Chapter 1

INTRODUCTION

1.1 Gravitational waves and their properties

According to Einstein's general theory of relativity, gravitation is curvature of spacetime. In this picture, gravitational waves are ripples in the curtature of spacetime that propagate with the speed of light. In general, due to the non-linear nature of gravity, it is not possible to separate the contribution to the spacetime curvature due to the gravitational waves from that due to the background geometry in a fully rigorous manner. However, in realistic astrophysical scenarios, where detectable gravitational waves are generated, the length scales over which the waves vary, roughly their wavelength, is very small compared to the length scale over which the background curvature varies. This difference in length scales allow the following highly accurate, though approximate split of the the metric $g_{\mu\nu}$ describing the spacetime outside the material source

$$g_{\mu\nu} = h_{\mu\nu}^{\mathrm{B}} + h_{\mu\nu}^{\mathrm{GW}} , \tag{1.1}$$

where $h_{\mu\nu}^{\mathrm{B}}$ and $h_{\mu\nu}^{\mathrm{GW}}$ represent the contributions to spacetime metric arising from the background material sources and the gravitational waves respectively. In this thesis, greek indices range from 0 to **3** while latin indices i, j, k, m... range from **1** to **3.** Following [1], one can argue that for all types of sources, in the local wave-zone whose radius is much smaller than the wavelength of the wave, $h = |h_{\mu\nu}^{\mathrm{GW}}| << 1$. The other contribution to $g_{\mu\nu}$, $h_{\mu\nu}^{\mathrm{B}}$ will be important only when one deals with the propagation effects. As regions of very strong curvature are very rare in the universe, waves from most of the sources will propagate to earth via pure geometric optics. For distances small compared to the Hubble distance, the universe is almost globally Lorentz in its background geometry. Then the geometric optics propagation will preserve the structure of $|h_{\mu\nu}^{\rm GW}|$. However, for sources at cosmological distances, the background geometry is Friedman-Robertson-Walker and the geometric optics propagation produces the same effects for gravitational radiation as for electromagnetic radiation. This implies, in this case, the length scales and the mass scales appearing in $|h_{\mu\nu}^{\rm GW}|$ will contain cosmological redshift corrections.

The arguments presented in the last paragraph allow us to use linearized approximation to general relativity where

$$g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + h_{\mu\nu}(x^*). \tag{1.2}$$

In the above the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $h_{\mu\nu}(x^\lambda)$ is a compact notation for $h_{\mu\nu}^{\rm GW}$ and $|h_{\mu\nu}| \ll 1$. The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ are of the order of h and the Riemann curvature tensor $R^\alpha_{\beta\gamma\delta}$ is of the order of h^2. Then the Bianchi identities yield,

$$R^\mu_{\ \nu\alpha\beta,\gamma\delta} + R^\mu_{\ \nu\beta\gamma,\alpha\delta} + R^\mu_{\ \nu\gamma\alpha,\beta\delta} = 0\,, \tag{1.3}$$

which on contracting over μ and β yields in vacuum

$$R^\mu_{\nu\alpha\beta,\mu} = 0 \tag{1.4}$$

Further, contracting over γ and δ yields

$$\Box_f R_{\mu\nu\alpha\beta} = 0\,, \tag{1.5}$$

where $\Box_f \equiv f^{\mu\nu}\partial_\mu\partial_\nu$ is the *D'*Alembertian. Under small coordinate changes $R_{\alpha\beta\gamma\delta}$ is invariant. Thus, in this approximation there exist curvature waves with an invariant meaning. The more familiar form is obtained in the transverse traceless (TT) gauge or synchronous coordinates. Under the transformation,

$$x'^{\mu} = x^{\mu} + \xi^{\mu}, \tag{1.6}$$

we obtain

$$h_{00}^{\rm TT} = h_{00}^{\rm old} - 2\xi_{0,0};\ h_{0i}^{TT} = h_{0i}^{old} - \xi_{0,i} - \xi_{i,0};\ h_{ij}^{\rm TT} = h_{ij}^{\rm old} - \xi_{i;j} - \xi_{j;i} \tag{1.7}$$

By simple time quadratures one can make

$$h_{00}^{\rm TT} = 0 = h_{0i}^{\rm TT}, \tag{1.8}$$

whence $R_{0i0j} = -\frac{1}{2}h_{ij,00}^{\rm TT}$ which can be integrated to yield,

$$h_{ij}^{\rm TT}(x^0, x^k) = -2\int_{-\infty}^{x^0} dx'^0 \int_{-\infty}^{x'^0} dx''^0\ R_{0i0j}(x''^0, x^k) \tag{1.9}$$

From the equations for $R_{\alpha\beta\gamma\delta}$ we get the following equations for $h_{ij}^{\rm TT}$:

$$\Box_f h_{ij}^{\rm TT} = 0;\ h_{ij,j}^{\rm TT} = 0;\ h_{jj}^{\rm TT} = 0. \tag{1.10}$$

From Eqs.(1.9) and (1.10) it is evident that $h_{ij}^{\rm TT}$ are the double time integrals of the components of the Riemann curvature tensor that propagate through spacetime with the velocity of light. From general counting, we see, that finally we are left with – $(10 - 4 - 3 - 1 = 2)$ – i.e two independent components for $h_{ij}^{\rm TT}$, corresponding to the two polarization states for the gravitational wave. This derivation is adapted from [2, 3].

We next rewrite the equation $R_{0i0j} = -\frac{1}{2}h_{ij,00}^{\rm TT}$ in a more familiar form as

$$\frac{\partial^2 h_{jk}^{\rm TT}}{\partial t^2} = -2\,R_{0j0k}\,, \tag{1.11}$$

to explain its physical significance. In general relativity, the Riemann curvature tensor is operationally defined by the equation of geodesic deviation which gives the relative acceleration between nearby test particles. Consider a region of spacetime where the gravitational waves are the only source of spacetime curvature; these waves will produce tiny oscillatory changes δx^i in the position of a test particle with respect to the origin of the coordinate system. These changes will satisfy the equation of geodesic deviation, written in this case as

$$m \frac{d^2\, \delta x^j}{d\, t^2} = \; -R^{\mathrm{GW}}_{0j0k} x^k \; = \frac{1}{2} \frac{\partial^2 h^{\mathrm{TT}}_{jk}}{\partial t^2} x^k \, . \tag{1.12}$$

As $|\delta x^j| << |x^j|$, x^j being the components of the separation vector for the test particle from the origin of the coordinate system, we can consider it as a constant. We can then readily integrate Eq.(1.12) to obtain

$$\delta x^j = \frac{1}{2}\, h^{\mathrm{TT}}_{jk}\, x^k \, . \tag{1.13}$$

Eq.(1.13) suggests that h^{TT}_{jk} can be defined as the 'dimensionless strain of space', as it is the ratio of the wave induced displacement of a free particle to its original distance with respect to the origin of the coordinate system. Consider a coordinate system oriented such that waves propagate in the z direction. The transverse and traceless nature of h^{TT}_{jk} implies that there are only two non-zero components for h^{TT}_{jk}. They are given by $h^{\mathrm{TT}}_{xy} = h^{\mathrm{TT}}_{yx}$, $h^{\mathrm{TT}}_{xx} = -h^{\mathrm{TT}}_{yy}$, corresponding to the two independent polarization states. We observe that the quantity $h_+ \equiv h^{\mathrm{TT}}_{xx} = -h^{\mathrm{TT}}_{yy}$, in Eq.(1.13), produces a force field with the orientation of a '+' sign, while $h_\times \equiv h^{\mathrm{TT}}_{xy} = h^{\mathrm{TT}}_{yx}$, produces one with the orientation of x sign as shown in Fig.1.1. The quantities h_+ and $h_\times$ are called the 'plus' and 'cross' gravitational wave polarizations.

1.2 Order of magnitude estimates

To obtain an order of magnitude estimate for the strength of gravitational waves $h = |h^{\mathrm{TT}}_{jk}|$ from astrophysical systems, we employ the quadrupole formalism for the gravitational wave generation. This formalism gives the h^{TT}_{jk} in the source's local wave-zone as,

$$h^{\mathrm{TT}}_{jk} = \frac{2G}{c^4\, R} \left\{ \frac{\partial^2 Q_{jk}}{\partial T^2} (T - R/c) \right\}^{\mathrm{TT}} \, . \tag{1.14}$$

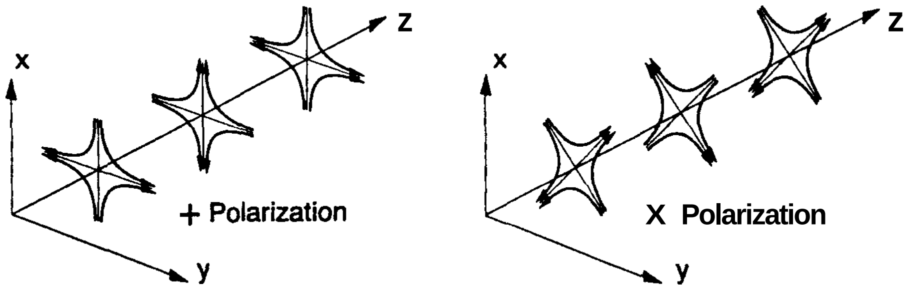

Figure 1.1: The lines of force associated with the two polarizations of a gravitational wave. This Figure is from [7]

