Chapter 6

Concluding remarks

In this Thesis we have observed statistical fluctuations of light emission intensity from a macroscopic Random Amplifying Medium (RAM) over the ensemble of different microscopic realizations of the randomness. These show a crossover from the Gaussian (narrow, light-tailed) distribution to the Lévy (broad, fat-tailed) distribution in the limit of high gain and strong scattering. We attribute these to the "amplification of fluctuations" arising from the presence of "larger-than-rare" events which dominate the intensity statistics in the limit of high gain and strong scattering. Thus, the high gain and the strong scattering act as a "fluctuation-amplifier" much the same way as tunneling through nano/meso-scaled metal-insulator-metal junction acts for the tunneling current (which depends exponentially on the barrier height). An important point in our optical study on the RAM is the continuous tunability of the Lévy exponent through the optical-pump intensity (gain) and the scattering parameters. These studies relate to the fluctuation aspect of the random lasers.

Yet another significant finding is the R-RAM (discrete active scatterers embedded in continuous passive medium) as distinct from the D-RAM (discrete passive scatterers embedded in continuous active medium) because of the reversal of roles of the scatterers and the propagation medium. For an R-RAM, we have demonstrated the competition between the effectiveness of the individual amplifying scattering events and the frequency of these multiple scattering events. The main observation of this work is the non-monotonic dependence of the overall gain on the refractive index mismatch for a suitable range of parameter values characterizing the R-RAM.

There is a subtle point involved in treating a RAM as a dielectric with complex refractive index ($n = n' - in'' : n'' > 0$), with the sign of imaginary part of the refractive index chosen to correspond to amplification rather than attenuation. This would, normally mean an absolute instability, e.g., for a closed laser cavity with such a medium. This would invalidate any treatment of such a medium by way of linear response theory. The important point to be noted here is that in our simulation the absolute instability (for which the gain diverges exponentially in time) is replaced by a convective instability where the wave propagates as it grows (in intensity) and thus convects away the energy. It is this which validates our treatment of a RAM. (Formally, the distinction between an absolute instability and a convective instability is that for the absolute instability, it is the frequency which is made complex while for the convective case, it is the wave vector which is made complex.)

Another interesting aspect which came out of Monte Carlo simulation of photon diffusion in a RAM is the "Lévy microscope effect", where the sum of intensity values is dominated by the largest event (or a few very large events). Yet another case of RAM that we term F-RAM (Fiber-RAM) was experimentally studied where the disorder was structured as consisting of random aggregate of active *fiber segments* (with exponential length distribution) embedded in passive bulk. This also gave an optical realization of the Arrhenius cascade model. Thus, our scattering system (in particular, the D-RAM) provides example of both the Lévy microscope and the Arrhenius cascade, with Lévy exponents which are tunable optically.

Finally, we would like to point out the future possibility of translating the statistical fluctuations over the spatial realizations of disorder to random fluctuations in time (temporal fluctuations). This can be realized, for instance, in an active but turbulent fluid medium (circulated through the cavity/cell). Such a time translation of randomness through turbulence will be a directly observable fluctuation due to diffusion in the presence of a random velocity field – as in Sinai fluctuations in a random random medium.

One other idea we would like to explore in future is the experimental realization of an R-RAM. The extreme parameters (refractive index mismatch and gain) required may involve the use of partial metallic coating on the dyed active, scattering microspheres to enhance multiple scattering within these scatterers (active) due to internal reflections.

Appendix A

Statistical Distributions

Random variable (r.v.)

A random variable (r.v.) X is defined as a function whose *domain* is the set S of all outcomes (ω), and *range* a set of numbers.

$$\omega \epsilon S \ : w \rightarrow X(\omega) \tag{A.1}$$

where, $X(\omega)$ indicates the number assigned to the specific outcome.

Probability distribution function (PDF)

The PDF (also known as the *cumulative distribution function*) of the r.v. X is :

$$F(x) = P\ (X \leq x) \tag{A.2}$$

defined for $-\infty < x < \infty$.

Thus, for a given x, $F(x)$ is the probability of the event ($X \leq x$), consisting of all outcomes ω such that $X(\omega) \leq x$.

PDF's have the following properties :

$$F(-\infty) \ = \ 0 \tag{A.3}$$

$$F(+\infty) \ = \ 1 \tag{A.4}$$

It is a nondecreasing function of x :

$$F(x_1) \leq F(x_2) \qquad : for \;\; x_1 < x_2 \tag{A.5}$$

In particular, for $x_1 < x_2$

$$F(x_2) - F(x_1) = P\,(x_1 < X \leq x_2) \tag{A.6}$$

It is continuous from the right :

$$F(x^+) = F(x) \qquad : for \;\; x_1 < x_2 \tag{A.7}$$

where,

$$F(x^+) \;\; = \;\; lim\, F(x + \epsilon) \tag{A.8}$$

$$F(x^-) \;\; = \;\; lim\, F(x - \epsilon) \qquad : \; \epsilon > 0, \; \epsilon \rightarrow 0 \tag{A.9}$$

Probability density function

The derivative :

$$f(x) = \frac{dF(x)}{dx} \tag{A.10}$$

of the PDF $F(x)$ is called the probability density function of the r.v. X (it is also known as *frequency function*).

