Chapter 3

The evolution of the orbital elements in the generalized quasi-Keplerian parameterization of the binary

3.1 Introduction

In this chapter, we compute the 2PN corrections to the rate of decay of the orbital elements of a compact binary, in quasi-elliptical orbit, *i.e.* the effect of the 4.5PN radiation reaction on a 2PN accurate conservative elliptical motion, extending the earlier computations [126, 128, 127, 129]. The basic ingredients we employ for the calculations are the far-zone energy and angular momentum fluxes in the harmonic coordinates computed in the previous chapter and a 2PN accurate description of the relative motion of the compact binary available in a generalized quasi-Keplerian parameterization given in the ADM coordinates [40, 41, 42]. Since the 2PN accurate orbital representation is in the ADM coordinates, we use the coordinate transformations connecting the harmonic and the ADM coordinates [142], to rewrite the far-zone fluxes in the ADM coordinates. The far-zone fluxes, in the ADM coordinates are averaged over an orbital period, extending the earlier computations at the 1PN and the 1.5PN order [126, 128, 127, 129]. The 2PN corrections to the rate of decay of the orbital elements are computed using heuristic arguments based on the conservation of energy and angular momentum to the 2PN order. The argument is that the energy and the angular momentum carried away from the binary by gravitational waves, theoretically computed to the 2PN order in chapter 1, should be balanced by a decrease of the 2PN accurate energy and angular momentum of the binary. Hence emission of gravitational radiation will result in the decay of the orbital elements of the generalized quasi-Keplerian representation, as they are expressed in terms of the conserved energy and the angular momentum of the binary. Though we are extending the computations of [126, 128, 127, 129] to obtain the 2PN corrections to the evolution of orbital elements, we have to take care of a new complication at this order. The complication arises due to the fact that the far-zone fluxes are computed in the harmonic or De-Donder coordinates, while the orbital representation is available only in the ADM coordinates. In the limit of $\eta \to 0$ our results reduce to the test particle results [61] to 2PN accuracy.

This chapter is organized as follows. In Section 3.2 we summarize the generalized quasi-Keplerian description of the bound orbits of the binary in the ADM coordinates. Section 3.3 deals with the transformation equations relating the De-Donder and the ADM gauges. In Section 3.4 we rewrite the expressions for the far-zone fluxes using the the generalized quasi-Keplerian representation for elliptical motion and average the fluxes over an orbital period of the binary. Section 3.5 deals with the evolution of some of the important orbital elements of the 2PN accurate representation. In section 3.6 we discuss different limiting cases and compare them with earlier results. Most of the results presented in this chapter have been published in Ref. [44].

3.2 The second post-Newtonian motion of compact binaries

Let $r(t)$, $\phi(t)$ be the planar relative motion of the two point masses in a bound binary. It is well know that the solution of the Newtonian equations of motion for a bound binary $(E < 0)$ can be expressed in the following form,

$$r = a\,(1 - e\cos u)\ , \tag{3.1a}$$

$$n(t - t_0) = u - e \sin u\,, \tag{3.1b}$$

$$\phi - \phi_0 = v\,, \tag{3.1c}$$

$$\text{where } v = 2 \tan^{-1} \left\{ \left(\frac{1+e}{1-e} \right)^{\frac{1}{2}} \tan\left(\frac{u}{2}\right) \right\}. \tag{3.1d}$$

The above description is known in the literature as the Keplerian representation. Here n, the mean motion is given by $n = \frac{2\pi}{P}$ where P is the orbital period. Also e is the eccentricity and a is the semi-major axis of the orbit. The auxiliary quantities u and v are called eccentric and true anomalies. Note that the parameters n, a and e are functions of the conserved energy and angular momentum per unit reduced mass μ of the binary. To avoid introducing additional notation following [40, 41, 42], in what follows, these are also denoted as E and $\mathbf{J} = |\mathbf{J}|$.

Damour and Deruelle found a remarkably simple parameterization for the solution to the 1PN accurate Damour-Deruelle equations of motion [104, 39]. This representation known in the literature as the quasi-Keplerian parametrization, is given by,

$$r = a\,(1 - e_r \cos u)\,, \tag{3.2a}$$

$$n(t - t_0) = u - e_t \sin u\,, \tag{3.2b}$$

$$\phi - \phi_0 = \left(1 + \frac{k}{c^2}\right) v\,, \tag{3.2c}$$

$$\text{where } v = 2 \tan^{-1} \left\{ \left(\frac{\ }{\ } \right)^{\frac{1}{2}} \tan\left(\frac{u}{2}\right) \right\}. \tag{3.2d}$$

Instead of a single eccentricity e as in the Newtonian case, there are three different eccentricities, e,, e_t and ed. Further the ϕ equation contains k, the periastron precession constant. As before all parameters are functions of E and J of the binary.

Damour and Schafer observed that in the ADM coordinates there exists an elegant and most Keplerian like representation to the second post-Newtonian motion of a binary system [40]. This generalized quasi-Keplerian description for the general binary orbits to the 2PN order, developed by Damour, Schafer, and Wex [40, 41, 42] is best suited for the calculation we propose to do in the following sections and we summarize it in what follows. Let $r_A(t_A)$, $\phi_A(t_A)$ be the planar relative motion of the two compact objects in usual polar coordinates associated with the ADM coordinates. The radial motion $r_A(t_A)$ is conveniently parameterized by

$$r_A = \text{a, } (1 - \text{e, } \cos \text{u}) \,, \tag{3.3a}$$

$$n(t_A - t_0) = u - e_t \sin \text{u} + \frac{\text{ft}}{\text{c}^4} \sin v + \frac{g_t}{\text{c}^4} (v - \text{u}) \,, \tag{3.3b}$$

where 'u' is the 'eccentric anomaly' parameterizing the motion and the constants a,, e,, e_t, n and t_0 are some 2PN semi-major axis, radial eccentricity, time eccentricity, mean motion, and initial instant respectively. The angular motion $\phi_A(t_A)$ is given by

$$\phi_A - \phi_0 = \left(1 + \frac{k}{c^2}\right) v + \frac{f_\phi}{c^4} \sin 2v + \frac{g_\phi}{c^4} \sin 3v \,, \tag{3.4a}$$

$$\text{where } v = 2 \tan\text{-'} \left\{ \left(\frac{1 + e_\phi}{1 - e_\phi} \right)^{\frac{1}{2}} \tan(\frac{u}{2}) \right\} . \tag{3.4b}$$

