Chapter 2

The second post-Newtonian corrections to the gravitational waveform and the far-zone fluxes

2.1 Introduction

In this chapter we apply the 2PN accurate, multipolar post-Minkowskian generation formalism of Blanchet, Damour and Iyer [46] to the specific case of an inspiraling compact binary of arbitrary mass ratio moving in a general orbit. The aim is to obtain 2PN corrections to the following quantities in terms of the dynamical variables of the binary. The quantities of interest are

$$I_{ij} = (I_{ij})_{\mathrm{N}} \left\{1 + O(\epsilon) + O(\epsilon^2) + ...\right\} , \tag{2.1a}$$

$$h^{TT}_{km} = (h^{TT}_{km})_{\mathrm{N}} \left\{1 + O(\epsilon^{0.5}) + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + ...\right\} , \tag{2.1b}$$

$$\frac{d\mathcal{E}}{dt} = (\frac{d\mathcal{E}}{dt})_{\mathrm{N}} \left\{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + ..\right\} , \tag{2.1c}$$

$$\frac{d\mathcal{J}}{dt} = (\frac{d\mathcal{J}}{dt})_{\mathrm{N}} \left\{1 + O(\epsilon) + O(\epsilon^{1.5}) + O(\epsilon^2) + ...\right\} , \tag{2.1d}$$

where $\epsilon \sim \mathrm{v}^2/\mathrm{c}^2 \approx \mathrm{Gm/c}^2\, r$. Here m is the total mass, r the distance between the bodies and v the relative velocity of the two bodies. In Eqs.(2.1), I_{ij} is the mass quadrupole moment for a system of two compact objects moving in general orbits while h^{TT}_{km} is the transverse-traceless (TT) part of the radiation-field, representing the deviation of the metric from the flat spacetime. $\mathrm{d}\mathcal{E}/\mathrm{dt}$ and $\mathrm{d}\mathcal{J}/\mathrm{dt}$ represent the far-zone energy and angular momentum fluxes due to the emission of gravitational radiation. Note that the suffix 'N' denotes the Newtonian contribution in all the above equations. For example $(\frac{d\mathcal{E}}{dt})_N$ is given by $\frac{8}{15}\frac{G^3m^2\mu^2}{c^5r^4}\{12v^2-11\dot{r}^2\}$, where μ is the reduced mass of the binary – defined in terms of the individual masses of the bodies m_1 and m_2 by $\mu = m_1 m_2/m^2$ – and $\dot{r} = dr/dt$. The construction of I_{ij} and other relevant mass and current moments is performed using the Blanchet-Damour-Iyer (BDI) formalism. In this formalism the gravitational waveform and the far-zone energy and angular momentum fluxes are given in terms of particular time derivatives of the mass and current multipole moments of the binary. These computations are done using the algebraic computing software, Maple [134] and published in Ref. [44].

In Section 2.2 we present a summary of the 2PN-accurate BDI generation formalism. Using this formalism, in Section 2.3, we compute the mass and current multipole moments for the binary in general orbits in terms of the binary's dynamical variables. The 2PN corrections to h^{TT}_{km} are obtained in Section 2.4. Section 2.5 deals with the computation of the far-zone energy and angular momentum fluxes. In Section 2.6 we exhibit various limiting cases of our results. The chapter ends with a few concluding remarks in Sec 2.7.

2.2 Summary of the 2PN-accurate generation formalism

The BDI formalism is a mathematically rigorous technique for computing the higher order post-Newtonian corrections to the mass and current multipole moments a relativistic binary, containing compact objects of arbitrary mass ratio. Below we present a brief and precise summary of this formalism, drawing heavily on sections in [18, 46] where the authors of the formalism themselves described their method. Consider the metric $g_{\alpha\beta}$ describing the gravitational field outside an isolated system. It is well known that in a suitable "radiative" coordinate system $X^\mu = (cT, X^i)$, X being the vector pointing from the source to the observer, the metric coefficients admit an asymptotic expansion in powers of R^{-1}, when $R = |\mathbf{X}| \to \infty$ with $T - R/c$

and N $\equiv \mathbf{X}/R$ being fixed ("future null infinity"). The transverse-traceless (TT) projection of the deviation of $g_{\alpha\beta}(X^\gamma)$ from the flat metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ defines the asymptotic waveform $h^{TT}_{km} \equiv (g_{km}(X) - \delta_{km})^{TT}$. Note that greek indices range from 0 to 3 while latin indices $i, j, k, m\ldots$ range from 1 to 3. The $1/R$ part of h^{TT}_{km} can be uniquely decomposed into multipoles:

$$h^{TT}_{km}(\mathbf{X},T) = \frac{4G}{c^2R}\mathcal{P}_{ijkm}(\mathbf{N})\sum_{\ell=2}^{\infty}\frac{1}{c^\ell \ell!}\Big\{N_{L-2}U_{ijL-2}(T_R) - \frac{2\ell}{(\ell+1)c}N_{aL-2}\varepsilon_{ab(i}V_{j)bL-2}(T_R)\Big\} + O\left(\frac{1}{R^2}\right) . \qquad (2.2)$$

The "radiative" multipole moments U_L and V_L (defined for $\ell \geq 2$) denote some functions of the retarded time $T_R \equiv T - R/c$, taking values in the set of symmetric trace-free (STF) three-dimensional cartesian tensors of order ℓ. Here L $\equiv i_1 \ldots i_\ell$ denotes a spatial multi-index of order ℓ, $N_{L-2} \equiv N_{i_1}\ldots N_{i_{\ell-2}}$, $X_{(ij)} \equiv \frac{1}{2}(X_{ij} + X_{ji})$ and

$$\mathcal{P}_{ijkm}(\mathbf{N}) = (\delta_{ik} - N_iN_k)(\delta_{jm} - N_jN_m) - \frac{1}{2}(\delta_{ij} - N_iN_j)(\delta_{km} - N_kN_m) \ , \qquad (2.3)$$

We summarize our notation here. In this thesis we have $-+++$ signature, greek indices run over 0, 1, 2, 3, latin indices vary from 1, 2, 3, the covariant metric is $g_{\mu\nu}$ and g $=$ det $(g_{\mu\nu})$. The relative separation $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $n^i = n_i = x^i/r$, $\partial_i = \partial/\partial x^i$, $x^L = x_L = x_{i_1}x_{i_2}\ldots x_{i_l}$ and $\partial_L = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$, where $L = i_1i_2\ldots i_l$ is a multi-index with l indices and $x_{L-1} = x_{i_1}x_{i_2}\ldots x_{i_{l-1}}$, etc.... The symmetric and tracefree (STF) part of a tensor T_L is denoted in any of the following ways $\hat{T}_L = T_{<L>} = \mathrm{STF}_L(T_L)$, e.g., $\hat{x}_{ij} = x_ix_j - \frac{1}{3}\delta_{ij}r^2$; $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$.

As indicated in Eq.(2.2), for slowly moving systems the multipole order is correlated with the post-Newtonian order. The coefficients in Eq.(2.2) have been chosen so that the moments U_L and V_L reduce, in the non-relativistic limit c $\rightarrow +\infty$ (or $\varepsilon \rightarrow 0$), to the ℓ-th time-derivatives of the usual Newtonian mass-type and current-type moments of the source. At the 2PN approximation, i.e., when retaining all terms of fractional order $\varepsilon^4 \sim c^{-4}$ with respect to the leading (Newtonian quadrupole) result, the waveform (2.2) reads

$$h_{km}^{TT} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\Big\{U_{ij} + \frac{1}{c}\left[\frac{1}{3}N_aU_{ija} + \frac{4}{3}\varepsilon_{ab(i}V_{j)a}N_b\right] + \frac{1}{c^2}\left[\frac{1}{12}N_{ab}U_{ijab} + \frac{1}{2}\varepsilon_{ab(i}V_{j)ac}N_{bc}\right] + \frac{1}{c^3}\left[\frac{1}{60}N_{abc}U_{ijabc} + \frac{2}{15}\varepsilon_{ab(i}V_{j)acd}N_{bcd}\right] + \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}U_{ijabcd} + \frac{1}{36}\varepsilon_{ab(i}V_{j)acde}N_{bcde}\right] + O(\varepsilon^5)\Big\}\,. \tag{2.4}$$

The far-zone energy flux is related the waveform by

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\rm far-zone} = 32\pi G \left(\frac{\partial h_{ij}^{TT}}{\partial T_R}\right)^2 R^2 d\Omega(\mathbf{N})\,. \tag{2.5}$$

At the 2PN approximation this yields (with $U^{(n)} \equiv d^nU/dT_R^n$)

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\rm far-zone} = \frac{G}{c^5}\Big\{\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left[\frac{1}{189}U_{ijk}^{(1)}U_{ijk}^{(1)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right] + \frac{1}{c^4}\left[\frac{1}{9072}U_{ijkm}^{(1)}U_{ijkm}^{(1)} + \frac{1}{84}V_{ijk}^{(1)}V_{ijk}^{(1)}\right] + O(\varepsilon^6)\Big\}\,. \tag{2.6}$$

The angular dependence of the waveform h_{km}^{TT} causes its wavefront to be not quite spherical and thereby enables the waves to carry off angular momentum. A general expression for the far-zone angular momentum flux in terms of the radiative multipole moments is available in the literature, see Eq.(4.23') of [135]. At 2PN approximation this equation reduces to

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm far-zone} = \frac{G}{c^5}\epsilon_{ipq}\Big\{\frac{2}{5}U_{pj}U_{qj}^{(1)} + \frac{1}{c^2}\left[\frac{1}{63}U_{pjk}U_{qjk}^{(1)} + \frac{32}{45}V_{pj}V_{qj}^{(1)}\right] + \frac{1}{c^4}\left[\frac{1}{2268}U_{pjkl}U_{qjkl}^{(1)} + \frac{1}{28}V_{pjk}V_{qjk}^{(1)}\right]\Big\}\,. \tag{2.7}$$

The explicit computation of Eqs.(2.4), (2.6) and (2.7) in terms of source variables are obtained using the 2PN accurate BDI formalism, which we describe briefly below.

The 2PN-accurate BDI gravitational wave generation formalism is a precise, modular treatment that allows one to compute the radiative moments entering Eqs.(2.4), (2.6) and (2.59) in terms of the source variables to an accuracy sufficient for obtaining the waveform with fractional accuracy $\mathbf{l/c^4}$. The latter requirement implies, in view of Eq.(2.4), that one should (at a minimum) compute: the mass-type quadrupole radiative moment $U_{i_1i_2}$ to $\mathbf{l/c^4}$ accuracy; the mass-type radiative octupole $U_{i_1i_2i_3}$ and the current-type radiative quadrupole $V_{i_1i_2}$ to $1/c^3$ accuracy; $U_{i_1i_2i_3i_4}$ and $V_{i_1i_2i_3}$ to $\mathbf{l/c^2}$ accuracy; $U_{i_1i_2i_3i_4i_5}$ and $V_{i_1i_2i_3i_4}$ to $1/c$ accuracy; and $U_{i_1i_2i_3i_4i_5i_6}$ and $V_{i_1i_2i_3i_4i_5}$ to the Newtonian accuracy. Note that these requirements are relaxed if one is only interested in getting the energy loss rate with 2PN-accuracy. In that case, Eq.(2.6) shows that one still needs $U_{i_1i_2}$ to $\mathbf{l/c^4}$ accuracy, but that it is enough to compute $U_{i_1i_2i_3}$ and $V_{i_1i_2}$ to $1/c^2$ accuracy, and $U_{i_1i_2i_3i_4}$ and $V_{i_1i_2i_3}$ to Newtonian accuracy.

In the BDI generation formalism, the link between the radiative multipoles U_L and V_L and the dynamical state of the material source is obtained in several steps. These various steps are briefly presented here. The BDI approach begins with Einstein's equations written in harmonic coordinates. The field h^{a}p, measuring the deviation of the "gothic" metric, from the Minkowski metric $\eta^{\alpha\beta}$ is defined to be h^{aP} $= \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta}$. Imposing the harmonic coordinate condition $\partial_\beta h^{\alpha\beta} = 0$ then leads to the field equations

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \Lambda^{\alpha\beta}(h) \equiv \frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{2.8}$$

where $\Box$ denotes the flat spacetime d'Alembertian operator, $T^{\alpha\beta}$ is the matter stress-energy tensor, and $\Lambda^{\alpha\beta}$ is an effective gravitational source containing the non-linearities of Einstein's equations. It is a series in powers of $h^{\alpha\beta}$ and its derivatives; both quadratic and cubic nonlinearities in $\Lambda^{\alpha\beta}$ which play an essential role in the BDI calculations.

