

Chapter 3

Active oriented filaments : shear stabilization and rheology

3.1 Introduction

In this chapter, we study theoretically the collective behaviour and mechanical response of *active* filaments (e.g., actin-myosin complexes, motor-microtubule complexes). Such a study has potential implications in understanding the mechanical response and movement of cells.

The surprising consequences of activity have been highlighted in Chapter 1; indeed the most dramatic manifestation is in movement [1]. A suspension of active filaments can exhibit steady states wherein the active particles move with respect to the solvent [2, 3] and hence generate local flows in the fluid. This would lead to a long-range hydrodynamic interaction between the filaments. When the concentration of the active particles is large, one might imagine that this interaction between filaments could result in a tendency to orientationally order. This however is not the case, since the oriented phase of active suspension is hydrodynamically unstable in the Stokesian regime [4]. This is because the flow generated by a small, long-wavelength splay or bend deformation imposed on an oriented configuration always acts instantaneously (in the Stokesian regime) to further increase the deformation [4].

Here we describe a way to stabilize this orientationally ordered phase, by externally imposing a shear flow. We find that an imposed shear, larger than a critical strain rate $\dot{\gamma}_c$, stabilizes the orientationally ordered phase, yielding a stability diagram controlled by two variables, a flow alignment parameter λ and the ratio of the shear to active stress. Having stabilized the orientationally ordered phase, we explore its rheological response. The rheological properties of such active particle systems show strong deviation from passive systems [3]: the orientationally ordered phase is characterised by a non-zero steady-state average of the deviatoric stress, which is a kind of yield stress. On approaching the orientationally ordered state from the isotropic fluid, a suspension of active *contractile* elements exhibits solid-like behaviour *without translational arrest*. Unlike in passive nematogenic systems, which exhibit *shear thinning* [5], as one approaches the nematic phase from the isotropic side, *active* filament suspensions exhibit shear thinning or thickening depending on the nature of activity, i.e., whether it is contractile or extensile. One way of quantifying the rheology of these active filament systems would be to prescribe phenomenological constitutive relations between the stress and the strain rate. Instead, our approach will be to derive the stress-strain rate relations from the underlying hydrodynamic equations.

In constructing the hydrodynamic equations, we need to prescribe how active filaments generate local mechanical stresses, which lead to hydrodynamic flows which in turn affect the neighbouring filaments. In section 3.2, we model this active mechanical stress and show

how the near field flows generated by activity depend on whether the filaments are contractile or extensile. In section 3.3, we review the rheology of the isotropic phase of such active suspensions. In section 3.4, we construct hydrodynamic equations for sheared suspensions of active filaments, analyse the stability of the orientationally ordered phase and obtain a non-equilibrium phase diagram. In section 3.5, we look at the rheological behaviour of the shear stabilized orientationally ordered phase. Finally in section 3.6, we discuss possible extensions of this work.

3.2 Modeling active stress

To determine the stress due to activity, we need to model the forces associated with the active particles. It is important to realize that the force density associated with each individual active filament in a fluid must integrate to zero; a consequence of Newton's Third Law. Thus the simplest model for an active Stokesian particle is an object which absorbs energy from its surroundings and dissipates it in the process of carrying out cyclical, *non-reciprocal* movements [6]. On timescales larger than this activity cycle time, the active particle can be represented by a permanent force dipole [4]. This gives rise to two distinct possibilities: (i) if the centre of the force dipole is such that it coincides with the centre of mass of the active particle, then by symmetry, there can be no net velocity at the centre of mass of the active particle, (ii) if the force dipole centre is asymmetrically disposed about the centre of mass then there will be net relative velocity at the centre of mass of the active particle which we denote as \mathbf{p} . To elaborate, consider the α th active filament, with axis \hat{p}_α , with point forces of equal magnitude on its ends, directed along $\pm \hat{p}_\alpha$ (Fig 3.1). In general, we keep the force centres asymmetrically disposed about the centre of mass of the active particle, $\vec{r}_\alpha(t)$, the ends being at $\vec{r}_\alpha + b\hat{p}_\alpha$ and $\vec{r}_\alpha - b'\hat{p}_\alpha$ respectively. For a collection of such active filaments, this leads to a force density given by,

$$\mathbf{F}_a = f \sum_\alpha \left[\delta(\vec{r} - \vec{r}_\alpha - b\hat{p}_\alpha) - \delta(\vec{r} - \vec{r}_\alpha + b'\hat{p}_\alpha) \right] \hat{p}_\alpha \quad (3.1)$$

Doing a Taylor expansion about \vec{r}_α ,

$$\begin{aligned} \mathbf{F}_a &= f \sum_\alpha \left[-(b+b')(\hat{p}_\alpha \cdot \vec{\nabla}) + \frac{b^2 - b'^2}{2} (\hat{p}_\alpha \cdot \vec{\nabla})^2 + \dots \right] \delta(\vec{r} - \vec{r}_\alpha) \hat{p}_\alpha \\ &= -(b+b') f \vec{\nabla} \cdot \sum_\alpha \hat{p}_\alpha \hat{p}_\alpha \delta(\vec{r} - \vec{r}_\alpha) + \frac{b^2 - b'^2}{2} f \vec{\nabla} \vec{\nabla} : \sum_\alpha \hat{p}_\alpha \hat{p}_\alpha \hat{p}_\alpha \delta(\vec{r} - \vec{r}_\alpha) + \dots \\ &= \nabla \cdot \boldsymbol{\sigma}^a, \end{aligned} \quad (3.2)$$