From the monotonicity of $F(x)$, follows that $f(x)$ is nonnegative,

$$f(x) \geq 0 \tag{A.11}$$

and from Eqs (A.3) and (A.4)

$$\int_{-\infty}^{\infty} f(x)\, dx = F(\infty) - F(-\infty) = 1 \qquad : Normalization \;\; condition \tag{A.12}$$

Integrating the Eq. (A.10) from $-\infty$ to x,

$$F(x) = \int_{-\infty}^{x} f(y)\, dy \tag{A.13}$$

Hence,

$$F(x_2) - F(x_1) = P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx \tag{A.14}$$

Moments of a random variable X

Moments (m_k) are defined as :

$$m_k = E(X^k) \equiv \ < X^k >= \int_{-\infty}^{\infty} x^k\ f(x)\,dx \tag{A.15}$$

thus, the first few moments are

$$m_0 = \int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{A.16}$$

$$m_1 = E(X) \equiv \ < X > \ = \int_{-\infty}^{\infty} x\ f(x)\,dx \tag{A.17}$$

$$m_2 = E(X^2) \equiv \ < X^2 > \ = \int_{-\infty}^{\infty} x^2\ f(x)\,dx \tag{A.18}$$

where, the first moment $< X >$ is the mean or expected value of r.v. X. If for a random variable, X, the second moment $< X^2 >$ is finite, then X is said to have a “narrow” probability density.

Central moments of a random variable X

Central moments (μ_k) are defined as :

$$\mu_k = E((X- < X >)^k) = \int_{-\infty}^{\infty} (x- < x >)^k\ f(x)\,dx \tag{A.19}$$

Thus, the first few central moments are :

$$\mu_0 = \int_{-\infty}^{\infty} f(x)\,dx \tag{A.20}$$

$$\mu_1 = \int_{-\infty}^{\infty} (x- < x >)\ f(x)\,dx = 0 \tag{A.21}$$

$$\mu_2 = \int_{-\infty}^{\infty} (x- < x >)^2\ f(x)\,dx =< X^2 > - < X >^2 \tag{A.22}$$

(A.23)

where, the second central moment $\mu_2 \equiv \sigma^2$, is defined as the *variance or dispersion* of the r.v. X. It represents the mean-squared fluctuations about the mean $< X >$. Its positive squareroot σ is called the *standard deviation*.

Standardized random variable X^*

The standardized r.v. corresponding to the r.v. X

$$X^* = \frac{X - < X >}{\sigma} \tag{A.24}$$

such that,

$$E(X^*) = 0 \quad and, \qquad Var(X^*) = 1 \tag{A.25}$$

(where, “Var” implies ”variance”).

Characteristic function of a random variable X

The characteristic function $\tilde{f}(k)$ of a r.v. X is the Fourier transform of its density function $f(x)$:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} exp\ (ikx)\ f(x)\ dx \equiv\ E(exp(ikx)) \tag{A.26}$$

This is, clearly, the expected value of the complex function $(exp(ikX))$ of the r.v. X.

Second characteristic function $\Phi(k)$ of the r.v. X is defined as :

$$\Phi(k) = ln\ \tilde{f}(k) \tag{A.27}$$

Thus,

$$\tilde{f}(0) = \int_{-\infty}^{\infty} f(x)\ dx = 1 \quad hence, \quad \Phi(0) = 0 \tag{A.28}$$

and since $f(x) \geq 0$,

$$|\tilde{f}(k)| = \left| \int_{-\infty}^{\infty} exp\ (ikx)\ f(x)\ dx \right| \leq \int_{-\infty}^{\infty} f(x)\ dx = 1 \tag{A.29}$$

Thus, $|\tilde{f}(k)| \leq 1$.

The density $f(x)$ can be expressed in terms of $\tilde{f}(k)$ by the integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} exp\ (-ikx)\ \tilde{f}(k)\ dk \tag{A.30}$$

known as inversion formula.

Given the function, $\tilde{f}(k)$ all the moments of r.v. X can be determined - therefore, it is also called the *moment generating function*.