The next step consists of constructing an iterative solution to Eq.(2.8) in an inner domain (or near-zone) that includes the material source but whose radius is much less than a gravitational wavelength. Defining source densities a $= \frac{T^{00}+T^{kk}}{c^2}$, $\sigma_i = \frac{T^{0i}}{c}$, $\sigma_{ij} = T^{ij}$, and the retarded potentials V $= -4\pi$ G $\Box_R^{-1}\sigma$, $V_i = -4\pi G \Box_R^{-1}\sigma_i$, and $W_{ij} = -4\pi G \Box_R^{-1}[\sigma_{ij} + (4\pi G)^{-1}(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V)]$, where $\Box_R^{-1}$ denotes the usual flat spacetime retarded integral, one obtains the inner metric $h_{\text{in}}^{\alpha\beta}$ to some intermediate accuracy $O(6,5,6)$. Here $h_{\text{in}}^{00} = -\frac{4}{c^2}$ V $+ \frac{4}{c^4}(W_{ii} - 2\text{V}^2) + O(6)$, $h_{\text{in}}^{0i} = -\frac{4}{c^3} V_i + O(5)$, and $h_{\text{in}}^{ij} = -\frac{4}{c^4} W_{ij} + O(6)$, where $O(n)$ means a term of order ,? in the post-Newtonian parameter $\varepsilon \sim v/c$. Also, $O(n_1, n_2, n_3)$ denotes an accuracy of $O(n_1)$, $O(n_2)$, $O(n_3)$ for h_{in}^{00}, h_{in}^{0i} and h_{in}^{ij}. From this, one constructs the inner metric with the higher accuracy $O(8,7,8)$ needed for subsequent matching as $h_{\text{in}}^{\alpha\beta} = \Box_R^{-1}[\frac{16\pi G}{c^4} \tau^{\alpha\beta}(\text{V},\text{W})] + O(8,7,8)$, where $\tau^{\alpha\beta}(V,\text{W})$ denotes the right-hand-side of Eq.(2.8) when retaining all the quadratic and cubic nonlinearities to the required post-Newtonian order in the near-zone, and given as explicit combinations of derivatives of V, V_i and W_{ij}.

The second step consists of constructing a generic solution of the vacuum Einstein equations (Eq.(2.8) with $T^{\alpha\beta} = 0$), in the form of a multipolar-post-Minkowskian expansion that is valid in an external domain which overlaps with the near-zone and extends into the far wave-zone. The construction of $h^{\alpha\beta}$ in the external domain is done algorithmically as a functional of a set of parameters, called the "canonical" multipole moments $M_{i_1 \ldots i_l}(\text{t}), S_{i_1 \ldots i_l}(\text{t})$ which are STF Cartesian tensors. Schematically, $h_{\text{ext}}^{\alpha\beta} = \mathcal{F}^{\alpha\beta}[M_L, S_L]$ where the functional dependence includes a non-local time dependence on the past "history" of $M_L(t)$ and $S_L(t)$.

The third, "matching" step consists of requiring that the inner and external metrics be equivalent (modulo a coordinate transformation) in the overlap between the inner and the external domains. This requirement determines the relation between the canonical moments and the inner metric (itself expressed in terms of the source variables). Performing the matching through 2PN order [52] thus determines

$M_L(t) = I_L[\overline{\tau}^{\alpha\beta}] + O(5)$, $S_L(t) = J_L[\overline{\tau}^{\alpha\beta}] + O(4)$, where the "source" moments I_L and J_L are given by some mathematically well-defined (analytically continued) integrals of the quantity $\overline{\tau}^{\alpha\beta}(V, W)$ which appeared as source of $h_{\rm in}^{\alpha\beta}$. When computing the source moments, all finite size effects, such as spin (which to 2PN accuracy can be added separately) and internal quadrupole effects are neglected.

In the last step, the computation of nonlinear effects in the wave-zone allows the construction of the radiative multipole moments U_L and V_L as some nonlinear functionals of "canonical" multipole moments M_L, S_L and therefore of the source multipole moments.

The final result for the 2PN-accurate generation formalism when working in an initially mass-centred coordinate system, i.e. such that the canonical mass dipole M_i vanishes for all times is given by Eqs.(2.6) and (2.7) of [46] which reads

$$U_{ij}(T_R) = I_{ij}^{(2)}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{11}{12}\right] I_{ij}^{(4)}(T_R - \tau) + O(\varepsilon^5) \,, \tag{2.9a}$$

$$U_{ijk}(T_R) = I_{ijk}^{(3)}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{97}{60}\right] I_{ijk}^{(5)}(T_R - \tau) + O(\varepsilon^5) \,, \tag{2.9b}$$

$$V_{ij}(T_R) = J_{ij}^{(2)}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{7}{6}\right] J_{ij}^{(4)}(T_R - \tau) + O(\varepsilon^4) \,, \tag{2.9c}$$

for the moments that need to be known beyond the 1PN accuracy, and

$$U_L(T_R) = I_L^{(\ell)}(T_R) + O(\varepsilon^3) \,, \tag{2.10a}$$

$$V_L(T_R) = J_L^{(\ell)}(T_R) + O(\varepsilon^3) \,, \tag{2.10b}$$

for the other ones. Eqs.(2.9) involve some integrals which are associated with tails; these integrals have in front of them the total mass-energy m of the source, and contain a quantity b which is an arbitrary constant (with the dimension of time) parametrizing a certain freedom in the construction of the radiative coordinate system (T,X). More precisely, the link between the (Bondi-type) radiative coordinates $X^\mu = (cT, \mathrm{X}^i)$ and the (harmonic) canonical coordinates $x^\mu_{\rm can} = (ct_{\rm can}, x^i_{\rm can})$ is given by Eq.(2.8) of [46] and displayed below

$$T_R = t_{\rm can} - \frac{r_{\rm can}}{c} - \frac{2Gm}{c^3}\ln\left(\frac{r_{\rm can}}{cb}\right) + O(\varepsilon^5) + O\left(1/r^2_{\rm can}\right) . \tag{2.11}$$

Except for the computation of U_{ij} which requires the knowledge of the mass quadrupole source moment I_{ij} with 2PN accuracy, the computation of the other multipole contributions to the waveform can be obtained from 1PN-accurate expressions of the mass-type and current-type source moments which have been obtained for all values of ℓ in Refs.[136] and [137] respectively, as explicit integrals extending only on the compact support of the material source. Note that there are no $1/c^3$ contributions in the source moments.

Finally we list expressions for the 2PN accurate mass multipole moments and 1PN accurate spin multipole moments obtained in [52]. The 2PN accurate mass multipole moment given by Eq.(2.17) of [46] reads,

$$\begin{aligned} I_L(t) \;=\; & \mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B \Bigg\{ \hat{x}_L\left[\sigma - \frac{4}{c^4}\sigma U_{ss} + \frac{4}{c^4}U\sigma_{ss}\right] + \frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2\ell+3)}\partial_t^2\sigma \\ & -\frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{2U}{c^2}\right)\sigma_i - \frac{2U_i}{c^2}\sigma\right. \\ & \left. +\frac{1}{\pi G c^2}\left(\partial_j U\partial_i U_j - \frac{3}{4}\partial_i U\partial_j U_j\right)\right] \\ & +\frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma - \frac{2(2\ell+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i \\ & +\frac{2(2\ell+1)\hat{x}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U\partial_j U\right] \\ & +\frac{\hat{x}_L}{\pi G c^4}\left[2U_i\partial_{ij}U_j - U_{ij}\partial_{ij}U - \frac{1}{2}(\partial_i U_i)^2\right. \\ & \left. +2\partial_i U_j\partial_j U_i - \frac{1}{2}\partial_t^2(U^2) + W_{ij}\partial_{ij}U\right]\Bigg\} \; + O(\varepsilon^5) . \end{aligned} \tag{2.12}$$

Though the 1PN accurate expression for the the current moment J_L was initially obtained in [137], we employ another equivalent form following [52]. The 1PN accurate current multipole moments are given by Eq.(2.13) of [46] which gives

$$\begin{aligned}J_L(t) \;=\; & \mathrm{FP}_{B=0}\,\varepsilon_{ab<i_\ell}\int d^3\mathbf{x}|\mathbf{x}|^B\Big\{\hat{x}_{L-1>a}\left(1+\frac{4}{c^2}U\right)\sigma_b+\frac{|\mathbf{x}|^2\hat{x}_{L-1>a}}{2c^2(2\ell+3)}\partial_t^2\sigma_b\\ & +\frac{1}{\pi Gc^2}\hat{x}_{L-1>a}\left[\partial_k U(\partial_b U_k-\partial_k U_b)+\frac{3}{4}\partial_t U\partial_b U\right]\\ & -\frac{(2\ell+1)\hat{x}_{L-1>ac}}{c^2(\ell+2)(2\ell+3)}\partial_t\left[\sigma_{bc}+\frac{1}{4\pi G}\partial_b U\partial_c U\right]\Big\}+O(\varepsilon^4)\,. \end{aligned} \tag{2.13}$$

The symbol $\mathrm{FP}_{B=0}$ in the above stands for "Finite Part at B = 0" and denotes a mathematically well-defined operation of analytic continuation. For more details see [46]. As emphasized in [46] though the above expression is mathematically well-defined, it is a non-trivial and long calculation to rewrite it explicitly in terms of the source variables only. In the next section we perform the above task, for binaries in general orbits and obtain 2PN accurate expressions for I_{ij}, 1PN accurate J_{ij}, I_{ijk} and I_{ijkl} in terms of the dynamical variables of the binary.

2.3 Mass and current moments of compact binaries on general orbits for 2PN generation

2.3.1 2PN mass quadrupole moment

The starting point for the computation of the 2PN accurate mass moment is the form of the moment given by Eq.(2.17) of [46], which is reproduced as Eq.(2.12) in the above section. In order to rewrite the mass moment explicitly in terms of the source variables we represent the stress energy tensor of the source as a sum of Dirac δ-functions.

$$T^{\mu\nu}(\mathbf{x},t)=\sum_{A=1}^{N} m_A\frac{dy_A^\mu}{dt}\frac{dy_A^\nu}{dt}\frac{1}{\sqrt{-g}}\frac{dt}{d\tau}\delta(\mathbf{x}-\mathbf{y}_A(t))\,. \tag{2.14}$$

where A denotes the A^{th} particle, m_A denotes the (constant) Schwarzschild mass of the Ath compact body and the summation is over the N particles in the system.

Evaluating this to 2PN accuracy we obtain for the source variables

$$\sigma(\mathbf{x},t) = \sum_{A=1}^{N} \mu_A(t)\left(1+\frac{\mathbf{v}_A^2}{c^2}\right)\delta(\mathbf{x}-\mathbf{y}_A(t)) \,, \tag{2.15a}$$

$$\sigma_i(\mathbf{x},t) = \sum_{A=1}^{N} \mathrm{P}_A(t) v_A^i \delta(\mathbf{x}-\mathbf{y}_A(t)) \,, \tag{2.15b}$$

$$\sigma_{ij}(\mathbf{x},t) = \sum_{A=1}^{N} \mathrm{P}_A(t) v_A^i v_A^j \delta(\mathbf{x}-\mathbf{y}_A(t)) \tag{2.15c}$$

where $v_A^i \equiv dy_A^i/dt$ and

$$\mu_A(t) = m_A\Big\{1+(d_2)_A+(d_4)_A\Big\} \,, \tag{2.16a}$$

$$d_2 \equiv \frac{1}{c^2}\left\{\frac{1}{2}\mathbf{v}^2 - V\right\} \,, \tag{2.16b}$$

$$d_4 \equiv \frac{1}{c^4}\left\{\frac{3}{8}\mathbf{v}^4+\frac{3}{2}U\mathbf{v}^2-4U_iv_i-2\Phi+\frac{3}{2}U^2+4U_{ss}\right\} . \tag{2.16c}$$

In the above V denotes the combination

$$V \equiv U + \frac{1}{2c^2}\partial_t^2 X \,, \tag{2.17}$$

the potential appearing naturally in the 1PN near-zone metric in harmonic coordinates. The subscript A appearing in Eq.(2.16a) indicates that one must replace the field point x by the position $\mathbf{y}_A$ of the Ath mass point, while discarding all the ill-defined (formally infinite) terms arising in the limit $\mathrm{x} \to \mathbf{y}_A$. For instance

$$(U)_A = G\sum_{B\neq A}\frac{\mu_B(t)(1+\mathbf{v}_B^2/c^2)}{|\mathbf{y}_A-\mathbf{y}_B|} \,, \tag{2.18a}$$

$$(U_{ss})_A = G\sum_{B\neq A}\frac{\mu_B(t)\mathbf{v}_B^2}{|\mathbf{y}_A-\mathbf{y}_B|} \,, \tag{2.18b}$$

$$(\Phi)_A = G\sum_{B\neq A}\frac{\mu_B(t)(1+\mathbf{v}_B^2/c^2)(U)_B}{|\mathbf{y}_A-\mathbf{y}_B|} \,, \tag{2.18c}$$

$$(X)_A = G\sum_{B\neq A}\mu_B(t)(1+\mathbf{v}_B^2/c^2)|\mathbf{y}_A-\mathbf{y}_B| \,. \tag{2.18d}$$

[Note that the second time derivative appearing in V, Eq.(2.17), must be explicated before making the replacement $\mathrm{x} \to \mathbf{y}_A(t)$.]

The terms in Eq.(2.12) fall into 3 types: compact terms, Y terms and W terms. The compact terms, where the 3-dimensional integral extends only over the compact support of the material sources; the Y terms involving three dimensional integral of the product of two Newtonian like potentials; and the W term involving three dimensional integrals of terms trilinear in source variables. The evaluation of these different terms proceeds exactly as in the circular case. In fact, if the time derivatives are not explicitly implemented the expression in the general case and the circular case would be identical. The difference obtains when the time derivatives are implemented using the equation of motion. In this section we need to use the general form of the Damour-Deruelle equations of motion rather than the restricted form of the circular orbit equations of motion relevant in [46].

