 1. \quad (2.67)$$

Thus,

$$C_1(p, u) = e^{\varphi(p, u, 0)} - \Gamma I_{4\tau}, \quad I_{4\tau} = \left[\int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau \right]_{\tau=0}. \quad (2.68)$$

Equations (2.66), and (2.68) give

$$\begin{aligned} \bar{\chi}(u, \tau) &= \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} e^{-\varphi(p, u, \tau)} I_4 dp + \frac{1}{2\pi} \int_{-\pi}^{\pi} [e^{\varphi(p, u, 0)} - \Gamma I_{4\tau}] e^{-\varphi(p, u, \tau)} dp, \\ I_4 &= \int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau, \\ I_{4\tau} &= \left[\int e^{\varphi(p, u, \tau)} \bar{\chi}(u, \tau) d\tau \right]_{\tau=0}, \\ \varphi(p, u, \tau) &= \frac{2i}{\Delta} \cos(p + \Delta\tau) \sin(u/2) + \Gamma\tau. \end{aligned} \quad (2.69)$$

Noting that $\langle x(\tau) \rangle = i \frac{\partial \bar{\chi}(u, \tau)}{\partial u} \Big|_{u=0}$, in order to calculate the mean displacement, we differentiate w. r. t. u and set $u = 0$. Using Eq.(2.61) we finally obtain(Apendex(2B))

$$\langle x(\tau) \rangle = \left[\frac{\partial \bar{\chi}(u, \tau)}{\partial u} \right]_{u=0} = 0. \quad (2.70)$$

Thus, the mean displacement is zero in the rotating wave approximation even for non-zero bias. To calculate mean-squared displacement $\langle x^2(\tau) \rangle = - \left[\frac{\partial^2 \bar{\chi}(u, \tau)}{\partial u^2} \right]_{u=0}$, we solve Eq.(2.69) by doubly differentiating it w. r. t. (u) setting $u = 0$ (Apendex(2c)), and get

$$\begin{aligned} \left[\frac{\partial^2 \bar{\chi}(u, \tau)}{\partial u^2} \right]_{u=0} &= \Gamma e^{-\Gamma \tau} \int e^{\Gamma \tau} \left[\frac{\partial^2 \bar{\chi}(u, \tau)}{\partial u^2} \right]_{u=0} d\tau - \frac{1}{(\Delta^2 + \Gamma^2)} (1 - e^{-\Gamma \tau} \cos \Delta \tau) \\ &+ \frac{\Gamma}{\Delta(\Delta^2 + \Gamma^2)} e^{-\Gamma \tau} \sin \Delta \tau - 2\pi \left[\int e^{\Gamma \tau} \left[\frac{\partial^2 \bar{\chi}(u, \tau)}{\partial u^2} \right]_{u=0} d\tau \right]_{\tau=0} e^{-\Gamma \tau}. \end{aligned} \quad (2.71)$$

Equation (2.71) is solved by the Laplace Transform method (Apendex(2c)). We finally obtain

$$\langle x^2(\tau) \rangle = \frac{\Gamma}{\Gamma^2 + \Delta^2} \tau + \left[\frac{\Delta^2 - \Gamma^2}{(\Delta^2 + \Gamma^2)^2} \right] \{1 - e^{-\Gamma \tau} \cos \Delta \tau\} - \frac{2\Gamma\Delta}{(\Delta^2 + \Gamma^2)^2} e^{-\Gamma \tau} \sin \Delta \tau. \quad (2.72)$$

The above equation (Eq.(2.72))is an important result of the present work. The mean squared displacement from above equation is plotted in Figs. 2.4,2.5 and 2.6.