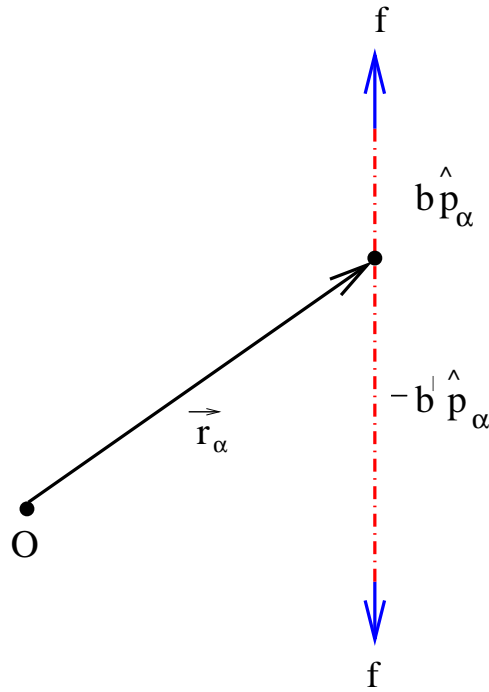


Figure 3.1: A force dipole corresponding to an active filament. If $b \neq b'$ then the active filament is polar else if $b = b'$ then it is apolar. \vec{r}_α is the position coordinate of the centre of mass of α th active particle, O being the origin of the coordinate system.

we immediately see that since \mathbf{F}_a is an internal force density, it is the divergence of an active stress, which to lowest order is given by,

$$\sigma_{ij}^a(\mathbf{r}, t) = f(b + b')c(\mathbf{r}, t)(p_i p_j - \delta_{ij}/3) = Wc(\mathbf{r}, t)Q_{ij} \quad (3.3)$$

Active filaments are therefore permanent stresslets. Note that this is the deviatoric part of the active stress. One can interpret Q_{ij} as the local nematic order parameter or alignment tensor associated with activity. As mentioned earlier, polar active particle suspension ($b \neq b'$) disturbs the fluid and induces a non-zero particle velocity with respect to the fluid, \mathbf{p} , at the centre of mass. Suspension of apolar active particles ($b = b'$) because of symmetry cannot move with respect to the fluid. However to lowest order in activity, both polar and apolar active particle suspension induces the same far-field fluid flow. This implies that the rheology of such active suspension should be the same for both the polar and the apolar case. The sign of W depends on the nature of elementary force dipole. For contractile filaments, $W < 0$ and such dipoles enhance an imposed flow, whereas for extensile filaments, $W > 0$ and they oppose the imposed flow (Fig. 3.2).

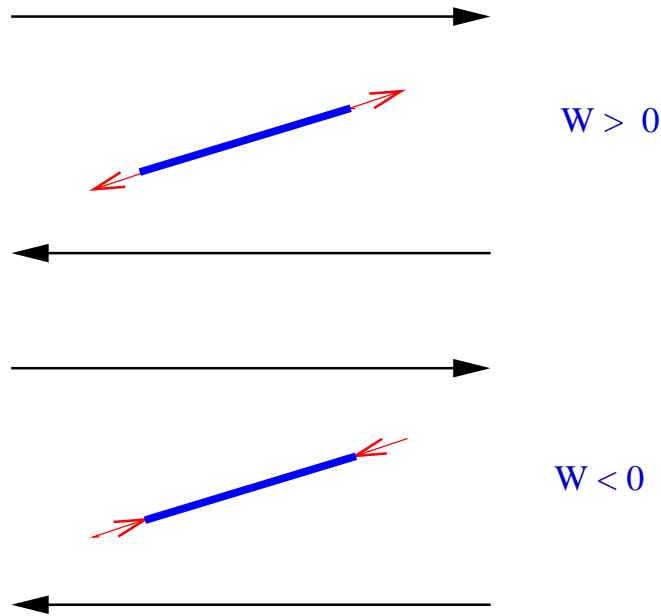


Figure 3.2: Rods with active force densities attached along their symmetry axis. (i) Extensile rod ($W > 0$), opposes the imposed flow. (ii) Contractile rod ($W < 0$), enhances the imposed flow.

3.3 Review of active hydrodynamics and rheology in the isotropic phase

3.3.1 Hydrodynamic equations in the isotropic phase

In the last section we argued that an active filament behaves like a permanent force dipole and a collection of such active filaments leads to a stress field in this suspension. The deviatoric active stress has the form given by equation (3.3). We will now review earlier work done on hydrodynamics and rheology of the isotropic phase of such active particle suspension [3].

The hydrodynamic variables describing the active particle suspension [3, 4] are: (i) the conserved hydrodynamic fluid momentum $\mathbf{g} = \rho_0 \mathbf{u}$, where \mathbf{u} is the hydrodynamic velocity in a fixed frame (and ρ_0 , the mean density), (ii) the conserved active-particle concentration $c(\mathbf{r}, t)$ and (iii) the alignment tensor \mathbf{Q} constructed from the local drift velocity \mathbf{p} associated with the average drift velocity of the active particle relative to the fluid. We will ignore the effects of fluctuations of concentration for the purpose of our present discussion. Including the effects of concentration fluctuations can lead to a clumping instability in the isotropic phase [7].

In analogy with the form of equation of motion for a *passive* nematogenic suspension, we construct the hydrodynamic equations for an active suspension in isotropic phase. The

equation of motion of \mathbf{Q} involves two pieces: (i) the dissipative part and (ii) the reactive part.

The dissipative part is proportional to the molecular field $-\frac{\delta F}{\delta \mathbf{Q}}$, where F is the Landau-deGennes free energy functional for passive nematogens [3, 5],

$$\begin{aligned} \frac{F}{k_B T} &= \int d^3r c_o \left[\alpha \text{Tr} \mathbf{Q}^2 + \beta \text{Tr} \mathbf{Q}^3 + \gamma (\text{Tr} \mathbf{Q}^2)^2 + \frac{K_1}{2} (\nabla_k Q_{ij})^2 \right. \\ &\quad \left. + \frac{K_2}{2} (\nabla_j Q_{ij}) (\nabla_k Q_{ik}) + K_3 (\epsilon_{ijk} Q_{il} \nabla_j Q_{kl}) \right]. \end{aligned} \quad (3.4)$$

The first three terms describe the N-I transition and the last three terms are the Frank free energy terms associated with bending, splay and twist distortions of the alignment tensor \mathbf{Q} . c_o is the equilibrium concentration of the nematogen in the suspension. In the expression above we have ignored the terms involving solely the concentration, i.e., term corresponding to entropy of mixing and that involving fluctuations of concentration.