We take up the compact terms first. They are given by

$$\begin{aligned} I_L^{(C)} &= \sum_{A=1}^{N} \Bigg\{ \tilde{\mu}_A \left[1 - \frac{4}{c^4} U^A_{ss} + \frac{4}{c^4} U^A (\mathbf{v}_A)^2 \right] \hat{y}^L_A \\ &+ \frac{1}{2(2\ell+3)c^2} \frac{d^2}{dt^2} (\tilde{\mu}_A \mathbf{y}_A^2 \hat{y}^L_A) + \frac{1}{8(2\ell+3)(2\ell+5)c^4} \frac{d^4}{dt^4} (\tilde{\mu}_A (\mathbf{y}_A^2)^2 \hat{y}^L_A) \\ &- \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2} \frac{d}{dt} \left(\left[\mu_A \left(1 + \frac{2U^A}{c^2} \right) v^i_A - \frac{2U^A_i}{c^2} \tilde{\mu}_A \right] \hat{y}^{iL}_A \right) \\ &- \frac{2(2\ell+1)}{(\ell+1)(2\ell+3)(2\ell+5)c^4} \frac{d^3}{dt^3} (\mu_A v^i_A \mathbf{y}_A^2 \hat{y}^{iL}_A) \\ &+ \frac{2(2\ell+1)}{(\ell+1)(\ell+2)(2\ell+5)c^4} \frac{d^2}{dt^2} (\mu_A v^i_A v^j_A \hat{y}^{ijL}_A) \Bigg\} , \end{aligned} \tag{2.19}$$

in which we have introduced for convenience $\tilde{\mu}_A \equiv \mu_A(1+\mathbf{v}_A^2/c^2)$. In the above form the moment depends not only on the position and velocity of the bodies but also on higher time derivatives. It is in the reduction of these derivatives that we need the 2PN accurate equation of motion for general orbits. We use a harmonic coordinate system in which the 2PN center of mass is at rest at the origin. Using the 2PN accurate center of mass theorem, in the center of mass frame, we can express the individual positions of the two bodies moving in general orbits in terms of their relative position $\mathrm{x} = \mathbf{y_1} - \mathbf{y_2}$ and velocity $\mathbf{v} = \mathbf{v_1} - \mathbf{v_2}$

$$\mathbf{y_1} = \left\{ X_2 + \frac{\eta \delta m}{2mc^2} \left[v^2 - \frac{Gm}{r} \right] + \frac{\chi_1}{c^4} \right\} \mathbf{x} + \frac{\chi_2}{c^4} \mathbf{v} , \tag{2.20a}$$

$$\mathbf{y_2} = \left\{-X_1 + \frac{\eta\delta m}{2mc^2}\left[v^2 - \frac{Gm}{r}\right] + \frac{\chi_1}{c^4}\right\}\mathbf{x} + \frac{\chi_2}{c^4}\mathbf{v}\,, \tag{2.20b}$$

where $\mathrm{r} = |\mathbf{y_1} - \mathbf{y_2}|$ is the harmonic separation between the two bodies. The explicit values of χ_1 and χ_2 are not needed in our calculations and hence not given above. The above equations are obtained by setting equal to zero the conserved mass dipole G for general orbits. Here we denote

$$m \equiv m_1 + m_2\ , \quad \delta m \equiv m_1 - m_2\ ,$$
$$X_1 \equiv \frac{m_1}{m}\ , \quad X_2 \equiv \frac{m_2}{m} = 1 - X_1,$$
$$\eta \equiv X_1 X_2 = \frac{m_1 m_2}{m^2} \equiv \frac{\mu}{m} \tag{2.21}$$

The 2PN accurate equations of motion is written down next for completeness, where finite-size effects, such as spin-orbit, spin-spin, or tidal interactions are ignored. [104, 38, 130]. For the relative motion we have

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}^{(1)}_{\mathrm{PN}} + \mathbf{a}^{(2)}_{\mathrm{2PN}} + O(\epsilon^{2.5})\,, \tag{2.22}$$

where the subscripts denote the nature of the term, Newtonian (N), post-Newtonian (PN), post-post-Newtonian (2PN), and the superscripts denote the order in ϵ. The explicit expressions for various terms mentioned above are given by

$$\mathbf{a}_N = -\frac{Gm}{r^2}\mathbf{n}\,, \tag{2.23a}$$

$$\mathbf{a}^{(1)}_{\mathrm{PN}} = -\frac{Gm}{c^2 r^2}\left\{\left[-2(2+\eta)\frac{Gm}{r} + (1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2\right]\mathbf{n} - 2(2-\eta)\dot{r}\mathbf{v}\right\}, \tag{2.23b}$$

$$\begin{aligned}\mathbf{a}^{(2)}_{\mathrm{2PN}} = & -\frac{Gm}{c^4 r^2}\left\{\left[\frac{3}{4}(12+29\eta)\frac{G^2 m^2}{r^2} + \eta(3-4\eta)v^4 + \frac{15}{8}\eta(1-3\eta)\dot{r}^4\right.\right.\\ & \left. -\frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2 - \frac{1}{2}\eta(13-4\eta)\frac{Gm}{r}v^2 - (2+25\eta+2\eta^2)\frac{Gm}{r}\dot{r}^2\right]\mathbf{n}\\ & \left. -\frac{1}{2}\left[\eta(15+4\eta)v^2 - (4+41\eta+8\eta^2)\frac{Gm}{r} - 3\eta(3+2\eta)\dot{r}^2\right]\dot{r}\mathbf{v}\right\},\end{aligned} \tag{2.23c}$$

where $\mathrm{n} = \mathbf{x}/r$ and $\dot{r} = dr/dt$.

We have on hand all the ingredients to compute **I,.** Though long and tedious the computation is straightforward and yields for the 2PN mass quadrupole:

$$
\begin{aligned}
I_{ij}^{[C]} = {} & \eta\, m\, \mathrm{STF}_{ij} \Big\{ x^{ij} + \\
& + \frac{1}{42c^2} \Big\{ x^{ij} \Big[29(1-3\eta)v^2 - 6(5-8\eta)\frac{Gm}{r} \Big] \\
& -24(1-3\eta) r\dot{r} x^i v^j + 22(1-3\eta) r^2 v^{ij} \Big\} \\
& + \frac{1}{1512c^4} \Big[v^4 (759 - 5505\eta + 10635\eta^2) \\
& + \frac{G^2m^2}{r^2}(1758 - 6468\eta + 1878\eta^2) \\
& + v^2 \frac{Gm}{r}(5818 - 16742\eta - 12166\eta^2) \\
& - \dot{r}^2 \frac{Gm}{r}(2038 - 6662\eta + 146\eta^2) \Big] x^{ij} \\
& + \frac{1}{378c^4} \Big[v^2 (123 - 1011\eta + 2199\eta^2) \\
& + \frac{Gm}{r}(68 + 434\eta - 2090\eta^2) \\
& + 30\dot{r}^2 (1 - 5\eta + 5\eta^2) \Big] r^2 v^{ij} \\
& - \frac{1}{378c^4} \Big[\frac{Gm}{r}(101 + 287\eta - 1655\eta^2) \\
& + v^2 (156 - 1212\eta + 2508\eta^2) \Big] r\dot{r} x^i v^j \Big\} . \qquad (2.24)
\end{aligned}
$$

The Y terms on the other hand are given by

$$
\begin{aligned}
I_{ij}^{[Y]} = {} & -\frac{2Gm_1m_2}{c^4} \Big\{ 2Y^{ij}_{v_1v_2} - Y^{ij}_{v_1v_1} \\
& -\frac{1}{2}\,{}_{v_1}Y^{ij}_{v_2} + 2\,{}_{v_2}Y^{ij}_{v_1} - \frac{1}{2}\partial_t^2 (Y^{ij}) \\
& -\frac{20}{21}\partial_t \Big[{}_{v_2}Y^{aij}_{a} - \frac{3}{4}\,{}_{v_1}Y^{aij}_{a} \Big] \\
& + \frac{5}{216}\partial_t^2 \Big[{}_{a}Y^{abij}_{b} \Big] \Big\} + (1 \leftrightarrow 2) , \qquad (2.25)
\end{aligned}
$$

where following [46]

$$ {}_{v_1}Y^{L}_{v_2} = v_1^a\, v_2^b\, {}_{a}Y^{L}_{b} , \qquad (2.26a) $$

$$ {}_{a}Y^{L}_{b} = \partial_{y_1^a} \partial_{y_2^b} Y^L , \qquad (2.26b) $$

$$ Y^L(\mathbf{y_1}, \mathbf{y_2}) = \frac{|\mathbf{y_1} - \mathbf{y_2}|}{l+1} \sum_{p=0}^{l} y_1^{<l-p} y_2^{p>} , \qquad (2.26c) $$

so that

$$
\begin{aligned}
Y^{ij}_{v_1 v_2} &\equiv 2 v_1^s v_2^k Y^{ij}_{sk} &\quad (2.27a)\\
Y^{ij}_{sk} &= \frac{1}{3}\frac{\partial}{\partial y_2^s}\frac{\partial}{\partial y_2^k} r_{12}\left(y_1^{ij} + y_1^{(i}y_2^{j)} + y_2^{ij}\right) &\quad (2.27b)\\
r_{12} &= |\mathbf{y_1} - \mathbf{y_2}|\,. &\quad (2.27c)
\end{aligned}
$$

The explication of all the above terms finally leads us to

$$
\begin{aligned}
I^{[Y]}_{ij} = & -\frac{2\,m\,\eta}{63\,c^4}\frac{Gm}{r}\,\mathrm{STF}_{ij}\left\{x^{ij}\left[(v^2-\dot{r}^2)(37-101\eta-50\eta^2)\right.\right.\\
& \left.+\frac{Gm}{r}(18-54\eta-3\eta^2)\right]\\
& -r^2 v^{ij}(118-92\eta+10\eta^2)\\
& \left.+r\dot{r}x^i v^j(82-362\eta+16\eta^2)\right\}\,. \qquad (2.28)
\end{aligned}
$$

The evaluation of the $I^{[W]}$ term, the new feature at 2PN level, was discussed in detail in [46]. The W term has been evaluated there for general orbits and we need to use the same result here. We have

$$
I^{[W]}_{ij} = -\frac{\eta\,m}{c^4}\frac{G^2\,m^2}{r^2}\mathrm{STF}_{ij}\left\{[2+5\eta]x^{ij}\right\}\,. \qquad (2.29)
$$

Adding up the compact i.e., C, Y and W contributions given by Eqs.(2.24), (2.28) and (2.29), we finally obtain the expression for the 2PN accurate mass quadrupole for a system of two bodies moving in general orbits. The final result is written below as a combination of the three possible combinations $x^{ij}, x^i v^j, v^{ij}$ with coefficients which include corrections beyond the Newtonian order at 1PN and 2PN orders:

$$
\begin{aligned}
I_{ij} = & \;\mu\,\mathrm{STF}_{ij}\Big\{x^{ij}\Big[1\\
& + \frac{1}{42\,c^2}\left((29-87\eta)v^2-(30-48\eta)\frac{Gm}{r}\right)\\
& + \frac{1}{c^4}\Big(\frac{1}{504}(253-1835\eta+3545\eta^2)v^4\\
& + \frac{1}{756}(2021-5947\eta-4883\eta^2)\frac{Gm}{r}\,v^2\\
& - \frac{1}{756}(131-907\eta+1273\eta^2)\frac{Gm}{r}\,\dot{r}^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{252}(355 + 1906\eta - 337\eta^2)\frac{G^2m^2}{r^2}\Big)\Big] \\
&- x^iv^j \Big[\frac{r\dot{r}}{42\,c^2}(24 - 72\eta) \\
&+ \frac{r\dot{r}}{c^4}\Big(\frac{1}{63}(26 - 202\eta + 418\eta^2)v^2 \\
&+ \frac{1}{378}(1085 - 4057\eta - 1463\eta^2)\frac{Gm}{r}\Big)\Big] \\
&+ v^{ij}\Big[\frac{r^2}{21\,c^2}(11 - 33\eta) \\
&+ \frac{r^2}{c^4}\Big(\frac{1}{126}(41 - 337\eta + 733\eta^2)v^2 \\
&+ \frac{5}{63}(1 - 5\eta + 5\eta^2)\dot{r}^2 \\
&+ \frac{1}{189}\,(742 - 335\eta - 985\eta^2)\frac{Gm}{r}\Big)\Big]\Big\}\,. \qquad (2.30)
\end{aligned}
$$

The above expression is identical to the one obtained by Will and Wiseman in the appendix E of [43] using the new improved version of the Epstein-Wagoner formalism. In their treatment the Epstein-Wagoner multipoles appear more naturally, using which they compute the STF mass quadrupole moment. Since the approach employed here and in [43] follow algebraically different routes, the above match provides a valuable check on the long and complicated algebra involved in the determination of the crucial mass quadrupole moment for 2PN generation.