In the above ϕ_0, k, e_ϕ are some constant, periastron precession constant, and angular eccentricity respectively. All the parameters n, k, a,, e_t, e,, e_ϕ, f_t, g_t, f_ϕ and g_ϕ are functions of the 2PN conserved energy and angular momentum per unit reduced mass μ, E and J. Their explicit functional forms, given in [41] are displayed below

$$a_r = -\frac{G m}{2 E} \left\{ 1 + \frac{1}{2 c^2}(7 - \eta) E + \frac{1}{c^4} \left[\frac{1}{4}(1 + 10\eta + \eta^2) E^2 + \frac{1}{2}(17 - 11\eta) \frac{E}{h^2} \right] \right\} , \tag{3.5a}$$

$$\begin{aligned} e_r^2 = {} & 1 + 2Eh^2 - \frac{1}{c^2}\left\{ 2(6 - \eta)E + 5(3 - \eta) E^2 h^2 \right\} + \frac{1}{c^4}\left\{ (26 + \eta + \eta^2) E^2 \right. \\ & \left. -2(17 - 11\eta) \frac{E}{h^2} + (80 - 55\eta + 4\eta^2) E^3 h^2 \right\} , \end{aligned} \tag{3.5b}$$

$$\begin{aligned} n = {} & \frac{(-2 E)^{\frac{3}{2}}}{Gm} \left\{ 1 + \frac{1}{4 c^2}(15 - \eta) E + \frac{1}{c^4} \left[\frac{1}{32}(555 + 30\eta + 11\eta) E^2 \right. \right. \\ & \left. \left. -\frac{3}{2}(5 - 2\eta) \frac{(-2E)^{\frac{3}{2}}}{h} \right] \right\} , \end{aligned} \tag{3.5c}$$

$$
\begin{aligned}
e_t^2 &= 1+2Eh^2+\frac{1}{c^2}\Big\{4(1-\eta)E+(17-7\eta)E^2h^2\Big\}+\frac{1}{c^4}\Big\{2(2+\eta+5\eta^2)E^2 \\
&\quad -(17-11\eta)\frac{E}{h^2}+(112-47\eta+16\eta^2)E^3h^2-3(5-2\eta)(1+2Eh^2)\frac{(-2E)^{\frac{3}{2}}}{h}\Big\}, \qquad (3.5d)
\end{aligned}
$$

$$
\begin{aligned}
f_t &= -\frac{1}{8\,h}\,\eta(4+\eta)(1+2Eh^2)^{\frac{1}{2}}(-2E)^{\frac{3}{2}}, && (3.5e)\\
g_t &= \frac{3}{2}(5-2\eta)\frac{(-2E)^{\frac{3}{2}}}{h}, && (3.5f)\\
k &= \frac{3}{h^2}\Big\{1+\frac{1}{2\,c^2}\Big[(5-2\eta)E+\frac{5}{2\,h^2}(7-2\eta)\Big]\Big\}, && (3.5g)\\
f_\phi &= \frac{1}{8}\frac{\eta}{h^4}(1-3\eta)(1+2E\,h^2), && (3.5h)\\
g_\phi &= -\frac{3}{32}\frac{\eta^2}{h^4}(1+2\,Eh^2)^{3/2}, && (3.5i)\\
e_\phi^2 &= 1+2Eh^2-\frac{1}{c^2}\Big\{12E+(15-\eta)E^2h^2\Big\}-\frac{1}{8\,c^4}\Big\{4(16-88\eta-9\eta^2)E^2 \\
&\quad -4(160-30\eta+3\eta^2)E^3h^2+(408-232\eta-15\eta^2)\frac{E}{h^2}\Big\}, && (3.5j)
\end{aligned}
$$

where $h=|\mathbf{J}|/(G\,\mathrm{m})$. Using these parametric equations of the motion, we compute $\dot r_A^2$, v_A^2 to the 2PN order in terms of $E, h^2, (1-$ e, cosu) using,

$$
\begin{aligned}
\frac{dt_A}{du} &= \frac{\partial t_A}{\partial u}+\frac{\partial t_A}{\partial v}\frac{dv}{du} && (3.6a)\\
\dot r_A^2 &= \left(\frac{dr_A}{du}/\frac{dt_A}{du}\right)^2 && (3.6b)\\
\dot\phi_A^2 &= \left(\frac{d\phi_A}{dv}\frac{dv}{du}/\frac{dt_A}{du}\right)^2 && (3.6c)\\
v_A^2 &= \dot r_A^2+r_A^2\dot\phi_A^2\,. && (3.6d)
\end{aligned}
$$