Its easy to see from equation (3.4) that to the lowest order in \mathbf{Q} , the molecular field goes as $-a \mathbf{Q}$, where a is positive and it approaches zero on approaching nematic phase as $a \propto (T - T_{N-I})$. The dissipative part also has term which is associated with frank free energy distortions which goes as $\nabla \nabla \mathbf{Q}$. The reactive part is given by [5],

$$\mathbf{R} = \lambda_o \mathbf{A} + \boldsymbol{\kappa} \cdot \mathbf{Q} + \mathbf{Q} \cdot \boldsymbol{\kappa}^T - 2 \left(\mathbf{Q} + \frac{1}{3} \mathbf{I} \right) \text{Tr} (\mathbf{Q} \cdot \boldsymbol{\kappa}), \quad (3.5)$$

where $\kappa_{ij} = \nabla_j u_i$ is the strain rate tensor. \mathbf{A} is the symmetrised strain rate tensor $\frac{1}{2}[\nabla \mathbf{u} + \nabla \mathbf{u}^T]$ and λ_o is the alignment tensor.

The linearized equations of motion for \mathbf{Q} is obtained simply by adding up the contributions due to both the reactive part and the dissipative part and retaining terms upto linear order. In the one constant approximation for the Frank distortions, it is given by,

$$\frac{\partial \mathbf{Q}}{\partial t} = -\frac{1}{\tau} \mathbf{Q} + K \nabla^2 \mathbf{Q} + \lambda_o \mathbf{A} + \dots, \quad (3.6)$$

where τ is the relaxation time, with $\tau^{-1} \propto a$ which follows from the fact that molecular field goes as $-a \mathbf{Q}$, as mentioned earlier.

The conservation of total momentum implies that the total (fluid + suspension) momentum density satisfies the continuity equation $\partial_t \mathbf{g} = -\nabla \cdot \boldsymbol{\sigma}$. The total stress, has three pieces : (i) reactive part, (ii) dissipative part and (iii) active part.

The *reactive stress*, $\boldsymbol{\sigma}^{OP}$ is given by [5],

$$\boldsymbol{\sigma}^{OP} = -3\mathbf{G} + \mathbf{G} \cdot \mathbf{Q} - \mathbf{Q} \cdot \mathbf{G} + \dots, \quad (3.7)$$

where $\mathbf{G} \equiv -\frac{\delta F}{\delta \mathbf{Q}} + \frac{1}{3} \mathbf{I} \text{Tr} \frac{\delta F}{\delta \mathbf{Q}}$ is the nematic molecular field constructed to be traceless. The mean deviatoric passive stress is zero in isotropic phase. Thus to leading order in \mathbf{Q} , $\boldsymbol{\sigma}^{OP}$ goes as

$a \mathbf{Q}$ with a being positive in the isotropic phase and $a \rightarrow 0$ on approaching the transition to the orientationally ordered phase. The *dissipative stress*, $\sigma^d = -\eta_0 \mathbf{A}$. The *active stress*, to leading order in \mathbf{Q} is given by $\sigma^a = W c_o \mathbf{Q} \equiv W_1 \mathbf{Q}$. Adding up all the contributions, the total stress in the isotropic phase is given by,

$$\boldsymbol{\sigma} = (a + W_1) \mathbf{Q} - \eta_0 \mathbf{A}. \quad (3.8)$$

This defines completely the linearized equation of motion for the momentum density \mathbf{g} .

3.3.2 Linear rheology of a suspension of isotropic active filaments

From equations (3.6) and (3.8), we can calculate the linear viscoelastic properties of the active filament suspension. It can be easily deduced that on applying an uniform oscillatory shear flow at frequency ω in the xy plane

$$\begin{aligned} \sigma_{xy}(\omega) &= - \left[\eta_0 - \frac{(a + W_1) \lambda_0}{-i\omega + \tau^{-1}} \right] A_{xy} \\ &= \frac{G'(\omega) - iG''(\omega)}{\omega} iA_{xy}, \end{aligned} \quad (3.9)$$

where $G'(\omega)$ is the storage modulus and $G''(\omega)$ is the loss modulus. From (3.9) it follows that,

$$-\frac{G''(\omega)}{\omega} = \eta_0 - \lambda_0 \frac{\tau(a + W_1)}{1 + \omega^2 \tau^2} \quad (3.10)$$

The zero-frequency response for the loss modulus is given by ,

$$-\frac{G''(\omega = 0)}{\omega} = \eta_0 - (a\tau + W_1\tau)\lambda_0 \quad (3.11)$$

Thus because of activity, there is either an enhancement or reduction $\eta_{act} \propto W_1\tau$ of the effective viscosity at zero shear rate. Equation (3.11) implies that as the system approaches a transition to the orientational order from the isotropic side, then for $W > 0$, there is a reduction of the effective viscosity. Note that for the passive case, as the system approaches the oriented state, then the effective viscosity is reduced by an amount $a\tau$. As $\tau \propto \frac{1}{a}$, for passive nematogen, the reduction of viscosity approaches a constant value. Due to active contribution for $W > 0$, the effective viscosity is further reduced leading to enhancement of *shear thinning* effect. For $W < 0$, the consequences are even more dramatic : Activity results in enhancement of the effective viscosity on approaching orientational order. This can lead to mitigation of *shear thinning* and if strong enough will lead to *shear thickening*.

From equation (3.9) it also follows,

$$G'(\omega) = \frac{\lambda_0 \omega^2 \tau^2 (a + W_1)}{1 + \tau^2 \omega^2} \quad (3.12)$$

This implies that for active suspension ,

$$G'(\omega\tau \gg 1) \simeq +\lambda_0 W_1, \quad (3.13)$$

whereas for a passive suspension, the storage modulus decreases as $\frac{\lambda_0}{\tau}$.

This result is indeed surprising; at equilibrium one would expect such strong viscoelastic behaviour near *translational* freezing and not near *orientational* ordering [3]. However the orientationally ordered phase of active particle suspension has been shown to be *always* unstable in the absence of any imposed shear [4]. In section 3.5 we would come back to the rheological consequences of the the oriented phase and the behaviour of the system on approaching orientational order, once we are able to stabilize the oriented phase.