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} f(x)\,[1 + ikx + \frac{(ikx)^2}{2!} + .. + \frac{(ikx)^n}{n!} + ...]\,dx \quad : (using \;\; exp(ikx) = \sum_{n=0}^{\infty} \frac{(ikx)^n}{n!}) \\
&= \int_{-\infty}^{\infty} f(x)\,dx + ik \int_{-\infty}^{\infty} x\; f(x)\,dx + \frac{(ik)^2}{2!} \int_{-\infty}^{\infty} x^2\; f(x)\,dx + .. + \frac{(ik)^n}{n!} \int_{-\infty}^{\infty} x^n\; f(x)\,dx + .. \\
&= 1 + ik < X > + \frac{(ik)^2}{2!} < X^2 > +.. + \frac{(ik)^n}{n!} < X^n > +.. \\
&= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} < X^n > \qquad \text{(A.31)}
\end{aligned}$$

Thus,

$$< X^n > \; = \; (-i)^n \frac{d^n \tilde{f}(0)}{dk^n} \quad : \quad n = 1, 2, 3.... \qquad : \quad Moment \;\; theorem \qquad \text{(A.32)}$$

Joint probability distribution function of random variables X and Y

The joint probability distribution function of the r.v. X and Y is defined by :

$$F_{XY}(x, y) = P\,\{X \leq x, Y \leq y\} \qquad \text{(A.33)}$$

In the study of several random variables, the distribution of each r.v. is called *marginal distribution*. Thus, $F_X(x)$ and $F_Y(y)$ are called the marginal distributions of r.v. X and Y. Clearly,

$$F_{XY}(x, \infty) = F_X(x) \quad and, \quad F_{XY}(x, -\infty) = 0 \qquad \text{(A.34)}$$

$$F_{XY}(\infty, y) = F_Y(y) \quad and, \quad F_{XY}(-\infty, y) = 0 \qquad \text{(A.35)}$$

$$F_{XY}(\infty, \infty) = 1 \qquad \text{(A.36)}$$

Joint Density function

Joint density function of the r.v. X and Y is given as :

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x\, \partial y} \qquad \text{(A.37)}$$

such that,

$$f(x,y) \geq 0 \qquad and, \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1 \tag{A.38}$$

Relationship between marginal and joint densities

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \tag{A.39}$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx \tag{A.40}$$
$$\tag{A.41}$$

Independent random variables

Two r.v. X and Y are called independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for any x and y, i.e., if

$$F_{XY}(x,y) = P\{X \leq x, Y \leq y\} = P\{X \leq x\}\, P\{Y \leq y\} = F_X(x)\, F_Y(y) \tag{A.42}$$

In terms of densities,

$$f_{XY}(x,y) = f_X(x)\ f_Y(y) \tag{A.43}$$

i.e., the joint probability density function of the independent r.v. is the product of their marginal densities.
Further, for independent r.v.

$$< XY > \;=\; < X > \;< Y > \tag{A.44}$$
$$Var(X+Y) \;=\; Var(X) + Var(Y) \tag{A.45}$$

Independent identically distributed random steps or variables (i.i.d.r.v.)

A random walk comprising of independent identically distributed random steps implies that
(a) Each step is independent of the previous steps
(b) The length and direction have the same probability density distributions for all steps.