For two special cases of interest, equation (2.72) gives: (A) On-resonance, i.e., $\Delta = \frac{\hbar\omega - \alpha}{E_0} = 0$,

$$\langle x^2(t) \rangle = \frac{E_0^2}{\hbar^2 \gamma} t, \quad (\text{diffusive}). \quad (2.73)$$

(B) Off-resonance and long time t ,

$$\langle x^2(t) \rangle = \frac{E_0^2 \gamma}{\hbar^2 \gamma^2 + (\hbar\omega - \alpha)^2} t, \quad (\text{diffusive - controllable}) \quad (2.74)$$

which indicates diffusion, but with a diffusion constant

$$D = \frac{E_0^2 \gamma}{2[\hbar^2 \gamma^2 + (\hbar\omega - \alpha)^2]}, \quad (2.75)$$

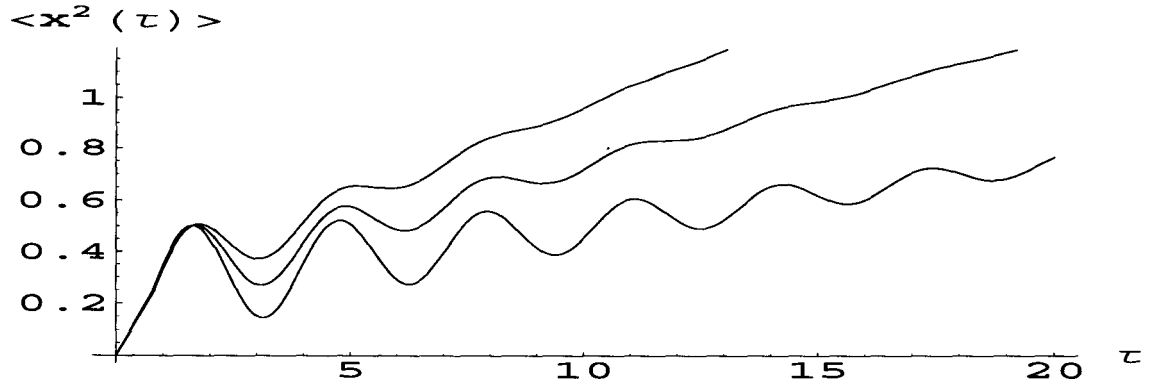


Figure 2.4: The effect of dimensionless damping Γ on mean-squared displacement. The top most curve is for high damping case $\Gamma = 0.3$, central for $\Gamma = 0.2$, and lowest for $\Gamma = 0.1$. With constant detuning parameter $\mathbf{A} = 2$. As the damping decreases, the oscillations in the mean-squared displacement increase, but after a long time oscillations vanish and the mean-squared displacement goes linearly with time as it should. Note that, $\Gamma = \gamma \frac{\hbar}{E_0}$, $r = t \frac{E_0}{\hbar}$, and $\mathbf{A} = \frac{\hbar\omega - \alpha}{E_0}$.