3.4 Active hydrodynamics of the orientationally ordered phase in a shear flow

On reducing the temperature towards T_{N-I} , or increasing the packing fraction, the isotropic phase of passive nematogens becomes unstable, leading to a first order phase transition to the orientationally ordered nematic phase. Surprisingly, this is not true for active suspensions – increasing the concentration, or lowering the temperature does not lead to an orientationally ordered state. As shown in [4], long-range uniaxial orientational order in active Stokesian suspension of polar active particles is *always* destroyed by a hydrodynamic instability. This is because a flow generated by a small, long wavelength splay or bend deformation imposed on the oriented configuration always acts instantaneously to further increase the deformation [4]. In this section, we will prevent this instability and stabilize the orientationally ordered phase by imposing a shear flow. Since the calculational route to obtain the instability and stabilization by shear is the same, we will display the calculation for the hydrodynamics of a sheared suspension of active particles and analyse the result with and without the imposed shear.

3.4.1 Hydrodynamic equations in the oriented phase with shear

The hydrodynamic variables for a suspension of active particles [4, 3] are (i) the total momentum density $\mathbf{g} = \rho\mathbf{u}$ of the particles + fluid, where \mathbf{u} is the hydrodynamic velocity field and ρ the density, (ii) the concentration $c(\mathbf{r}, t)$ of active particles, and (iii) the orientational order parameter field. For *polar* systems, the order parameter is the polarisation vector $\mathbf{p}(\mathbf{r}, t)$ of the force dipoles associated with the active particles. Active systems being out of thermal

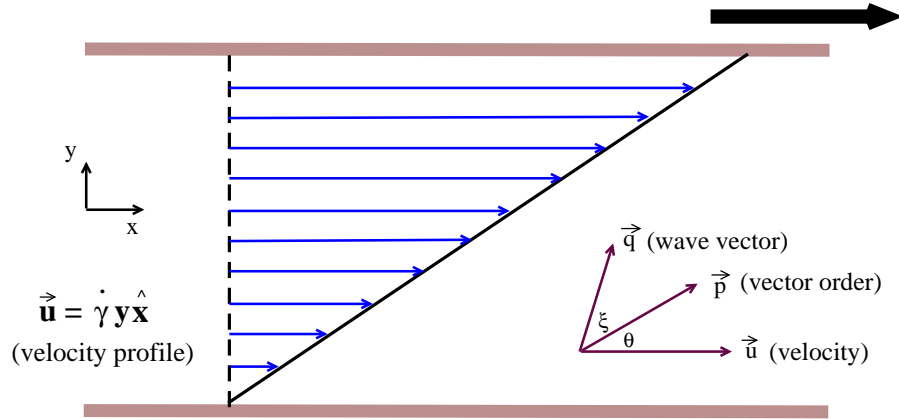


Figure 3.3: Suspension of active vector ordered particles subject to an imposed velocity along the \hat{x} -axis, $u_0 = \dot{\gamma}y\hat{x}$ with a gradient along the \hat{y} -axis. The shear alignment angle θ and the angle ξ between the wavevector \mathbf{q} and the ordering direction are also shown.

equilibrium, a local polarisation always implies a local average drift velocity of the active particle relative to the fluid. We thus take \mathbf{p} to be the velocity field of the active particles relative to the fluid. *Apolar* nematic orientational order is characterised by a traceless symmetric tensor $\mathbf{Q} \equiv \mathbf{S}(\frac{\mathbf{p}\mathbf{p}}{p^2} - (\mathbf{1}/d)\mathbf{I})$, where \mathbf{p} is now the director and S is the scalar nematic order parameter [20]. We focus on the polar case, and indicate differences where appropriate for the apolar case. Momentum conservation in the Stokesian limit means $\nabla \cdot \boldsymbol{\sigma} = 0$, where the total stress tensor $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}^a + \boldsymbol{\sigma}^r + \boldsymbol{\sigma}^d$ is the sum of contributions from activity, order-parameter gradients and viscous dissipation, and overall incompressibility of the suspension means $\nabla \cdot \mathbf{u} = 0$, $\rho = \rho_0$, the mean density. Impose a planar shear flow (Fig. 3.3) along the \hat{x} -axis, with a velocity gradient along \hat{y} , giving rise to an imposed velocity field $\mathbf{u}_0 = \dot{\gamma}y\hat{x}$.

The equations for the active polar order parameter \mathbf{p} read

$$\partial_t \mathbf{p} + (\mathbf{u} \cdot \nabla) \mathbf{p} - \frac{1}{2} (\nabla \times \mathbf{u}) \times \mathbf{p} + \left[\lambda_1 (\mathbf{p} \cdot \nabla) \mathbf{p} + \lambda_2 (\nabla \cdot \mathbf{p}) \mathbf{p} + \lambda_3 \nabla |\mathbf{p}|^2 \right]$$

$$= \frac{\lambda}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \cdot \mathbf{p} - \zeta \nabla c(\mathbf{r}, t) + \Gamma \mathbf{h} + \dots \quad (3.14)$$

In (3.14), the first three terms on the left are the material derivative of \mathbf{p} (co-moving and co-rotating with the suspension) and the square brackets (as well as the ζ term on the right) contain symmetry-allowed *polar* contributions [1] (of which the λ_1 term alone [7] is active), ruled out in apolar nematohydrodynamics, whether passive [20, 19] or active [4]. The first term on the right, together with the $\nabla \times \mathbf{u}$ term on the left, lead to flow-alignment [20]. The relaxation dynamics is contained in the order-parameter molecular field $\mathbf{h} = c(r, t) [\alpha \mathbf{p} - \beta |\mathbf{p}|^2 \mathbf{p} + K \nabla^2 \mathbf{p}]$ which favours a fixed length for \mathbf{p} and assigns an elastic cost to inhomogeneities in the one-Frank-constant approximation.

Conservation of active particles states

$$\partial_t c = -\nabla \cdot [c(\mathbf{u} + \mathbf{p})]. \quad (3.15)$$

The reactive stress from order-parameter gradients is

$$\boldsymbol{\sigma}^r = -\frac{\lambda}{2} (\mathbf{p} \mathbf{h} + (\mathbf{h} \mathbf{p})^T) + \Pi \mathbf{I} \quad (3.16)$$

where Π is a generalised pressure, and the viscous stress is,

$$\boldsymbol{\sigma}^d = \frac{\eta}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \equiv \eta \mathbf{A}, \quad (3.17)$$

where η the bare shear viscosity of the suspension.