The coordinate system (X, T) is centered on the source, with R $= |\mathbf{X}|$, the distance to the source. The superscript TT pictorially means algebraically project out and keep only the part that is transverse to the radial direction of propagation and traceless. Q_{jk} is the source quadrupole moment which is symmetric and tracefree, evaluated at the retarded time $(T - R/c)$. By differentiating with respect to time Eq.(1.14), squaring the result and then integrating over all directions $N = \mathbf{X}/R$, one obtains the total power in the gravitational waves emitted by the source,

$$\mathcal{L}_{\mathrm{N}} = \frac{G}{5c^5}\left\{\frac{\partial^3 Q_{jk}}{\partial T^3}\right\} \tag{1.15}$$

This is the famous Einstein quadrupole formula [4]. To get an order of magnitude estimate for $h = |h_{jk}^{\mathrm{TT}}|$, following [5], we use Eq.(1.14) which gives

$$h \sim \frac{G}{c^4}\frac{\ddot{Q}}{R}. \tag{1.16}$$

Thus the strong sources must be highly nonspherical. Writing $Q \sim M L^2$, $Q \sim 2 M v^2 \sim 4 E_{\mathrm{kin}}^{\mathrm{ns}}$, where M is the mass, L the size, v the internal kinetic energy and $E_{\mathrm{kin}}^{\mathrm{ns}}$ the nonspherical kinetic energy of the source. This implies

$$h \sim \frac{1}{c^2}\frac{4G\left(\frac{E_{\mathrm{kin}}^{\mathrm{ns}}}{c^2}\right)}{R}. \tag{1.17}$$

.According to the virial theorem, any gravitationally induced kinetic energy should be comparable to the source's potential energy. This implies that sources of gravitational waves should have a large potential energy, which is possible only if the source is very compact. They should also be highly dynamical to have large values for Q. Thus we infer that strong sources of gravitational waves are highly compact, dynamical concentrations of large amounts of matter. For $E_{kin}^{ns} \sim M_\odot$, thumb rule values for h are in the range $10^{-21} - 10^{-22}$. The gravitational wave source cannot emit strongly at periods smaller than the light travel time which is $\sim \frac{4\pi GM}{c^3}$ implying that the frequencies at which these sources radiate are

$$f \leq \frac{c^3}{4\pi GM} \sim 10^4 \frac{M_\odot}{M} \, \text{Hz} \tag{1.18}$$

Gravitational waves from astrophysical systems that exist today lie between $10^{-4} - 10^4$ Hz. However waves from the processes in the early universe could range from $10^{-18} - 10^8$ Hz.

1.3 Astrophysical sources of gravitational waves and their detection

Astrophysical gravitational waves are extremely weak by the time they reach the Earth. This is why, though Einstein predicted gravitational waves as a consequence of his general theory of relativity back in 1916, only in the late 1950's experiments were attempted to detect them. There are four gravitational wave frequency bands that are actively being explored experimentally and theoretical studies have also identified plausible sources in each each of these bands. These frequency bands are

- The high frequency band that lies between 1 to 10^4 Hz in which the earth based gravitational wave detectors operate.
- The low frequency band lying between 1 Hz to 10^{-4} Hz, which is the realm of the space based detectors.
- The very low frequency band, 10^{-7} to 10^{-9} Hz.
- The extremely low frequency band, 10^{-15} to 10^{-18} Hz.

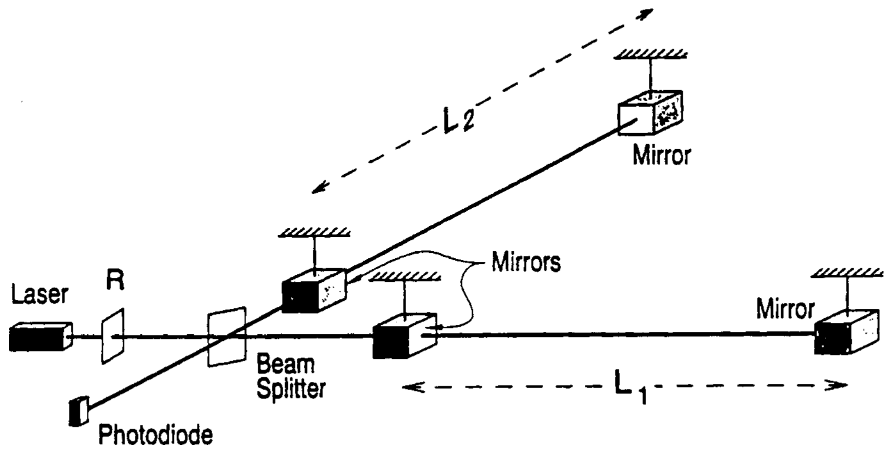

Figure 1.2: Schematic diagram of a laser interferometer gravitational wave detector. This Figure is adapted from [7]

In what follows, we will discuss in detail the detectors and sources for the high and low frequency bands and will make passing remarks about the very low and extremely low frequency bands.

The important sources in the high frequency band are stellar collapses forming neutron stars and black holes, both in our galaxy and the nearby galaxies, spinning neutron stars having slightly imperfect crust, accretion on to spinning neutron stars and black holes, gravitational radiation-reaction driven instabilities operating during the first few years after the formation of a neutron stars and the coalescence of compact binaries containing neutron stars and black holes (with masses lying between a few $M_\odot$ to $10^4 M_\odot$), occuring in distant galaxies. The stochastic gravitational waves from the big bang, phase transitions in the early universe and vibrating loops of cosmic strings may also exist in this range [5, 6].

The most promising and versatile type of gravitational wave detector in the high frequency band is an laser interferometer. It consists of four mirror-endowed masses that hang from vibration-isolated supports and the associated optical systems for monitoring the separations between the masses, as shown in Fig.(1.2). The passing gravitational wave changes the round-trip travel time of the constant-phase fronts differentially in each of the interferometer's arms, resulting in a fringe shift at the output. Thus by monitoring the fringe shifts, caused by the changing arm-length difference $\Delta L(t)$, one can infer the action of passing gravitational waves. In general, when the gravitational waves are present, the output of the interferometer will be a linear combination of the two polarizations:

$$\frac{\Delta L(t)}{L} = F_+ h_+(t) + F_\times h_\times(t) \equiv h(t)\,, \tag{1.19}$$

where $L_1 \sim L_2 = \mathrm{L}$ and F_+ and $F_\times$ are of the order of unity and depend in a quadrupolar manner on the direction to the source and the orientation of the detector. At present laser interferometers are capable of measuring $\Delta L \sim 10^{-16}$ cm. Since L $\sim \Delta L/h$, one is led to interferometers with arm-length L $\sim$ 1 to 10 km in order to detect waves producing strain in the range of 10^{-21} to 10^{-22}. However interferometers are plagued by non-Gaussian noise which can be eliminated only by cross correlating the outputs of two, three or more interferometers that are networked together at widely separated sites. This is why an international network consisting of three km-scale interferometers at three widely separated sites is now being constructed for the direct detection of gravitational waves.

The three km-scale interferometers are

- The United States Laser Interferometer Gravitational wave Observatory (LIGO) project which will support interferometers with 4 km arms at two different sites at Hanford, Washington and Livingston, Louisiana in the U.S.A [7].

- The French-Italian Virgo project, named after the Virgo cluster of galaxies consists of a single laser interferometer with **3** km arm-length and under construction in Pisa, Italy [8].

These are designed to accommodate future expansions without the necessity of any major facilities upgrade. They will be operational by the year 2002.

At present, there are two more sub kilometer arm-length interferometers that are being constructed. One is a British-German collaboration constructing a 600-meter interferometer near Hanover, Germany called GEO-600 [9]. The other one is TAMA-300 constructed by Japan near Tokyo with a 300-meter arm-length [10]. These are important test-beds for interferometer techniques and in future may lead to kilometer-scale interferometers, thus enhancing all sky-coverage. It is expected that two LIGOs and VIRGO operating as a coordinated international network will be able to detect and locate burst sources which will last for a few minutes. They will be able to monitor $h_+(t)$ and $h_\times(t)$ in the frequency range 10 Hz to 1000 Hz.

At frequencies above 1000 Hz, the interferometer's photon shot noise due to photon counting statistics becomes a serious obstacle to wave detection. Resonant mass detectors which are future variants of Weber's original bar detectors [11] are good candidates in these frequency domains. A resonant bar detector can be modelled as a pair of masses connected by a spring. The effect of a transitory metric perturbation is felt as a tidal relative force between a pair of masses or across the object. The gravitational wave impulses would set the pair of masses vibrating about their common center of mass. That vibration, persisting long after the brief gravitational wave has passed, would register as an oscillatory acceleration in the attached sensor [12]. Supernovae within the galaxy and quasi-normal modes of black holes in the Virgo cluster may be detected by these instruments.