Appendix B

Simulation code for a D-RAM

```
//_____________________________________________________________________
/* MONTE CARLO SIMULATION of photon diffusion in a direct or D-RAM.
 * The program traces the path of a spontaneous photon emitted by a
 * dye molecule (picked randomly from a uniform distribution) till
 * it finally exits the medium. */
//_____________________________________________________________________

#include<stdio.h>
#include<math.h>
#include<stdlib.h>
#include "ran_double.c"

//_____________________________________________________________________
      /* function prototype declarations */
//_____________________________________________________________________

double ran_double (long *seed);
  /* prototype declaration for random no. generation */

main ()
{
//_____________________________________________________________________
                        /* type declarations */
//_____________________________________________________________________
  int i, j, k, nscat, kmark1 = 31, n_scat_eff;

  double x1, y1, z1, x_dye, y_dye, z_dye, xi, yi, zi, x_current,
         y_current, z_current, x_latest, y_latest, z_latest, X0, X1,
         Y0, Y1, Z0, Z1;
```

```
  double L, L_scat, denom, L_total, L_eff_total, I_total, I_eff_total,
         avg_n_scat_eff, avg_L_eff_total, avg_I_eff_total,
         ln_avg_I_eff_total;

  double theta, phi, theta_h, phi_h, RAN3, RAN4, P, Q, R, Q1, r, s,
         l_g;

  double lambda_pump, number_density, dia, pi, c[100], m[100], n[100];

  long int seed1;

  FILE *fp, *fp1, *fp2;

  char stringA[100] = "_microns_refsphere_2.7";
  char stringB[100] = "DIA_";
  char stringC[100] = "_lg_0.5";
  char stringD[100] = "_scatter_kmark1";
  char string[100], string1[100], string2[100], string3[100],
       string4[100], string5[100];
//______________________________________________________________________
  /* values of input parameters */
//______________________________________________________________________

  pi = 4 * atan (1);

  printf ("Enter the value of seed No. 1\n");
  scanf ("%ld", &seed1);

  X0 = 0.0, Y0 = 0.0, Z0 = 0.0; /* in cm */
  X1 = 1.0, Y1 = 1.0, Z1 = 1.0; /* in cm */

//______________________________________________________________________
  /*  polystyrene scatterer specifications and parameter definitions.
   *  program used for the finding the values : "calc_anisotropy1.m" */
//______________________________________________________________________

  lambda_pump = 633e-7; /* in cm */
  number_density = 1e10; /* per cubic cm */

  for (dia = 0.06; dia < 0.40; dia += 0.04)
  {
      sprintf (string, "%4.2f", dia);
      strcpy (string1, stringA);
```

```
strcat (string, string1);
strcpy (string2, stringB);
strcat (string2, string);
fp = fopen (string2, "r");

for (j = 0; j <= 0; j++)
     fgets (string3, 100, fp);

strcpy (string4, string2);
strcat (string4, stringC);
fp1 = fopen (string4, "w");
strcpy (string5, string4);
strcat (string5, stringD);
fp2 = fopen (string5, "w");

for (k = 0; k <= 32; k++)
{
   fscanf (fp, "%*lf%*lf%lf%lf%lf", &c[k], &m[k], &n[k]);
   denom = number_density * n[k];
   L_scat = 1 / denom;

   /*
     c[k] : ref index mismatch (n_sphere - n_bulk), m[k] :
     anisotropy parameter (g) and n[k]: scattering cross-section
   */

   //______________________________________________________________
   /*
     A dye molecule at some position (x1,y1,z1) inside the pumped
     volume is picked randomly from a uniform distribution
   */
   //______________________________________________________________

    L_eff_total = 0.0; /* initialization of variables */
    I_eff_total = 0.0;
    n_scat_eff = 0;

    for (i = 1; i <= 500000; i++)
    {

        /*
          position coordinates of a dye molecule picked randomly
          from a uniform distribution
        */
```

```
x_dye = ran_double (&seed1);