The subscript 'A ' present in Eqs.(3.6) is a reminder that the expressions refer to the ADM gauge. We have

$$
\begin{aligned}
\dot r_{\mathrm{A}}^2 &= \Big\{-1+\frac{2}{(1-e_r\cos u)}+\frac{2}{(1-\mathrm{e},\ \cos u)^2}E\,h^2\Big\}(-2E) \\
&\quad +\frac{1}{c^2}\Big\{-3+9\eta+\frac{1}{(1-\mathrm{e},\ \cos \mathrm{u})}[38-30\eta] \\
&\quad -\frac{}{(1-\mathrm{e},\ \cos u)^2}[40-20\eta-(36-28\eta)\ Eh^2] \\
&\quad -\frac{}{(1-e_r\cos u)^3}\Big[(64-24\eta)\ Eh^2\Big]\Big\}E^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{c^4}\left\{4-19\eta+16\eta^2-\frac{1}{(1-e_r\cos u)}\left[168-326\eta+98\eta^2-\frac{1}{Eh^2}(34-22\eta)\right]\right.\\
&+\frac{1}{(1-e_r\cos u)^2}\left[496-712\eta+164\eta^2-\left(213-298\eta+85\eta^2\right)Eh^2\right]\\
&-\frac{1}{(1-e_r\cos u)^3}\left[212-332\eta+80\eta^2-\left(800-932\eta+188\eta^2\right)Eh^2\right]\\
&-\frac{1}{(1-e_r\cos u)^4}\left[528-528\eta+96\eta^2\right]Eh^2\\
&+\frac{1}{(1-e_r\cos u)^5}\left[32+8\eta^2\right]\eta E^2h^4)(-E)^3, \qquad (3.7a)
\end{aligned}
$$

$$
\begin{aligned}
v_A^2 =& \left\{-1+\frac{2}{(1-e_r\cos u)}\right\}(-2E)\\
&-\frac{1}{c^2}\left\{3-97-\frac{1}{(1-e_r\cos u)}[38-30\eta]+\frac{1}{(1-e_r\cos u)^2}[40-20\eta]\right.\\
&\left.+8\frac{1}{(1-e_r\cos u)^3}\eta Eh^2\right\}E^2\\
&-\frac{1}{c^4}\left\{4-19\eta+16\eta^2-\frac{1}{(1-e_r\cos u)}\left[168-326\eta+98\eta^2\right.\right.\\
&\left.-\frac{1}{Eh^2}(34-22\eta)\right]+\frac{1}{(1-e_r\cos u)^2}\left[428-6687+164\eta^2\right]\\
&-\frac{1}{(1-e_r\cos u)^3}\left[212-332\eta+80\eta^2-(76-84\eta)\eta Eh^2\right]\\
&\left.-\frac{1}{(1-e_r\cos u)^4}[80-128\eta]\eta Eh^2+72\frac{1}{(1-e_r\cos u)^5}\eta^2E^2h^4\right\}(-E)^3. \qquad (3.7b)
\end{aligned}
$$

These expressions for $\dot{r}_A^2$ and v_A^2 are consistent with Eqs.(6) and (7) of [141].

3.3 The transformation between De-Donder (harmonic) and ADM gauges

As pointed out earlier, the far-zone fluxes obtained in the last chapter are in the harmonic coordinates, whereas, the 2PN accurate orbital description given by Eqs.(3.3), (3.4), and (3.5) are in the ADM coordinates. For the purpose of averaging the far-zone fluxes using the the 2PN accurate orbital representation, we need to go from the De-Donder(harmonic) to the ADM gauge, and rewrite the expressions for the far-zone fluxes in the ADM coordinates. These follow straightforwardly from the transformation equations in [142] and we list below the transformation equations, relating the harmonic(De-Donder) variables to the corresponding ADM variables:

$$\begin{aligned}
\mathbf{r}_{\rm D} &= \mathbf{r}_{\rm A} + \frac{Gm}{8\,c^4\,r}\left\{\left[\left(5v^2 - \dot{r}^2\right)\eta + 2\,(1 + 12\eta)\,\frac{Gm}{r}\right]\mathbf{r}\right. \\
&\quad \left. -18\,\eta\, r\dot{r}\,\mathbf{v}\right\}\,, && (3.8a) \\
t_{\rm D} &= t_{\rm A} - \frac{Gm}{c^4}\,\eta\,\dot{r}\,, && (3.8b) \\
\mathbf{v}_{\rm D} &= \mathbf{v}_{\rm A} - \frac{Gm\dot{r}}{8\,c^4\,r^2}\left\{\left[7v^2 + 38\,\frac{Gm}{r} - 3\dot{r}^2\right]\eta + 4\,\frac{Gm}{r}\right\}\mathbf{r} \\
&\quad -\frac{Gm}{8\,c^4\,r}\left\{\left[5v^2 - 9\dot{r} - 34\,\frac{Gm}{r}\right]\eta - 2\,\frac{Gm}{r}\right\}\mathbf{v}\,, && (3.8c) \\
(\mathbf{L}_N)_{\rm D} &= (\mathbf{L}_N)_{\rm A}\left\{1 + \frac{Gm}{4\,c^4\,r}\left[(2 + 29\eta)\,\frac{Gm}{r} + 4\,\eta\,\dot{r}^2\right]\right\}, && (3.8d) \\
r_{\rm D} &= r_{\rm A} + \frac{Gm}{8\,c^4}\left\{5\,\eta v^2 + 2\,(1 + 12\eta)\,\frac{Gm}{r} - 19\,\eta\dot{r}^2\right\}\,, && (3.8e) \\
v_{\rm D}^2 &= v_{\rm A}^2 - \frac{Gm}{4\,c^4\,r}\left\{\left[5v^4 - 2v^2\dot{r}^2 - 3\dot{r}^4\right]\eta\right. \\
&\quad \left. -\left[2\,(1 + 17\eta)\,v^2 - (4 + 38\eta)\,\dot{r}^2\right]\frac{Gm}{r}\right\}, && (3.8f) \\
\dot{r}_{\rm D}^2 &= \dot{r}_{\rm A}^2 - \frac{Gm}{2\,c^4\,r}\,\dot{r}^2\left\{15\left(v^2 - \dot{r}^2\right)\eta + (1 + 2\eta)\,\frac{Gm}{r}\right\}\,. && (3.8g)
\end{aligned}$$