To determine the active stress, we make use of the fact that the simplest active particle, on long timescales, is a permanent force dipole (equation 3.3) [3, 4]. To leading order, the deviatoric part of the stress coming from activity is given by,

$$\boldsymbol{\sigma}^a(\mathbf{r}, t) = W c(\mathbf{r}, t) \left(\frac{\mathbf{p} \mathbf{p}}{p^2} - \frac{1}{d} \mathbf{I} \right), \quad (3.18)$$

where d is the spatial dimension, and the magnitude and *sign* of W characterise the nature of the elementary force dipoles [3, 4].

Here we use this hydrodynamic description to study the effect of shear flow on the ordering and stability of active particle suspensions.

3.4.2 Homogeneous steady state

The instability of the orientationally ordered phase [4] and its stabilisation due to shear is most simply seen in two dimensions ($d = 2$); we therefore present detailed calculations in $d = 2$, and merely state results in $d = 3$.

We look for homogeneous steady states of (3.14), (3.15); the steady state concentration is a uniform $c(\mathbf{r}, t) = c_0$. The amplitude and phase of the steady state active vector order parameter are, respectively,

$$p_0^2 = \frac{\alpha}{\beta} + \frac{\dot{\gamma}}{2\beta} \sqrt{\lambda^2 - 1} \quad (3.19)$$

$$\tan \theta = \sqrt{\frac{\lambda - 1}{\lambda + 1}}. \quad (3.20)$$

The flow alignment parameter λ can take values between 1 and ∞ , corresponding to $0 \leq \theta \leq \pi/4$. Note that while p_0 increases with shear rate, the phase θ is independent of it.

3.4.3 Dynamics of fluctuations

To determine the stability of this homogeneous steady state, we set $\mathbf{p}(\mathbf{r}, t) = p_0(\cos \theta, \sin \theta) + \delta\mathbf{p}(\mathbf{r}, t)$, $\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_0 + \delta\mathbf{u}(\mathbf{r}, t)$ and $c(\mathbf{r}, t) = c_0 + \delta c(\mathbf{r}, t)$, where the perturbations are assumed small. It is convenient to decompose $\delta\mathbf{p}$ and $\delta\mathbf{u}$ parallel and perpendicular to the ordering direction \mathbf{p}_0 , e.g., $\delta\mathbf{p} = (\delta p_{\parallel}, \delta p_{\perp})$,

(i) *Order parameter fluctuations* : To linear order in fluctuations and lowest order in spatial derivatives, the hydrodynamics of the order parameter fluctuations read,

$$\begin{aligned} \delta\dot{p}_{\parallel} &+ \lambda_1 p_0 \nabla_{\parallel} \delta p_{\parallel} + \lambda_2 p_0 \nabla \cdot \delta\mathbf{p} + 2\lambda_3 p_0 \nabla_{\parallel} \delta p_{\parallel} + \dot{\gamma} y \partial_x \delta p_{\parallel} - \frac{\dot{\gamma}}{2} \delta p_{\perp} \\ &+ \frac{\dot{\gamma} y}{2} (\cos \theta \nabla_{\parallel} \delta p_{\parallel} - \sin \theta \nabla_{\parallel} \delta p_{\perp}) \\ &= \lambda p_0 \nabla_{\parallel} \delta u_{\parallel} + \frac{\lambda \dot{\gamma}}{2} (\sin 2\theta \delta p_{\parallel} + \cos 2\theta \delta p_{\perp}) - \left(2\alpha + \frac{3}{2} \dot{\gamma} \sqrt{\lambda^2 - 1} \right) \delta p_{\parallel} \\ &- \zeta \nabla_{\parallel} \delta c \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} \delta\dot{p}_{\perp} &+ \lambda_1 p_0 \nabla_{\parallel} \delta p_{\perp} + 2\lambda_3 p_0 \nabla_{\perp} \delta p_{\parallel} + \frac{\dot{\gamma}}{2} (1 + y \cos \theta \nabla_{\perp}) \delta p_{\parallel} + \dot{\gamma} y \left(\partial_x - \frac{1}{2} \sin \theta \nabla_{\perp} \right) \delta p_{\perp} \\ &+ \frac{p_0}{2} (\nabla_{\perp} \delta u_{\parallel} - \nabla_{\parallel} \delta u_{\perp}) \\ &= \frac{\lambda \dot{\gamma}}{2} \cos 2\theta \delta p_{\parallel} - \frac{\lambda \dot{\gamma}}{2} \sin 2\theta \delta p_{\perp} + \frac{\lambda p_0}{2} (\nabla_{\perp} u_{\parallel} + \nabla_{\parallel} u_{\perp}) - \frac{\dot{\gamma}}{2} (\sqrt{\lambda^2 - 1}) \delta p_{\perp} \\ &- \zeta \nabla_{\perp} \delta c \end{aligned} \quad (3.22)$$

Since we expect δp_{\parallel} to be massive, we may solve for δp_{\parallel} in terms of the other modes. Thus to lowest order,

$$\delta p_{\parallel}(t) = F \delta p_{\perp} - D_0 \nabla_{\parallel} \delta c + \delta p_{\parallel}(0) \exp(-Pt) + \exp(-Pt) \int_0^t \frac{d}{dt'} \left[\frac{\dot{\gamma}}{2} \delta p_{\perp} - \zeta \nabla_{\parallel} \delta c \right] \exp(Pt') dt' \quad (3.23)$$

where, $P = 2\alpha + \dot{\gamma}(\lambda^2 - 1)$, $F = \dot{\gamma}/P$ and $D_0 = \zeta/P$. For $t \gg P^{-1}$, δp_{\parallel} is slaved to these other modes. The resulting equation for δp_{\perp} , may be recast in fourier space,

$$\begin{aligned} & \left[\partial_t + iq_{\parallel}\lambda_1 p_0 + 2iq_{\perp}F\lambda_3 p_0 + \dot{\gamma}\sqrt{\lambda^2 - 1} \right] \delta p_{\perp}(q) + \frac{\dot{\gamma}}{2} (F \cos \theta - \sin \theta) \frac{\partial}{\partial q_y} (q_{\perp} \delta p_{\perp}(q)) \\ & = \left[\frac{ip_0 q^2}{2\rho_0 q_{\parallel}} (1 + \lambda \cos 2\xi) \right] \delta g_{\perp}(q) + \left[\frac{i\dot{\gamma}q_{\parallel}}{2} D_0 + \frac{i\lambda\dot{\gamma}q_{\parallel}}{2} D_0 \cos 2\theta \right] \delta c_q, \end{aligned} \quad (3.24)$$

Since the instability we are interested in has maximum growth rate [4] at zero wavenumber, we ignore the term $q_i \frac{\partial}{\partial q_j}$ operating on the fields, whose effect can be shown to vanish for $q \rightarrow 0$.

