4 Heat transport through disordered harmonic chain

Since the seminal paper of Anderson [11], the physics of localization in disordered systems has now been studied for over half a century [12, 15, 19, 160]. Recently, there has been a renewed interest in this field with a lot of work on some open questions such as, the effect of interactions on localization [161–163], and the metal-insulator transition in two dimensions [164]. A number of recent experiments have also reported detailed studies on localization in varied systems such as heat conduction in an isotopically disordered nanotube [165] , electrons in a disordered carbon nanotube [166], photons in a waveguide [167] and sound localization in elastic networks [168]. The field is thus still filled with interesting questions and puzzles. In this chapter, we report our recent results on heat energy transport in disordered harmonic lattices. We consider quenched disorder. We study random mass harmonic lattices with scalar displacements in one dimension. The subject of heat conduction in disordered chains is now quite old; still the dependence of the heat current on boundary conditions and the bath spectral properties has been quite puzzling and has not been clearly understood. Here we have critically examined this problem both analytically and through numerics and believe that we now have a clear understanding of the issue [169]. Our general method of study is again through LEGF.

It is well known that all the eigenstates of an electron in a one-dimensional (uncorrelated) disordered potential are localized [12]. The electrical current, thus, decays exponentially with wire length, making it an insulator. In contrast, in phononic systems, for example, a disordered harmonic chain, the long wavelength modes are extended and can conduct a significant amount of heat. How good a heat conductor then is the disordered harmonic chain? The obvious question to ask is the system size (N) dependence of the disorder averaged steady state heat current, which we will denote by $\langle J \rangle$. It is expected that this has the form $\langle J \rangle \sim 1/N^{1-\alpha}$ so that the conductivity scales as $\kappa \sim N^{\alpha}$. The present chapter is organised as follows. In Sec. (4.1), we briefly review earlier work on heat conduction in a one dimensional isotopically disordered harmonic chain (IDHC)[14]. Then, in Sec. (4.2), we introduce the model of interest and give numerical results. Next, we do a qualitative analysis to understand better our numerical results in Sec. (4.3). In Sec. (4.4), we provide heuristic arguments and numerical evidence to show that $\langle J_n \rangle \sim 1/N^{n-1/2}$ for a finite (n) number of pinning centers with $2 \leq n \ll N$. In

Sec. (4.5), we comment on the nature of size dependence in case of quantum mechanical heat conduction in IDHC at low temperatures. We complete the chapter with a short discussion in Sec .(4.6) enlightening on experimental verification of our results and physical understanding of the phenomena.

4.1 Previous results

In an important work on the localization of normal modes in the IDHC, Matsuda and Ishii [15] (MI) showed that normal modes with frequencies $\omega \lesssim \omega_d$ were extended. For a harmonic chain of length N, given the average mass $m = \langle m_l \rangle$, the variance $\sigma^2 = \langle (m_l - m)^2 \rangle$ and interparticle spring constant k, it was shown that

$$\omega_d \sim \left(\frac{km}{N\sigma^2} \right)^{1/2} \tag{4.1}$$

They also evaluated expressions for thermal conductivity of a finite disordered chain connected to (a) white noise baths and (b) baths modeled by semi-infinite ordered harmonic chains (Rubin's model of bath). In the following we will also consider these two models of baths and refer to them as model(a) and model(b). For model(a) MI used fixed boundary conditions (BC) and the limit of weak coupling to baths, while for case (b) they considered free BC and this was treated using the Kubo formalism. They found $\alpha = 1/2$ in both cases, a conclusion which we will show is incorrect. The other two important theoretical papers on heat conduction in the disordered chain are those by Rubin and Greer [16] (RG)who considered model(b) and of Casher and Lebowitz [17] (CL) who used model(a) for baths. RG obtained a lower bound $\langle J \rangle \geq 1/N^{1/2}$ and gave numerical evidence for an exponent $\alpha = 1/2$ and this was later proved rigorously by Verheggen [170]. On the other hand, for model(a), CL found a rigorous bound $\langle J \rangle \geq 1/N^{3/2}$ and simulations by Visscher with the same baths supported the corresponding exponent $\alpha = -1/2$. In a more recent work [18], Dhar gave a unified treatment of the problem of heat conduction in disordered harmonic chains connected to baths modeled by generalized Langevin equations and showed that models(a,b) were two special cases. An efficient numerical scheme was proposed and used to obtain the exponent α and it was established that $\alpha = -1/2$ for model(a) (with fixed BC) and $\alpha = 1/2$ for model(b) (with free BC). It was also pointed out that in general, α depended on the spectral properties of the baths.