The subscript 'D' denotes quantities in the De-Donder (harmonic) coordinates. Note that in all the above equations the differences between the two gauges are of the 2PN order. **As** there is no difference between the harmonic and the ADM coordinates to 1PN accuracy, in Eqs.(3.8), for the 2PN terms, no suffix is used. The 2PN extension of the evolution of the orbital elements thus requires more technical care than the 1PN case due to the differences in the ADM and harmonic coordinates given by Eqs.(3.8). Finally using the above equations we have verified that the expressions given by Eqs.(2.20), relating the individual locations of the two bodies to the centre of mass coordinate are consistent with the corresponding choice in ADM coordinates, given by Eqs.(A5) - (A8) of [42].

3.4 2PN corrections to $< d\mathcal{E}/dt >$ and $< d\mathcal{J}/dt >$

Starting from Eqs.(2.57) and (2.61) for the far-zone fluxes in the harmonic coordinates obtained in the previous chapter, we use Eqs.(3.8), to obtain $d\mathcal{E}/dt$ and $d\mathcal{J}/dt$ in the ADM coordinates. For economy of presentation, we write the results in the following manner, $(Flux)_{\rm A} = (Flux)_{\rm O}+$ 'Corrections', where $(Flux)_{\rm A}$ represent the far-zone flux in the ADM coordinates. $(Flux)_{\rm O}$ is a short hand notation for expressions on the r.h.s of Eqs.(2.57) and (2.61), where v^2, $\dot{r}$, r are the ADM variables $v_{\rm A}^2$, $\dot{r}_{\rm A}$, $r_{\rm A}$ respectively. For example, the Newtonian part of $(d\mathcal{E}/dt)_{\rm O}$ will be 15 c $r_{\rm A}$ 1 $- 11\dot{r}_{\rm A}^2\}$. The 'Corrections' represent the differences at the 2PN order, that arise due to the change of the coordinate system, given by Eqs.(3.8). As the two coordinates are different at the 2PN order, the 'Corrections' come only from the leading Newtonian terms in Eqs.(2.57) and (2.61).

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\rm A} = \left(\frac{d\mathcal{E}}{dt}\right)_{\rm O} - \frac{G^4 m^3 \mu^2}{15 c^9 r_{\rm A}^5}\Big\{\left[(48+336\eta)v_{\rm A}^2 - (36+232\eta)\dot{r}_{\rm A}^2\right]\frac{Gm}{r_{\rm A}} + \left[360 v_{\rm A}^4 - 1840 v_{\rm A}^2\dot{r}_{\rm A}^2 + 1424\dot{r}_{\rm A}^4\right]\eta\Big\}, \tag{3.9a}$$

$$\left(\frac{d\mathcal{J}}{dt}\right)_{\rm A} = \left(\frac{d\mathcal{J}}{dt}\right)_{\rm O} + \frac{G^3 m^2 \mu^2 (\tilde{\mathbf{L}}_N)_{\rm A}}{5 c^9 r_{\rm A}^4}\Big\{\left[(4+68\eta)v_{\rm A}^2 - (8+76\eta)\frac{Gm}{r_{\rm A}} + (2+82\eta)\dot{r}_{\rm A}^2\right]\frac{Gm}{r_{\rm A}} + (363 v_{\rm A}^2\dot{r}_{\rm A}^2 - 50 v_{\rm A}^4 - 363\dot{r}_{\rm A}^4)\,\eta\Big\}. \tag{3.9b}$$

Note that all the variables on the r.h.s of Eqs.(3.9) are in the ADM coordinates. In the circular limit, energy and angular momentum fluxes are again related as in Eqs.(2.67), via the corresponding 'v^2' in the ADM coordinates given by

$$v_{\rm A}^2 = \frac{Gm}{r_{\rm A}}\left\{1 - (3-\eta)\frac{Gm}{c^2 r_{\rm A}} + \frac{1}{8}(42 - 5\eta + 8\eta^2)\frac{G^2 m^2}{c^4 r_{\rm A}^2}\right\}. \tag{3.10}$$

From this point onwards, in this section, we work exclusively in the ADM gauge and hence we drop the subscript 'A' for the ease of presentation. We now have all the ingredients needed to calculate the 2PN corrections in $< d\mathcal{E}/dt >$ and $< d\mathcal{J}/dt >$. We explain in detail, the procedure to compute $< d\mathcal{E}/dt >$ and only display the final expression for $< d\mathcal{J}/dt >$, as the procedure is the same in both the cases. Starting from Eqs.(3.9), (2.57), and *(2.61)* which give the far-zone fluxes as functions of v^2, $\dot{r}^2$, and Gm/r, we use the *2PN* accurate orbital representation, to rewrite $d\mathcal{E}/dt$ as a polynomial in $(1 - e_r \cos u)^{-1}$. This polynomial is of the form

$$\frac{dl}{dt} = \frac{du}{ndt} \sum_{N=2}^{8} \frac{\alpha_N(E,h)}{(1 - e_r \cos u)^{(N+1)}}, \tag{3.11}$$

where for the convenience we have factored out du/ndt given by

$$\begin{aligned}\frac{du}{ndt} &= \frac{1}{(1 - e_r \cos u)}\Big\{1 - \frac{E}{c^2}(8 - 3\eta)\Big(1 - \frac{1}{(1 - e_r \cos u)}\Big) \\ &\quad + \frac{1}{2c^4}\Big[E^2\Big((56 - 63\eta + 6\eta^2) \\ &\quad - \frac{1}{Eh^2}(17 - 11\eta)(1 - \frac{1}{(1 - e_r \cos u)}) - \frac{1}{(1 - e_r \cos u)}(184 - 159\eta + 24\eta^2) \\ &\quad + \frac{1}{(1 - e_r \cos u)^2}(68 - 76\eta + 17\eta^2) - \frac{2Eh^2}{(1 - e_r \cos u)^3}\eta(4 + \eta)\Big) \\ &\quad + \frac{3}{h}(-2E)^{3/2}(5 - 2\eta)\Big]\Big\}. \end{aligned} \tag{3.12}$$