y_dye = ran_double (&seed1);
z_dye = ran_double (&seed1);
x_current = x_dye;
y_current = y_dye;
z_current = z_dye;

//________________________________________________________
/*
  direction of travel of the spontaneously emitted photon:
  phi is generated from a uniform distribution ranging
  (0 to 2*pi) and theta from 0 to pi
*/
//________________________________________________________

theta = acos (-1 + 2 * ran_double (&seed1));
theta_h = theta;

phi = (ran_double (&seed1)) * 2 * pi;
phi_h = phi;

//________________________________________________________
/*
  The length travelled by the photon before being
  scattered i.e. "l". It is picked from an exponential
  distribution
*/
//________________________________________________________

RAN3 = ran_double (&seed1);
L = -L_scat * log (RAN3);

//________________________________________________________
/* The coordinates of the new position of the photon
  (i.e. the position of the scatterer) is calculated. It
  is checked whether or not that coordinate lies within
  the cuvette i.e. photon is within the sample or not
*/
//________________________________________________________

x1 = L * sin (theta) * cos (phi);
y1 = L * sin (theta) * sin (phi);
z1 = L * cos (theta);
```

```
x_latest = x_current + x1;
y_latest = y_current + y1;
z_latest = z_current + z1;

L_total = 0.0;
nscat = 0;

while (x_latest > X0 && x_latest < X1 && y_latest > Y0
       && y_latest < Y1 && z_latest > Z0 && z_latest < Z1)
{
    L_total += L;
    nscat++;

    x_current = x_latest;
    y_current = y_latest;
    z_current = z_latest;

    RAN4 = ran_double (&seed1);
    L = -L_scat * log (RAN4);

    phi = (ran_double (&seed1)) * 2 * pi;

    /*
      new theta selected so that it follows Henyey
      Greenstein distribution
    */

    P  = 1 + m[k] * m[k];
    Q1 = (1 - m[k] * m[k])/(1+m[k]-2*m[k]*
         ran_double (&seed1));
    Q  = Q1 * Q1;
    R  = 2 * m[k];
    theta = acos ((P - Q)/R);

    /*
      Transformation from the local co-ordinate frame to
      the global co-ordinate frame.
    */
    xi = L * sin (theta) * cos (phi);
    yi = L * sin (theta) * sin (phi);
    zi = L * cos (theta);

    x1 = xi*cos(theta_h)*cos (phi_h)-yi*sin (phi_h)+
```

```
                zi*sin (theta_h)*cos (phi_h);
            y1 = xi*cos(theta_h)*sin(phi_h)+yi*cos(phi_h)+
                 zi*sin(theta_h)*sin (phi_h);
            z1 = -xi*sin(theta_h)+zi*cos(theta_h);

            x_latest = x_current + x1;
            y_latest = y_current + y1;
            z_latest = z_current + z1;

            r = sqrt (x1 * x1 + y1 * y1 + z1 * z1);
            s = sqrt (x1 * x1 + y1 * y1);
            theta = atan2 (s, z1);
            phi = atan2 (y1, x1);
            if (phi < 0)
                phi = 2 * pi + phi;
            if (phi > 2 * pi)
                phi = 2 * phi - 2 * pi;
            if (theta > pi)
                theta = 2 * pi - theta;
            theta_h = theta;
            phi_h = phi;
        }

        l_g = 0.5; // l_g : gain length (cm)
        I_total = exp (L_total / l_g);

        if (k == kmark1)
        {
            fprintf (fp2, "%d\t%g\t%d\t%g\t%g\n", i - 1, c[k],
                     nscat, L_total, I_total);
            fflush (fp2);
        }

        I_eff_total += I_total;
        L_eff_total += L_total;
        n_scat_eff += nscat;
    }

    avg_n_scat_eff = 1.0 * n_scat_eff / (i - 1);
    avg_L_eff_total = 1.0 * L_eff_total / (i - 1);
    avg_I_eff_total = 1.0 * I_eff_total / (i - 1);
    ln_avg_I_eff_total = log (avg_I_eff_total);

    fprintf (fp1, "%d\t%g\t%g\t%g\t%g\t%d\t%g\t%g\t%g\t%g\n",
```

```
                    i - 1, c[k], L_scat, L_eff_total, avg_L_eff_total,
                    n_scat_eff, avg_n_scat_eff, I_eff_total,
                    avg_I_eff_total, ln_avg_I_eff_total);
          fflush (fp1);
    }
  }
  fclose (fp);
  fclose (fp1);
  fclose (fp2);
}
```