For ground based detectors, there exists a low frequency limit due to the seismic noise generated by gravity gradients associated with seismic activities. Space based detectors can avoid this low frequency limit and search for gravitational waves at much lower frequencies. We now turn to the low frequency band where the proposed Laser Interferometer Space Antenna (LISA) will operate [13]. It consists of three space-crafts forming an equilateral triangle that is inclined at an angle of 60^0 to the Earth's orbital plane. It will be arranged to form a Michelson interferometer with

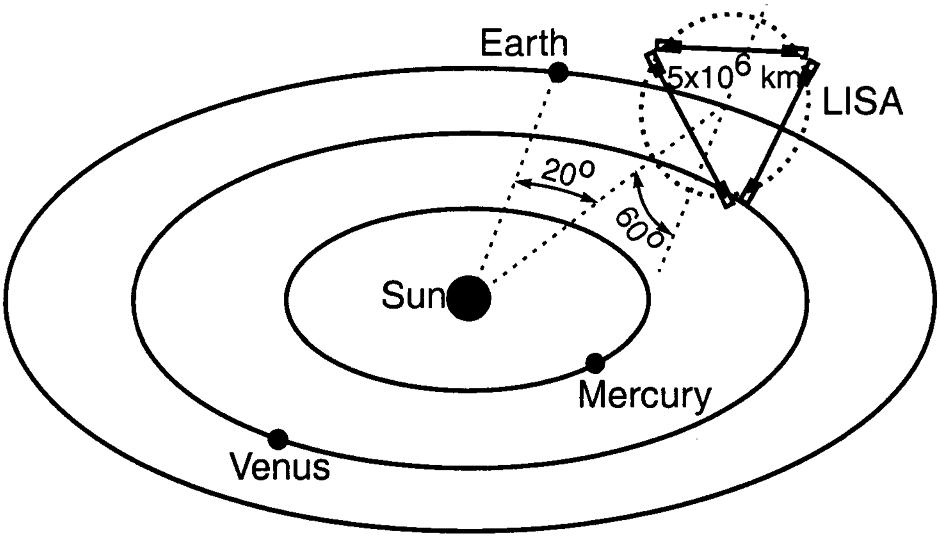

Figure 1.3: Schematic diagram for LISA's orbital configuration. This Figure is from [13]

5 x 10^6 km baseline in a solar orbit at one A.u. as shown in Fig. 1.3. It will have a peak sensitivity in the $1 - 10$ mHz band. The European Space Agency has adopted LISA as the third "cornerstone" mission in its Horizon 2000+ program. LISA will surely detect waves from short period binaries containing main sequence stars, white dwarfs, neutron stars and black holes in our galaxy. It will also observe coalescence of supermassive black hole binaries ($M \sim 10^5 - 10^7 M_\odot$) in distant galaxies, capture of solar mass compact objects by massive black holes, solar oscillations and stochastic waves from processes in the early universe. The cutoff at lower frequencies is due to the difficulty in isolating the spacecraft from forces due to fluctuations in solar wind and cosmic rays. Finally, Doppler tracking of spacecraft by microwaves is also used to put bounds on strengths of various low frequency waves from specific sources like cosmic strings. Finally, we consider the two remaining wave bands. The first one is the very low frequency band lying between 10^{-7} and 10^{-8} Hz [14]. This band is explored by the timing of millisecond pulsars. At 95 % confidence level, the power density in these gravitational waves $\Omega_g(4 \text{ x } 10^{-9}\text{Hz}), 6 \text{ x } 10^{-8}\text{H}^{-2}$ where H is the Hubble constant relative to $100\,\text{km}\,\text{sec}^{-1}\,\text{Mpc}^{-1}$. This range include stochastic

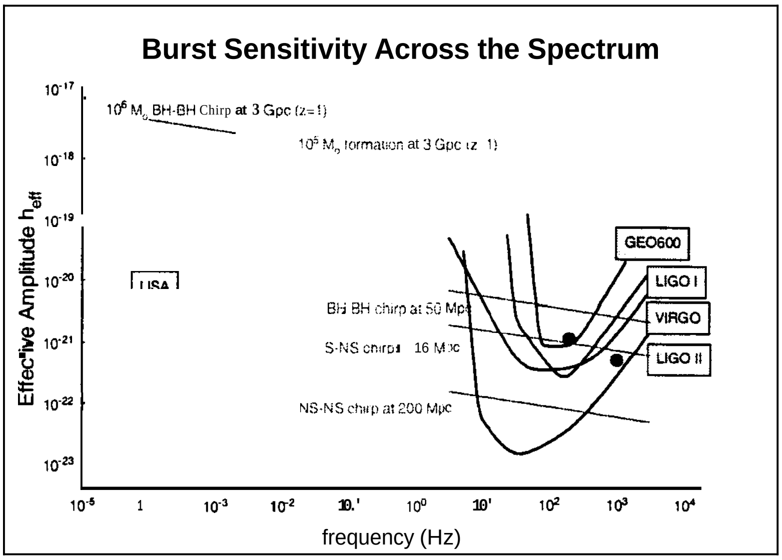

Figure 1.4: The burst sources for gravitational waves. The burst sensitivities of five major interferometers are shown. Here projected broad-band noise sensitivities to bursts are compared with the effective amplitudes h_{eff} of the waves from various sources. Note that $h_{eff} = h\,(n)^{1/2}$, where h and n are the amplitude and the number of cycles of gravitational wave in the respective frequency bands. LIGO I and LIGO II correspond to initial and advanced versions. This Figure is taken from [15]

waves from cosmic strings, phase transitions in early universe and the big bang. The second band which lies between 10^{-15} and 10^{-18} Hz, referred to as the extremely low frequency band earlier, should contribute to anisotropies in the cosmic microwave background radiation. If all the anisotropies in COBE measurements were due to gravitational waves it would correspond to $\Omega_g(10^{-18}\,\text{Hz}) \sim 10^{-9}$.

1.4 Relativistic inspiraling binaries, phasing and gravitational wave astronomy

It is clear from Fig.1.4 that inspiraling compact binaries are the most promising sources of gravitational radiation for earth based interferometers. It also turned out to be the best understood of all gravitational wave sources. The famous Hulse-Taylor binary pulsar PSR 1913+16 is an example of inspiraling compact binary containing two neutron stars. As the two neutron stars are moving fast and relatively close, according to general relativity they should emit moderate amounts of gravitational radiation, causing the binary to lose energy. The orbital period of the binary should decrease by Kepler's third law, and this has been observed. The radio observations of PSR 1913+16 for the last twenty five years has yielded a verification of the quadrupole formula for the radiation damping to better than 0.4% [16, 17]. This is considered as sufficient evidence, though indirect, for the existence of the gravitational radiation and the Nobel Prize in 1993 certifies the quality of the evidence. At present the neutron stars in PSR 1913+16 are slowly spiraling inwards and emitting gravitational waves with $f \sim 10^{-4}$ Hz, too small to detect even by LISA. However, after about hundred million years, this inspiral will bring waves into the LIGO/VIRGO high frequency band. As the neutron stars continue their inspiral, over a time of about fifteen minutes the gravitational waves will sweep through the interferometer's sensitive frequency bandwidth between ten to thousand hertz. It is these last stages of inspiral and subsequent merger of compact binaries involving neutron stars and black holes that the LIGO/VIRGO network seeks to monitor. The estimates based on the quadrupole approximation suggest that the number of cycles of gravitational waves in the sensitive bandwidth of LIGO/VIRGO will be about sixteen thousand for neutron star-neutron star binaries, about three thousand for neutron star-black hole binaries and about six hundred for black hole-black hole binaries [18]. The current estimates of neutron star-neutron star binary coalescences based on the very few known systems like PSR 1913+16 project an event rate of three per year to 200 Mpc [19, 20] while the estimates based on the evolution of the progenitor main sequence binaries, suggest roughly three events per year to 70 Mpc [21]. A similar event rate exists for neutron star-black hole and black hole-black hole binary coalescences [22].

Objects like neutron stars and black holes have strong internal self-gravity. This implies that most of the non-gravitational effects like magnetic fields, interstellar medium, etc., which usually plague binary star systems will be negligible compared to the strong gravitational effects. Inspiraling binaries are also clean systems as tidal interactions and mass transfer between the two objects can be neglected during most of the inspiral. Thus, the model of inspiraling compact binaries consists of two non-spinning point masses moving under their gravitational influence in either circular or non-circular (though decaying) orbit. Their evolution should be appropriately described within the framework of general relativity. As the two-body problem in Einstein's relativity is not exactly solvable, post-Newtonian approximation methods have to be employed. A post-Newtonian (PN) approximation is an expansion of corrections to Newtonian gravitational theory in terms of a small parameter $\epsilon \sim (v/c)^2 \sim (Gm/rc^2)$, where m , v and r are the total mass, orbital velocity and the separation of the binary system. For example, consider the PN equations of motion for spinless, point masses. Schematically it reads,

$$\mathbf{a} \equiv \frac{d^2\mathbf{x}}{dt^2} \approx -\frac{G\,m\mathbf{x}}{r^3}[1+O(\epsilon)+O(\epsilon^2)+O(\epsilon^{2.5})+O(\epsilon^3)+O(\epsilon^{3.5})+O(\epsilon^4)+O(\epsilon^{4.5})+\ldots]\,, \tag{1.20}$$

where x and r $= |\mathbf{x}|$ denote the separation vector and distance between the bodies, and m $= m_1 + m_2$ denotes the total mass. The quantity ϵ is the small expansion parameter. The symbols $O(\epsilon)$ and $O(\epsilon^2)$ represent post-Newtonian (PN), post-post-Newtonian (2PN) corrections and so on. Gravitational radiation reaction first appears at $O(\epsilon^{2.5})$ beyond Newtonian gravitation, or at 2.5PN order, generally called the Newtonian radiation reaction term. In this manner terms at $O(\epsilon^{4.5})$ will be referred to as the 2PN radiation reactions terms. Similarly, starting from the leading order contribution to the gravitational wave luminosity $\mathcal{L}$ given by Eq.(1.15), we can write schematically the higher order corrections in the luminosity as

$$\mathcal{L} = \mathcal{L}_N \left\{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2)\right\}\,, \tag{1.21}$$

where $\mathcal{L}_N$ is given by Eq.(1.15) and the higher order corrections are denoted by

$O(\epsilon), O(\epsilon^{1.5})$ and $O(\epsilon^2)$. In this thesis, following the accepted terminology, the terms of $O(\epsilon)$ will be called the 1PN terms though they are of absolute order $O(\epsilon^{3.5})$. Similarly, in the above equation and in equations similar to that, $O(\epsilon^{1.5})$ and $O(\epsilon^2)$ terms will be denoted as 1.5PN and 2PN terms.