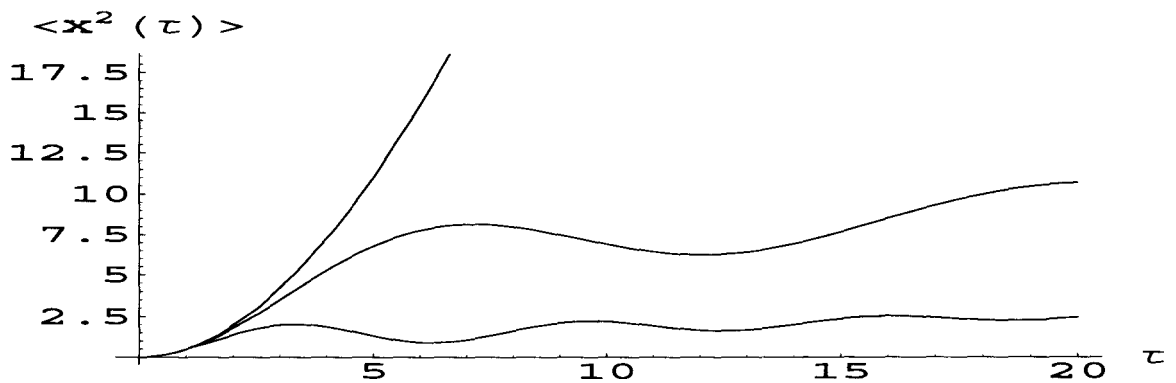


Figure 2.5: Shows the effect of detuning (\mathbf{A}) on mean-squared displacement. The top most curve is for the resonance case, detuning $\mathbf{A} = 0$; central for $\mathbf{A} = 0.5$; and the lowest for $\mathbf{A} = 1$. Damping parameter $\Gamma = 0.08$. As the detuning goes up, the oscillations in the mean-squared displacement increase, but the latter has the same overall evolution, namely, short time τ^2 rise, followed by oscillations, and finally the mean-squared displacement goes linearly with time.

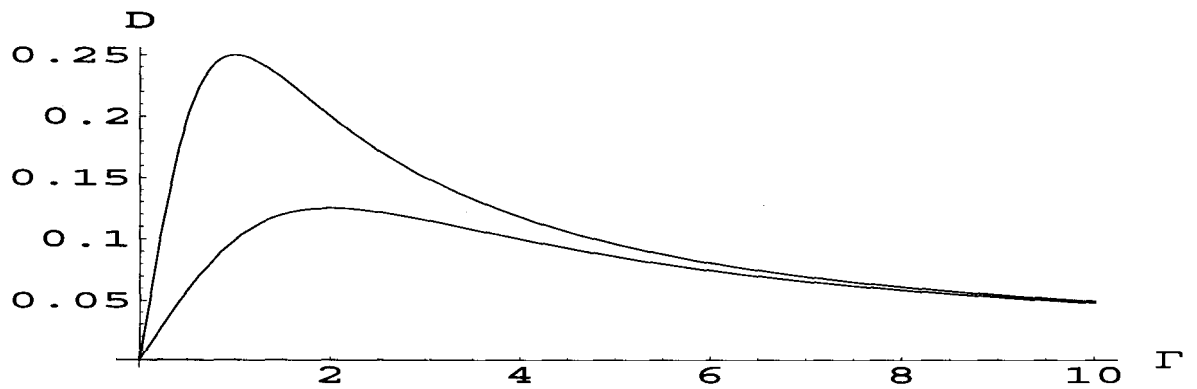


Figure 2.6: Diffusion coefficient D in the long time (classical) regime as a function of the damping parameter Γ (scaled system-bath coupling parameter). It is a maximum for the detuning parameter \mathbf{A} equal to damping parameter Γ , i.e., when the drive frequency $\omega_c = \gamma + \alpha/\hbar$.

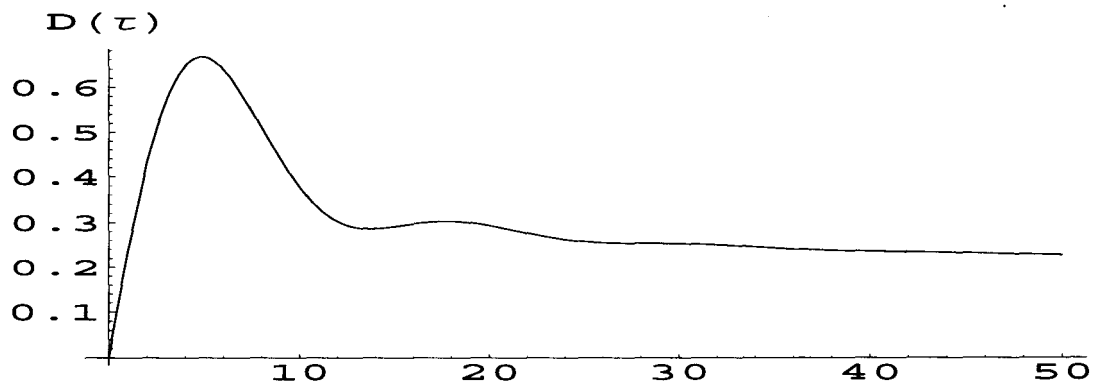


Figure 2.7: Diffusion coefficient D as a function of scaled time τ . It oscillates initially and in the long time limit it takes on a constant value. Here, we have taken $\Gamma = 0.1$, and $\mathbf{A} = 0.5$.