2.3.2 The other relevant mass and current moments

In this section we list the higher order mass and current multipole moments, required to compute the 2PN contributions to the gravitational waveform and the associated far-zone energy and angular momentum fluxes. They are straightforwardly obtained by explicating the point particle limits of the more general expressions, given by Eqs.(2.12) and (2.13).

$$
\begin{aligned}
I_{ijk} &= -(\mu\frac{\delta m}{m})\mathrm{STF}_{ijk}\Big\{ \\
&\qquad x^{ijk}\Big[1 + \frac{1}{6\,c^2}\Big((5 - 19\eta)v^2
\end{aligned}
$$

$$
\begin{aligned}
&-(5-13\eta)\frac{Gm}{r}\Big)\Big] \\
&-x^{ij}v^{k}\left[\frac{r\dot r}{c^2}(1-2\eta)\right] \\
+\ & x^{i}v^{jk}\left[\frac{r^2}{c^2}(1-2\eta)\right]\Bigg\}\,,
\end{aligned} \tag{2.31}
$$

$$
\begin{aligned}
I_{ijkl} &= \mu\,\mathrm{STF}_{ijkl}\Bigg\{ \\
& x^{ijkl}\Big[(1-3\eta) \\
& +\frac{1}{110\,c^2}\Big((103-735\eta+1395\eta^2)v^2 \\
& -(100-610\eta+1050\eta^2)\frac{Gm}{r}\Big)\Big] \\
& -v^{i}x^{jkl}\left\{\frac{72\,r\,\dot r}{55\,c^2}(1-5\eta+5\eta^2)\right\} \\
& +v^{ij}x^{kl}\left\{\frac{78\,r^2}{55\,c^2}(1-5\eta+5\eta^2)\right\}\Bigg\}\,,
\end{aligned} \tag{2.32}
$$

$$
I_{ijklm} = -(\mu\frac{\delta m}{m})\,(1-2\eta)\mathrm{STF}_{ijklm}\left\{x^{ijklm}\right\} \tag{2.33}
$$

$$
I_{ijklmn} = \mu(1-5\eta+5\eta^2)\mathrm{STF}_{ijklmn}\left\{x^{ijklmn}\right\} \tag{2.34}
$$

$$
\begin{aligned}
J_{ij} &= -(\mu\frac{\delta m}{m})\mathrm{STF}_{ij}\epsilon_{jab}\Bigg\{ \\
& x^{ia}v^{b}\Big[1+\frac{1}{28\,c^2}\Big((13-68\eta)v^2 \\
& +(54+60\eta)\frac{Gm}{r}\Big)\Big] \\
& +v^{ib}x^{a}\left[\frac{r\,\dot r}{28\,c^2}(5-10\eta)\right]\Bigg\}\,,
\end{aligned} \tag{2.35}
$$

$$
\begin{aligned}
J_{ijk} &= \mu\,\mathrm{STF}_{ijk}\epsilon_{kab}\Big\{x^{aij}v^{b}\Big[(1-3\eta) \\
&+\ \frac{1}{90\,c^2}\Big((41-385\eta+925\eta^2)v^2 \\
&+\ (140-160\eta-860\eta^2)\frac{Gm}{r}\Big)\Big] \\
&+\ \frac{7\,r^2}{45\,c^2}(1-5\eta+5\eta^2)\,x^{a}v^{ijb} \\
&+\ \frac{10\,r\,\dot r}{45\,c^2}(1-5\eta+5\eta^2)x^{ai}v^{bj}\Big\}\,,
\end{aligned} \tag{2.36}
$$

$$J_{ijkl} = -\left(\mu\,\frac{\delta m}{m}(1-2\eta)\right)\mathrm{STF}_{ijkl}\left\{\epsilon_{lab}\,x^{aijk}v^b\right\}\,, \tag{2.37}$$

$$J_{ijklm} = \left(\mu\,(1-5\eta+5\eta^2)\right)\,\mathrm{STF}_{ijklm}\left\{\epsilon_{mab}\,x^{aijkl}v^b\right\}\,. \tag{2.38}$$

The mass and the current moments listed above, agree with Eqs.(E3) of [43]. For the case of circular orbits, the above mass and current moments reduce to Eqs.(4.4) of [46].

2.4 The 2PN contribution to the waveform

2.4.1 The Blanchet-Damour-Iyer waveform for binaries in general orbits

The 2PN-accurate waveform is given by Eq.(2.4) in terms of the "radiative" multipole moments U_L and V_L which are in turn linked to the source moments I_L and J_L by Eqs.(2.9) and (2.10). The latter equations involve some tail integrals and therefore yield a natural decomposition of the waveform into two pieces, one which depends on the state of the binary at the retarded instant $T_R \equiv T - R/c$ only (we qualify this piece as "instantaneous"), and another which is a ***priori*** sensitive to the binary's dynamics at all previous instants $T_R - \tau \leq T_R$ (we refer to this piece as the "tail" contribution). More precisely, we decompose

$$h^{TT}_{km} = (h^{TT}_{km})_{\rm inst} + (h^{TT}_{km})_{\rm tail}\,. \tag{2.39}$$

In this section, we compute explicitly the above instantaneous part of the 2PN accurate gravitational waveform i.e., the transverse-traceless (TT) part of the 2PN accurate far-zone field for two compact objects of arbitrary mass ratio, moving in a general orbit. It is given by [46]:

$$\begin{aligned}(h^{TT}_{km})_{\rm inst} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\Big\{I^{(2)}_{ij} &+ \frac{1}{c}\left[\frac{1}{3}N_a I^{(3)}_{ija} + \frac{4}{3}\varepsilon_{ab(i}J^{(2)}_{j)a}N_b\right] \\ &+ \frac{1}{c^2}\left[\frac{1}{12}N_{ab}I^{(4)}_{ijab} + \frac{1}{2}\varepsilon_{ab(i}J^{(3)}_{j)ac}N_{bc}\right] \\ &+ \frac{1}{c^3}\left[\frac{1}{60}N_{abc}I^{(5)}_{ijabc} + \frac{2}{15}\varepsilon_{ab(i}J^{(4)}_{j)acd}N_{bcd}\right]\end{aligned}$$

$$+ \quad \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}I^{(6)}_{ijabcd} + \frac{1}{36}\varepsilon_{ab(i}J^{(5)}_{j)acde}N_{bcde}\right]\Bigg\}\,, \tag{2.40}$$

where R is the Cartesian observer-source distance and N_a's are the components of $\mathrm{N} = \mathbf{X}/R$, the unit normal in the direction of the vector X, pointing from the source to the observer. The transverse traceless projection operator projecting orthogonal to X, is given by Eq.(2.3), which we reproduce below:

$$\mathcal{P}_{ijkm}(\mathbf{N}) = (\delta_{ik} - N_iN_k)(\delta_{jm} - N_jN_m) - \frac{1}{2}(\delta_{ij} - N_iN_j)(\delta_{km} - N_kN_m)\,. \tag{2.41}$$

Evaluating the appropriate time derivatives of the multipole moments and performing the relevant contractions with N as required by Eq.(2.40), some details of which are given in Section 2.4.3, we obtain explicit expression for $\left(h^{TT}_{km}\right)_{inst}$ in terms of the source variables as shown below. Note that all the computations from here onwards are performed, using MAPLE [134].

$$\begin{aligned}\left(h^{TT}_{km}\right)_{inst} &= \frac{2G\mu}{c^4R}P_{ijkm}\Bigg\{\xi^{(0)}_{ij} + \frac{1}{c}\frac{\delta m}{m}\xi^{(0.5)}_{ij} \\ &+ \frac{1}{c^2}\,\xi^{(1)}_{ij} + \frac{1}{c^3}\frac{\delta m}{m}\xi^{(1.5)}_{ij} + \frac{1}{c^4}\,\xi^{(2)}_{ij}\Bigg\}\,,\end{aligned} \tag{2.42}$$

where the various ξ_{ij}'s are given by

$$\xi^{(0)}_{ij} = 2\left(v_{ij} - \frac{Gm}{r}n_{ij}\right), \tag{2.43a}$$

$$\xi^{(0.5)}_{ij} = \left\{3(\mathbf{N.n})\frac{Gm}{r}\left[2n_{(i}v_{j)} - \dot{r}n_{ij}\right] + (\mathbf{N.v})\left[\frac{Gm}{r}n_{ij} - 2v_{ij}\right]\right\}, \tag{2.43b}$$

$$\begin{aligned}\xi^{(1)}_{ij} &= \frac{1}{3}\Bigg\{(1-3\eta)\Bigg[(\mathbf{N.n})^2\frac{Gm}{r}\left(\left(3v^2 - 15\dot{r}^2 + 7\frac{Gm}{r}\right)n_{ij} + 30\dot{r}n_{(i}v_{j)} - 14v_{ij}\right) \\ &+(\mathbf{N.n})(\mathbf{N.v})\frac{Gm}{r}\left[12\dot{r}n_{ij} - 32n_{(i}v_{j)}\right] + (\mathbf{N.v})^2\left[6v_{ij} - 2\frac{Gm}{r}n_{ij}\right]\Bigg] \\ &+\left[3(1-3\eta)v^2 - 2(2-3\eta)\frac{Gm}{r}\right]v_{ij} + 4\frac{Gm}{r}\dot{r}(5+3\eta)n_{(i}v_{j)} \\ &+\frac{Gm}{r}\left[3(1-3\eta)\dot{r}^2 - (10+3\eta)v^2 + 29\frac{Gm}{r}\right]n_{ij}\Bigg\}\,,\end{aligned} \tag{2.43c}$$

$$\begin{aligned}\xi^{(1.5)}_{ij} &= \frac{1}{12}(1-2\eta)\Bigg\{(\mathbf{N.n})^3\frac{Gm}{r}\Bigg[\left(45v^2 - 105b^2 + 90\frac{Gm}{r}\right)\dot{r}n_{ij} - 96\dot{r}v_{ij} \\ &- \left(42v^2 - 210\dot{r}^2 + 88\frac{Gm}{r}\right)n_{(i}v_{j)}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&-(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})\,\frac{G\,m}{r}\Big[\Big(27v^2-135\dot{r}^2+84\frac{G\,m}{r}\Big)\,n_{ij}+336\dot{r}n_{(i}v_{j)}-172v_{ij}\Big]\\
&-(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})^2\frac{G\,m}{r}\Big[48\dot{r}n_{ij}-184n_{(i}v_{j)}\Big]+(\mathbf{N}.\mathbf{v})^3\Big[4\frac{G\,m}{r}n_{ij}-24v_{ij}\Big]\Big\}\\
&-\frac{1}{12}(\mathbf{N}.\mathbf{n})\,\frac{G\,m}{r}\Big\{\Big[(69-66\eta)v^2-(15-90\eta)\dot{r}^2-(242-24\eta)\frac{G\,m}{r}\Big]\dot{r}n_{ij}\\
&-\Big[(66-36\eta)v^2+(138+84\eta)\dot{r}^2\\
&-(256-72\eta)\frac{G\,m}{r}\Big]n_{(i}v_{j)}+(192+12\eta)\dot{r}v_{ij}\Big\}\\
&+\frac{1}{12}\,(\mathbf{N}.\mathbf{v})\Big\{\Big[(23-10\eta)v^2-(9-18\eta)\dot{r}^2-(104-12\eta)\frac{G\,m}{r}\Big]\frac{G\,m}{r}n_{ij}\\
&-(88+40\eta)\,\frac{G\,m}{r}\,\dot{r}n_{(i}v_{j)}-\Big[(12-60\eta)v^2-(20-52\eta)\frac{G\,m}{r}\Big]v_{ij}\Big\}\,,
\end{aligned}
\tag{2.43d}
$$

$$
\begin{aligned}
\xi^{(2)}_{ij}\;=\;&\frac{1}{120}(1-5\eta+5\eta^2)\Big\{240\,(\mathbf{N}.\mathbf{v})^4v_{ij}-(\mathbf{N}.\mathbf{n})^4\\
&\frac{G\,m}{r}\Big[\Big(90v^4+(318\frac{G\,m}{r}-1260\dot{r}^2)v^2+344\frac{G^2\,m^2}{r^2}+1890\dot{r}^4\\
&-2310\frac{G\,m}{r}\dot{r}^2\Big)n_{ij}\\
&+\Big(1620v^2+3000\frac{G\,m}{r}-3780\dot{r}^2\Big)\dot{r}n_{(i}v_{j)}-\Big(336v^2-1680\dot{r}^2+688\frac{G\,m}{r}\Big)v_{ij}\Big]\\
&-(\mathbf{N}.\mathbf{n})^3(\mathbf{N}.\mathbf{v})\frac{G\,m}{r}\Big[\Big(1440v^2-3360\dot{r}^2+3600\frac{G\,m}{r}\Big)\dot{r}n_{ij}\\
&-\Big(1608v^2-8040\dot{r}^2+3864\frac{G\,m}{r}\Big)n_{(i}v_{j)}-3960\dot{r}v_{ij}\Big]\\
&+120(\mathbf{N}.\mathbf{v})^3(\mathbf{N}.\mathbf{n})\frac{G\,m}{r}\Big(3\dot{r}n_{ij}-20n_{(i}v_{j)}\Big)\\
&+(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})^2\frac{G\,m}{r}\Big[\Big(396v^2-1980\dot{r}^2+1668\frac{G\,m}{r}\Big)n_{ij}+6480\dot{r}n_{(i}v_{j)}\\
&-3600v_{ij}\Big]\Big\}-\frac{1}{30}\,(\mathbf{N}.\mathbf{v})^2\Big\{\Big[(87-315\eta+145\eta^2)v^2\\
&-(135-465\eta+75\eta^2)\dot{r}^2-(289-905\eta+115\eta^2)\frac{G\,m}{r}\Big]\frac{G\,m}{r}\,n_{ij}\\
&-\Big(240-660\eta-240\eta^2\Big)\dot{r}n_{(i}v_{j)}\\
&-\Big[(30-270\eta+630\eta^2)v^2-60(1-6\eta+10\eta^2)\frac{G\,m}{r}\Big]v_{ij}\Big\}\\
&+\frac{1}{30}(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})\frac{G\,m}{r}\Big\{\Big[(270-1140\eta+1170\eta^2)v^2\\
&-(60-450\eta+900\eta^2)\dot{r}^2-(1270-3920\eta+360\eta^2)\frac{G\,m}{r}\Big]\dot{r}n_{ij}
\end{aligned}
$$