To understand why higher order corrections in Eqs.(1.20) and (1.21) are required to describe the dynamics of compact binaries during their late stages of inspiral, let us consider typical numbers involved in the problem. When a neutron-star binary enters the bandwidth of the LIGO/VIRGO detectors, say at frequency $f \sim 10\,\mathrm{Hz}$, the distance between the stars is $r \sim 500\,\mathrm{km}$. About fifteen minutes later, $f \sim 100\,\mathrm{Hz}$ and $r \sim 100\,\mathrm{km}$. Within the next three seconds the stars will merge and the frequency and the distance will be roughly 1000 Hz and 20 km respectively. In these fifteen minutes of observation, the binary executes about sixteen thousand orbital rotations. The orbital velocity of the two objects during this stage will be about 30% of the speed of light. Though the gravitational wave signal is extremely weak and buried deep in the detector noise, the large number of precisely predictable cycles in the detector bandwidth brings the characteristic signal strength to the realm of the measurable. The method of matched filtering will be employed to detect and extract information of the binaries from the inspiral waveforms [1, 23]. In this technique one cross correlates the noisy output of a detector with theoretical templates (theoretical expressions for $h_+(t)$ and $h_\times(t)$ computed a *priori* using some approximation to general relativity). For this technique to be successful, the templates must remain in phase with the exact – general relativistic – waveform as long as possible. If the signal and template lose phase with each other even by a cycle in the ten thousand, as the waves sweep through the bandwidth of the detector, their cross-correlation will be significantly reduced and one may lose the event altogether. Recently, the contributions to the accumulated number of gravitational wave cycles $\mathcal{N}$, arising from the various post-Newtonian corrections for the compact binaries were computed [18]. They have shown that for a neutron star binary, the 1PN, 1.5PN and 2PN corrections contribute around four hundred cycles, (minus) two hundred cycles and nine cycles respectively, to the accumulated number of cycles in the sensitive bandwidth of LIGO/VIRGO. Since, even the 2PN corrections contribute more than nine cycles to $\mathcal{N}$, one is forced to a description of the evolution of the binary system, using the best available theory of gravity, to substantially higher accuracy than that provided by the lowest order Newtonian approximation.

In our modelling of compact binaries as point masses, we have neglected the intrinsic rotation (spin) of the two bodies. The spins of the two bodies will induce some spin-orbit and spin-spin contributions both in the gravitational waveforms and in the equations of motion of the binary. In the equations of motion for the compact objects, the spin-orbit coupling enters at the order $\epsilon^{1.5}$ and the spin-spin corrections at the order ϵ^2. The spin dependent terms involve a dimensionless vector measuring the rotation rate of each body,

$$\chi = \frac{c\,\mathbf{S}}{Gm^2}\,, \tag{1.22}$$

where S denotes the intrinsic spin. We have $|\chi| \leq 1$ for black holes, and $|\chi|$ lies in the range $0.63 - 0.74$ for neutron stars, depending on the equation of state of nuclear matter inside the star. In the case of spinning compact objects in circular orbits, precessional, non-precessional and dissipative effects on the gravitational waveform due to the spin-orbit and the spin-spin interactions have been studied extensively [24, 25, 26, 27]. Further, the deformation due to the possibly rapid rotation of the bodies may lead to some intrinsic Newtonian quadrupole moment for each body. However, these moments are expected to be quasi-stationary and consequently not contribute to the gravitational radiation, though they will appear in the binary's equations of motion [28].

Though the post-Newtonian methods will be crucial for the construction of theoretical templates for inspiraling compact binaries, near the coalescence, PN methods should be replaced by a fully relativistic investigation of the hydrodynamics of the two merging neutron stars [29], or by the exact (albeit also numerical) computation of the dynamics of two black hole horizons [30].

The information content in these gravitational wave events is of excellent quality. If they are detected with a suitably high signal to noise ratio they should allow one to do astronomy. For instance, some of the astrophysical measurements that are possible with gravitational waves are the following [5, 28]:

- They can provide precise measurements of the masses of the objects, possibly of their spins and probably, in the case of neutron stars, of their radii.
- These waves allow one to measure cosmological distance directly and provide a cleaner determination of the Hubble's constant and the deceleration parameter.
- They will allow us to test the non-linear structure of radiative gravitation.
- To perform new tests of the existence of a scalar component to gravitation.
- To probe black hole physics, *e.g.* no hair theorems.

Estimates of the rate of such coalescence events are about a few per year upto 200 Mpc. Advanced LIGO which would look upto cosmological distances [5] would get to numbers of hundreds per year.

1.5 The construction of search templates

1.5.1 Theoretical issues

The construction of the theoretical search templates for gravitational radiation from inspiraling binaries, which will be cross correlated with the noisy output of detectors may be done in two steps. The two steps are generally referred to respectively as the "wave generation problem" and the "radiation reaction problem" [2].

The wave generation problem deals with the computation of the gravitational waveforms generated by the binary (at the leading order in $1/R$, where R is the distance of the binary) when the orbital phase and frequency of the binary take some specific values. In other words it deals with the construction of the 'plus' and 'cross' gravitational wave polarizations.

The radiation reaction problem consists of determining the evolution of the orbital elements (the orbital phase and parameters like frequency, eccentricity) as a function of time. The parameters describing the orbit vary in a non-linear manner with respect to time, as the orbit evolves under the action of gravitational radiation reaction forces. In principle, the evolution of the orbital elements should be determined from the knowledge of the radiation reaction forces acting locally on the orbit. However, the radiation reaction forces are at present not known with sufficient accuracy (only the relative first post-Newtonian corrections are known [31, 32, 33, 34]). Therefore, in practice, the evolution of orbital elements is determined ***assuming*** energy and angular momentum balance and the far-zone expressions for energy and angular momentum fluxes.

None of the above problems can be solved exactly for binaries consisting of two compact objects of comparable masses. They are treated by a combination of approximation methods like Post Minkowskian approximation and Post Newtonian approximation whose main features [3] we list in the next two subsections.

1.5.2 The post-Minkowskian approximation

The post-Minkowskian Approximation (PMA) is an expansion in $\gamma_i = GM/c^2R$ or $\gamma_e = GM/c^2D$ where M, L, D are the characteristic mass, size and separation respectively. Loosely, it is an expansion in G and hence it is also called weak field, non-linear or fast motion approximation. It makes crucial use of the geometry of Minkowski spacetime and its causality properties. The equations in this scheme reduce to a hierarchy of wave equations on Minkowski background which are solved by retarded potentials. The basic complication is in the non-linear iteration.

1.5.3 The post-Newtonian approximation

The post-Newtonian approximation is an expansion in $\beta \sim v/c \sim L/\lambda \sim L/c/P$ where v, L, λ and P are the characteristic velocity, size, wavelength and period respectively. Loosely it is an expansion in $1/c$ and is also called slow motion expansion. It uses Newtonian concepts like absolute space, with an Euclidean metric and absolute time. It uses Newtonian techniques and in this viewpoint, Einstein's theory provides small numerical corrections to Newtonian theory. The equations in this scheme are a hierarchy of Poisson equations which are solved by instantaneous potentials.

1.5.4 The equation of motion of compact objects in general relativity

We now discuss the issues associated with the equations of motion of compact objects in general relativity, heavily depending on [35]. The topic of equations of motion (EOM) for compact binary systems received careful scrutiny in the years following the discovery of the binary pulsar PSR 1913+16. There have been three different approaches to the complete kinematical description of a two-body system upto the level where radiation damping first occurs (2.5PN) due to Damour and his co-workers [36], Schafer [37] and Grischuk and Kopejkin [38] respectively.

Damour's method explicitly discusses the external motion of two condensed bodies without ambiguities. The method employs the best techniques to treat various sub-problems [36]. The final EOM at 2.5PN level are expressed only in terms of instantaneous positions, velocities and spins in a given harmonic coordinate system and given explicitly in [36]. The two mass parameters in these formulas are the Schwarzschild masses of the two condensed bodies.

The conservative part of the EOM upto 2PN (excluding the secular 2.5PN terms) are not deducible from an conventional Lagrangian (function of positions and velocities) in harmonic coordinates, but only from a generalized Lagrangian (depending on accelerations). This Lagrangian is invariant under the Poincaré group and thus allows one to construct ten Noetherian quantities that would be conserved during the motion. These include the 'Energy', 'Angular Momentum' and 'Center of Mass'.

Damour and Deruelle [39] observed that the above 1PN accurate equations of motion allow a remarkably simple parametrization, structurally related to the Keplerian parametrization of the Newtonian equations of motion. This quasi-Keplerian representation uses, instead of a single eccentricity e, three different eccentricities. The above construction has been generalized to the 2PN order by Damour, Schafer and Wex [40, 41, 42] which is referred as the generalized quasi-Keplerian representation in the literature. The EOM for the general case is given in [36] and crucially used in the following studies of generation [43, 44] and radiation reaction [45].

