Chapter 3

THERMODYNAMIC MOTIVATION

3.1 A STATISTICAL ANALOGY

We start this section by quoting Perelman. In his paper on "The entropy formula for the Ricci flow and its geometric application" [1], Perelman remarks (section 5.3): "The interplay between statistical physics and (pseudo) Riemannian geometry occurs in the subject of black hole thermodynamics developed by Hawking et al. Unfortunately, this subject is beyond my understanding at the moment". In this chapter we will provide a thermodynamic motivation for addressing issues in black hole physics using the Ricci flow.

Recall that the Ricci flow [2, 3, 1, 4] is an evolution equation for Riemannian geometries that tends to smooth and homogenize them. This flow is a (degenerate) parabolic differential equation, and is very similar to the heat equation. This suggests a thermodynamic analogy in which evolution of a geometry along the Ricci flow is like the approach of a physical system to thermodynamic equilibrium.

From a physical point of view, the most striking property of the Ricci flow (which it shares with the heat equation) is its tendency to lose memory of initial conditions. This is very similar to the approach of an isolated physical system to thermal equilibrium.

3.2 EXAMPLES OF GRADIENT FLOWS

In this section we give some examples to make the thermodynamic motivation clearer.

1. The first instance is that of a thermally insulated circular copper wire of circumference

L, which initially has a temperature distribution $T(x)$, $(T(0) = T(L), T(x) > 0)$ evolves according to the heat equation

$$\frac{\partial T}{\partial t} = \frac{d^2 T}{dx^2} \tag{3.1}$$

(by choice of units, we set the diffusion constant to one) and tends to a constant, losing memory of the details of the initial distribution $T(x)$. The final state is characterized by a single number, the uniform final temperature T_f and not an entire function $T(x)$. This "information loss" is very reminiscent of entropy increase and indeed, one can view the heat equation in this light. Consider the functional $S[T(x)] = \int_0^L dx a(T)$, where $a(T) = -T \log T$. Using the heat equation (3.1), integrating by parts and dropping a divergence we see that $S[T(x)]$ is monotonic along the flow. $\frac{dS}{dt} = \int dx a'(T) dT/dt = \int dx a'(T) d^2T/dx^2 = -\int dx a''(T)(dT/dx)^2 \geq 0$ The last inequality follows since $a(T)$ is a convex function $a'' < 0$.

2. Next we consider the equation of motion of a particle inside a viscous medium

$$\frac{dp}{dt} = -\gamma \frac{dq}{dt} - \frac{\partial V}{\partial q} \tag{3.2}$$

where p is the momentum and q is the position of the particle which is in a viscous medium experiencing a viscous force that is proportional to the velocity where γ is the proportionality constant. V is an external potential. In the over damped limit (the low Reynold's number limit) we can neglect the inertial term and take the LHS to be zero. Then the equation (3.2) becomes

$$\frac{dq}{dt} = -\frac{1}{\gamma} \frac{\partial V}{\partial q} \tag{3.3}$$

which shows that the trajectory of the particle will be decided by the gradient of the potential V and it will settle at the minimum of the potential where $\frac{\partial V}{\partial q} = 0$ which is the fixed point of the equation (3.3).

3. Another example is that of the theory of superconductors. We write the free energy functional of a superconductor as

$$F = \int d^3x \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 \right) \tag{3.4}$$

and the evolution of ψ is

$$\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \alpha\psi + \beta\,|\psi|^2\,\psi. \tag{3.5}$$

We expect that given any initial condition $\psi(x)$, the system will relax following equation (3.5) and will approach the minimum of the free energy (3.4).

In the three examples chosen above, the first was purely entropic, the second was purely energetic and the third was a flow along the gradient of the free energy.

3.3 A GRADIENT FORMULATION FOR THE RICCI FLOW

Following a similar argument as above we may think of considering the Ricci flow equation as the over damped limit of the Einstein's equation in GR where q is replaced by the metric functions h_{ij} and the RHS of the equation (3.3) is given by the the Ricci tensor R_{ij}.

Can we in some sense regard the Ricci flow as representing the increase of "entropy" ? that is to ask, " Is there a functional of the metric which increases monotonically along the Ricci flow?"

The answer is provided by Perelman's gradient formulation of the Ricci flow which is described in detail in the appendix B.

We briefly describe this formulation. Consider the Ricci flow, supplemented by a diffeomorphism $\mathcal{L}_\xi h_{ab} = D_a\xi_b + D_b\xi_a$ generated by a vector field ξ^a, which is itself a gradient $\xi_a = D_a f$, where f is a scalar function on Σ.

$$\frac{dh_{ab}}{d\tau} = -2R_{ab} + 2D_aD_b f. \tag{3.6}$$

Perelman gave a gradient formulation for this flow by considering the following "entropy functional" which depends on the pair (h_{ab}, f), of tensor fields on Σ.

$$\mathcal{F}_{\mathcal{P}}(h_{ab}, f) := \int_\Sigma d^3x\,\sqrt{h}\,(\exp f)\,[\,R + (Df)^2]. \tag{3.7}$$

(Here f is reversed in sign from Perelman's equations). There is a subtlety in the variation however: the degrees of freedom in the metric can be split up into a conformal factor and a conformal structure $[h_{ab}(x)]$, where the square brackets signify a conformal class. In the variation of $\mathcal{F}_{\mathcal{P}}$, the conformal structure is freely varied, but the conformal factor is subject to the constraint that the "distorted volume" $\sqrt{h}(\exp f)$ is held fixed. Performing a variation $\delta h_{ab} = \frac{dh_{ab}}{d\tau}\delta\tau$ and using the notation $\frac{dh^{ab}}{d\tau} = h^{ac}h^{bd}\frac{dh_{cd}}{d\tau}$, we find using a standard geometric identity for the variation of R,

$$\frac{dR}{d\tau} = -R_{ab}\frac{dh^{ab}}{d\tau} + D_a D_b \frac{dh^{ab}}{d\tau} - D^2(h_{ab}\frac{dh^{ab}}{d\tau}) \tag{3.8}$$

and after dropping some divergences.