$$
\begin{aligned}
&-\Big[(186-810\eta+1450\eta^2)v^2+(990-2910\eta-930\eta^2)\dot r^2\\
&-(1242-4170\eta+1930\eta^2)\frac{G\,m}{r}\Big]n_{(i}v_{j)}\\
&+\Big[1230-3810\eta-90\eta^2\Big]\dot r v_{ij}\Big\}\\
&+\frac{1}{60}(\mathbf{N}.\mathbf{n})^2\frac{G\,m}{r}\Big\{\Big[(117-480\eta+540\eta^2)v^4\\
&-(630-2850\eta+4050\eta^2)v^2\dot r^2-(125-740\eta+900\eta^2)\frac{G\,m}{r}\,v^2\\
&+(105-1050\eta+3150\eta^2)\dot r^4+(2715-8580\eta+1260\eta^2)\frac{G\,m}{r}\,\dot r^2\\
&-(1048-3120\eta+240\eta^2)\frac{G^2\,m^2}{r^2}\Big]n_{ij}\\
&+\Big[(216-1380\eta+4320\eta^2)v^2+(1260-3300\eta-3600\eta^2)\dot r^2\\
&-(3952-12860\eta+3660\eta^2)\frac{G\,m}{r}\Big]\,\dot r\,n_{(i}v_{j)}\\
&-\Big[(12-180\eta+1160\eta^2)v^2+(1260-3840\eta-780\eta^2)\dot r^2\\
&-(664-2360\eta+1700\eta^2)\frac{G\,m}{r}\Big]v_{ij}\Big\}\\
&-\frac{1}{60}\Big\{\Big[(66-15\eta-125\eta^2)v^4\\
&+(90-180\eta-480\eta^2)v^2\dot r^2-(389+1030\eta-110\eta^2)\frac{G\,m}{r}\,v^2\\
&+(45-225\eta+225\eta^2)\dot r^4+(915-1440\eta+720\eta^2)\frac{G\,m}{r}\,\dot r^2\\
&+(1284+1090\eta)\frac{G^2\,m^2}{r^2}\Big]\,\frac{G\,m}{r}\,n_{ij}\\
&-\Big[(132+540\eta-580\eta^2)v^2+(300-1140\eta+300\eta^2)\dot r^2\\
&+(856+400\eta+700\eta^2)\frac{G\,m}{r}\Big]\,\frac{G\,m}{r}\,\dot r\,n_{(i}v_{j)}\\
&-\Big[(45-315\eta+585\eta^2)v^4+(354-210\eta-550\eta^2)\,\frac{G\,m}{r}\,v^2\\
&-(270-30\eta+270\eta^2)\frac{G\,m}{r}\,\dot r^2\\
&-(638+1400\eta-130\eta^2)\,\frac{G^2\,m^2}{r^2}\Big]v_{ij}\Big\}\,.
\end{aligned}
\tag{2.43e}
$$

The "tail" contribution reads

$$
(h^{TT}_{km})_{\rm tail}=\frac{2G}{c^4R}\frac{2Gm}{c^3}\mathcal{P}_{ijkm}\int_0^{+\infty}\quad d\tau\quad\Big\{\ln\left(\frac{\tau}{2b_1}\right)I^{(4)}_{ij}(T_R-\tau)
$$

$$+\frac{1}{3c}\ln\left(\frac{\tau}{2b_2}\right) N_a I^{(5)}_{ija}(T_R-\tau)$$
$$+\frac{4}{3c}\ln\left(\frac{\tau}{2b_3}\right)\varepsilon_{ab(i}N_b J^{(4)}_{j)a}(T_R-\tau)\Big\}\ , \tag{2.44}$$

where we have used for simplicity the notation

$$b_1 \equiv b\,e^{-11/12}\ , \qquad b_2 \equiv b\,e^{-97/60}\ , \qquad b_3 \equiv b\,e^{-7/6} \tag{2.45}$$

We do not discuss the "tail" terms in this thesis. Some details of these tail terms may be found in [46, 43].

The first check on the above waveform is its circular limit, which matches with the waveform computed earlier in [46]. The next check of the waveform in the general case is performed by computing the far-zone energy flux using

$$\frac{d\mathcal{E}}{dt} = \frac{c^3 R^2}{32\pi G}\int\left(\dot{h}^{TT}_{km}\dot{h}^{TT}_{km}\right) d\Omega(\mathbf{N})\,. \tag{2.46}$$

The expression for $d\mathcal{E}/dt$ thus obtained is identical to the far-zone energy flux directly obtained from multipole moments Eq.(2.57). Of course, these checks do not uniquely fix the expressions in Eq.(2.43) and equivalent expressions are possible leading to the same transverse traceless parts as discussed below.

The above expressions for the waveform, computed using STF multipole moments differ from the corresponding expressions obtained by Will and Wiseman (Eqs.(6.10), (6.11) of [43]), using the Epstein-Wagoner multipole moments at 1.5PN and 2PN orders. Though the two expressions are totally different looking at these orders, even in the circular limit, it is possible to show that they are equivalent. The equivalence is established by showing that the difference between the two expressions, at 1.5PN and 2PN orders has a vanishing transverse-traceless, (TT) part. The easiest way of verifying this is to show that the 'plus' and 'cross' polarizations of the difference in the two expressions vanish at 1.5PN and 2PN orders [138]. In section 2.4.2, we present the difference – at 1.5PN and 2PN orders –, between our waveform expression computed directly using the STF multipoles and the Will-Wiseman one computed using the EW multipoles and verify their equivalence. Finally we note that the statement in the appendix E of [43] should more precisely read that, STF multipole moments presented there yield an expression for the waveform *equivalent* to Eqs.(6.10) and (6.11) of [43], and not *identical* to it [138].

2.4.2 The equivalence to Will-Wiseman waveform

The expression for the gravitational waveform, obtained by Will and Wiseman [43] differs from our waveform expression at the 1.5PN and the 2PN orders. We give below the difference in the waveform expressions at these orders and show that the two polarization states, h_+ and $h_\times$ of the difference are zero at 1.5PN and 2PN orders.

$$
\begin{aligned}
\{(h^{TT}_{km})^{(1.5)}_{BDI} - (h^{TT}_{km})^{(1.5)}_{WW}\} &= \frac{1}{3\,c^3}\,\mathcal{P}_{ijkm}\,\frac{\delta m}{m}\,\frac{G\,m}{r}\,(1-2\eta)\Big\{3\,(\mathbf{N}.\mathbf{n})^3\dot{r}v_{ij} \\
&\quad -(\mathbf{N}.\mathbf{v})(\mathbf{N}.\mathbf{n})^2\Big[v_{ij}+6\dot{r}n_{(i}v_{j)}\Big] \\
&\quad +(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})^2\Big[2\,n_{(i}v_{j)}+3\,\dot{r}\,n_{ij}\Big] \\
&\quad -(\mathbf{N}.\mathbf{v})^3 n_{ij}+3\,(\mathbf{N}.\mathbf{n})\dot{r}\Big[v^2 n_{ij}+v_{ij}-2\,\dot{r}\,n_{(i}v_{j)}\Big] \\
&\quad +(\mathbf{N}.\mathbf{v})\Big[v^2 n_{ij}+v_{ij}-2\dot{r}n_{(i}v_{j)}\Big]\Big\}, \qquad (2.47a) \\
\{(h^{TT}_{km})^{(2)}_{BDI} - (h^{TT}_{km})^{(2)}_{WW}\} &= \frac{1}{15\,c^4}\,\mathcal{P}_{ijkm}\,\frac{G\,m}{r}\Big\{(1-5\eta+5\eta^2)\Big[12(\mathbf{N}.\mathbf{v})^4 n_{ij} \\
&\quad -3\,(\mathbf{N}.\mathbf{n})^4\Big(3v^2-15\dot{r}^2+\frac{G\,m}{r}\Big)v_{ij} \\
&\quad +6(\mathbf{N}.\mathbf{n})^3(\mathbf{N}.\mathbf{v})\Big(\Big[3v^2-15\dot{r}^2+\frac{G\,m}{r}\Big]\,n_{(i}v_{j)}-9\dot{r}v_{ij}\Big) \\
&\quad -6(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})^3\Big(9\dot{r}n_{ij}+4n_{(i}v_{j)}\Big) \\
&\quad -3(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})^2\Big(\Big[3v^2-15\dot{r}^2+\frac{G\,m}{r}\Big]\,n_{ij} \\
&\quad -36\dot{r}n_{(i}v_{j)}-4v_{ij}\Big]-(\mathbf{N}.\mathbf{v})^2\Big[\Big((51-185\eta+55\eta^2)v^2 \\
&\quad -(117-375\eta-15\eta^2)\dot{r}^2-(39-125\eta-5\eta^2)\,\frac{G\,m}{r}\Big)n_{ij} \\
&\quad -24\Big(1-5\eta+5\eta^2\Big)\dot{r}n_{(i}v_{j)}+12\Big(1-5\eta+5\eta^2\Big)v_{ij}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+2\,(\mathbf{N}.\mathbf{v})\,(\mathbf{N}.\mathbf{n})\Big[27\left(1-5\eta+5\eta^2\right)\dot{r}\,v^2\,n_{ij}\\
&+\Big((39-125\eta-5\,\eta^2)[v^2-\frac{G\,m}{r}]\\
&-(171-645\eta+255\,\eta^2)\dot{r}^2\Big)n_{(i}v_{j)}\\
&+27\left(1-5\eta+5\eta^2\right)\dot{r}\,v_{ij}\Big]\\
&-(\mathbf{N}.\mathbf{n})^2\Big[(1-5\eta+5\eta^2)\Big(-9\,v^4+45\dot{r}^2v^2\\
&-3\,v^2\,\frac{G\,m}{r}\Big)(n_{ij}-2\,\dot{r}\,n_{(i}v_{j)})\\
&+\Big((30-80\eta-50\,\eta^2)v^2-(72-150\eta-240\,\eta^2)\dot{r}^2\\
&-(42-140\eta+10\,\eta^2)\frac{G\,m}{r}\Big)\,v_{ij}\Big]\\
&-\Big[\Big((39-125\,\eta-5\,\eta^2)\,(\frac{G\,m}{r}-v^2)\\
&+(117-375\eta-15\,\eta^2)\,\dot{r}^2\Big)\left(v^2\,n_{ij}-2\,\dot{r}\,n_{(i}v_{j)}+v_{ij}\right)\Big]\Big\}\,.
\end{aligned}
\tag{2.47b}
$$

The two independent polarization states of the gravitational wave h_+ and $h_\times$ are given by $h_+=\frac{1}{2}\left(p_i\,p_j-q_i\,q_j\right)h^{TT}_{ij}$ and $h_\times=\frac{1}{2}\left(p_i\,q_j+p_j\,q_i\right)h^{TT}_{ij}$, where p and q are the two polarization vectors, forming along with N an orthogonal triad [46, 47, 43]. Note that there is no need to apply the TT projection before contracting on p and **q.** Consequently, we write the difference in the waveform at the 1.5PN and the 2PN orders as

$$
\{(h^{TT}_{ij})_{WW}-(h^{TT}_{ij})_{BDI}\}=\zeta_1\,v_{ij}+\zeta_2\,n_{ij}+\zeta_3\,n_{(i}v_{j)}\,. \tag{2.48}
$$

The polarization states h_+ and h,, for Eqs.(2.48) are given by,

$$
\begin{aligned}
h_+ &= \frac{1}{2}\left(p_i\,p_j-q_i\,q_j\right)\left(\zeta_1\,v_{ij}+\zeta_2\,n_{ij}+\zeta_3\,n_{(i}v_{j)}\right),\\
&= \frac{\zeta_1}{2}\left((\mathbf{p}.\mathbf{v})^2-(\mathbf{q}.\mathbf{v})^2\right)+\frac{\zeta_2}{2}\left((\mathbf{p}.\mathbf{n})^2-(\mathbf{q}.\mathbf{n})^2\right)\\
&\quad+\frac{\zeta_3}{2}\left((\mathbf{p}.\mathbf{n})(\mathbf{p}.\mathbf{v})-(\mathbf{q}.\mathbf{n})(\mathbf{q}.\mathbf{v})\right), \qquad (2.49a)\\
h_\times &= \frac{1}{2}\left(p_i\,q_j+p_j\,q_i\right)\left(\zeta_1\,v_{ij}+\zeta_2\,n_{ij}+\zeta_3\,n_{(i}v_{j)}\right)\\
&= \zeta_1\,(\mathbf{p}.\mathbf{v})\,(\mathbf{q}.\mathbf{v})+\zeta_2\,(\mathbf{p}.\mathbf{n})\,(\mathbf{q}.\mathbf{n})+\frac{\zeta_3}{2}\left((\mathbf{p}.\mathbf{n})\,(\mathbf{q}.\mathbf{v})+(\mathbf{p}.\mathbf{v})\,(\mathbf{q}.\mathbf{n})\right).
\end{aligned}
$$

(2.49b)

For the explicit computation of Eqs.(2.49), we use the standard convention adopted in [46, 47, 43], which gives, $\mathbf{p} = (0,1,0)$, $\mathbf{q} = (-\cos i, 0, \sin i)$, $\mathbf{N} = (\sin i, 0, \cos i)$, $\mathbf{n} = (\cos\phi, \sin\phi, 0)$, and $\mathbf{v} = (\dot{r}\cos\phi - r\,w\sin\phi,\ \dot{r}\sin\phi + r\,w\cos\phi,\ 0)$, where $\mathbf{n}$ and $\mathbf{v}$ are the unit separation vector, and the velocity vector respectively, ϕ is the orbital phase angle, such that the orbital angular velocity $w = d\phi/dt$ and i is the inclination angle of the source.