(ii) *Velocity fluctuations*: Fluctuations in the hydrodynamic velocity field, $\delta \mathbf{u} \equiv \delta \mathbf{g}/\rho_0$ are governed by $\nabla \cdot \boldsymbol{\sigma} = 0$, i.e., the Stokes equation. Thus the equation of fluctuation of the hydrodynamic velocity field, to linear order is given by,

$$0 = -\nabla_j \Pi \delta_{ij} + \eta \nabla^2 \delta g_i - W \left(\frac{c_0}{p_0^2} \nabla_k [\delta(p_i p_k)] - c_0 p_i p_k \nabla_k \left[\delta \left(\frac{1}{p^2} \right) \right] - \frac{p_i p_k}{p_0^2} \nabla_k \delta c + \frac{1}{3} \nabla_i \delta c \right). \quad (3.25)$$

This maybe simplified by the following steps of tedious algebra. First, resolve \mathbf{p}_{\parallel} and \mathbf{p}_{\perp} in the xy plane, $p_i p_k = p_0^2 [\cos^2 \theta \delta_{ix} \delta_{kx} + \sin^2 \theta \delta_{iy} \delta_{ky} + \cos \theta \sin \theta \delta_{ix} \delta_{iy} + \cos \theta \sin \theta \delta_{kx} \delta_{ky}]$ and $\delta(p_i p_k) = p_0 (\cos \theta \delta_{ix} + \sin \theta \delta_{iy}) \delta p_k + p_0 (\cos \theta \delta_{kx} + \sin \theta \delta_{ky}) \delta p_i$, and use the variation $\delta \left(\frac{1}{p^2} \right) \approx -\frac{2\delta p_{\parallel}}{p_0^3}$. Then fourier transforming, we obtain,

$$\begin{aligned} 0 = & -iq_{\parallel} \Pi \delta_{ij} - \nu q^2 \delta g_i - \frac{iAc_0}{p_0} \left[q_k \delta p_k (\cos \theta \delta_{ix} + \sin \theta \delta_{iy}) + \delta p_i (q_x \cos \theta + q_y \sin \theta) \right] \\ & + \frac{2iAc_0 \delta p_{\parallel}}{p_0} \left[q_x \cos^2 \theta \delta_{jx} + q_y \sin^2 \theta \delta_{jy} + \cos \theta \sin \theta (q_y \delta_{jx} + q_x \delta_{jy}) \right] + \frac{iq_i A}{3} \delta c_q \\ & - iA \delta c_q \left[q_x \cos^2 \theta \delta_{jx} + q_y \sin^2 \theta \delta_{jy} + \cos \theta \sin \theta (q_y \delta_{jx} + q_x \delta_{jy}) \right]. \end{aligned} \quad (3.26)$$

Since the fluid is incompressible, we can eliminate the generalized pressure, Π by operating both sides of (3.26) with the transverse projection operator, $\delta_{ij} - \frac{q_i q_j}{q^2}$. Now rearranging the terms we obtain,

$$0 = -\eta q^2 \delta g_{\perp}(q) + \left(\frac{iWq_{\parallel}^2 q_{\perp}}{q^2} \right) \delta c_q + \left[\frac{2iWc_0 q_{\parallel} q_{\perp}}{p_0 q^2} \right] q_{\perp} \delta p_{\perp}(q) - \frac{iWc_0 q_{\parallel}}{p_0} \delta p_{\perp}(q). \quad (3.27)$$

(iii) *Concentration fluctuations*: Using equation (3.15) we obtain the Fourier transformed equation for concentration fluctuations. To linear order it is given by,

$$\partial_t \delta c_q = -iq_{\parallel} p_0 \delta c_q - ic_0 (q_{\perp} - Fq_{\parallel}) \delta p_{\perp}(q). \quad (3.28)$$

3.4.4 Fluctuation spectrum

We are now in a position to compute the full fluctuation spectrum. While we have analysed the linear stability of the orientationally ordered phase in the basis spanned by $\delta\mathbf{u}$, $\delta\mathbf{p}$ and δc , we find that the origin of instability and its recovery by the shear flow can be understood even in the absence of the concentration equation. Thus, to make the subsequent analysis more transparent, we drop the concentration fluctuation terms at the outset.

Following [4], we express the fluctuation spectrum in terms of the splay fluctuation $\Phi = \nabla_{\perp}\delta p_{\perp}$, the in-plane expansion rate $\Theta = \nabla_{\perp}\delta g_{\perp}$, and the angle ξ made by \mathbf{q} with the ordering direction:

$$\left[\partial_t + iq_{\parallel}\lambda_1 p_0 + \dot{\gamma} \sqrt{\lambda^2 - 1} \left(1 - \frac{1}{4\lambda} \right) + O(\dot{\gamma}^2) \right] \Phi_q - \left[\frac{ip_0 q^2}{2\rho_0 q_{\parallel}} (1 + \lambda \cos 2\xi) \right] \Theta_q = 0 \quad (3.29)$$

and

$$0 = -\eta q^2 \Theta_q - \left[\frac{iWc_0 q_{\parallel} \cos 2\xi}{p_0} \right] \Phi_q \quad (3.30)$$

This implies that the splay fluctuations Φ_q have a growth rate

$$\Omega = \frac{Wc_0}{2\eta} \cos 2\xi (1 + \lambda \cos 2\xi) - \dot{\gamma} \sqrt{\lambda^2 - 1} \left(1 - \frac{1}{4\lambda} \right) + O(\dot{\gamma}^2). \quad (3.31)$$