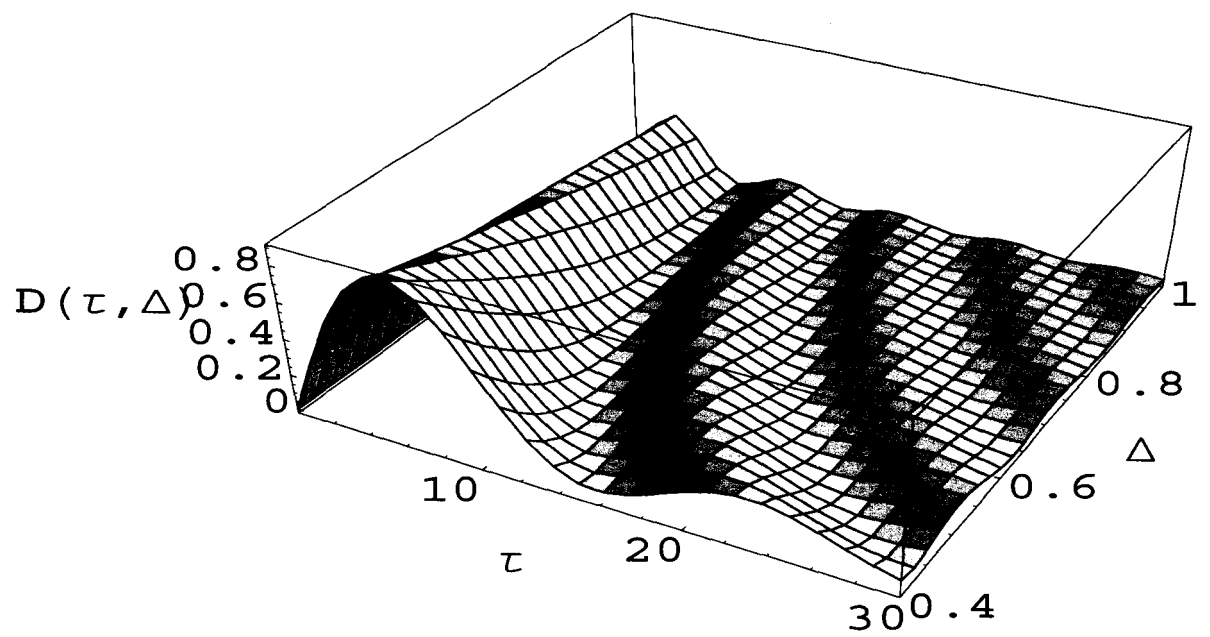


Figure 2.8: Diffusion coefficient D vs scaled time τ and detuning parameter Δ . It is clear that after the initial oscillations, diffusion coefficient takes on a constant classical value in the long time limit. Here $\Gamma = 0.01$.

tunable with the external drive. This is one of the main results of this work. The energy-level spacing ' α ' between the sites can be controlled by the external electrostatic field \mathbf{E} as $\alpha = e\mathbf{E}\cdot\mathbf{a}$, where \mathbf{a} is the lattice vector[14]. Thus α and w act as control parameters in an experiment. Diffusion coefficient becomes maximum at

$$\omega_c = \gamma + \alpha/\hbar. \quad (2.76)$$

All our analytical expressions specialize correctly to the earlier exact results in the proper limits.