Schafer's [37] approach on the other hand is based on the Hamiltonian approach to the interaction of spinless point particles with the gravitational wave field. The Hamiltonian formulation is best done in the Arnowitt-Deser-Misner (ADM) coordinates in which two metric coefficients satisfy hyperbolic equations (evolution) while the remaining eight are of elliptic type (constraints). It uses a different gauge that allows an elegant separation of conservative and damping effects. One recovers the damping force acting on the Hamiltonian subsystem of instantaneously interacting particles coming from its interaction with the dynamical degrees of freedom of the gravitational field. In this approach point masses are used as sources and the divergences cured by a well-defined regularization procedure.

The last approach due to Grischuk and Kopejkin [38] on the other hand is based on (a) PNA scheme (b) assumption that bodies are non-rotating 'sphericallysymmetric' fluid balls. The symmetry is in the coordinate sense. The EOM of the center of mass of each body is obtained by integration of the local PN EOM. These are explicitly calculated retaining all higher derivatives that appear. One then reduces the higher derivatives by EOM and obtains the final results. Formally collecting the various relativistic corrections into a 'effective mass', one can have a PN proof of effacement of internal structure and provide a plausibility argument for validity of 'weak field formulas' for compact objects.

The fact that three independent methods give formally identical equations of motion at the 2PN order is a strong confirmation of the validity of the numerical coefficients in the EOM. The EOM along with the quasi-Keplerian parametrization mentioned above form the basis for the "timing formula", employed to make accurate radio observations of the binary pulsars. The damping terms can be considered as perturbation to a Lagrangian system which is multiperiodic – a radial period and a angular period corresponding to periastron precession – and leads to the observed secular acceleration effect in the binary pulsar. No balance argument is involved at any stage.

The PN accurate equations of motion is also employed at two crucial stages in the construction of the 'ready to use' search templates for the detection of gravitational radiation from compact binaries. The first instance is related to the construction of accurate mass and current multipole moments. In general, the post-Newtonian accurate expressions for the mass and current multipole moments of the binary will be written not only in terms of the individual positions and velocities of the masses, but also in their higher time derivatives. The post-Newtonian accurate equations of motion are used to reduce the functional dependence of the multipole moments only to the positions and velocities of the masses in the binary. In the other instance, the PN equations of motion are employed to compute the time derivatives of the multipole moments appearing in the expressions for the gravitational waveform and the far-zone fluxes, to the desired accuracy.

1.5.5 The 2PN challenge

Six years back the theorists were asked to derive the gravitational waveform and the resulting radiation back reaction on the orbit at least to 2PN, or second post-Newtonian order, beyond the quadrupole approximation. This implies the correction terms of $O(\epsilon^{4.5})$ in the far-zone fluxes and the terms of $O(\epsilon^4)$ in the expressions for h_+ and h,, which are corrections of $O(\epsilon^2)$ with respect to leading order terms in these expressions. Furthermore, due to the extreme complexity of the calculations at such high PN order, independent calculations were called for, in order to inspire confidence in the final formulae. The challenge was was taken up by two teams of relativists, one composed of Blanchet, Damour and Iyer and the other composed of Will and Wiseman. All the relevant results obtained using the two different methods agreed precisely. As the end point of their calculation, they obtained 'ready to use' search templates for gravitational waves from inspiraling non-spinning compact binaries of arbitrary mass ratio moving in quasi-circular orbits [18, 46, 47, 43]. These templates are employed by GRASP, the data analysis package of the LIGO collaboration [48], to search for gravitational waves when interferometers become operational.

1.5.5.1 The Blanchet-Damour-Iyer approach

We give the following brief summary of the Blanchet-Damour-Iyer (BDI) approach following [35]. The BDI approach employs the multipole expansion methods in combination with the PMA scheme to obtain the desired goal. This technique was developed by Blanchet, Damour and Iyer in a long series of papers [49]. Multipole expansion is most conveniently implemented by using symmetric trace free (STF) tensors rather than tensor spherical harmonics. The Multipole Post Minkowskian method exploits the computational advantage of working in De-Donder, Lorentz or harmonic gauge. In this gauge Einstein's equations can be conveniently written in terms of a wave operator of flat space with source terms that include the non-linear gravitational stress energy. This permits a solution in terms of flat space Green functions and decomposition of the solution in terms of STF tensors which are eigen-functions of the flat space D'Alembertian.

The BDI approach builds on a Fock type derivation [50] using the double-expansion method of Bonnor [51]. This approach makes a clean separation of the near-zone and the wave-zone effects. It is mathematically well-defined, algorithmic and provides corrections to the quadrupolar formalism in the form of compact support integrals or more generally well-defined analytically continued integrals. The BDI scheme has a modular structure: the final results are obtained by combining an 'external zone module' with a 'radiative zone module' and a 'near zone module'. For dealing with strongly self-gravitating material sources like neutron stars or black holes one needs to use a 'compact body module' together with an 'equation of motion module'. It correctly takes into account all the non-linear effects.

The general approach to solve the generation problem may be broken up into the following steps:

1. Integrate the Einstein field equations in the vacuum exterior region D_e by means of a Multipolar Post-Minkowskian series. The exterior solution is parametrized by moments M_L and S_L called the ***algorithmic*** moments. The mass monopole and dipole moments as well as the current dipole moment are necessarily constant to satisfy the harmonic gauge condition.

2. In the far wave-zone rewrite the solution in suitable coordinates to find the observable moments of the radiative field that a detector would measure. This involves going over from the harmonic coordinates to the radiative or Bondi coordinates to correct for the logarithmic deviation of the true light cones from the flat line cones in

the wave-zone D,. In these coordinates we have the gravitational waveform and the far-zone fluxes as post-Newtonian expansions in U_L and V_L, where U_L and V_L are the 'mass' and 'current' type radiative moments. The relation between the observable or radiative moments U_L and V_L to the algorithmic moments is also obtained in this step. The structure of these relations embody the fact that the gravitational field in higher approximations depends on the 'history' of the source and that propagation of radiation is not only along light cones but also inside them.

3. Finally one needs to relate the field in D_e to the inner field in the source. To this end one does two things: Re-expand the external post-Minkowskian field in a post-Newtonian expansion. Integrate the non-vacuum field equations in the near-zone D_i by means of a post-Newtonian expansion using as source variables $\mathrm{a} = (T^{00}+T^{ss})/c^2$, $\mathrm{a}, = T^{0i}/c$ and $\sigma_{ij} = T^{ij}$. This choice simplifies the 1PN solution and hence the subsequent iterations. Starting with the source terms at the lowest order one solves for the gravitational field $h^{\mu\nu}$. This solution for $h^{\mu\nu}$ is then used in the relevant non-linear terms to generate a more accurate source term at the next order. This, in turn, determines a h to higher accuracy. To 2PN accuracy the solutions are determined in terms of potentials V, V_i and W_{ij} which are retarded integrals associated with sources a, σ_i and $\sigma_{ij} + (1/4\pi G)(\partial_i V \partial_j V - (1/2)\delta_{ij}\partial_k V \partial_k V)$ respectively.

4. One finally matches the two solutions in the exterior near-zone $D_i \cap D_e$ to relate the algorithmic moments to the source properties.

The end results of this approach are the expressions for the 2PN accurate mass multipole moment I_L and the 1PN accurate current multipole moments, given by Eqs.(2.17) and (2.13) of [52]. These are also reproduced as Eqs.(2.12) and (2.13) in chapter 2. These equations are the starting point for our computations in this thesis.

It should be noted that, in generation problems, as one goes to higher orders of approximation two independent complications arise. Though algebraically involved in principle the first is simpler: contributions from higher multipoles. The second complication is not only algebraically tedious but technically more involved: contributions from higher nonlinearities e.g for 2PN generation cubic nonlinearities need to be handled.

1.5.5.2 The Will-Wiseman approach

The other method developed by Epstein-Wagoner-Thorne-Will-Wiseman builds on a Landau-Lifshiftz [53] type treatment to derive post- Newtonian corrections to the lowest order quadrupole formula [54]. The combined use of an effective stress energy tensor for the gravitational field (with non-compact support) and of formal post-Newtonian expansions led to the appearance of divergent integrals. The presense of the divergent integrals and the lack of a clear separation between the near-zone and the wave-zone were unsatisfactory features of this scheme until recently. A couple of years back, Will and Wiseman [43] have provided a resolution to this problem by taking literally the statement that the solution is a *retarded integral i.e.,* an integral over the entire past null cone of the field point. A careful evaluation of the far-zone contributions, then shows that all integrations are indeed convergent and finite and moreover the tail terms are also correctly recovered. Using this treatment, Will and Wiseman have computed the 2PN accurate waveform and energy flux for *general orbits.*

1.5.6 The perturbation approach

In the case of gravitational radiation from a *test particle* orbiting *a* Schwarzchild black hole we know the exact predictions of general relativity numerically [55], obtained by employing a perturbation about the curved background created by the black hole. Though this method is applicable only when the ratio of the binary masses is very small, it is applicable for fully relativistic situations $v \sim c$. Poisson used the black hole perturbation theory to analytically compute gravitational radiation in this limit within the post-Newtonian framework to the 2PN order [56]. Extending Poisson's work, Sasaki and his collaborators developed a more systematic approach, to obtain the analytical expressions for the gravitational waveforms and the luminosity to the 4PY order [57, 58]. These expressions are in excellent agreement with the numerical results at this order [59]. Recently, the method has been extended to the case of a rotating black hole [60] where they obtained the energy and angular momentum luminosities to the 2.5PN order from a particle in circular orbit with small inclination angle. For slightly eccentric orbits around a Kerr black hole, similar results are obtained to the 2.5PN order [61]. For circular orbits around a Kerr black hole, the calculations have been performed to the 4PN order [62]. The extension of the method to the case of spinning particles has been accomplished and the luminosity to the 2.5PN order has been obtained for circular orbits, including the effect of spin-spin coupling [63]. Recently, analytical results accurate to the 5.5PN order, *i.e.* corrections of $O[(v/c)^{11}]$ for a test particle in a circular orbit around a Schwarzschild black hole is also obtained [64].