Here we apply the same formulation as developed in [18] to understand in detail the role of BCs' (and more generally the presence of pinning potentials) on heat transport in the IDHC connected to either white noise [model(a)] or Rubin baths [model(b)]. We show that with the same kind of pinning, the exponent α is the same for the two different bath models. The pinning potentials strongly scatter low frequency waves and hence can be expected to lower the heat current. Surprisingly, we find that even the exponent α changes with the number of pinning centers. We also provide expressions for the asymptotic value of $\langle J \rangle$ for various cases.

4.2 Model and results for zero, one and two pins

The Hamiltonian of the IDHC considered here is

$$H = \sum_{l=1}^{N} \frac{1}{2} m_l \dot{x}_l^2 + \sum_{l=1}^{N-1} \frac{1}{2} k (x_{l+1} - x_l)^2 + \frac{1}{2} k' (x_1^2 + x_N^2) \ , \tag{4.2}$$

where $\{x_l\}$ denotes the displacement of the particle at lattice site l. The random masses $\{m_l\}$ are chosen from a uniform distribution between $(m - \Delta)$ to $(m + \Delta)$. The strength of onsite potentials at the boundaries is k'. The particles at two ends are connected to heat baths at temperature T_L and T_R. The heat reservoirs are modelled by generalised Langevin equations [18, 33, 46]. The steady state classical heat current through the chain is given by:

$$\begin{aligned} J &= \frac{k_B(T_L - T_R)}{4\pi} \int_{-\infty}^{\infty} d\omega \mathcal{T}_N(\omega), \quad \text{where} \qquad (4.3) \\ \mathcal{T}_N(\omega) &= 4\Gamma(\omega)^2 |G_{1N}(\omega)|^2, \quad \hat{G}(\omega) = \hat{Z}^{-1}/k \\ \text{and} \ \ \hat{Z}(\omega) &= [-\omega^2 \hat{M} + \hat{\Phi} - \hat{\Sigma}(\omega) \ ]/k, \\ \text{with} \ \ \Phi_{lm} &= (k + k')\delta_{l,m} - k\delta_{l,m-1} \quad \text{for} \ \ l = 1, \\ &= -k\delta_{l,m+1} + 2k\delta_{l,m} - k\delta_{l,m-1} \quad \text{for} \ \ 1 < l < N, \\ &= (k + k')\delta_{l,m} - k\delta_{l,m+1} \quad \text{for} \ \ l = N, \\ \Sigma_{lm}(\omega) &= \delta_{l,m}[\Sigma_{11}(\omega)\delta_{l,1} + \Sigma_{NN}(\omega)\delta_{l,N}], \\ M_{lm} &= m_l \delta_{l,m}, \end{aligned}$$

where $\hat{M}$ and $\hat{\Phi}$ are respectively the mass and force matrix for the harmonic chain and $\hat{G}$ is the Green's function of the chain connected to baths. The self-energy correction in the Green's function $\hat{\Sigma}$, coming from the baths, is a $N \times N$ matrix whose only non-zero elements are $\Sigma_{11} = \Sigma_{NN} = \Sigma(\omega)$ and $\Gamma(\omega) = Im[\Sigma]$. For white noise baths $\Sigma(\omega) = -i\gamma\omega$ where γ is the coupling strength with the baths, while in case of Rubin's baths $\Sigma(\omega) = k\{1 - m\omega^2/2k - i\omega(m/k)^{1/2}[1 - m\omega^2/(4k)]^{1/2}\}$. We have assumed that the RG bath has spring constant k and equal masses m. We note that $\mathcal{T}_N(\omega)$ is the transmission coefficient of phonons through the disordered chain. To extract the asymptotic N dependence of $\langle J \rangle$ we need to determine the Green's function element $G_{1N}(\omega)$. It is convenient to write the matrix elements $Z_{11} = -m_1\omega^2/k + 1 + k'/k - \Sigma/k = -m_1\omega^2/k + 2 - \Sigma'$ where $\Sigma' = \Sigma/k - k'/k + 1$ and similarly $Z_{NN} = -m_N\omega^2/k + 2 - \Sigma'$. Following the techniques used in [17, 18] we have