It is a straightforward algebra to show that the coefficients $\alpha_N(E,h)$ in Eq.(3.11) take the form

$$\alpha_{\rm N}(E,h) = \frac{\eta^2}{Gc^5}(-E)^5 \beta_{\rm N}(E,h), \tag{3.13}$$

where $\beta_{\rm N}(E,h)$ for $N = 1, 2, \ldots 8$ are given by

$$\begin{aligned}\beta_2 &= -\frac{256}{15} + \frac{1}{105c^2}(298\,24 - 154\,88\eta)E \\ &\quad + \frac{1}{c^4}\Big\{-\frac{1}{315}(791\,168 - 874\,624 + 179456\eta^2)E^2 \\ &\quad + \frac{128}{5}(17 - 11\eta)\frac{E}{h^2} + \frac{1}{5}(640 - 256\eta)\frac{(-2E)^{\frac{3}{2}}}{h}\Big\}, \end{aligned} \tag{3.14a}$$

$$\begin{aligned}\beta_3 &= \frac{512}{15} - \frac{1}{35c^2}(263\,68 - 19968\eta)E \\ &\quad + \frac{1}{c^4}\Big\{\Big[\frac{2716928}{315} - \frac{13040896}{945}\eta + \frac{538496}{135}\eta^2\Big]E^2 \\ &\quad - \frac{896}{15}(17 - 11\eta)\frac{E}{h^2} - \frac{1}{5}(1280 - 512\eta)\frac{(-2E)^{\frac{3}{2}}}{h}\Big\}, \end{aligned} \tag{3.14b}$$

$$\beta_4 = -\frac{5632}{15}Eh^2 + \frac{1}{c^2}\Big\{\frac{1}{7}(1024 - 3072\eta)E + \frac{512}{105}(1729 - 930\eta)\,E^2h^2\Big\}$$

$$
\begin{aligned}
& +\frac{1}{c^4}\left\{(\frac{46840064}{2835}+\frac{3537664}{945}\eta-\frac{2315648}{315}\eta^2)E^2\right. \\
& -\frac{128}{105}(86403-89968\eta+20923\eta^2)\,E^3h^2-\frac{256}{15}(17-11\eta)\frac{E}{h^2} \\
& \left.-\frac{1}{5}(7040-2816\eta)(-2E)^{\frac{5}{2}}h\right\}\,, \qquad (3.14c)
\end{aligned}
$$

$$
\begin{aligned}
\beta_5 &= -\frac{512}{105\,c^2}(3232-1395\eta)\,E^2\,h^2+\frac{1}{c^4}\left\{-\left[\frac{14200576}{2835}-\frac{38656}{189}\eta-\frac{219904}{63}\eta^2\right]E^2\right. \\
&\left.+\frac{256}{945}\left[148\,648\,8-1545569\eta+343813\eta^2\right]E^3h^2\right\}, \qquad (3.14d) \\
\beta_6 &= -\frac{512}{35\,c^2}(687-620\eta)E^3h^4-\frac{1}{c^4}\left\{\frac{256}{945}\left[1221526-1333624\eta+319739\eta^2\right]E^3h^2\right. \\
&\left.-\frac{512}{105}\left[51396-91541\eta+27508\eta^2\right]E^4h^4\right\}, \qquad (3.14e) \\
\beta_7 &= -\frac{512}{945\,c^4}\left\{748\,032-1385005\eta+387911\eta^2\right\}E^4h^4\,, \qquad (3.14f) \\
\beta_8 &= -\frac{4096}{315\,c^4}\left\{2501-202\,34\eta+8404\eta^2\right\}E^5h^6\,.
\end{aligned}
$$

$$(3.14g)$$

To the 1PN order Eqs.(3.14) agree with Eqs.(4.15) of [126]. The far-zone energy flux $(d\mathcal{E}/dt)$ is a periodic function of time with period $\mathbf{P}=2\pi/n$. Averaging $(d\mathcal{E}/dt)$, given by Eqs.(3.11), (3.13) and (3.14) over one time period P, we obtain

$$
<\frac{d\mathcal{E}}{\mathrm{dt}}>=\frac{1}{\mathbf{P}}\int_0^P\frac{d\mathcal{E}}{dt}\Big(t\Big)\,dt=\frac{1}{2\,\pi}\int_0^{2\,\pi}\left(\frac{ndt}{du}\right)\frac{d\mathcal{E}}{dt}\Big(u\Big)du\,. \qquad (3.15)
$$