Appendix C

Simulation code for an R-RAM

```
//____________________________________________________________________
/* MONTE CARLO SIMULATION of photon diffusion in an R-RAM.
 * The program traces the path of a spontaneous photon emitted by a
 * dye molecule (picked randomly from a uniform distribution) till
 * it finally exits the medium. */
//____________________________________________________________________

#include<stdio.h>
#include<math.h>
#include<stdlib.h>
#include "ran_double.c"

//____________________________________________________________________
              /* function prototype declarations */
//____________________________________________________________________

double ran_double(long *seed);
        /* prototype declaration for random no. generation */

main()
{
//____________________________________________________________________
                /* type declarations */
//____________________________________________________________________
    int   i,j,k;

    unsigned int nscat;

    double x1,y1,z1,x_initial,y_initial,z_initial,xi,yi,zi,x_in,y_in,
           z_in, X0, X1, Y0, Y1, Z0, Z1,x_current,y_current,z_current,
           x_latest, y_latest,z_latest;
```

```
    double theta,phi,theta_h,phi_h,r,s,RAN3,RAN4,amp_factor,
           ln_amp_factor,ln_Ii,avg_ln_Ii,avg_ln_Ii_eff;

    double c[1000],m[1000],n[1000],l[1000];

    double number_density,denom,L_scat,L,P,Q1,Q,R,pi,N_scat,No_of_runs;

    double dia,avg_n_scat,avg_n_scat_eff,lambda_pump;

    long int seed ;

    FILE *fp, *fp1 ;

    char stringA[100] = "_microns_lg_0.11mic_neg_img";
    char stringB[100] = "DIA_";
    char stringC[100] = "_inv_nd_6_10-11";
    char string[100], string1[100], string2[100], string3[100],
         string4[100];

//______________________________________________________________________
               /* values of input parameters */
//______________________________________________________________________

   pi = 4 * atan(1);

   printf("Enter the value of seed \n");
   scanf("%ld",&seed);

   X0 = 0.0, Y0 = 0.0, Z0 = 0.0 ;      /* in cm */
   X1 = 1.0, Y1 = 1.0, Z1 = 1.0 ;     /* in cm */

//______________________________________________________________________
   /* polystyrene scatterer specifications and parameter definitions.
      program used for the finding the values : "calc_anisotropy1.m"
   */
//______________________________________________________________________

    lambda_pump = 800e-7;          /* in cm */
    number_density = 6e11;       /* per cubic cm */
    No_of_runs = 500000.0;

    for(dia=0.14;dia<0.16;dia+=0.04)
    {
```

```
sprintf(string,"%4.2f",dia);
strcpy(string1,stringA);
strcat(string,string1);
strcpy(string2,stringB);
strcat(string2,string);
printf("%s\n",string2);
fp = fopen(string2,"r");

for(j=0;j<=0;j++)
   fgets(string3,100,fp);

strcpy(string4,string2);
strcat(string4,stringC);
fp1 = fopen(string4,"w");

for(k=0;k<=78;k++)
{
   fscanf(fp,"%*lf%*lf%lf%lf%lf%lf%*lf",&c[k],&m[k],&n[k],&l[k]);
   l[k] = -l[k];
   denom = number_density * n[k] ;
   // m[k]=g,n[k]=sigma_scat,l[k]=sigma_gain
   L_scat = 1/denom ;

   //___________________________________________________________
   /* A dye molecule at some position (x1,y1,z1) inside the
      pumped volume is picked randomly from a uniform
      distribution
   */
   //___________________________________________________________

   avg_ln_Ii_eff = 0.0;
   avg_n_scat_eff=0.0;

   for(i=1;i<=500000;i++)
   {
        x_in = ran_double(&seed);
        x_initial = x_in * X1;
        y_in = ran_double(&seed);
        y_initial = y_in * Y1;
        z_in = ran_double(&seed);
        z_initial = z_in * Z1;

        x_current = x_initial ;
        y_current = y_initial ;
```

```
 z_current = z_initial ;

 //_______________________________________________________
 /* direction of travel of the spontaneously emitted
    photon : phi is generated from a uniform distribution
    ranging (0 to 2*pi) and theta is so generated that it
    lies b/w 0 to pi
 */
 //_______________________________________________________

 theta = acos(-1+2*ran_double(&seed)) ;
 theta_h = theta;

 // random no. generated is scaled from
 phi = (ran_double(&seed))*2*pi ;
 phi_h = phi;          // (0 to 1) TO (0 to 360 degrees)

 //_______________________________________________________
 /*  The length travelled by the photon before being
   scattered i.e. "l" . It is picked from an exponential
   distribution
 */
 //_______________________________________________________

RAN3 = ran_double(&seed);
L = - L_scat*log(RAN3);

 //_______________________________________________________
 /*  The coordinates of the new position of the photon
   (i.e. the position of the scatterer) is calculated. It
   is checked whether or not that coordinate lies within
   the cuvette i.e. photon is within the sample or not
 */
 //_______________________________________________________

x1 = L*sin(theta)*cos(phi);
y1 = L*sin(theta)*sin(phi);
z1 = L*cos(theta);

x_latest = x_current + x1;
y_latest = y_current + y1;
z_latest = z_current + z1;

nscat = 0;
```

```
while (x_latest > X0 && x_latest < X1 && y_latest > Y0 &&
        y_latest < Y1 && z_latest > Z0 && z_latest < Z1)
 {
     nscat++;

     x_current = x_latest;
     y_current = y_latest;
     z_current = z_latest;

     RAN4 = ran_double(&seed);
     L = -L_scat*log(RAN4);

     phi = (ran_double(&seed))*2*pi;

     /* new theta selected so that it follows Henyey
        Greenstein distribution
     */

     P = 1 + m[k]*m[k];
     Q1 = (1-m[k]*m[k])/(1+m[k]-2*m[k]*ran_double(&seed));
     Q = Q1*Q1;
     R = 2*m[k];
     theta = acos((P - Q)/R);

     /*
       Transformation from the local co-ordinate frame to
       the global co-ordinate frame.
     */
     xi = L*sin(theta)*cos(phi);
     yi = L*sin(theta)*sin(phi);
     zi = L*cos(theta);

     x1 = xi*cos(theta_h)*cos(phi_h)-yi*sin(phi_h)+
          zi*sin(theta_h)*cos(phi_h);
     y1 = xi*cos(theta_h)*sin(phi_h)+yi*cos(phi_h)+
          zi*sin(theta_h)*sin(phi_h);
     z1 = -xi*sin(theta_h)+zi*cos(theta_h);

     x_latest = x_current + x1;
     y_latest = y_current + y1;
     z_latest = z_current + z1;

     r = sqrt(x1*x1 + y1*y1 + z1*z1);
```

```
                s = sqrt(x1*x1 + y1*y1);
                theta = atan2(s,z1);
                phi = atan2(y1,x1);
                if(phi<0)
                    phi = 2*pi + phi;
                if(phi>2*pi)
                    phi = 2*phi - 2*pi;
                if(theta>pi)
                    theta = 2*pi - theta;

                theta_h = theta;
                phi_h = phi;
            }

            N_scat = (double)nscat;
            amp_factor = (n[k]+l[k])/n[k];
            ln_amp_factor = log(amp_factor);
            ln_Ii = N_scat*ln_amp_factor;
            avg_ln_Ii = ln_Ii/No_of_runs;
            avg_ln_Ii_eff+=avg_ln_Ii;
            avg_n_scat=N_scat/No_of_runs;
            avg_n_scat_eff+=avg_n_scat;
        }

        fprintf(fp1,"%d\t%g\t%g\t%g\t%g\t%g\n",i-1,c[k],L_scat,
                ln_amp_factor,avg_n_scat_eff,avg_ln_Ii_eff);
        fflush(fp1);
    }
}
fclose(fp);
fclose(fp1);

}
```