$$\begin{aligned} |G_{1N}(\omega)|^2 &= k^{-2}|\Delta_N(\omega)|^{-2} \ \ \text{with} \qquad (4.4) \\ \Delta_N(\omega) &= D_{1,N} - \Sigma'(D_{2,N} + D_{1,N-1}) + \Sigma'^2 D_{2,N-1} \end{aligned}$$

where $\Delta_N(\omega)$ is the determinant of $\hat{Z}$ and the matrix elements $D_{l,m}$ are given by the following product of (2×2) random matrices $\hat{T}_l$:

$$\hat{D} = \begin{pmatrix} D_{1,N} & -D_{1,N-1} \\ D_{2,N} & -D_{2,N-1} \end{pmatrix} = \hat{T}_1 \hat{T}_2 .... \hat{T}_N \quad \text{where} \quad \hat{T}_l = \begin{pmatrix} 2 - m_l\omega^2/k & -1 \\ 1 & 0 \end{pmatrix} \tag{4.5}$$

We note that the information about bath properties and boundary conditions are now contained entirely in $\Sigma'(\omega)$ while $\hat{D}$ contains the system properties. It is known that $|D_{l,m}| \sim e^{cN\omega^2}$ for $|l-m| \sim N$ [15], where c is a constant, and so we need to look only at the low frequency ($\omega \lesssim 1/N^{1/2}$) form of Σ' . We now proceed to examine various cases. For model(a) free BC correspond to $k' = 0$ and so $\Sigma' = 1 - i\gamma\omega/k$ while for model(b) free boundaries correspond to $k' = k$ and this gives, at low frequencies, $\Sigma' = 1 - i(m/k)^{1/2}\omega$. Other values of k' correspond to pinned boundary sites with an onsite potential $k_o x^2/2$ where $k_o = k'$ for model(a) and $k_o = k' - k$ for model(b). The main difference, from the unpinned case, is that now $Re[\Sigma'] \neq 1$. The arguments of [18] then immediately give $\alpha = 1/2$ for free BC and $\alpha = -1/2$ for fixed BC for both bath models. In fact for the choice of parameters $\gamma = (mk)^{1/2}$ we expect, for large system sizes, the actual values of the current to be the same for both bath models. This can be seen in Fig. (4.1) where we show the system size dependence of the current for the various cases. The current was evaluated numerically using Eq. (4.3) and averaging over many realizations ($\sim 4 - 100$). We also show the exact asymptotic forms for the current which we will discuss later. Note that for free BC, the exponent $\alpha = 1/2$ settles to its asymptotic value at relatively small values ($N \sim 10^3$) while, with pinning, we need to examine much longer chains ($N \sim 10^5$). We also find that the presence of a single pinning centre in the IDHC is sufficient to change the value of α from $1/2$ to $-1/2$ [see Fig. (4.2)]. These results clearly show that, for both models(a,b), the exponent α is the same and is controlled by the presence or absence of pinning in the IDHC.

4.3 Qualitative analysis and asymptotic expressions

As mentioned before only modes $\omega \lesssim \omega_d$ are involved in conduction. It was noted in [18] that in this low frequency regime we can approximate $\langle \mathcal{T}_N(\omega) \rangle$ by the transmission coefficient of the ordered chain $\mathcal{T}_N^O(\omega)$. We can understand it analysing the transfermatrix T_l of disordered chain. We write $(1,1)$ term of the transfer matrix as, $[T_l]_{11} = 2 - \langle m \rangle \omega^2 - \delta m_l \omega^2$. In $\omega \to 0$, we can now approximate $\langle \hat{T} \rangle = \hat{T}^O$ where $\hat{T}^O$ is the transfer matrix of the ordered chain of mass $\langle m \rangle$. We then obtain

$$\langle J \rangle \sim (T_L - T_R) \int_0^{\omega_d} \mathcal{T}_N^O(\omega) d\omega \ . \tag{4.6}$$