A straightforward but lengthy computation shows that h_+ and $h_\times$ given by Eqs.(2.49) vanish, both at the 1.5PN and the 2PN orders. This establishes the equivalence of our waveform expression, Eqs.(2.42) and (2.43) with the Will-Wiseman one given by Eqs.(6.10) and (6.11) of [43].

2.4.3 STF tensors and formulas for the waveform computations

We present details of the scheme, employed to compute the contributions to h_{jk} from various multipole moments, as required by Eqs.(2.40), (2.42) and (2.43). Our scheme proceeds in steps. In the first step, we write down schematically, the form of the desired time derivative of the STF multipole moment, using the compact notation $\{\}$, introduced by Blanchet and Damour [49]. Here $\{\}$ denotes the un-normalized minimum number of terms, required to make the expression symmetric in all the indicated indices. The second step involves peeling, where by observation and counting, we rewrite the expression obtained in the step 1, as STF on the free indices – i and j in our case –. In step 3, we contract, the final expression of step 2 with appropriate number of **N**'s as required by Eq.(2.40). The actual evaluation of the result of step 3 is performed using Maple [134]. In all the formulae, S_L, denotes the symmetric version of the object under consideration; e.g. $S_L = I^{(n)}_{(L)}$ if the object is $I^{(n)}_L$ and $S_L = J^{(m)}_{(L)}$ if the object is $J^{(m)}_L$; – the object in the formula is obvious from the context.

The un-normalized symmetric blocks .

$$\delta_{\{ij}S_{a\}} = \delta_{ij}\,S_a + \delta_{ia}\,S_j + \delta_{ja}\,S_i\,, \tag{2.50a}$$

$$\begin{aligned}\delta_{\{ij}S_{ab\}} &= \delta_{ij}S_{ab} + \delta_{ia}S_{jb} + \delta_{ib}S_{aj} + \\ &\quad \delta_{ja}S_{ib} + \delta_{jb}S_{ai} + \delta_{ab}S_{ij}\,,\end{aligned} \tag{2.50b}$$

$$\delta_{\{ij}\delta_{ab\}} = \delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{aj}\,, \tag{2.50c}$$

$$\begin{aligned}\delta_{\{ij}S_{abc\}} &= \delta_{ij}S_{abc} + \delta_{ia}S_{jbc} + \delta_{ib}S_{ajc} + \delta_{ic}S_{abj} \\ &\quad +\delta_{ja}S_{ibc} + \delta_{jb}S_{aic} + \delta_{jc}S_{abi} \\ &\quad +\delta_{ab}S_{ijc} + \delta_{ac}S_{ibj} + \delta_{bc}S_{aij}\,,\end{aligned} \tag{2.50d}$$

$$\begin{aligned}\delta_{\{ij}\delta_{ab}S_{c\}} &= \Big\{\Big[\delta_{ja}\delta_{bc} + \delta_{jb}\delta_{ac} + \delta_{jc}\delta_{ab}\Big]S_i \\ &\quad +\Big[\delta_{ia}\delta_{bc} + \delta_{ib}\delta_{ac} + \delta_{ic}\delta_{ab}\Big]S_j \\ &\quad +\Big[\delta_{ij}\delta_{bc} + \delta_{ib}\delta_{jc} + \delta_{ic}\delta_{bj}\Big]S_a \\ &\quad +\Big[\delta_{ij}\delta_{ac} + \delta_{ia}\delta_{jc} + \delta_{ic}\delta_{ja}\Big]S_b \\ &\quad +\Big[\delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}\Big]S_c\Big\}\,,\end{aligned} \tag{2.50e}$$

$$\begin{aligned}\delta_{\{ij}S_{abcd\}} &= \Big\{\delta_{ij}S_{abcd} + \delta_{ia}S_{jbcd} + \delta_{ib}S_{ajcd} + \delta_{ic}S_{abjd} \\ &\quad +\delta_{id}S_{abcj} + \delta_{ja}S_{ibcd} + \delta_{jb}S_{aicd} + \delta_{jc}S_{abid} \\ &\quad +\delta_{jd}S_{abci} + \delta_{ab}S_{ijcd} + \delta_{ac}S_{bdij} + \delta_{ad}S_{bcij} \\ &\quad +\delta_{bc}S_{adij} + \delta_{bd}S_{acij} + \delta_{cd}S_{abij}\Big\}\,,\end{aligned} \tag{2.50f}$$

$$\begin{aligned}\delta_{\{ij}\delta_{ab}S_{cd\}} &= \Big\{\Big[\delta_{ij}\delta_{ab} + \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{aj}\Big]S_{cd} + \Big[\delta_{ij}\delta_{ac} + \delta_{ia}\delta_{jc} + \delta_{ic}\delta_{aj}\Big]S_{bd} \\ &\quad +\Big[\delta_{ij}\delta_{dc} + \delta_{ic}\delta_{jb} + \delta_{ib}\delta_{jc}\Big]S_{ad} + \Big[\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb} + \delta_{ib}\delta_{ac}\Big]S_{jd} \\ &\quad +\Big[\delta_{cj}\delta_{ab} + \delta_{ca}\delta_{jb} + \delta_{cb}\delta_{aj}\Big]S_{id} + \Big[\delta_{ij}\delta_{ad} + \delta_{ia}\delta_{jd} + \delta_{id}\delta_{aj}\Big]S_{cb} \\ &\quad +\Big[\delta_{ij}\delta_{db} + \delta_{id}\delta_{jb} + \delta_{ib}\delta_{dj}\Big]S_{ca} + \Big[\delta_{id}\delta_{ab} + \delta_{ia}\delta_{db} + \delta_{ib}\delta_{ad}\Big]S_{cj} \\ &\quad +\Big[\delta_{dj}\delta_{ab} + \delta_{da}\delta_{jb} + \delta_{db}\delta_{aj}\Big]S_{ci} + \Big[\delta_{ij}\delta_{cd} + \delta_{ic}\delta_{jd} + \delta_{id}\delta_{cj}\Big]S_{ab} \\ &\quad +\Big[\delta_{ai}\delta_{cd} + \delta_{ic}\delta_{ad} + \delta_{id}\delta_{ac}\Big]S_{jb} + \Big[\delta_{aj}\delta_{cd} + \delta_{ca}\delta_{jd} + \delta_{ad}\delta_{cj}\Big]S_{ib} \\ &\quad +\Big[\delta_{ib}\delta_{cd} + \delta_{ic}\delta_{db} + \delta_{id}\delta_{bc}\Big]S_{aj} + \Big[\delta_{bj}\delta_{cd} + \delta_{bd}\delta_{jc} + \delta_{jd}\delta_{bc}\Big]S_{ai}\end{aligned}$$

$$+\left[\delta_{cd}\delta_{ab}+\delta_{ca}\delta_{db}+\delta_{cb}\delta_{ad}\right]S_{ji}\Big\}\,, \tag{2.50g}$$

$$\begin{aligned}\delta_{\{ij}\delta_{ab}\delta_{cd\}} &= \Big\{\left[\delta_{ij}\delta_{ab}+\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja}\right]\delta_{cd}+\left[\delta_{ij}\delta_{ac}+\delta_{ia}\delta_{jc}+\delta_{ic}\delta_{ja}\right]\delta_{bd}\\ &\quad+\left[\delta_{ij}\delta_{cb}+\delta_{ic}\delta_{jb}+\delta_{ib}\delta_{jc}\right]\delta_{ad}+\left[\delta_{ic}\delta_{ab}+\delta_{ia}\delta_{cb}+\delta_{ib}\delta_{ca}\right]\delta_{jd}\\ &\quad+\left[\delta_{cj}\delta_{ab}+\delta_{ca}\delta_{jb}+\delta_{cb}\delta_{ja}\right]\delta_{id}\Big\}\,.\end{aligned} \tag{2.50h}$$

The STF **tensors.**

$$\mathrm{STF}_{ija}(I_{ija}) = S_{ija}-\frac{1}{5}\delta_{\{ij}S_{a\}tt}\,, \tag{2.51a}$$

$$\mathrm{STF}_{ijab}(I_{ijab}) = S_{ijab}-\frac{1}{7}\delta_{\{ij}S_{ab\}tt}+\frac{1}{35}\delta_{\{ij}\delta_{ab\}}S_{sstt}\,, \tag{2.51b}$$

$$\mathrm{STF}_{ijabc}(I_{ijabc}) = S_{ijabc}-\frac{1}{9}\delta_{\{ij}S_{abc\}tt}+\frac{1}{63}\delta_{\{ij}\delta_{ab}S_{c\}sstt}\,, \tag{2.51c}$$

$$\begin{aligned}\mathrm{STF}_{ijabcd}(I_{ijabcd}) &= S_{ijabcd}-\frac{1}{11}\delta_{\{ij}S_{abcd\}pp}+\frac{1}{99}\delta_{\{ij}\delta_{ab}S_{cd\}ttqq}\\ &\quad-\frac{1}{693}\delta_{\{ij}\delta_{ab}\delta_{cd\}}S_{ppqqtt}\,.\end{aligned} \tag{2.51d}$$

The ‘Peeling’.

$$\mathrm{STF}_{ija}(I_{ija}) = \mathrm{STF}_{ij}\Big\{S_{ija}-\frac{2}{5}\delta_{ia}S_{jtt}\Big\}\,, \tag{2.52a}$$

$$\begin{aligned}\mathrm{STF}_{ijab}(I_{ijab}) &= \mathrm{STF}_{ij}\Big\{S_{ijab}-\frac{1}{7}\Big[2\delta_{ia}S_{jbtt}+2\delta_{ib}S_{jatt}+\delta_{ba}S_{jitt}\Big]+\\ &\quad\frac{2}{35}\Big[\delta_{ia}\delta_{jb}S_{ttss}\Big]\Big\}\,,\end{aligned} \tag{2.52b}$$

$$\begin{aligned}\mathrm{STF}_{ijabc}(I_{ijabc}) &= \mathrm{STF}_{ij}\Big\{S_{ijabc}-\frac{1}{9}\Big[2\Big(\delta_{ia}S_{jbcpp}+\delta_{ib}S_{jacpp}+\delta_{ic}S_{abjpp}\Big)\\ &\quad+\Big(\delta_{ab}S_{ijcpp}+\delta_{ac}S_{ijbpp}+\delta_{bc}S_{ijapp}\Big)\Big]\\ &\quad+\frac{2}{63}\Big[\Big(\delta_{ja}\delta_{bc}+\delta_{jb}\delta_{ac}+\delta_{jc}\delta_{ab}\Big)S_{ippqq}\\ &\quad+\Big(\delta_{ib}\delta_{jc}S_{appqq}+\delta_{ia}\delta_{jc}S_{bppqq}+\delta_{ia}\delta_{jb}S_{cppqq}\Big)\Big]\Big\}\,,\end{aligned} \tag{2.52c}$$

$$\begin{aligned}\mathrm{STF}_{ijabcd}(I_{ijabcd}) &= \mathrm{STF}_{ij}\Big\{S_{ijabcd}-\frac{1}{11}\Big[2\Big(\delta_{ia}S_{jbcdpp}+\delta_{ib}S_{jacdpp}\\ &\quad+\delta_{ic}S_{abjdpp}+\delta_{id}S_{abcjpp}\Big)\\ &\quad+\Big(\delta_{ad}S_{bcijpp}+\delta_{ab}S_{dcijpp}\\ &\quad+\delta_{ac}S_{bdijpp}+\delta_{bc}S_{adijpp}+\delta_{bd}S_{acijpp}+\delta_{cd}S_{baijpp}\Big)\Big]\\ &\quad+\frac{1}{99}\Big[2\Big(\delta^{ia}_{ia}\delta^{jb}_{jb}S^{cdppqq}+\delta^{ia}_{ia}\delta^{jc}_{jc}S^{bdppqq}+\end{aligned}$$

$$
\begin{aligned}
&\delta_{ic}\delta_{jb}S_{adppqq} + \delta_{ia}\delta_{jd}S_{cbppqq} + \\
&\delta_{id}\delta_{jb}S_{acppqq} + \delta_{ic}\delta_{jd}S_{bappqq}\Big) \\
&+2\Big(\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb} + \delta_{ib}\delta_{ca}\Big)S_{jdppqq} \\
&+2\Big(\delta_{id}\delta_{ab} + \delta_{ia}\delta_{db} + \delta_{ib}\delta_{da}\Big)S_{jcppqq} \\
&+2\Big(\delta_{ia}\delta_{cd} + \delta_{ic}\delta_{da} + \delta_{id}\delta_{ca}\Big)S_{jbppqq} \\
&+2\Big(\delta_{ib}\delta_{cd} + \delta_{ic}\delta_{db} + \delta_{id}\delta_{bc}\Big)S_{jappqq} \\
&+\Big(\delta_{cd}\delta_{ab} + \delta_{ca}\delta_{db} + \delta_{da}\delta_{cb}\Big)S_{jippqq}\Big] \\
&-\frac{2}{693}\Big[\Big(\delta_{ia}\delta_{jb}\delta_{cd} + \delta_{ia}\delta_{jc}\delta_{bd} \\
&+\delta_{ic}\delta_{jb}\delta_{ad}\Big) + \Big(\delta_{ic}\delta_{ab} + \delta_{ia}\delta_{cb} + \delta_{ib}\delta_{ac}\Big)\delta_{jd}\Big]S_{ppqqtt}\Big\} .
\end{aligned}
\tag{2.52d}
$$