1.5.7 Issues in data analysis

The availability of highly accurate inspiral waveforms will allow one firstly to employ pattern matching techniques like the matched filtering for the detection of weak gravitational radiation from the compact binaries [1, 23]. One of the issues for the data analysis of the compact binary inspiral search is the determination of the number of templates to use in the matched filtering. This has been studied extensively [65, 66, 67, 68, 69]. A recent analysis [69] indicates that with 2PN accurate inspiral waveforms and using a one-step matched filtering search, for binaries with components more massive than $0.2\,M_{\odot}$ while losing no more than 10 % of the events due to coarseness of template spacing, the number of search templates will be roughly

10^5 and 10^7 for LIGO and VIRGO respectively. This implies roughly 10^{11} and 10^{12} flops (floating point operations per second) for data analysis to keep up with data acquisition for LIGO and VIRGO. This clearly shows that one step search would be computationally expensive. An alternative strategy could be to use several templates banks in a hierarchy such that the information provided by a coarsely spaced bank of templates at a lower level is used to restrict the search region in a more finely spaced template bank at a higher level. This kind of hierarchical search investigated using 1PN accurate search templates imply that the reduction in the number of templates compared to a one step search will be of the order of twenty to thirty [70, 71].

The next important issue in data analysis after detection of the gravitational wave signal is the estimation of parameters characterizing the event and possible error bounds on the measured values. The metric defined on the space of waveforms is known as the Fisher information metric and its inverse is the covariance matrix, whose diagonal and off-diagonal elements are, in the limit of large signal-to-noise ratio, the variances in the measured values of the parameters and the correlation coefficients among different parameters, respectively [72]. The estimates of errors for various PN signals can be found in [73, 74, 67, 75]. Using the 2PN inspiral waveforms, the covariance matrix based errors in the estimation of various parameters appearing in the 2PN waveform have been computed [76]. These estimates suggest that each binary's chirp mass $M_{\rm chirp} = \left(\frac{m_1^3 m_2^3}{m_1+m_2}\right)^{1/5}$ can be measured to an accuracy of a few tenths of a percent [76]. Bounds on the parameters computed using the covariance matrix are called Cramer-Rao bounds. These are only a lower limit on the expected errors and realistic errors are much larger than this. One brute force way of estimating the errors is to carry out numerical simulations mimicking the actual detection process. The results from such an analysis indicate that the covariance matrix indeed underestimates the errors of various parameters by factors of two to three for moderate signal-to-noise ratios [75]. Recently, the discrepancy between the above two methods has also been explained [77]. The standard PN templates also produce large biases as they are just an approximation to the fully general relativistic signal [67].

The final theoretical issue stemming from the data analysis requirements relates to the order of the PN approximation one needs for the purpose of detection and parameter estimation discussed above. Based on the test particle results, which are numerically exact and analytically approximate to a very high PN orders, it was felt that very high post-Newtonian order (may be as high as $(v/c)^9$ beyond the leading order) might be required for reasonably accurate signal extraction [78]. This prompted Damour, Iyer and Sathyaprakash [79] to critically examine the following question: To what order must the orbital phasing to be computed, in order to guarantee that the systematic errors due to the neglect of the higher PN corrections are smaller than the statistical errors due to noise in a given detector? They constructed a new class of approximate waveforms called P-approximants, using systematic use of the Padé approximation. Their calculations suggest that with the P-approximant 3PN accurate theoretical templates, the loss in number of detectable events will be smaller than 1% and also these templates will give significantly smaller biases less than 0.5% in the parameter estimation.

1.5.8 The 3PN challenge

At present three independent groups are tackling the formidable problem of constructing 'ready to use' search templates for compact binaries accurate to the third post-Newtonian order, the corrections $O(\epsilon^3)$ to the quadrupole approximations [80]. The following summarizes the status [35].

The 3PN challenge crucially requires the equations of motion to 3PN accuracy and the situation is now under investigation. The gravitational field is computed using the standard post-Newtonian theory from the stress-energy tensor appropriate for point particles (*i.e.,* involving delta functions). A careful procedure for regularization is needed to handle the infinite self-field of point masses at this order. Work is in progress to obtain the 3PN contributions by different techniques. These include the MPM method supplemented by Hadamard 'partie-finie' [81], the Epstein-Wagoner-Thorne-Will-Wiseman method [82] and also the Hamiltonian formalism [83]. As mentioned above upto 2.5PN three distinct computational techniques led to a unique EOM. Though, preliminary investigations have even raised questions about whether this sort of uniqueness will persist at 3PN [84], the work of Blanchet, Faye and Ponsot [81] indicate that unique results may obtain.

The French group headed by Blanchet has made substantial progress in tackling the "wave generation problem" and the "radiation reaction problem" at the 3PN order. Recently, Blanchet constructed the multipole expansion (in general relativity) of the gravitational field generated by a slowly-moving isolated source [85] and recovered previously obtained expressions of the source multipole moments [52] in the appropriate limits. Earlier, he determined the radiation field at large distances from the source due to the so called 'tails of tails' of the gravitational radiation and also the non-linear self interaction of quadrupole waves [86, 87]. All this contribute to the 3PN order beyond the quadrupole approximation.

It is expected that the 3PN accurate 'ready to use' templates for compact binaries in quasi-circular orbits will be ready by the year 2001.

1.6 The radiation reaction problem

As mentioned earlier, the extremely high phasing accuracy requirement makes mandatory the control of reactive terms way beyond the Newtonian. This has prompted on the one hand, work on generation aspects to compute the far-zone flux of energy and angular momentum carried by gravitational waves and on the other, work on the radiation reaction aspects to compute the effect on the orbital motion of the emission of gravitational radiation.

The idea of a damping force associated with an interaction that propagates with a finite velocity was first discussed in the context of electromagnetism by Lorentz [88]. He obtained it by a direct calculation of the total force acting on a small extended particle due to its self-field. The answer was incorrect by a numerical factor and the correct result was first obtained by Planck [89] using a 'heuristic' argument based on energy balance which prompted Lorentz [90] to re-examine his self-field calculations and confirm Planck's result,

$$F^i = \frac{2}{3}\,\frac{e^2}{c^3}\,\ddot{v}^i\,, \tag{1.23}$$

where v_i is the velocity of the particle. The relativistic generalization of the radiation reaction by Abraham [91] based on arguments of energy and linear momentum balance preceded by a few years the direct relativistic self-field calculation by Schott [92] and illustrates the utility of this heuristic, albeit less rigorous, approach [93].

The argument based on energy balance proceeds thus: A non-accelerated particle does not radiate and satisfies Newton's (conservative) equation of motion. If it is accelerated, it radiates, loses energy and this implies damping terms in the equation of motion. Equating the work done by the reactive force on the particle in a unit time interval to the negative of the energy radiated by the accelerated particle in that interval (Larmor's formula) the reactive acceleration is determined and one is led to the Abraham-Lorentz equation of motion for the charged particle. The direct method of obtaining radiation damping, on the other hand, is based on the evaluation of the self-force. Starting with the momentum conservation law for the electromagnetic fields one rewrites this as Newton's equation of motion by decomposing the electromagnetic fields into an 'external field' and a 'self-field'. Expanding the self-field in terms of potentials, solving for them in terms of retarded fields and finally making a retardation expansion, one obtains the required equation of motion when one goes to the point particle limit [94].

As in the electromagnetic case, the approach to gravitational radiation damping has been based on the balance methods, the reaction potential or a full iteration of Einstein's equation. The first computation in general relativity was by Einstein [95] who derived the loss in energy of a spinning rod by a far-zone energy flux computation. The same was derived by Eddington [96] by a direct near-zone radiation damping approach. He also pointed out that the physical mechanism causing damping was the effect discussed by Laplace [97], that if gravity was not propagated instantaneously, reactive forces could result. An useful development was the introduction of the radiation reaction potential by Burke [98] and Thorne [99] using the method of matched asymptotic expansions. In this approach, one derives the equation of motion by constructing an outgoing wave solution of Einstein's equation in some convenient gauge and then matching it to the near-zone solution. Restricting attention only to lowest order Newtonian terms and terms sensitive to the outgoing (ingoing) boundary conditions and neglecting all other terms, one obtains the required result. The first complete direct calculation à la Lorentz of the gravitational radiation reaction force was by Chandrasekhar and Esposito [100]. Chandrasekhar and collaborators [101, 102] developed a systematic post-Newtonian expansion for extended perfect fluid systems and put together correctly the necessary elements like the Landau-Lifshiftz pseudotensor, the retarded potentials and the near-zone expansion. These works established the balance equations to Newtonian order, albeit for weakly self-gravitating fluid systems. The revival of interest in these issues following the discovery of the binary pulsar and the applicability of these very equations to binary systems of compact objects follows from the works of Damour [93, 103] and Damour and Deruelle [104].