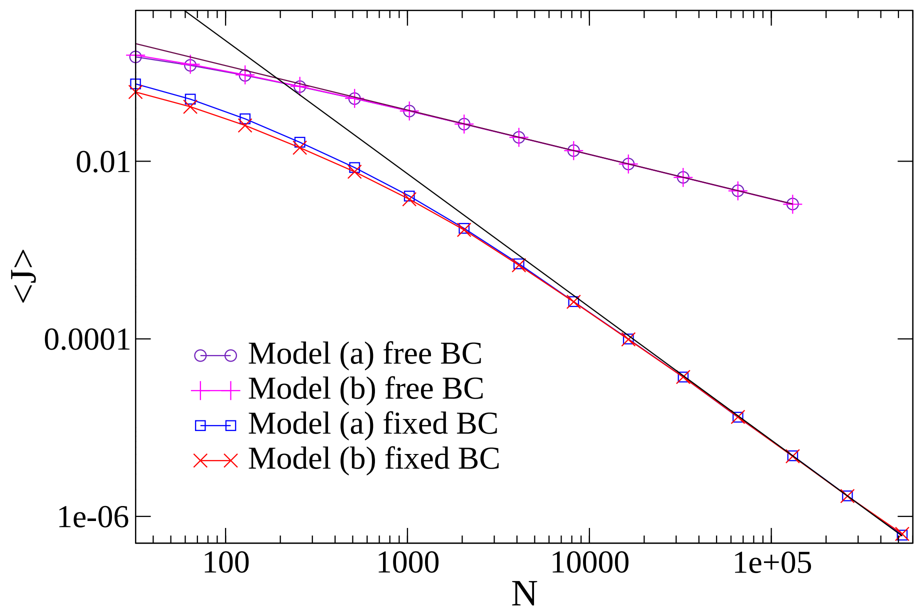

FIGURE 4.1: Plot of $\langle J \rangle$ versus N for free BC ($n = 0$) and fixed BC ($n = 2$). Results are given for both models(a,b) of baths. The two straight lines correspond to the asymptotic expressions given in Eqs. (4.10,4.11). We used parameters $m = 1,\ \Delta = 0.5,\ k = 1,\ \gamma = 1,\ T_L = 2,\ T_R = 1$ and $k_o = 1$. The error in the measurements is much smaller than the size of the symbols.

For model(a), $\mathcal{T}_N^O$ in the limit $N \to \infty$ is effectively given by [42]:

$$\mathcal{T}^O(\omega) = \frac{\gamma\omega^2\sqrt{4mk - m^2\omega^2}}{k'^2 + (\gamma^2 + m(k - k'))\omega^2} \ . \tag{4.7}$$

We then find, for free BC ($k' = 0$) $\mathcal{T}^O(\omega) \sim 1$ while for fixed BC ($k' \neq 0$), $\mathcal{T}^O(\omega) \sim \omega^2$. Using Eq. (4.6) then immediately gives the asymptotic N dependence for the two BCs'. Our results are valid even in the weak coupling limit $\gamma << 1$ and this means that the result given by MI for model(a) in the weak coupling limit is incorrect. Our numerics supports this conclusion. The transmission coefficient of the ordered chain in the presence of a single pinning site can also be computed using the same method of [42]. For a single pin at any of the boundary, it is given as

$$\begin{aligned} \mathcal{T}_\infty^O(\omega) &= \frac{2\gamma\omega^2\sqrt{4mk - m^2\omega^2}}{\sqrt{4\omega^4\Lambda^2 + k'\Omega(k'\Omega + 4\omega^2\Lambda)}}, \\ \text{with} \quad \Lambda &= \gamma^2 + km, \quad \Omega = k' - m\omega^2. \end{aligned} \tag{4.8}$$

We again find $\mathcal{T}^O \sim \omega^2$ for $k' \neq 0$ at one site. For model(b), the transmission coefficient of the ordered chain, pinned at the two boundary sites with $k_o = k' - k$ is given effectively by (as $N \to \infty$):

$$\mathcal{T}^O(\omega) = \frac{2k^2 \sin^2 q}{2k^2 \sin^2 q + k_o^2} \ , \tag{4.9}$$

where $\omega = 2(k/m)^{1/2}\sin(q/2)$. As expected, for $k_o = 0$ we have $\mathcal{T}^O = 1$ while for $k_o \neq 0$, $\mathcal{T}^O \sim \omega^2$. The above qualitative analysis thus shows that the effect of introducing pinning potentials is to pinch the band of conducting modes (between $0 - \omega_d$) from the zero frequency side and thus lower $\langle J \rangle$.