2.3 Discussion

We have studied the quantum diffusion on a dynamically disordered lattice described by a tight-binding one-band Hamiltonian in the presence of static bias and harmonic drive. The usual site-diagonal gaussian white noise is replaced by a set of Lindblad operators that project on to the sites and cause decoherence. With the Lindblad master equation we reproduce several known exact results based on the Gaussian white noise stochastic models. An interesting new result obtained by us is the nonlinear enhancements of the diffusion coefficient $\gamma E_0^2/2(\hbar\gamma^2 + (\text{fiw} - \alpha)^2)$ with increasing drive amplitude, and its variation with the frequency-detuning relative to the inter-site energy gap which is tunable by the static bias. A physical realization of the above model is the Stark-Wannier (SW) ladder of states localized in the potential wells of a heterostructure superlattice (SL), subjected simultaneously to an electrostatic field \mathbf{E} (the bias) normal to plane of the layers and an electromagnetic excitation (the drive). Unlike the usual atomic lattices, for the SW states in the SL, the energy-level spacing ' α ' between the sites can be controlled by the strong external electrostatic field \mathbf{E} as $\alpha = e\mathbf{E}\cdot\mathbf{a}$, where \mathbf{a} is the lattice vector.¹³ Thus the mean displacement is controllable. Our expressions correctly specialize to the known exact results in the limit of zero drive and zero bias.

2.4 Appendix(2A)

The various s-domain integrals and their derivatives for calculating the mean-squared displacement in Section 2.2.2 (Eq.2.52) are:

$$I = \frac{1}{2\hbar} \int_{-\tau}^{\pi} e^{-\phi(p,u)} I_2 dp, \quad I_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\phi(p,u)} dp, \quad I_2 = \int e^{\phi(p,u)} dp, \quad I_{2p} = [I_2]_{p=0}, \quad I_3 = I_1 I_{2p}.$$

Here, $\phi(p, u) = -\nu p + i f \cos p$, $\nu = \frac{s+\Gamma}{\delta}$, $f = \frac{2}{\delta} \sin(u/2)$, $z = \frac{\sinh \pi \nu}{\delta}$.

$$I_{1u} = z, \quad I'_{1u} = \frac{i\nu^2}{6(\nu^2 + 1)}, \quad I_{2up} = -\frac{1}{\nu}, \quad I'_{2u} = \frac{ie^{-\nu p}(\sin p - \nu \cos p)}{\delta(\nu^2 + 1)},$$

$$I'_{2up} = \frac{-i\nu}{\delta(\nu^2 + 1)}, \quad I_{3u} = -z/\nu, \quad I'_{3u} = -z \frac{2i\nu}{\delta(\nu^2 + 1)}, \quad I_u = -1/\nu, \quad I'_u = 0, \quad I''_{1u} = -\frac{z}{\delta^2} \frac{\nu^2 + 2}{\nu^2 + 4},$$

$$I''_u = \frac{1}{\delta^2 \nu (\nu^2 + 1)}, \quad I''_{2pu} = \frac{\nu^2 + 2}{\nu \delta^2 (\nu^2 + 4)}, \quad I''_{3u} = \frac{2z}{\nu \delta^2} \frac{\nu^2 + 2}{\nu^2 + 4} + \frac{2\nu^3 z}{\delta^2 (\nu^2 + 1)^2}.$$

2.5 Appendix(2B)

On differentiating the equation(2.67) with respect to (u) and putting $u = 0$, we get

$$\begin{aligned} \frac{\partial u \bar{\chi}(u, \tau)}{\partial u} \Big|_{u=0} &= \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} [e^{-\Gamma\tau} (-\frac{i}{\Delta} \cos(p + \Delta\tau)) I_{4u} + e^{-\Gamma\tau} I'_{4u}] dp \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} [e^{-\Gamma\tau} (\frac{i}{\Delta} \cos p - \frac{i}{\Delta} \cos(p + \Delta\tau))] dp - \frac{\Gamma}{2\pi} \int_{-\pi}^{\pi} [e^{-\Gamma\tau} (-\frac{i}{\Delta} \cos(p + \Delta\tau)) I_{4\tau u} \\ &+ e^{-\Gamma\tau} I'_{4\tau u}] dp. \end{aligned} \quad (2.77)$$