$$\frac{d\mathcal{F}_{\mathcal{P}}}{d\tau} = \int_{\Sigma} d^3x\,\sqrt{h}\,(\exp f)[-R_{ab} + D_a D_b f]\frac{dh^{ab}}{d\tau}. \tag{3.9}$$

Thus $\frac{dh_{ab}}{d\tau}$ in (3.6) is twice the gradient of $\mathcal{F}_{\mathcal{P}}$ subject to the constraint of preserving the distorted volume. It follows that $\mathcal{F}_{\mathcal{P}}$ is non decreasing along the flow

$$\frac{d\mathcal{F}_{\mathcal{P}}}{d\tau} \geq 0. \tag{3.10}$$

Hence the use of the word "entropy" for the Perelman functional. The evolution equation for f is not independent, but determined by the constraint on the distorted volume.

3.4 THERMODYNAMIC APPROACH TO PENROSE INEQUALITY

Recall the Penrose inequality which is a condition on the initial data of GR.

Let $(\mathcal{M}, g_{ab})$ $(a, b = 0, 1, 2, 3)$ be an asymptotically flat, globally hyperbolic spacetime (with signature $(-, +, +, +)$) and (Σ, h_{ij}, K_{ij}) be a spatial slice in $\mathcal{M}$. The induced metric of Σ is h_{ij} $(i, j = 1, 2, 3)$ and its extrinsic curvature is K_{ij}. An initial data set for the spacetime $(\mathcal{M}, g_{ab})$ is the pair (h_{ij}, K_{ij}) along with a local mass density ρ, and a local current density J^a.

The fields on Σ must obey the constraint equations

$$R - K^{ab}K_{ab} + K^2 = 16\pi\rho \tag{3.11}$$

$$D_a(K^{ab} - Kh^{ab}) = 8\pi J^b \tag{3.12}$$

where R is the Ricci scalar of the metric h_{ab}, K is the trace of the extrinsic curvature K^{ab}, and D_a is the covariant derivative operator with respect to the metric h_{ab}. Furthermore, ρ and J^a must satisfy the local energy condition

$$\rho \geq (J^a J_a)^{1/2}. \tag{3.13}$$

If we consider an initial data set having the topology $\mathbf{R}^3$ and with $K_{ab} = 0$ i.e. the case of time symmetric initial data, then equations (3.11-3.13) imply that

$$R \geq 0 \tag{3.14}$$

and the Penrose inequality reduces to the Riemannian Penrose inequality, which states that

$$A \leq 16\pi M^2 \tag{3.15}$$

where A is the area of the outermost minimal surface in Σ and M is its ADM mass. Thermodynamically we can interpret (3.15) as saying that the Schwarzschild space (which saturates (3.15)) maximizes its entropy for a fixed energy. Any other initial data set can be "flowed" towards this equilibrium state of maximum entropy. We can now hope to find a geometric flow with the following properties

1. The condition in equation (3.14) is preserved under the flow so that we do not violate the local energy condition during the flow.

2. The area A of the outermost apparent horizon does not decrease under the flow.

3. The ADM mass M of the initial data set does not increase under the flow.

We immediately notice that the existence of such a geometric flow will prove the PI for the special case of time symmetric initial data. The flow that first comes to our mind is RF as this has the property that it preserves the positivity of the scalar curvature R which is the first of the three properties mentioned above. With this in mind, we wish to go further and study first the evolution of the area A and the mass M under RF and then possibly the evolution of other geometric quantities of interest.

Bibliography

[1] Grisha Perelman 2002 The entropy formula for the Ricci flow and its geometric applications *Preprint* math.DG/0211159.

[2] D. Friedan 1980 *Phys. Rev. Lett.* **45**, 1057.

[3] R. S. Hamilton 1982 Three-manifolds with positive Ricci curvature *J. Differential Geom.* 17.

[4] Huai-Dong Cao and Bennett Chow 1999 Recent developments on the Ricci flow *Bull. Amer. Math. Soc.* 36 59-74.

[5] Peter Topping 2006 London Mathematical Society Lecture Notes Series *Cambridge University Press*.

[6] Joseph Samuel 2006 The Ricci Flow in Physics *Unpublished Lecture Notes Based on his talk given in the International Workshop and Conference on Geometric methods in Topology*.

[7] John Milnor 2003 Towards the Poincare Conjecture and the Classification of 3-Manifolds *Notices of the AMS* **50** 1226.

[8] John Milnor 1969 Morse theory *Annals of Mathematical Studies* **51**.

[9] Michael T. Anderson 2004 Geometrization of 3-Manifolds via the Ricci Flow *Notices of the AMS* **51** 184.

[10] Bennett Chow and Dan Knopf 2004 The Ricci Flow: An Introduction *American Mathematical Society*.

[11] L. Ni 2004 The entropy formula for linear heat equation *J. Geom. Anal.* **14** 369-374.

[12] C. Villani 2003 Topics in optimal transportation *Graduate Studies in Mathematics, AMS* **58**.

[13] Theodore Frankel 1997 The Geometry of Physics *Cambridge University Press*.