Our asymptotic analysis also allows us to make predictions, on the dependence of $\langle J \rangle$, on various system parameters such as mass variance, spring constant etc. From Eq. (4.6) and the forms of $\mathcal{T}^O(\omega)$ in various cases we get:

$$\langle J \rangle_{Fr} = A\ c\ \frac{k_B(T_L - T_R)}{\pi}\Big(\frac{km}{N\sigma^2}\Big)^{1/2} \qquad (4.10)$$

$$\langle J \rangle_{Fi} = A'\ c'\ \frac{k_B(T_L - T_R)}{\pi}\Big(\frac{km}{N\sigma^2}\Big)^{3/2}, \qquad (4.11)$$

where $c = 2\gamma(mk)^{1/2}/(\gamma^2 + mk),\ 1$ for model(a), model(b) respectively. For fixed boundaries we have $c' = \gamma(mk)^{1/2}/{k_o}^2,\ mk/{k_o}^2$ for model(a), model(b) respectively. A, A' are constant numbers. We find that for model(b) our numerical results agrees with an exact expression for $\langle J \rangle_{Fr}$ due to Papanicolau (apart from a factor of 2π) and this gives $A = \pi^{3/2}\int_0^\infty dt\ [t\sinh(\pi t)]/[(t^2 + 1/4)^{1/2}\cosh^2(\pi t)] \approx 1.08417$ (see [[170]]). We note that this differs from the expression given in [15]. For fixed boundaries we find numerically that $A' \approx 17.28$ and the fit is shown in Fig. (4.1).

4.4 More than two pins

Till now, using numerical results and heurestic arguments, we have arrived at the result that for a IDHC, in the absence of any pinning potential $\alpha = 1/2$ while the presence of a one or two pinned sites changes the exponent to $\alpha = -1/2$. This is true both for white noise and Rubin's bath. It is natural to now ask as to what happens in the presence of more number of pinning centers. It is expected that more pinning centers will lead to enhanced scattering of low frequency phonons and decrease the heat current but it is not obvious as to whether the exponent α changes. For a finite fraction of sites on the lattice having pinning potentials, it is known that $\langle J \rangle \sim e^{-cN}$ [163]. Here we investigate the case with a finite number, say n, of pinning sites. Numerically it becomes difficult to determine α for $n > 4$ as, with more pins, the heat current becomes very small at large system sizes and numerical errors become significant. In Fig. (4.2) we show numerical results for $n = 3, 4$, where the extra pinning potentials with $k_o = 1$ are placed in the bulk of the chain with equal separations. We find $\alpha \approx -1.03, -1.38$ respectively for $n = 3, 4$, which are clearly different from the $n = 1, 2$ value $\alpha = -0.5$. Let us now see what our earlier heuristic arguments give, for $n = 3$. We again find that the low frequency behaviour of $\Delta_N(\omega)$ are similar for the disordered and ordered lattices. Let us therefore find the form of Δ_N for the ordered case. Let $N = 2M + 1$ with the 1^{st}, $(M + 1)^{\mathrm{th}}$ and N^{th} sites being pinned. Except for $\hat{T}_{M+1} = \hat{T}'$ all the