3.4.5 Instability in the absence of external shear

In the absence of imposed shear, $\dot{\gamma} = 0$. From equation (3.31) we see that the splay fluctuations Φ_q have a growth rate,

$$\Omega = \frac{Wc_0}{2\eta} \cos 2\xi (1 + \lambda \cos 2\xi). \quad (3.32)$$

From this relation one can understand the generic instability when $\dot{\gamma} = 0$: the oriented phase is *always* unstable, either to splay or to bend fluctuations, depending on the sign of W [4]. This can be seen *even at* $q = 0$, where the growth rate $\Omega_{+}(0) > 0$ for $-\pi/4 > \xi > \pi/4$ when $W > 0$, and $\Omega_{-}(0) > 0$ for $\pi/4 > \xi > 3\pi/4$, when $W < 0$ (Fig. 3.4). The dispersion curve Ω_{\pm} is shown in Fig. 3.5 : fluctuations with wavenumber $q < q_0 \propto |W|$ grow in time; this sets the scale over which orientational order can be stabilised. Note that *both* the polar phase, as noted by [4], and the apolar nematic as well, are generically unstable in the absence of shear. This is clear, since the polar terms containing $\{\lambda_i\}$ and ζ in (3.14), do not appear in the growth rate equation (3.32).

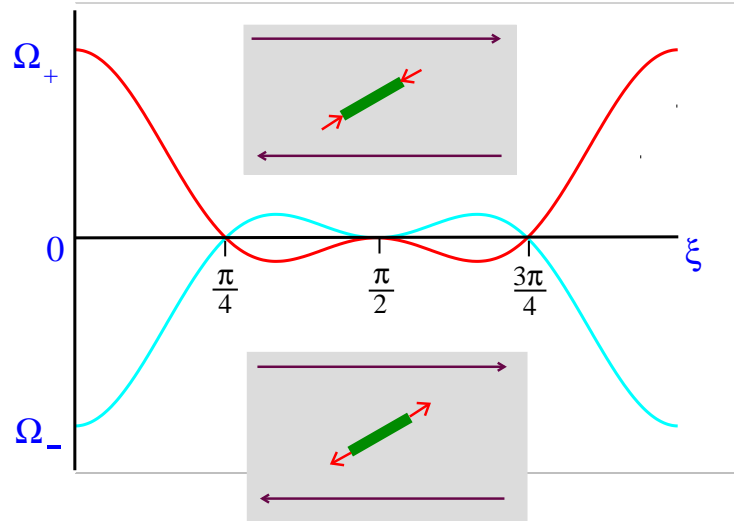


Figure 3.4: The growth rates $\Omega_{\pm}(q = 0)$ versus 2ξ (angle between perturbing and the ordering direction), for either sign of activity W . Inset shows the induced flow lines arising from active rods subject to a flow.

3.4.6 Non-equilibrium stability diagram in $d = 2$

The imposed shear flow represented by a horizontal line $\dot{\gamma} = \text{const.}$ in Fig. 3.5, cuts this dispersion curve at $q^*(\dot{\gamma})$, suggesting that fluctuations whose scale is *smaller* than $1/q^*$ are the first to be stabilised by the shear flow. Our estimate of q^* is qualitative : the crossover from Ω -dominated to $\dot{\gamma}$ -dominated at nonzero q cannot strictly be evaluated in our treatment, since we dropped the $q_x \partial q_y$ terms on the grounds that they wouldn't matter at $q = 0$ where the instability is fastest. As $\dot{\gamma}$ is increased, $q^*(\dot{\gamma})$ decreases, till at $\dot{\gamma} = \dot{\gamma}_c$, this cutoff scale moves to zero, as $q^* = (\dot{\gamma} - \dot{\gamma}_c)^{1/2}$. At this shear rate $\dot{\gamma}_c$, the oriented phase is completely stabilised by the shear flow. This defines a stability boundary as a function of λ and W .

The stability phase diagram is best represented by defining a dimensionless *active* Peclet number, $Pe_a = 2\eta\dot{\gamma}/|W|c_0$, as the ratio of the imposed shear rate to the typical shear-rate produced around the active particles. As one crosses from the unstable to the stable region in the plane of Pe_a and flow-alignment parameter λ (Fig. 3.6), the orientational order parameter sets in at the value given by equation. (3.19), which in effect gives a discontinuous transition since the order parameter in the hydrodynamically unstable region is zero. For polar active particles, the shear-stabilised oriented phase has a nonzero drift of the particles with respect

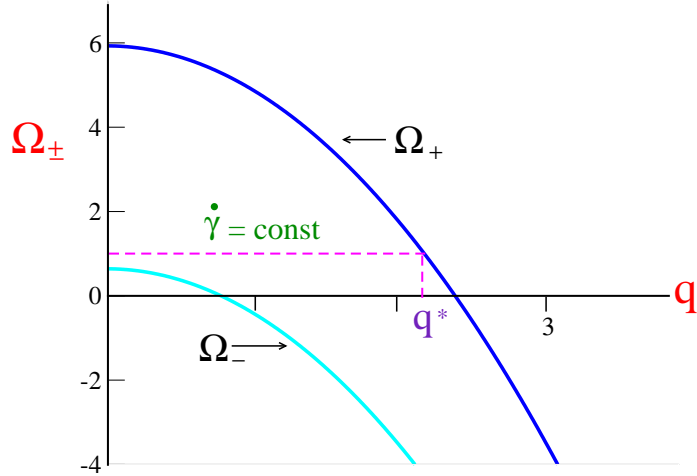


Figure 3.5: Dispersion curves Ω_{\pm} versus q , for ξ corresponding to maximum growth rate. The horizontal line corresponding to a frequency $\dot{\gamma} \sqrt{\lambda^2 - 1} \left(1 - \frac{1}{4\lambda}\right)$ defines the scale q^* beyond which fluctuations are stabilised. As $\dot{\gamma}$ increases towards $\dot{\gamma}_c$, the stabilisation scale goes to zero as $(\dot{\gamma} - \dot{\gamma}_c)^{1/2}$.

to the solvent. Note that the critical shear rate required to stabilise the oriented phase $\dot{\gamma}_c$ is larger for positive W than negative.