Appendix D

Reflection and Transmission by an amplifying slab

Consider a wave normally incident on a plane-parallel amplifying slab (of complex refractive index $n_2 = n_2' - in_2'' : n_2'' > 0$ and thickness d) embedded in a passive (or nonabsorbing) medium (of real refractive index n_1) Fig. D.1. For this simple slab geometry one can readily find the reflection coefficient (R) and the transmission coefficient (T).

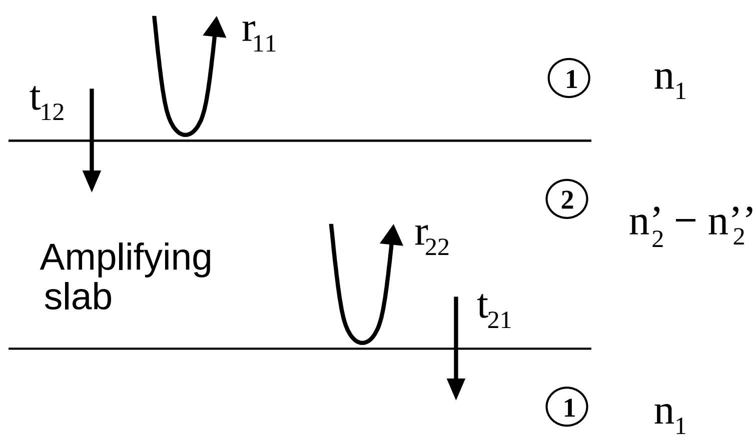

Figure D.1: *Schematic of light reflection and transmission from an amplifying slab.*

The main purpose of this somewhat elementary exercise is to show that, with proper identification of R and T, our results for the slab geometry provide a qualitative understanding of the results of our Monte Carlo simulation of the emission intensity from an R-RAM (see chapter 5). To this end, we regard the case of a slab geometry (with normal incidence) to be an extreme case of anisotropic scattering by a single active scatterer in three dimension.

Here, we identify the reflection coefficient R (the backscattering) with the scattering cross-section σ_s (for the isolated three dimensional scatterer) and the transmission coefficient T with the forward scattering. Thus, for the amplifying scatterer, $R+T$ corresponds to the gain $(\sigma_s + \sigma_{na})/\sigma_s$. With this identification, we can now use the analytical elementary results derived below for R and T, for an amplifying slab to understand the results of our simulation qualitatively.

In order to derive analytical expressions for R and T, for a slab geometry under normal incidence, consider the multiple scattering of a unit amplitude beam. For the multiply scattered beam, we obtain the amplitude reflection coefficient (r) as:

$$r = r_{11} + t_{12}e^{i2kd}r_{22}t_{21} + t_{12}e^{i4kd}{r_{22}}^3t_{21} + t_{12}e^{i6kd}{r_{22}}^5t_{21} + ... \quad \text{(D.1)}$$

$$= r_{11} + \frac{r_{22}t_{12}t_{21}\eta^2}{[1-(\eta r_{22})^2]} \quad \text{(D.2)}$$

where,

$$\eta = e^{ikd} = e^{i\omega d(n_2'-in_2'')/c} = e^{i\omega dn_2'/c}e^{\omega dn_2''/c} \quad \text{(D.3)}$$

$$r_{11} = \frac{n_2-n_1}{n_1+n_2} = \frac{(n_2'-n_1)-in_2''}{(n_1+n_2')-in_2''} \quad \text{(D.4)}$$

$$r_{22} = \frac{n_1-n_2}{n_1+n_2} = \frac{(n_1-n_2')+in_2''}{(n_1+n_2')-in_2''} = -r_{11} \quad \text{(D.5)}$$

$$|r_{11}|^2 = r_{11}{r_{11}}^* = \frac{(n_2'-n_1)^2+(n_2'')^2}{(n_2'+n_1)^2+(n_2'')^2} = |r_{22}|^2 = r_{22}{r_{22}}^* \quad \text{(D.6)}$$

$$t_{12} = \frac{2n_1}{n_1+n_2} = \frac{2n_1}{(n_1+n_2')-in_2''} \quad \text{(D.7)}$$

$$t_{21} = \frac{2n_2}{n_1+n_2} = \frac{2n_2}{(n_1+n_2')-in_2''} \quad \text{(D.8)}$$