The integrals in Eq.(3.15) are the Laplace second integrals for the Legendre polynomials [143] which yield,

$$
\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{\left\{1-e_r\cos u\right\}^{N+1}}=\frac{1}{(1-e_r^2)^{\frac{N+1}{2}}}P_{\mathrm{N}}\left(\frac{1}{\sqrt{(1-e_r^2)}}\right), \qquad (3.16)
$$

where P_{N} is Legendre polynomial. Using Eq.(3.16) in Eq.(3.15), we obtain $< d\mathcal{E}/dt >$ in terms of E and e,:

$$
\begin{aligned}
<\frac{d\mathcal{E}}{dt}> &= \frac{1024}{5}\frac{\mu\,\eta}{G\,m\,c^5}\frac{(-E)^5}{(1-e_r^2)^{\frac{7}{2}}}\left\{1+\frac{73}{24}e_r^2+\frac{37}{96}e_r^4\right. \\
&+\frac{1}{168}\frac{(-E)}{c^2\,(1-e_r^2)}\left[13-6414e_r^2-\frac{274\,05}{4}e_r^4-\frac{537\,7}{16}e_r^6\right. \\
&\left.-\left(840+\frac{6419}{2}e_r^2+\frac{5103}{8}e_r^4-\frac{259}{8}e_r^6\right)\,\eta\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(-E)^2}{c^4}\Bigg[\frac{1}{16\,(1-e_r^2)^{\frac{5}{2}}}\Big((480-192\eta)+(500-200\eta)e_r^2 \\
&-(2255-902\eta)e_r^4+(1090-436\eta)e_r^6+(185-74\eta)e_r^8\Big) \\
&-\frac{1}{(1-e_r^2)^2}\Big(\frac{253\,937}{4536}-\frac{180\,65}{504}\eta+10\eta^2 \\
&-\Big(\frac{879\,749}{4536}-\frac{301\,37}{72}\eta-\frac{1877}{48}\eta^2\Big)\,e_r^2 \\
&-\Big(\frac{513337}{6048}-\frac{531871}{672}\eta+\frac{1139}{192}\eta^2\Big)\,e_r^4 \\
&+\Big(\frac{2494795}{8064}+\frac{4823}{128}\eta-\frac{383}{96}\eta^2\Big)\,e_r^6 \\
&+\Big(\frac{283685}{16128}-\frac{13147}{2688}\eta+\frac{37}{192}\eta^2\Big)\,e_r^8\Big)\Bigg]\Bigg\}
\end{aligned}
\tag{3.17}
$$

Following exactly a similar procedure, we obtain the 2PN correction to $< d\mathcal{J}/dt >$. The final result we obtain is:

$$
\begin{aligned}
< \frac{d\mathcal{J}}{dt} > \;=\;& \frac{4}{5}\,\frac{\mu\,\eta}{c^5}\,\frac{(-2E)^{\frac{7}{2}}}{(1-e_r^2)^3}\Bigg\{8-e_r^2-7e_r^4 \\
&-\frac{(-E)}{168\,c^2}\Big[(2920+7056\eta)+(19738+14434\,\eta)\,e_r^2+(127+1330\,\eta)\,\mathrm{ef}\Big] \\
&-\frac{(-E)^2}{c^4}\Bigg[\frac{1}{(1-e_r^2)^{\frac{1}{2}}}\Big(240-96\eta-(30-12\eta)e_r^2-(210-84\eta)e_r^4\Big) \\
&-\frac{1}{(1-e_r^2)}\Big(\frac{299623}{1134}-\frac{22025}{252}\sim+\frac{351}{\mathrm{T}}\;0 \\
&-\Big(\frac{131\,6273}{864}-\frac{815\,597}{336}\eta-\frac{292\,07}{96}\eta^2\Big)\,e_r^2 \\
&-\Big(\frac{290\,1133}{6048}-\frac{124403}{96}\eta-\frac{7187}{48}\eta^2\Big)\,e_r^4 \\
&+\Big(\frac{7526}{63}-\frac{2869}{224}\eta+\frac{143\,5}{96}\eta^2\Big)\,e_r^6\Big)\Bigg]\Bigg\}.
\end{aligned}
\tag{3.18}
$$

To the 1PN order, Eqs.(3.17) and (3.18) agree with [126, 127] as required. For the special case of circular orbits, $e_r = 0$ and we observe that, $< d\mathcal{E}/dt > = w < d\mathcal{J}/dt >$ to the 2PN order, where w, the mean angular frequency of the relative motion, defined by $w = n(1+k)$ is given by

$$
\omega = \frac{(-2\,E)^{\frac{3}{2}}}{G\,m}\left\{1-\frac{1}{4\,c^2}(9+\eta)E+\frac{1}{32\,c^4}(2811-1170\eta+11\eta^2)E^2\right\}\,. \tag{3.19}
$$

It is not very difficult to trace the origin of the two types of terms in Eqs.(3.17) and (3.18) at the 2PN order. It is related to the fact that 'Corrections' in Eqs.(3.9), arising from the transformation equations connecting the harmonic and the ADM coordinates have a different functional form than the 2PN contributions to the corresponding far-zone fluxes in the harmonic coordinates. For example, in the far-zone energy flux, 'Corrections' contain a common factor ($G^4 m^3/r^5$), unlike the 2PN contributions in harmonic coordinates which have only ($G^3 m^2/r^4$) as the common factor (*c.f* Eqs.(2.57) and (3.9)). These different functional forms, after the averaging procedure give rise to the two different types of terms in Eqs.(3.17) and (3.18).