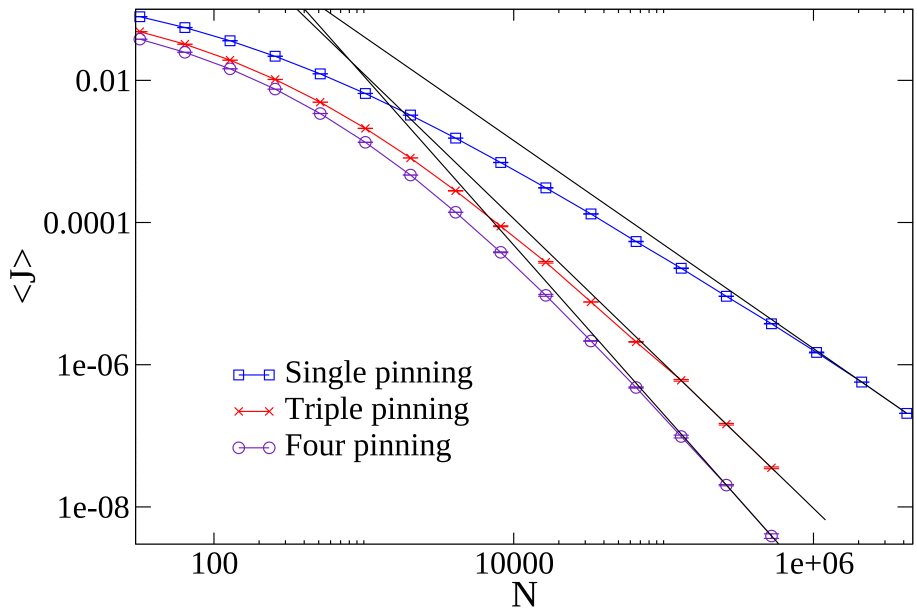

FIGURE 4.2: Plot of $\langle J \rangle$ versus N for $n = 1, 3, 4$ pinning centers for model(b). Parameters are same as in Fig. (4.1) and for these parameters model(a) results are almost indistinguishable for $N > 10^3$. The straight lines have slopes -1.47, -2.03 and -2.38. The error-bars shown are for disorder average and are of the same order as numerical errors.

other $\hat{T}_l$s' are identical and given by $\hat{T}$, say. The $(1,1)$ entry of $\hat{T}'$ is $2 + k_0/k - m_{(M+1)}\omega^2/k$ with k_0 being the strength of pinning at $(M+1)^{\text{th}}$ site and all other entries of $\hat{T}'$ are same as $\hat{T}$. If we denote $\hat{D}_N = \hat{T}^N$ and $\hat{D}'_N = \hat{T}^M \hat{T}' \hat{T}^M = \hat{D}_M \hat{T}' \hat{D}_M$, then using the fact that for the ordered lattice $D_{1N} = \sin q(N+1)/\sin(q)$, where $\cos(q) = 1 - m\omega^2/(2k)$, and carrying out the matrix multiplications above we find that at low frequencies D'_{1N} is larger than D_{1N} by a factor $\sim 1/\sin(q) \sim 1/\omega$. This means that $\mathcal{T}_N$ for the 3-pin case will have an extra factor of ω^2 compared to the 2-pin case. Correspondingly one expects, using Eq. (4.6), an exponent $\alpha = -3/2$. The argument can be extended to the case of $n \geq 2$ pins (two of which are in the boundaries) in which case we get

$$\alpha = 3/2 - n \; . \tag{4.12}$$

Our numerical results for $n = 3, 4$ [see Fig. (4.2)] are consistent with this prediction though we are not able to verify the precise value of the exponent.

Here we generalise an arguement orginally applied by Casher and Lebowitz [17] for IDHC with two pinning centers (fixed boundaries) connected with white noise baths. First we notice from Eqs.(4.3,A.7) that in the low ω regime where the IDHC conducts significantly, the dominant term in the denominator of the current expressions is $|D_{1,N}|^2$. We again consider the three pinning centers in the chain of length $N = 2M + 1$. To find an crude lower bound to the disorder averaged current we need to average over independent variables m_l. Next we take direct product of Eq.(4.5) and find that $(1,1)$ entry of the left side is $|D_{1,N}|^2$. Defining $Q_l = T_l \otimes T_l$ and $\langle Q_l \rangle = Q$, we find

$$\begin{aligned} & \langle \hat{D} \otimes \hat{D} \rangle = \langle Q_1 Q_2 .. Q'_{(M+1)} .. Q_N \rangle \\ = & \; U \underbrace{U^{-1} Q U} U^{-1} Q ... U \underbrace{U^{-1} Q' U} U^{-1} ... U \underbrace{U^{-1} Q U} U^{-1} \\ = & \; U \hat{\lambda}^M \underbrace{U^{-1} Q' U} \hat{\lambda}^M U^{-1} \;, \end{aligned} \tag{4.13}$$

where U is a (4×4) matrix construted from the eigenvectors of Q and $\hat{\lambda}$ is a (4×4) diagonal matrix formed by the eigenvalues of Q. Eigenvalues of Q have the form