The contractions with $\mathbf{N}_L$

$$
\mathrm{STF}_{ija}(I^{(3)}_{ija})\, N_a = \mathrm{STF}_{ij}\Big\{S^{(3)}_{ija}\, N_a - \frac{2}{5}\, N_i S^{(3)}_{jtt}\Big\} , \tag{2.53a}
$$

$$
\mathrm{STF}_{ijab}(I^{(4)}_{ijab})N_{ab} = \mathrm{STF}_{ij}\Big\{S^{(4)}_{ijab}N_{ab} - \frac{1}{7}\Big[4N_{ia}S^{(4)}_{jatt} + S^{(4)}_{ijtt}\Big] + \frac{2}{35}N_{ij}S^{(4)}_{ttss}\Big\} , \tag{2.53b}
$$

$$
\begin{aligned}
\mathrm{STF}_{ijabc}(I^{(5)}_{ijabc})N_{abc} &= \mathrm{STF}_{ij}\Big\{S^{(5)}_{ijabc}N_{abc} - \frac{6}{9}N_{ibc}S^{(5)}_{jbcpp} - \frac{1}{3}S^{(5)}_{ijcpp}N_c + \\
&\frac{6}{63}S^{(5)}_{ippqq}N_j + \frac{6}{63}N_{ija}S^{(5)}_{appqq}\Big\} ,
\end{aligned}
\tag{2.53c}
$$

$$
\begin{aligned}
\mathrm{STF}_{ijabcd}(I^{(6)}_{ijabcd})N_{abcd} &= \mathrm{STF}_{ij}\Big\{S^{(6)}_{ijabcd}\, N_{abcd} - \frac{8}{11}N_{ibcd}\, S^{(6)}_{jbcdpp} - \frac{6}{11}S^{(6)}_{ijcdpp}\, N_{cd} \\
&+\frac{12}{99}N_{ijcd}\, S^{(6)}_{cdppqq} + \frac{24}{99}N_{id}\, S^{(6)}_{jdppqq} + \frac{3}{99}S^{(6)}_{ijppqq} \\
&-\frac{12}{693}N_{ij}\, S^{(6)}_{ppqqtt}\Big\} .
\end{aligned}
\tag{2.53d}
$$

The current multipole moments.

$$
\epsilon_{pq(i}\hat{J}_{j)pL} = \mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\hat{J}_{jpL}\Big\}
$$

$$
\epsilon_{pq(i}\hat{J}^{(2)}_{j)p}\, N_q = \mathrm{STF}_{ij}\Big\{\epsilon_{pqi}S^{(2)}_{jp}\, N_q\Big\} , \tag{2.54a}
$$

$$
\epsilon_{pq(i}\hat{J}^{(3)}_{j)pa}\, N_{qa} = \mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\Big[S^{(3)}_{jpa}\, N_{qa} - \frac{1}{5}S^{(3)}_{ptt}\, N_{qj}\Big]\Big\} , \tag{2.54b}
$$

$$\epsilon_{pq(i}\hat{J}^{(4)}_{j)pab}\,N_{qab} = \mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,\Big[S^{(4)}_{jpab}\,N_{qab} - \frac{1}{7}\Big(2\,S^{(4)}_{pbtt}\,N_{qjb} + S^{(4)}_{pjtt}\,N_q\Big)\Big]\Big\}, \tag{2.54c}$$

$$\epsilon_{pq(i}\hat{J}^{(5)}_{j)pabc}\,N_{qabc} = \mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\Big[S^{(5)}_{jpabc}\,N_{qabc} - \frac{1}{3}\Big(S^{(5)}_{pbctt}\,N_{qjbc} + S^{(5)}_{jpctt}\,N_{qc}\Big) + \frac{1}{21}\Big(S^{(5)}_{pttvv}N_{qj}\Big)\Big]\Big\}. \tag{2.54d}$$

The explicit computations of the above equations require the following identities, which are easily derived, using the rules governing the product of ϵ's. The identities are

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q y_j \tilde{L}_p\Big\} = \mathrm{STF}_{ij}\Big\{-(\mathbf{N}.\mathbf{v})y_{ij} + (\mathbf{N}.\mathbf{n})r\,y_i v_j\Big\}, \tag{2.55a}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q v_j \tilde{L}_p\Big\} = \mathrm{STF}_{ij}\Big\{-(\mathbf{N}.\mathbf{v})y_i v_j + (\mathbf{N}.\mathbf{n})\,r\,v_{ij}\Big\}, \tag{2.55b}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_{qj} \tilde{L}_p\Big\} = \mathrm{STF}_{ij}\Big\{(\mathbf{N}.\mathbf{n})\,r\,N_i v_j - (\mathbf{N}.\mathbf{v})\,y_i N_j\Big\}, \tag{2.55c}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q y_p \tilde{L}_j\Big\} = \mathrm{STF}_{ij}\Big\{-(\mathbf{N}.\mathbf{v})\,y_{ij} + (\mathbf{N}.\mathbf{n})\,r\,y_i v_j + (r\dot{r})N_i y_j - r^2 N_i v_j\Big\}, \tag{2.55d}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q v_p \tilde{L}_j\Big\} = \mathrm{STF}_{ij}\Big\{(\mathbf{N}.\mathbf{n})\,r\,v_{ij} - (\mathbf{N}.\mathbf{v})y_i v_j - r\dot{r}\,N_i v_j + v^2\,N_i y_j\Big\}, \tag{2.55e}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_{qj} v_p (\tilde{\mathbf{L}}.\mathbf{N})\Big\} = \mathrm{STF}_{ij}\Big\{[v^2 - (\mathbf{N}.\mathbf{v})^2]N_i y_j + [(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v}) - \dot{r}]rN_i v_j + [\dot{r}(\mathbf{N}.\mathbf{v}) - v^2(\mathbf{N}.\mathbf{n})]rN_{ij}\Big\}, \tag{2.55f}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_{qj} y_p (\tilde{\mathbf{L}}.\mathbf{N})\Big\} = \mathrm{STF}_{ij}\Big\{[\dot{r} - (\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})]rN_i y_j + [(\mathbf{N}.\mathbf{n})^2 - 1]r^2 N_i v_j + [(\mathbf{N}.\mathbf{v}) - \dot{r}(\mathbf{N}.\mathbf{n})]r^2 N_{ij}\Big\}, \tag{2.55g}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q y_{jp} (\tilde{\mathbf{L}}.\mathbf{N})\Big\} = \mathrm{STF}_{ij}\Big\{[\dot{r} - (\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})]r y_{ij} + [(\mathbf{N}.\mathbf{n})^2 - 1]r^2 y_i v_j + [(\mathbf{N}.\mathbf{v}) - \dot{r}(\mathbf{N}.\mathbf{n})]r^2 N_i y_j\Big\}, \tag{2.55h}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q y_j v_p (\tilde{\mathbf{L}}.\mathbf{N})\Big\} = \mathrm{STF}_{ij}\Big\{[v^2 - (\mathbf{N}.\mathbf{v})^2]y_{ij} + [(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v}) - \dot{r}]r y_i v_j + [\dot{r}(\mathbf{N}.\mathbf{v}) - v^2\,(\mathbf{N}.\mathbf{n})]rN_i y_j\Big\}, \tag{2.55i}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q y_p v_j (\tilde{\mathbf{L}}.\mathbf{N})\Big\} = \mathrm{STF}_{ij}\Big\{[\dot{r} - (\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})]\,r\,y_i v_j + [(\mathbf{N}.\mathbf{n})^2 - 1]\,r^2\,v_{ij}$$

$$+[(\mathbf{N}.\mathbf{v}) - \dot{r}\,(\mathbf{N}.\mathbf{n})]r^2\,N_i v_j\Big\}\,, \tag{2.55j}$$

$$\mathrm{STF}_{ij}\Big\{\epsilon_{pqi}\,N_q v_{jp}(\tilde{\mathbf{L}}.\mathbf{N})\Big\} \;=\; \mathrm{STF}_{ij}\Big\{[v^2 - (\mathbf{N}.\mathbf{v})^2]y_i v_j + [(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v}) - \dot{r}]r v_{ij} + [\dot{r}(\mathbf{N}.\mathbf{v}) - v^2(\mathbf{N}.\mathbf{n})]r N_i v_j\Big\}\,, \tag{2.55k}$$

where $L_p = \epsilon_{pkl}\,y_k\,v_l$.

2.5 The far-zone fluxes

2.5.1 The Energy flux

In Section 2.2, we have seen that Eq.(2.6) gives the far-zone energy flux to 2PN order, in terms of the STF "radiative" moments of the gravitational field. As in the case of gravitational waveform, the 2PN accurate relations connecting the "radiative" multipole moments U_L and V_L to the source moments I_L and J_L are employed to split the 2PN-accurate far-zone energy flux into an "instantaneous" contribution and a "tail" one. In this thesis, we deal only with the instantaneous contribution, which is given by [135, 46]

$$_{\text{far-zone}} = \frac{G}{c^5}\Big\{\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}I^{(5)}_{ijkm}I^{(5)}_{ijkm} + \frac{1}{84}J^{(4)}_{ijk}J^{(4)}_{ijk}\right]\Big\}\,. \tag{2.56}$$

Here $I_L^{(n)}$ denotes the n^{th} time derivative of STF multipole moment of rank L. Evaluating the relevant time derivatives of the multipole moments in Eq.(2.56), using the post-Newtonian equations of motion to the appropriate order we obtain

$$\left(\frac{d\mathcal{E}}{dt}\right)^{\text{inst}}_{\text{far-zone}} = \dot{\mathcal{E}}_N + \dot{\mathcal{E}}_{1PN} + \dot{\mathcal{E}}_{2PN}\,, \tag{2.57a}$$

$$\dot{\mathcal{E}}_N = \frac{8}{15}\frac{G^3\,m^2\,\mu^2}{c^5\,r^4}\left\{12v^2 - 11\dot{r}^2\right\}\,, \tag{2.57b}$$

$$\dot{\mathcal{E}}_{1PN} = \frac{8}{15}\frac{G^3\,m^2\,\mu^2}{c^7\,r^4}\Big\{\frac{1}{28}\Big[(785 - 852\eta)v^4 - 2(1487 - 1392\eta)v^2\dot{r}^2 - 160(17 - \eta)\frac{Gm}{r}\,v^2$$

$$
\begin{aligned}
& + \quad 3(687-620\eta)\dot{r}^4 + 8(367-15\eta)\frac{Gm}{r}\dot{r}^2 \\
& + \quad 16(1-4\eta)\frac{G^2m^2}{r^2}\Bigg]\Bigg\}\,, \qquad (2.57c) \\
\dot{\mathcal{E}}_{2PN} = & \frac{8}{15}\frac{G^3m^2\mu^2}{c^9r^4}\Bigg\{\frac{1}{42}(1692-5497\eta+4430\eta^2)v^6 \\
& - \quad \frac{1}{14}(1719-10278\eta+6292\eta^2)v^4\dot{r}^2 \\
& - \quad \frac{1}{21}(4446-5237\eta+1393\eta^2)\frac{Gm}{r}v^4 \\
& + \quad \frac{1}{14}(2018-15207\eta+7572\eta^2)v^2\dot{r}^4 \\
& + \quad \frac{1}{7}(4987-8513\eta+2165\eta^2)\frac{Gm}{r}v^2\dot{r}^2 \\
& + \quad \frac{1}{756}(281473+81828\eta+4368\eta^2)\frac{G^2m^2}{r^2}v^2 \\
& - \quad \frac{1}{42}(2501-20234\eta+8404\eta^2)\dot{r}^6 \\
& - \quad \frac{1}{63}(33510-60971\eta+14290\eta^2)\frac{Gm}{r}\dot{r}^4 \\
& - \quad \frac{1}{252}(106319+9798\eta+5376\eta^2)\frac{G^2m^2}{r^2}\dot{r}^2 \\
& + \quad \frac{2}{63}(-253+1026\eta-56\eta^2)\frac{G^3m^3}{r^3}\Bigg\}\,. \qquad (2.57d)
\end{aligned}
$$

Eqs.(2.57) are in exact agreement with the results of Will and Wiseman [43] using the new improved Epstein-Wagoner approach. Circular and radial infall limits of Eqs.(2.57) are in agreement with earlier results [18, 46, 139, 43] and discussed further in section 2.6.

The tail contribution, on the other hand is given by

$$
\left(\frac{d\mathcal{E}}{dt}\right)^{\mathbf{tail}}_{\mathbf{far-zone}} = \frac{2G}{5c^5}\frac{2Gm}{c^3}I^{(3)}_{ij}(T_R)\int_0^{+\infty}d\tau\,\ln\left(\frac{\tau}{2b_1}\right)I^{(5)}_{ij}(T_R-\tau)\,, \qquad (2.58)
$$

where $b_1 \equiv b\,e^{-11/12}$. A detailed discussion of the tail terms and its implications has been given by Blanchet and Schafer [128], and we do not discuss it any further in this thesis.