We display below $< d\mathcal{E}/dt >$ and $< d\mathcal{J}/dt >$ in terms of $G\,m/a_r$ and e,, which can easily be obtained from Eqs.(3.17) and (3.18), using E written in terms of G m/a_r and e_r to the 2PN order. The required equation for E is obtained from Eqs.(3.5) for a, and e_r by inverting them for E and h^2 respectively, order by order. Eliminating h^2 from the expression for E we finally get,

$$E = -\frac{c^2}{2}\zeta\left\{1-\frac{1}{4}(7-\eta)\zeta+\frac{1}{8}\left[(25-2\eta+\eta^2)-2\,\frac{(17-11\eta)}{(1-e_r^2)}\right]\zeta^2\right\}, \quad (3.20)$$

where $\zeta = G\,m/c^2 a_r$. Using the above expression for E, Eq.(3.17) becomes

$$\begin{aligned}
< \frac{d\mathcal{E}}{dt} > \;=\; & \frac{1}{15}\frac{c^5}{G}\eta^2\frac{\zeta^5}{(1-e_r^2)^{\frac{13}{2}}}\Big\{\Big[(96+292e_r^2+37e_r^4)(1-e_r^2)^3\Big] \\
& -\frac{1}{56}\zeta(1-e_r^2)^2\Big[(468\,32+672\,0\eta)+(198\,664+376\,32\eta)e_r^2 \\
& -(153\,30-280\,56\eta)e_r^4-(127\,53-207\,2\eta)e_r^6\Big] \\
& +\zeta^2\Big[\frac{1}{6048}(1-e_r^2)\Big((224\,053\,12+122\,492\,16\eta) \\
& +(912\,416\,00+973\,409\,76\eta+290\,304\eta^2)e_r^2 \\
& -(977\,677\,44-731\,619\,00\eta-239\,500\,8\eta^2)e_r^4 \\
& -(757\,105\,2+606\,592\,8\eta-280\,627\,2\eta^2)e_r^6 \\
& +(680\,528\,7-148\,921\,2\eta+223\,776\eta^2)e_r^8\Big)
\end{aligned}$$

$$-\frac{3}{2}(1-e_r^2)^{\frac{5}{2}}\Big((96+292e_r^2+37e_r^4)(5-2\eta)\Big)\Big]\Big\}\,, \tag{3.21}$$

while Eq.*(3.18)* gets transformed to,

$$\begin{aligned}
<\frac{d\mathcal{J}}{dt}> \;=\;& \frac{4}{5}\mu\,\eta c^2\frac{\zeta^{\frac{7}{2}}}{(1-e_r^2)^4}\Big\{(8+7e_r^2)(1-e_r^2)^2\\
&-\frac{1}{336}\zeta(1-e_r^2)\Big[(193\,84+470\,4\eta)+(176\,80+147\,28\eta)e_r^2\\
&-(142\,79-338\,8\eta)e_r^4\Big]\\
&+\zeta^2\Big[\frac{1}{181\,44}\Big((381\,349\,6+314\,114\,4\eta+725\,76\eta^2)\\
&-(346\,264\,8-137\,197\,26\eta-815\,724\eta^2)e_r^2\\
&-(112\,754\,91-786\,483\eta-139\,784\,4\eta^2)e_r^4\\
&+(357\,872\,4-121\,329\,9\eta+238\,896\eta^2)e_r^6\Big)\\
&-\frac{3}{2}(1-e_r^2)^{\frac{3}{2}}(5-2\eta)\,(8+7e_r^2)\Big]\Big\}\,.
\end{aligned} \tag{3.22}$$

3.5 The evolution of the orbital elements

In this section, we compute the *2PN* corrections to the evolution of orbital elements due to the emission of gravitational radiation. We describe the procedure to compute the rate of decrease of the orbital period of the binary in some detail and display the final expressions for the rate of decay of other elements namely, $< da_r/dt >$ and $< de_r/dt >$. Employing the heuristic argument, based on the energy and the angular momentum conservation to the *2PN* order, the rate of decrease of the orbital period, $\mathbf{P}$ of the two compact objects moving in quasi-elliptical orbits is computed. The *2PN* accurate orbital period, $\mathbf{P}=2\pi/n$ given in [40, 41, 42] reads as

$$P=\frac{2\,\pi\,G\,m}{(-2E)^{\frac{3}{2}}}\Big\{1-\frac{1}{4c^2}(15-\eta)E-\frac{3}{32\,c^4}\Big[(35+30\eta+3\eta^2)E^2-16\,(5-2\eta)\frac{(-2E)^{\frac{3}{2}}}{h}\Big]\Big\}\,. \tag{3.23}$$

Differentiating Eq.(3.23) with respect to t and equating dE/dt to $(-<d\mathcal{E}/dt>/\mu)$ and dh/dt to $(-<d\mathcal{J}/dt>/Gm\mu)$ we find

$$
\begin{aligned}
\dot{P} &= \frac{6\,\pi\,G\,m}{(-2E)^{\frac{5}{2}}}\left\{1-\frac{1}{12\,c^2}(15-\eta)E+\frac{1}{32\,c^4}(35+30\eta+3\eta^2)E^2\right\} < \frac{d\mathcal{E}}{dt} > \\
&\quad -\frac{3\,\pi}{c^4\,h^2}(5-2\eta) < \frac{d\mathcal{J}}{dt} > \;. \qquad (3.24)
\end{aligned}
$$

Note that, in the above equation we need $< d\mathcal{J}/dt >$ to only the Newtonian accuracy. Using in Eq.(3.24), $< d\mathcal{E}/dt >$ given by Eq.(3.17) and the Newtonian part of Eq.(3.18) for $< d\mathcal{J}/dt >$, we get