Thus, the reflection coefficient is :

$$R = rr^*$$

$$= |r_{11}|^2 + \frac{{r_{11}}^*r_{22}t_{12}t_{21}\eta^2}{[1-(\eta r_{22})^2]} + \frac{r_{11}{r_{22}}^*{t_{12}}^*{t_{21}}^*(\eta^2)^*}{[1-(\eta r_{22})^2]^*} + \frac{|r_{22}|^2|t_{12}|^2|t_{21}|^2e^{4\omega dn_2''/c}}{[1-(\eta r_{22})^2][1-(\eta r_{22})^2]^*} \quad \text{(D.9)}$$

Similarly, the amplitude transmission coefficient (t) is given as:

$$t = t_{12}e^{ikd}(1 + e^{i2kd}{r_{22}}^2 + e^{i4kd}{r_{22}}^4 + ...)t_{21} \quad \text{(D.10)}$$

$$= \frac{\eta t_{12}t_{21}}{[1-(\eta r_{22})^2]} \quad \text{(D.11)}$$

Thus, the transmission coefficient is :

$$T = tt^* = \frac{|t_{12}|^2 |t_{21}|^2 e^{2\omega d n_2''/c}}{[1-(\eta r_{22})^2][1-(\eta r_{22})^2]^*} \tag{D.12}$$

It is now easily verified that :

(1) $R+T=1$, for a passive slab ($n_2''=0$) and finite d;

(2) $R+T>1$, for an active slab (finite n_2'') and finite d.

In figures D.2(a),(b), and (c), we have plotted $R+T$ and R versus refractive index mismatch ($\Delta n = n_2' - n_1$) for low, intermediate and high gains, respectively.

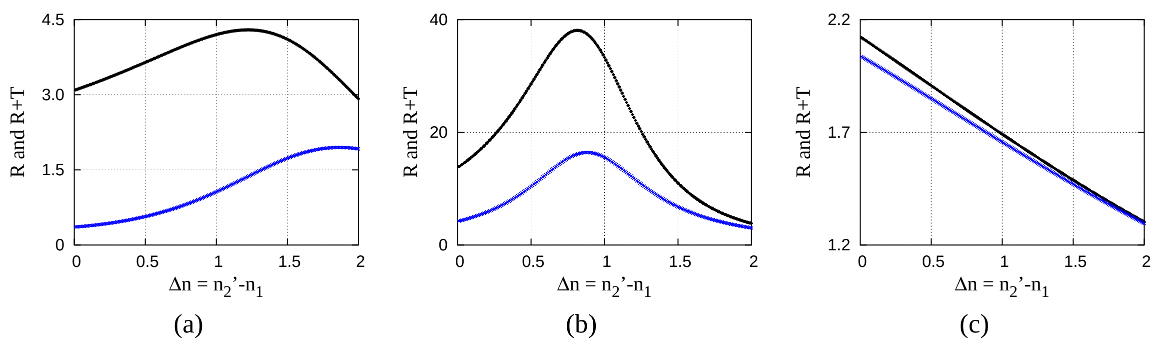

Figure D.2: *$R+T$ (black) and R (blue) as function of refractive index mismatch (in the real parts), for slab thickness = 0.04 μm, λ = 0.8 μm and n_2' = 3.0, with (a) l_g = 0.05 μm, (b) l_g = 0.03 μm, (c) l_g = 0.01 μm.*

In all the plots, in the limit of small mismatch and low gain, both $R+T$ (gain) and the reflection coefficient R (scattering) increase with increase in mismatch. This clearly indicates that the effectiveness of an individual amplifying scattering event increases with increase in mismatch. Recalling that the reflection coefficient (R) is analogous to the scattering cross-section (σ_s) for the R-RAM in our simulation, an increase in R with mismatch will imply an enhancement of multiple scattering off the active scatterers constituting the R-RAM. We should, therefore, expect, a fortiori, further enhancement of the gain with the increase in the refractive mismatch. This is indeed as is observed (see Figs 5.4(a) and 5.5(a)), for low gain. Next, in the limit of very high gain, our analytical results for the slab give $R+T$ and R decreasing with the increasing mismatch. Again recalling, that decreasing R would correspond to decrease in multiple scattering for our many scatterer system (R-RAM), we should expect, a fortiori, a much more pronounced decrease in gain with increasing mismatch in the limit of very high gain as, indeed, observed in our simulations (see Figs 5.4(c) and 5.5(c)).

Thus, properly interpreted, our analytical expressions for the simple case of this amplifying slab does rationalize the results of our simulations on R-RAM.