2.5.2 The angular momentum flux

Following [135], the far-zone angular momentum flux to 2PN accuracy written in terms of the STF radiative multipole moments of the gravitational field reads:

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\mathbf{far-zone}} - \frac{G}{c^5}\epsilon_{ipq}\left\{\frac{2}{5}U_{pj}U^{(1)}_{qj} + \frac{1}{c^2}\left[\frac{1}{63}U_{pjk}U^{(1)}_{qjk} + \frac{32}{45}V_{pj}V^{(1)}_{qj}\right] + \frac{1}{c^4}\left[\frac{1}{2268}U_{pjkl}U^{(1)}_{qjkl} + \frac{1}{28}V_{pjk}V^{(1)}_{qjk}\right]\right\}. \tag{2.59}$$

As before, rewriting the radiative moments in terms of the source moments using Eqs.**(2.9)** and (2.10) allows us to separate the instantaneous and tail contributions and we discuss them independently. The "instantaneous" contribuuion is given by, [135]

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{\mathbf{inst}}_{\mathbf{far-zone}} - \frac{G}{c^5}\epsilon_{ipq}\left\{\frac{2}{5}I^{(2)}_{pj}I^{(3)}_{qj} + \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{pjk}I^{(4)}_{qjk} + \frac{32}{45}J^{(2)}_{pj}J^{(3)}_{qj}\right] + \frac{1}{c^4}\left[\frac{1}{2268}I^{(4)}_{pjkl}I^{(5)}_{qjkl} + \frac{1}{28}J^{(3)}_{pjk}J^{(4)}_{qjk}\right]\right\}. \tag{2.60}$$

Computing the required time derivatives of the STF moments, using the post-Newtonian equations of motion to the appropriate order, we obtain

$$\left(\frac{d\mathcal{J}}{dt}\right)^{\mathbf{inst}}_{\mathbf{far-zone}} = \dot{\mathcal{J}}_N + \dot{\mathcal{J}}_{1PN} + \dot{\mathcal{J}}_{2PN}\,, \tag{2.61a}$$

$$\dot{\mathcal{J}}_N = \frac{8}{5}\frac{G^2\,m\,\mu^2}{c^5\,r^3}\tilde{\mathbf{L}}_{\mathbf{N}}\left\{2v^2 - 3\dot{r}^2 + 2\,\frac{Gm}{r}\right\}, \tag{2.61b}$$

$$\begin{aligned}\dot{\mathcal{J}}_{1PN} &= \frac{8}{5}\frac{G^2\,m\,\mu^2}{c^7\,r^3}\tilde{\mathbf{L}}_{\mathbf{N}}\left\{\frac{1}{84}\left[(307-548\eta)v^4\right.\right.\\ &- 6(74-277\eta)v^2\dot{r}^2 - 4(58+95\eta)\frac{Gm}{r}\,v^2\\ &+ 3(95-360\eta)\dot{r}^4 + 2(372+197\eta)\frac{Gm}{r}\,\dot{r}^2\\ &\left.\left.- 2(745-2\eta)\frac{G^2m^2}{r^2}\right]\right\},\end{aligned} \tag{2.61c}$$

$$\dot{\mathcal{J}}_{2PN} = \frac{8}{5}\frac{G^2\,m\,\mu^2}{c^9\,r^3}\tilde{\mathbf{L}}_{\mathbf{N}}\left\{\frac{1}{504}\left[(2665-12355\eta+12894\eta^2)v^6\right.\right.$$

$$
\begin{aligned}
&- \quad 3(2246 - 12653\eta + 15637\eta^2)v^4\dot{r}^2 \\
&+ \quad (165 - 491\eta + 4022\eta^2)\frac{Gm}{r}\,v^4 \\
&+ \quad 3(3575 - 16805\eta + 15680\eta^2)v^2\dot{r}^4 \\
&+ \quad (21853 - 21603\eta + 2551\eta^2)\frac{Gm}{r}\,v^2\dot{r}^2 \\
&- \quad 2(10651 - 10179\eta + 3428\eta^2)\frac{G^2m^2}{r^2}\,v^2 \\
&- \quad 28(195 - 815\eta + 485\eta^2)\dot{r}^6 \\
&- \quad (22312 - 41\,398\,\eta + 96\,95\,\eta^2)\dot{r}^4\,\frac{G\,m}{r} \\
&+ \quad 2(8436 - 25102\eta + 4587\eta^2)\frac{G^2m^2}{r^2}\,\dot{r}^2\Bigg] \\
&+ \quad \frac{1}{2268}(170362 + 70461\eta + 1386\eta^2)\frac{G^3m^3}{r^3}\Bigg\}\,, \qquad (2.61\text{d})
\end{aligned}
$$

where $\tilde{\mathbf{L}}_{\mathbf{N}} = \mathbf{r} \times \mathbf{v}$. The 1PN contribution is in agreement with the earlier results of Junker and Schafer [127]. The 2PN contribution is new and together with the energy flux obtained in the earlier section forms the starting point for the computation the 2PN radiation reaction for compact binary systems – 4.5PN terms in the equation of motion – [45], using the refined balance method proposed by Iyer and Will [33, 34]. The tail terms, in the angular momentum flux are given by

$$
\left(\frac{d\mathcal{J}_i}{dt}\right)^{\text{tail}}_{\text{far-zone}} - \frac{2G}{5c^5}\frac{2Gm}{c^3}\,\epsilon_{ijk}\,I^{(3)}_{kl}(T_R)\int_0^{+\infty} d\tau\,\ln\left(\frac{\tau}{2b_1}\right) I^{(4)}_{jl}(T_R-\tau)\,. \qquad (2.62)
$$

We will not be discussing the tails terms here, as they are extensively studied by Schafer and Rieth [129].

2.6 Limits

All the complicated formulae, discussed in the earlier sections take more simpler forms for quasi-circular orbits. For compact binaries like PSR 1913+16, quasi-circular orbits should provide a good description close to the inspiral phase, since gravitational radiation reaction would have reduced the present eccentricity, to vanishingly small values. In this context 'quasi' implies the slow inspiral caused by the radiation reaction. The quasi-circular orbit is characterized by $\ddot{r} = \dot{r} = O(\epsilon^{2.5})$. The 2PN equations of motion, become

$$\mathbf{a} \equiv \frac{d\mathrm{v}}{dt} = \frac{d^2\mathbf{x}}{dt''} = -\omega_{2\mathrm{PN}}^2 \mathbf{x} + O(\epsilon^{2.5}) . \tag{2.63a}$$

with $\omega_{2\mathrm{PN}}$, the 2PN accurate orbital frequency, is given by

$$\omega_{2\mathrm{PN}}^2 \equiv \frac{Gm}{r^3} \left\{ 1 - (3-\eta)\gamma + \left(6 + \frac{41}{4}\eta + \eta^2 \right) \gamma^2 \right\} , \tag{2.64}$$

where $\gamma = \mathrm{Grn/c^2\,r}$. Note that Eqs.(2.63) imply as usual, that $v \equiv |\mathbf{v}| = \omega_{2\mathrm{PN}}\,\mathrm{r} + O(\epsilon^{2.5})$, so that from Eq.(2.64) we get

$$v^2 = \frac{Gm}{r} \left\{ 1 - (3-\eta)\gamma + \left(6 + \frac{41}{4}\eta + \eta^2 \right) \gamma^2 \right\} \tag{2.65}$$

Substituting $\dot{r} = 0$ in Eqs.(2.57), and using Eq.(2.65) we obtain the 2PN corrections to the far-zone energy flux for compact binaries of arbitrary mass ratio, moving in a quasi-circular orbit

$$\left(\frac{d\mathcal{E}}{dt} \right)^{\mathrm{inst}}_{\mathbf{far-zone}} = \frac{32}{5} \frac{c^5}{G} \eta^2 \gamma^5 \left\{ 1 - \gamma \left(\frac{2927}{336} + \frac{5}{4}\eta \right) + \gamma^2 \left(\frac{293383}{9072} + \frac{380}{9}\eta \right) \right\} . \tag{2.66}$$

Eq.(2.66) is consistent with results of [18, 46, 43].

The energy and angular momentum fluxes are not independent but related in the case of circular orbits. The precise relation may be written following [140] as:

$$\left(\frac{d\mathcal{E}}{dt} \right)_{\mathbf{far-zone}} = \mathrm{v}^2\, \mathcal{J}, \tag{2.67a}$$

$$\text{where } \left(\frac{d\mathcal{J}}{dt} \right)_{\mathbf{far-zone}} \equiv \tilde{\mathbf{L}}_{\mathbf{N}} \dot{\mathcal{J}} , \tag{2.67b}$$

where v^2 defined in terms of $\mathrm{G}\, m/r$, is given by Eq.(2.65) [46].

The other limiting case we compare to corresponds to the case of radial infall of two compact objects of comparable masses. Equations representing the head-on infall can be obtained from the expressions for the general orbit by imposing the restrictions, $\mathrm{x} = \mathrm{z}\mathrm{n}$, $\mathrm{v} = \dot{z}\hat{\mathbf{n}}$, $\mathrm{r} = z$ and $\mathrm{v} = \dot{r} = \dot{z}$. We consider two different cases, following Simone, Poisson, and Will [139]. In case (A), the radial infall proceeds from rest at infinite initial separation, which implies that the conserved energy $E(z) = E(\infty) = 0$. In case (B), the radial infall proceeds from rest at finite initial separation z_0, which implies

$$E(z = z_0) = -\mu c^2 \gamma_0 \left\{1 - \frac{1}{2}\gamma_0 + \frac{1}{2}\left[1 + \frac{15}{2}\eta\right]\gamma_0^2\right\}, \tag{2.68}$$

where $\gamma_0 = \mathrm{G m/z_0 c^2}$. Inverting $E(z)$ for $\dot{z}^2$ and using Eq.(2.68) we obtain

$$\begin{aligned} \dot{z} &= -c\left\{2(\gamma - \gamma_0)\left[1 - 5\gamma\left(1 - \frac{\eta}{2}\right) + \gamma_0\left(1 - \frac{9\eta}{2}\right)\right.\right. \\ &\quad + \gamma^2\left(13 - \frac{81\eta}{4} + 5\eta^2\right) - \gamma\gamma_0\left(5 - \frac{173\eta}{4} + 13\eta^2\right) \\ &\quad \left.\left. + \gamma_0^2\left(1 - \frac{5\eta}{4} + 8\eta^2\right)\right]\right\}^{\frac{1}{2}}, \end{aligned} \tag{2.69}$$

where $\gamma = \mathrm{G}\, m/z\, c^2$. Using the radial infall restrictions and Eq.(2.69) in Eqs.(2.57) we obtain for the case B),the far-zone radiative energy flux

$$\begin{aligned} \left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{inst}}_{\mathrm{far-zone}} &= \frac{16}{15} c^5 \eta^2 \gamma^5 \left\{1 - x - \frac{-}{7}\, 43 - \frac{111}{2}\eta\right. \\ &\quad \left. - x(116 - 131\eta) + x^2\left(71 - \frac{135\,\eta}{2}\right)\right]\gamma \\ &\quad - \frac{1}{3}\left[\frac{1127}{9} + \frac{803}{12}\eta - 112\,\eta^2\right. \\ &\quad + \frac{x}{7}\left(\frac{4471}{9} - \frac{15481\eta}{3} + 2864\,\eta^2\right) \\ &\quad - \frac{x^2}{7}\left(1870 - \frac{38521\,\eta}{6} + \frac{8800\,\eta^2}{3}\right) \\ &\quad \left. + x^3\left(83 - \frac{1183\,\eta}{4} + \frac{872\,\eta^2}{7}\right)\right]\gamma^2\right\}, \end{aligned} \tag{2.70}$$

where $x = \gamma_0/\gamma$. For the case (A), the expressions for $\dot{z}$ and $d\mathcal{E}/dt$ are obtained by setting $\gamma_0 = 0$ in Eqs.(2.69) and (2.70). Eq.(2.70), along with corresponding one for case (A) are in agreement with [139].

2.7 Conclusions

In this chapter using the multipolar post-Minkowskian generation formalism of Blanchet, Damour and Iyer, we have computed the 2PN contributions to the mass quadrupole moment for two compact objects of arbitrary mass ratio moving in general orbits. Using this moment and other required multipole moments to lower post-Newtonian orders obtained using BDI formalism, we have computed the 2PN contributions to the gravitational waveform and the associated far-zone energy and angular momentum fluxes. We have also shown equivalence of our waveform with the one obtained by Will-Wiseman, providing a valuable check on the complicated and lengthy algebra. In the next chapter the expressions for the far-zone fluxes will be used to compute the evolution of the orbital elements of the 2PN accurate generalized quasi-Keplerian representation for elliptic orbits. Using the the gravitational waveform obtained here as one of the inputs, we will calculate the 2PN contributions to the 'plus' and 'cross' gravitational wave polarizations in chapter 4. Also, using the 2PN corrections to far-zone energy and angular momentum fluxes, in chapter 5 we compute the 2PN radiation reaction, *i.e* the 4.5PN terms in the equations of motion, using the refined balance method proposed by Iyer and Will [33, 34].