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1/2} e^{i\theta}, \quad \lambda_3 = \lambda^{-1/2} e^{-i\theta}, \quad \lambda_4 = 1,$$

with $\lambda > 1$. In large N and $\omega^2 \to 0$ limit the dominant contribution of $\langle |D_{1N}|^2 \rangle$ comes from λ_1. We find $\langle |D_{1N}|^2 \rangle \simeq \omega^{-2} e^{cN\omega^2}$ where c is a numerical constant. We then use it in the current expressions (4.3),

$$\begin{aligned} \langle J \rangle & \simeq (T_L - T_R) \int_0^\infty \Big\langle \frac{1}{|D_{1,N}|^2} \Big\rangle \omega^2 d\omega \\ & \geq (T_L - T_R) \int_0^\infty \frac{\omega^2}{\langle |D_{1,N}|^2 \rangle} d\omega \\ & \geq (T_L - T_R) \int_0^\infty \omega^4 e^{-cN\omega^2} \\ & \geq (T_L - T_R) \; \mathrm{O}[N^{-\frac{5}{2}}] \;. \end{aligned} \tag{4.14}$$

The above analysis can be carried on for more than three pinning centers and we find a general formula for the lower bound of disorder averaged thermal current in IDHC with finite n pinning centers.

$$\langle J \rangle \geq \;\; (T_L - T_R) \; \mathrm{O}[N^{-n+\frac{1}{2}}] \;\; \text{for } 2 \leq n << N. \tag{4.15}$$

4.5 Quantum regime

For a Hamiltonian of the form of Eq.(5.1) where now $\{x_l, p_l\}$ are Heisenberg operators, the steady state quantum heat current through the IDHC in the linear response regime is given by:

$$J_q = \frac{k_B(T_L - T_R)}{4\pi} \int_{-\infty}^{\infty} d\omega \mathcal{T}_N(\omega) \big(\frac{\hbar\omega}{2k_B T}\big)^2 \mathrm{cosech}^2 \big(\frac{\hbar\omega}{2k_B T}\big),$$

where $\mathcal{T}_N(\omega)$ is same as Eq.(4.3) and $T = (T_L + T_R)/2$. Following our derivation for the classical system we see that the asymptotic N dependence of $\langle J_q \rangle$ is determined by $\mathcal{T}_N(\omega)$ which is here exactly the same as the classical case. For any fixed temperature, however small, at suficiently large system sizes we will have $\hbar\omega_d << k_B T$, and hence within this cut-off frequency the factor $(\hbar\omega/k_B T)^2 \mathrm{cosech}(\hbar\omega/k_B T)^2 \rightarrow 1$. Hence for large system sizes we always get the classical result. The approach to the asymptotic behaviour though will be different.

4.6 Discussion

In real experiments heat baths usually have a finite bandwidth making the noise correlated, as in Rubin's model. Here we have shown that for heat conduction in the IDHC these noise correlations do not affect the exponent α (note that a bath for which $\Sigma(\omega)$ depends nonlinearly on ω at small frequencies can affect α). We have elucidated the role of boundary conditions and shown that the actual value of α depends on the number of pinned sites. Our results are also valid for bond disorder. We have provided explicit expressions for the currents which, apart from giving the system size-dependence, also give the dependence on various other parameters such as mass variance, coupling to baths etc. We also emphasize that heat conduction through IDHC is non-diffusive. Our physical understanding is as follows. In the presence of mass or bond disorder phonons are scattered coherently giving rise to localization and low transmission. Long wavelength phonons with $\omega \lesssim \omega_d$ [see Eq.(1)] are relatively unaffected and dominate heat conduction in such disordered materials. Now the introduction of pinning centers causes strong scattering of even the low frequency modes and, as we have shown, significantly reduces the current. We obtain the surprising and nontrivial result that the exponent α giving the system size dependence of current changes linearly with the number of pinning centers. There are now experimental measurements of heat conduction in one-dimensional systems such as nanotubes and nanowires [165, 171] and molecular wires [172]. At low temperatures one can neglect anharmonic effects and it will be interesting to see if our prediction of the strong reduction of heat current, by substrate potentials at localized points on a disordered wire, can be observed. While our results are for a simple classical model we expect the effect of pinning to be quite generic and should be true for systems with more complicated phonon dispersions. One relevant question is that how do quadratic pinning potentials at the boundaries affect the size dependence of the disorder averaged steady state current in higher dimensions? It is also important to check the fluctuations in the thermal conductance with different realizations of disorder. We find that for zero, single and double pinnings in the IDHC, the relative fluctuations in thermal conductance decay with system sizes. But relative thermal fluctuations become relevant even for longer length for heat conduction in the IDHC with more than two pinning centers.