As in the electromagnetic case, the computation of the reactive acceleration assuming balance equations is simpler than the computation of the damping terms by a direct near-field iteration. The computation of the energy and angular momentum fluxes at the lowest Newtonian order (quadrupole equation) requires the equation of motion at only Newtonian order. Assuming the balance equations one can infer the lowest order (2.5PN) radiation damping whose direct computation, as mentioned before, requires a 2.5PN iteration of the near-zone equations. Similarly, the computation of the 1PN corrections to the lowest order quadrupole luminosity requires the 1PN accurate equations of motion, but is potentially equivalent to the 3.5PN terms in the equation of motion. This motivated Iyer and Will (IW) [33, 34] to propose a refinement of the text-book [105] treatment of the energy balance method used to discuss radiation damping. This generalization uses both energy and angular momentum balance to deduce the radiation reaction force for a binary system made of non-spinning structureless particles moving on general orbits. Starting from the 1PN conserved dynamics of the two-body system, and the radiated energy and angular momentum in the gravitational waves, and taking into account the arbitrariness of the 'balance' upto total time derivatives, they determined the 2.5PN and 3.5PN terms in the equations of motion of the binary system. The part not fixed by the balance equations was identified with the freedom still residing in the choice of the coordinate system at that order. Thus, starting from the far-zone flux formulas, one deduces a formula that is suitable for evolving general orbits of compact binaries of arbitrary mass ratio that includes 1PN corrections to the dominant Newtonian radiation reaction terms. Blanchet [31, 32], on the other hand, obtained the post-Newtonian corrections to the radiation reaction force from first principles using a combination of post-Minkowskian, multipolar and post-Newtonian schemes together with techniques of analytic continuation and asymptotic matching. By looking at "antisymmetric" waves – a solution of the d'Alembertian equation composed of retarded wave minus advanced wave, regular all over the source – and matching, one obtains a radiation reaction tensor potential that generalizes the Burke-Thorne reaction potential [106], in terms of explicit integrals over matter fields in the source. The *validity* of the balance equations upto 1.5PN is also proved. By specializing this potential to two-body systems, Iyer and Will [34] checked that this solution indeed corresponds to a unique and consistent choice of coordinate system. This provides a delicate and non-trivial check on the validity of the 1PN reaction potentials and the overall consistency of the direct methods based on iteration of the near-field equations and indirect methods based on energy and angular momentum balance.

Work on radiation reaction in the test particle case has focussed on understanding the evolution of Carter constant in Kerr geometry. Ryan [107] has investigated the effect of gravitational radiation reaction, first on circular, and later even for non-equatorial orbits around a spinning black hole. Kennefick and Ori [108] developed a computational scheme in which the radiation reaction force is determined by the 'physical retarded' radiation field rather than the radiative field. This allows them to determine the evolution of all associated constants of motion. Capon and Schutz [109] have looked at a 'local expression' for radiation reaction by evaluating its self field as an integral over the particle world line. Recently Mino, Sasaki and Tanaka [110] have derived the leading order correction to the equation of motion of a particle which presumably describes the effect of gravitational radiation reaction by two methods: One approach is analogous to the DeWitt and Brehme [111] method in the case of electromagnetic radiation, where the conservation law of the total (matter + e.m. field) stress-energy tensor is integrated across a tube surrounding the particle world-line, giving the equations of motion including radiation reaction. The other method uses, on the other hand, asymptotic matching. Quinn and Wald [112] have discussed an axiomatic approach to gravitational radiation reaction and their results are consistent with those of Ref.[110]. Gergely, Perjes and Vasuth [113] have included the spin effects on gravitational radiation reaction using the BDI approach and their results are in accordance with those of Refs.[26, 107]

We end this section by making the following remarks on the IW method [45]. Though the IW method is valid for a large class of coordinate systems, it cannot by itself fix the particular expression for the reactive force in a given coordinate system. Thus in order to solve a practical problem (in which we erect a particular coordinate system), the method is in principle insufficient by itself, but it provides an extremely powerful check of other methods based on first principles.

1.7 Inspiraling binaries in quasi-elliptical orbits

In the previous sections we have restricted our attention to inspiraling binaries in quasi-circular orbits. This is due to the fact that quasi-circularity is an excellent approximation to late stages of inspiral for compact binaries which are formed with large initial separation, as the gravitational radiation reaction forces tend to circularize rapidly the orbital motion. To see how this happens, let us consider the following expression which relates the instantaneous orbital frequency $w = 2\pi/P$, P being the period of the binary to the eccentricity e at the Newtonian order, following [114],

$$\frac{e^2}{(1-e^2)^{19/6}} \left(1 + \frac{121}{304}\right)^{145/121} \sim \omega^{-19/9} \tag{1.24}$$

When e << 1, the above equation gives $e^2 \sim \omega^{-19/9}$. For example, the eccentricity of the Hulse-Taylor binary pulsar PSR 1913+16 is presently $e_0 = 0.617$, and the orbital frequency is $\omega_0 = 2.25 \times 10^{-4}$ Hz. When gravitational waves from PSR 1913+16 reaches $f = \omega/\pi = 10$ Hz, from Eq.(1.24) its eccentricity will be $\sim 10^{-6}$. This is very small even when compared to the high-order relativistic effects. Therefore 'circularity' of the orbit is justified during the late inspiral stages for binaries like PSR 1913+16.

Galactic binaries like PSR 1913+16, in general, will be in circular orbits by the time they reach the final inspiral stage. However, there exist astrophysical scenarios where compact binaries will have non-negligible eccentricity during these stages. For example, Shapiro and Teukolsky investigated a scenario [115] where black hole binaries – with total masses in the range $10M_\odot \leq M \leq$ (a few) $\times 10^2 M_\odot$ – are formed in the galactic nuclei prior to the formation of a supermassive black hole. Their simulation starts with a cluster of compact objects – neutron stars and black holes – residing at the center of a galactic nucleus. Coulomb scattering and dissipative processes will drive such a cluster to a high density, high redshift state. Once the central redshift becomes sufficiently large, relativistic instability sets in, and the core undergoes catastrophic collapse to form a supermassive black hole. Quinlan and Shapiro [116] have shown that during the final year of the evolution of such a cluster, just prior to the catastrophic collapse, there can be $100 - 10^4$ evolving black hole binaries in eccentric orbits driven by gravitational radiation reaction, with masses in the range $10 - 100\, M_\odot$. Recently, Flanagan and Hughes [117] suggested that intermediate mass black hole binaries – with total masses in the range $50M_\odot \leq M \leq$ (a few) $\times 10^3 M_\odot$ – may well be the first sources to be detected by LIGO and VIRGO. These systems may be seen via merger and ringdown waveforms for $M > 60M_\odot$ for initial LIGO and $M > 200M_\odot$ for advanced LIGO. Below these limits, inspiral waveforms will be useful. However, following [116], it is clear that the systems are not on circular orbits and general eccentric inspiral waveforms are required for such binaries.

The next possibility we consider, involves compact objects orbiting 10^6 to $10^7 M_\odot$ black holes, that seem fairly common in galactic nuclei. In this case the compact objects could be scattered into very eccentric orbits orbits via gravitational deflections by other stars. However, by the time gravitational radiation reaction becomes the dominant orbital driving force, there is not enough inspiral remaining to fully circularize these orbits. Hills and Bender [118] have argued that the event rates for the above process are very encouraging and the chances of such signals being observed by Laser Interferometric Space Antenna, LISA [13] appear very good.

The final class of sources we discuss, where eccentricity will be important is the speculative but exciting possibility involving MAssive Compact Halo Objects (MACHOs). The MACHO collaboration's analysis of the photometry of 8.5 million stars in the Large Magellanic Cloud suggests that $0.6^{+0.3}_{-0.2}$ of the halo consists of MAssive Compact Halo Objects of mass $0.5^{+0.3}_{-0.2}\,M_\odot$ in the standard spherical flat rotation halo model [119]. At present we do not know what MACHOs are. There is only tentative evidence suggesting that MACHOs are white dwarfs. However, only future observations will make clear whether white dwarf MACHOs exists or not. Recently, Nakamura et. al. suggested that MACHOs may be primordial black holes of mass $\sim 0.5\,M_\odot$ [120, 121]. These are formed in the early universe. They indicated that there should be at least $\sim 10^{11}$ primordial black holes in the halo and some of them would be in binaries due to three body interactions. Depending on the various values of the matter energy density and the Hubble parameter, they estimated that 8% to 0.9% of these MACHO black holes can be in binaries with semi-major axis $\sim 2 \times 10^{14}$ cm. The event rate for these binaries turned out to be to be several per year within 15 Mpc. It is suggested that the initial LIGO/VIRGO interferometers will be able to observe coalescence of these binaries and it is quite plausible that eccentricity will not be negligible before they coalesce. More recently, Hiscock [122] has shown that low frequency gravitational waves from black hole MACHO binaries would form a strong stochastic background in the frequency range 10^{-5} Hz $< f <$ 10-'Hz, where the proposed space-based interferometers like LISA and OMEGA [123] are the most sensitive. These binaries will be in highly eccentric orbits, which greatly increases their efficiency as sources of gravitational radiation, and spreads that radiation over a large number of harmonics of orbital frequency.