We have defined (Eq.(2.61))

$$\bar{\chi}(u, \tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varrho(q, u, \tau) dq = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n} \rho_{mn}(\tau) e^{i\delta(m-n)\tau} e^{-im(p-u/2)} e^{in(p+u/2)} dp.$$

The value of $\bar{\chi}(u, \tau)$ at $u = 0$ comes out to be

$$\bar{\chi}(0, \tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n} \rho_{mn}(\tau) e^{i\delta(m-n)\tau} e^{i(n-m)p} dp = \sum_{m,n} \delta_{mn} \rho_{mn}(\tau) e^{i\delta(m-n)\tau} = 1. \quad (2.78)$$

The integrals in Eq.(2.78) have the values

$$I_{4u} = \int e^{\Gamma\tau} \bar{\chi}(0, \tau) d\tau = \frac{e^{\Gamma\tau}}{\Gamma},$$

$$I'_{4u} = \frac{i}{\Delta} \frac{\Gamma e^{\Gamma\tau}}{\Delta^2 + \Gamma^2} \left[\cos(p + \Delta\tau) + \frac{\Delta}{\Gamma} \sin(p + \Delta\tau) \right] + \int e^{\Gamma\tau} \chi(0, \tau)' d\tau.$$

The notation I_{4u} means $[I_4]_{u=0}$. Inserting the calculated integrals, we obtain

$$\chi'(0, \tau) = \Gamma e^{-\Gamma\tau} \left[\int e^{\Gamma\tau} \chi'(0, \tau) - \left(\int e^{\Gamma\tau} \chi'(0, \tau) \right)_{\tau=0} \right],$$

$$e^{\Gamma\tau} \chi'(0, \tau) = \Gamma \int e^{\Gamma\tau} \chi'(0, \tau) - \Gamma e^{\Gamma\tau} \left(\int e^{\Gamma\tau} \chi'(0, \tau) \right)_{\tau=0}.$$

Defining $g(\tau) = e^{\Gamma\tau} \chi'(0, \tau) d\tau$, the above equation becomes

$$g(\tau) = \Gamma \int g(\tau) d\tau + \text{constant}.$$

Which readily gives $\langle x(\tau) \rangle = \text{constant}$. Since $\langle x(\tau = 0) \rangle = 0$ we have

$$\langle x(\tau) \rangle = 0. \quad (2.79)$$

2.6 Appendix(2C)

In equation (2.71), we define $e^{\Gamma\tau} \left[\frac{\partial^2 \bar{\chi}(u, \tau)}{\partial u^2} \right]_{u=0} = f(\tau)$. With this, $f(\tau)$ takes the following form

$$f(\tau) = \Gamma \int f(\tau) d\tau - \frac{\Gamma e^{-\Gamma\tau}}{\Delta^2 + \Gamma^2} + \frac{\Gamma \cos \Delta\tau}{\Delta^2 + \Gamma^2} + \frac{\Delta \sin \Delta\tau}{\Delta(\Delta^2 + \Gamma^2)} + \text{constant}. \quad (2.80)$$

Differentiating the above integral equation, we get

$$\frac{df(\tau)}{d\tau} = \Gamma f(\tau) - \frac{\Gamma e^{\Gamma\tau}}{\Delta^2 + \Gamma^2} + \frac{\Gamma \cos \Delta\tau}{\Delta^2 + \Gamma^2} - \frac{\Delta \sin \Delta\tau}{(\Delta^2 + \Gamma^2)}. \quad (2.81)$$

With the initial condition $f(\tau = 0) = 0$, i.e., mean-squared displacement is zero at time $\tau = 0$, the above equation can be readily solved to get the mean-squared displacement as given in Eq.(2.72)