3.4.7 Shear flow induced stabilisation in $d = 3$

The calculation of the stability diagram is more tedious in $d = 3$; however since the spirit is the same, we merely quote results and point out differences. To start with, we note that \mathbf{p}_0 , the steady state orientational conformation in the presence of the shear flow $\mathbf{u}_0 = \dot{\gamma}y\hat{\mathbf{x}}$, is the same as (3.19) and (3.20), i.e., it still lies in the plane of the imposed velocity and its gradient (the xy plane). We then decompose the fluctuations in an appropriate orthonormal basis, viz., $\delta\mathbf{p} = (\delta p_{\parallel}, \delta p_{\perp}, \delta p_z)$, where $\delta p_{\parallel} = \delta\mathbf{p} \cdot \mathbf{p}_0$, $\delta p_{\perp} = \delta\mathbf{p} \cdot (\hat{\mathbf{z}} \times \mathbf{p}_0)$ and $\delta p_z = \delta\mathbf{p} \cdot \hat{\mathbf{z}}$ (similarly for $\delta\mathbf{g}$). As before we ignore concentration fluctuations.

Once again, δp_{\parallel} is massive, and we rewrite the order parameter fluctuations in terms of the in-plane splay $\Phi_{\perp} = \nabla_{\perp} \delta p_{\perp}$ and $\Phi_z = \nabla_z \delta p_z$. Similarly invoking incompressibility, we rewrite the momentum fluctuations as the in-plane expansion rate $\Theta_{\perp} = \nabla_{\perp} \delta g_{\perp}$ and $\Theta_z = \nabla_z \delta g_z$. Eliminating the momentum fluctuations via force-balance, we find that the resulting linearised dynamical equations for Φ_{\perp} and Φ_z give rise to an eigenvalue spectrum

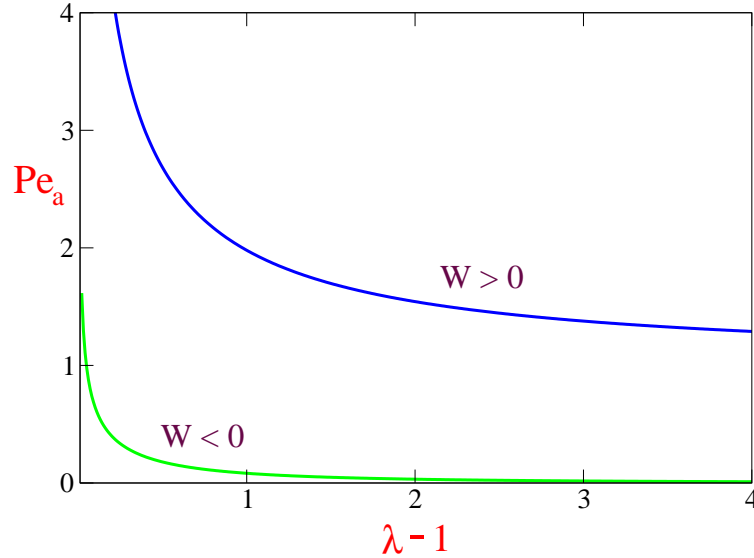


Figure 3.6: Stability diagram in the plane of Active Peclet number Pe_a and flow-alignment parameter λ .

qualitatively resembling (figure 3.5). Thus even in $d = 3$, one may define a stability phase diagram in the $Pe_a - \lambda$ plane; a large enough shear rate stabilises an orientationally ordered phase.

3.5 Rheology in the oriented phase

Once we have stabilized the oriented phase by an externally imposed shear, we can explore the rheology of the active oriented phase and highlight the differences with passive nematic.

Recall that in section (3.3) we had shown that a suspension of active particles have a nonzero, macroscopic and anisotropic mean stress. This is sharp contrast to the orientationally ordered state for a passive nematic. A passive nematic, bound by Pascal's law has purely *isotropic* mean stress, i.e., pressure. The deviatoric stress for the passive suspension always integrates to zero.

Another notable qualitative difference of the active oriented suspension with respect to the passive nematics is in the form of normal stresses. For the passive case, the normal stresses are always positive, ensuring that they are rheologically stable. However for an ac-

tive oriented suspension, the sign of the normal stress depends on the sign of W - active oriented matter can be rheologically unstable.

The viscoelastic behaviour of active suspension also differs qualitatively from their passive counterparts: While the $G'(\omega\tau \gg 1)$ for a passive nematogen decreases as $\frac{\lambda_a}{\tau}$ and approaches zero near the equilibrium isotropic to nematic transition, for the active case, $G'(\omega\tau \gg 1) \simeq W$. This is indeed surprising. At equilibrium one would expect such strong viscoelastic behaviour from a fluid or suspension near translational freezing, not near orientational ordering.

Note that the vector-ordered phase has a non-zero drift velocity v_d , with a magnitude proportional to $c_0 p_0$ (3.19). This macroscopic *particle current* will result in a counter *solvent flow* of the same magnitude. Experimental realisations in a planar shear flow geometry will necessarily have to contend with finite boundaries; it is therefore important to specify boundary conditions for the active order parameter \mathbf{p} . This is especially important in the polar case, since the local polarisation implies a local average drift velocity of the active particle relative to the fluid. Assuming the active particles cannot penetrate the walls, \mathbf{p} must be tangent to the confining walls (the homogeneous boundary conditions of liquid crystal physics). In plane Couette flow, however, this is at odds with the flow-alignment requirement of \mathbf{p} pointing at an angle to the suspension velocity as in equation (3.20). This conflict must be resolved by a boundary layer in \mathbf{p} at the walls.

3.6 Conclusions

In conclusion, we have shown how to stabilise the orientationally ordered phase of an active particle suspension by imposing a uniform shear flow. We have determined the nonequilibrium phase diagram in the plane of “active Peclet number” and flow-alignment parameter. Further we have shown how the rheology of suspension of active particles in the oriented phase is in sharp contrast to equilibrium orientationally ordered phase.

The unusual rheological features of the oriented phase would have vital consequence, especially in understanding the dynamics of reorientation of endothelial cells subject to shear flow [24].

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