1.8 The issues addressed in this thesis

In this thesis, we investigate issues related to gravitational radiation from compact binaries moving in *general orbits.* The final aim is to construct 'ready to use' 2PN accurate theoretical templates for inspiraling binaries of arbitrary mass ratio moving in quasi-elliptical orbits. The issues tackled here will form the basic inputs to the above construction.

1.8.1 The second post-Newtonian corrections to the gravitational waveform and the far-zone fluxes

In chapter 2, we apply the second post-Newtonian accurate BDI generation formalism to the case of inspiraling binaries in general orbits. Pioneering work for extending the Einstein quadrupole formula to post-Newtonian order is due to Epstein and Wagoner [124]. Wagoner and Will [125] applied the above result to the case of inspiraling binaries moving in general orbits and obtained the 1PN corrections to the gravitational wave luminosity. Later Blanchet and Schafer [126] and Junker and Schafer [127] obtained the 1PN corrections to the far-zone energy and angular momentum fluxes for general orbits, using the MPM approach to gravitational wave generation. In this chapter we extend the above computations to the 2PN order. We first give a precise and concise description of the MPM approach of BDI. We then compute the post-post-Newtonian (2PN) accurate mass quadrupole moment for compact binaries of arbitrary mass ratio moving in general orbits, in terms of the binary's dynamical variables, using the BDI formalism. Following [46], we split the 2PN accurate formulae for the gravitational waveform and the far-zone fluxes into "instantaneous" and "tail" parts. The "instantaneous" contribution depends only on the state of the binary at the retarded instant $T_R \equiv (T - R/c)$ while the "tail" contribution is *a priori* sensitive to the binary's dynamics at all previous instants $T_R - \tau \leq T_R$. The "tails" are caused by the backscatter of outgoing radiation off the background spacetime curvature and appears at $O(\epsilon^{1.5})$ beyond the quadrupole approximation. Using the mass quadrupole moment and other multipole moments to the required order for binaries in general orbits, we also compute the 2PN "instantaneous" corrections to the far-zone energy and angular momentum for binaries in general orbits. We also compute the 2PN "instantaneous" contributions to the gravitational waveform (the transverse-traceless (TT) part of the radiation-field, representing the deviation of the metric from the flat spacetime) using STF multipole moments of the BDI formalism. We observe that though our result for the far-zone energy flux matches precisely with that obtained by Will and Wiseman, our expressions for the waveform differ from the corresponding expressions obtained by them, using the Epstein-Wagoner multipole moments at 1.5PN and 2PN orders. Though the two expressions are totally different looking at these orders, even in the circular limit, we show that they are equivalent. The equivalence is established by showing that the difference between the two expressions, at 1.5PN and 2PN orders has a vanishing transverse-traceless part. We also exhibit various limiting cases of our results. The expressions for the 2PN corrections to the waveform and the far-zone fluxes for binaries in *general* orbits obtained in this chapter will form one of the basic inputs to tackle the "wave generation problem" and the "radiation reaction problem" for the construction of theoretical templates for binaries in *quasi-eccentric* orbits.

1.8.2 The evolution of the orbital elements under 2PN radiation reaction

In chapter **3** we investigate the "radiation reaction problem" for eccentric binaries. Here the aim is to obtain the 2PN "instantaneous" corrections to the evolution of the orbital elements like the orbital phase and parameters like frequency and eccentricity. As mentioned earlier, these computations are done *assuming* energy and angular momentum balance and the far-zone expressions for the energy and angular momentum fluxes, averaged over an orbit. This naturally requires, for the elliptical binaries, a convenient solution to the 2PN accurate equations of motion. As emphasised in [79], the gravitational wave observations of inspiraling compact binaries, is analogous to the high precision radio-wave observations of binary pulsars. The latter makes use of an accurate relativistic 'timing formula' based on the solution – in quasi-Keplerian parametrization – to the relativistic equation of motion for a compact binary in elliptical orbit [39]. In a similar manner, the former demands accurate 'phasing' i.e. an accurate mathematical modelling of the continuous time evolution of the gravitational wave phase which in turn depends of orbital parameters like frequency and eccentricity. A very elegant 2PN accurate generalized quasi-Keplerian parametrization for elliptical orbits in the Arnowit, Deser and Misner (ADM) coordinates has been implemented by Damour, Schafer, and Wex [40, 41, 42]. This representation is thus the most natural and best suited for our purpose to parametrize the dynamical variables that enter the expressions for the far-zone fluxes. We first obtain the 2PN corrections to the far-zone fluxes averaged over an orbit extending computations performed at 1PN and 1.5PN orders [126, 127, 128], taking due care of a new complication at this order that the far-zone fluxes are computed in the harmonic or De-Donder coordinates, while the orbital representation is available in the ADM coordinates. To obtain the evolution of the orbital elements, we start from the 2PN accurate expressions for the orbital elements in terms of the conserved energy and angular momentum given in [41, 42] and compute the time variation of these orbital elements. One ends up with a result, in terms of the time variation of the 'conserved' energy and angular momentum. By a heuristic argument, one replaces these by the corresponding ***average*** far-zone fluxes obtained previously. In the limit of $\eta \to 0$ our results reduce to the test particle results [61] to 2PN accuracy. These results along with the 'tail' contributions to the evolution of orbital elements, computed in [128, 129] form the basic set of equations from where one can numerically evaluate the evolution of the orbital phase and other orbital elements as a function of time under the effects of gravitational radiation reaction forces.

1.8.3 The second post-Newtonian gravitational wave polarizations

In chapter 4 we address the "wave generation problem" which deals with the computation of the gravitational wave polarizations h_+ and h,, at the leading order in $1/R$, when the orbital phase and other parameters of the binary orbit take some specific values. This problem and the 'radiation reaction' problem to the lowest order was investigated by Lincoln and Will [130]. They used the method of osculating orbital elements from celestial mechanics and the 2.5PN accurate Damour-Deruelle equations of motion [131, 93, 36], to study the evolution of general orbits and obtained 1PN accurate expressions for h_+ and $h_\times$ for quasi-circular orbits. Later Moreno-Garrido, Mediavilla and Buitrago obtained polarization waveforms for binaries in elliptical orbits at Newtonian order with and without radiation reaction and studied the effects of orbital parameters on the spectrum of the wave's amplitude [132, 133]. Analytic expressions for the gravitational wave amplitude emitted by an elliptic binary are obtained to 1.5PN order by Junker and Schafer, Blanchet and Schafer [127, 128]. As mentioned earlier, the 2PN accurate treatment of polarization waveforms and associated time-evolution of orbital phase for inspiraling compact binaries moving in quasi-circular orbits was given by Blanchet, Iyer, Will and Wiseman [47]. For the above calculation they employed the 2PN accurate expressions for $h^{\rm TT}_{ij}$, the transverse traceless part of the radiation field representing the deviation of the metric from the flat spacetime and $\left(\frac{d\mathcal{E}}{dt}\right)$, the far-zone energy flux obtained independently using two different formalisms [52, 18, 46, 43]. As indicated earlier, in the limiting case of a test particle orbiting a Schwarzschild black hole, very high accuracy has been achieved and the polarization waveforms are obtained to 4PN order by Tagoshi and Sasaki [58]. In this chapter we compute all the 'instantaneous' 2PN contributions to h_+ and $h_\times$ for two compact objects of arbitrary mass ratio moving in elliptical orbits, using the 2PN corrections to $h_{ij}^{\rm TT}$ for the general orbits and the generalized quasi-Keplerian representation for the 2PN motion. The expressions for h_+ and $h_\times$ obtained here represent gravitational radiation from an elliptical binary during that stage of inspiral when orbital parameters are essentially the same over a few orbital periods, in other words when the gravitational radiation reaction is negligible. We investigate the effect of eccentricity and orbital inclination on the amplitude of the Newtonian part of h_+ and h,. We observe that orbital inclination changes the magnitudes of $|h_+|^2$ and $|h_\times|^2$ appreciably. The reduction in $|h_+|^2$ and $|h_\times|^2$ for small and medium eccentricities, is small compared to that for higher eccentricities, when the inclination angle is varied from 0 to $\pi/2$. We compute $\left(\frac{h_\times}{h_+}\right)^2$ at the Newtonian order and conclude that this ratio may be used as a good indicator for the orbital inclination, for very small to very high values of eccentricity. The modulation of h_+ and $h_\times$ due to the precession of the periastron, which occurs at the 1PN order is also explicitly shown. We recover in the circular limit the results of [47] modulo the tail terms.

1.8.4 The second post-Newtonian gravitational radiation reaction for two-body systems

In chapter 5, we deduce the gravitational radiation reaction to the second post-Newtonian order beyond the quadrupole approximation – 4.5PN terms in the equations of motion – using the refined balance method proposed by Iyer and Will [33, 34]. We critically explore the features of their construction and illustrate them by contrast with other possible variants. We observe that in terms of the number of arbitrary parameters and the corresponding gauge transformations, the IW scheme exhibits remarkable stability for a variety of choices for the ambiguity in energy and angular momentum. The different choices merely give different numbers of degenerate equations, indicating the validity and soundness of the method. We also show that the far-zone formulae and the balance equations by themselves do not constrain the reactive acceleration to be a power series in the individual binary masses m_1 and m_2 as assumed by Iyer and Will, but are consistent with a more general form of the reactive acceleration. The equations of motion are valid for general binary orbits and for a class of coordinate gauges. The limiting cases of circular orbits and radial infall are also investigated [45].