$$
\begin{aligned}
\dot{P} &= -\frac{192}{5}\,\pi\,\eta\,\frac{\zeta^{\frac{5}{2}}}{(1-e_r^2)^{\frac{7}{2}}}\left\{1+\frac{73}{24}e_r^2+\frac{37}{96}e_r^4\right. \\
&\quad -\frac{1}{161\,28}\zeta\frac{1}{(1-\mathrm{ef})}\Big[(598\,56+309\,12\eta)+(431352+134\,848\eta)e_r^2 \\
&\quad +(168\,210+556\,08\eta)e_r^4-(717\,9-207\,2\eta)e_r^6\Big] \\
&\quad +\zeta^2\frac{1}{(1-e_r^2)^2}\left[\frac{1}{580\,608}\Big((763\,955\,2+607\,737\,6\,\eta+483\,840\,\eta^2)\right. \\
&\quad +(263\,832\,80+814\,273\,20\eta+251\,596\,8\eta^2)e_r^2 \\
&\quad -(190\,546\,44-825\,636\,06\eta-170\,553\,6\eta^2)e_r^4 \\
&\quad -(145\,177\,2-532\,202\,4\eta-935\,424\eta^2)e_r^6 \\
&\quad +(159\,698\,7-193\,374\eta+745\,92\,\eta^2)e_r^8\Big) \\
&\quad \left.\left.-\frac{1}{64}(5-2\eta)\,(1-e_r^2)^{\frac{3}{2}}\left(64+296\,e_r^2+65\,e_r^4\right)\right]\right\}. \qquad (3.25)
\end{aligned}
$$

Finally, inserting the expressions for e_r^2 and $G\,m/a_r$ in terms of E and $\boldsymbol{h}^2$ in Eq.(3.25) we obtain

$$
\begin{aligned}
\dot{P} &= -\frac{\pi\,\eta}{5\,c^5}\,\frac{1}{(-E)h^7}\left\{425+732\,Eh^2+148E^2h^4\right. \\
&\quad +\frac{1}{c^2\,h^2}\left[\frac{403\,41}{8}+\frac{381\,35}{4}\,Eh^2+\frac{722\,37}{14}\,E^2h^4\right. \\
&\quad \left.+\frac{498\,3}{7}E^3h^6-\left(\frac{5635}{2}+\frac{481\,25}{6}\,Eh^2+535\,4\,E^2h^4+\frac{140\,6}{3}\,E^3h^6\right)\eta\right] \\
&\quad +\frac{1}{c^4}\left[\frac{1}{672}\left(291\,982\,55-309\,096\,90\eta+690\,606\,0\eta^2\right)\frac{1}{h^4}\right. \\
&\quad +\frac{1}{432}\left(293\,418\,53-505\,570\,59\eta+187\,777\,80\eta^2\right)\frac{E}{h^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\,(637\,5 - 255\,0\eta)\,\frac{(-2E)^{\frac{3}{2}}}{h} \\
&+\frac{1}{252}\left(864\,965\,0 - 219\,467\,70\,\eta + 137\,502\,75\,\eta^2\right)\,E^2 \\
&-(319\,5 - 127\,8\eta)\,(-2E)^{\frac{5}{2}}h \\
&+\frac{1}{84}\left(166\,451\,5 - 206\,289\,3\eta + 171\,217\,2\eta^2\right)\,E^3h^2 \\
&+\frac{1}{2}\,(975 - 390\eta)\,\,(-2E)^{\frac{7}{2}}h^3 \\
&+\frac{1}{42}\left(163\,085 - 693\,68\eta + 445\,48\eta^2\right)\,E^4h^4\Big]\Big\}\,. \qquad (3.26)
\end{aligned}
$$

In the expression above, $\dot{P}$ is given as a function of the masses and of the 2PN-conserved energy and angular momentum. This expression for $\dot{P}$ is independent of the coordinate system used to derive it. Since **P** is a measurable quantity, one would have liked to express $\dot{P}$ in terms of other directly observable parameters like the orbital period and some convenient eccentricity as in the 1PN case [126]. However at present, to 2PN accuracy we do not have any such suitable and convenient choice and therefore we leave the expression for $\dot{P}$ in terms of the 2PN accurate E and h^2.

Similarly, using the definition of a_r and e_r in terms of E and h^2 and following the method described above, we obtain after a rather long but straightforward calculation

$$
\begin{aligned}
<\frac{da_r}{dt}> \;=\; & -\frac{2}{15}\,\eta\,c\frac{\zeta^3}{(1-e_r^2)^{\frac{11}{2}}}\Big\{(1-e_r^2)^2\left(96 + 292e_r^2 + 37e_r^4\right) \\
& -\frac{1}{56}\zeta(1-e_r^2)\Big[(280\,16 + 940\,8\eta) + (160\,248 + 431\,20\eta)e_r^2 + \\
& (346\,50 + 209\,16\eta)e_r^4 - (550\,1 - 103\,6\,\eta)e_r^6\Big] \\
& +\zeta^2\frac{1}{(1-e_r^2)^{\frac{11}{2}}}\Big[\frac{1}{604\,8}\Big((137\,748\,16 + 585\,129\,6\eta + 290\,304\eta^2) \\
& +(428\,878\,40 + 874\,684\,80\eta + 188\,395\,2\eta^2)e_r^2 \\
& -(396\,797\,28 - 824\,068\,08\eta - 221\,886\,0\eta^2)e_r^4 \\
& -(449\,753\,4 - 103\,086\eta - 123\,832\,8\eta^2)e_r^6 \\
& +(262\,800\,9 - 632\,718\eta + 839\,16\eta^2)e_r^8\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{3}{2}(1-e_r^2)^{\frac{3}{2}}\Big((5-2\eta)(96+292\,e_r^2+37\,e_r^4)\Big)\Big]\Big\}\,, && (3.27)\\
< \frac{de_r}{dt} > \;=\; & -\frac{1}{15}\,\frac{c^3}{G}\,\frac{\eta}{m}\,\frac{\zeta^4\,e_r}{(1-e_r^2)^{\frac{9}{2}}}\Big\{(304+121e_r^2)(1-e_r^2)^2 \\
& -\frac{1}{56}\zeta(1-e_r^2)\Big[(133\,640+374\,08\eta)+(108\,984+336\,84\eta)e_r^2 \\
& -(252\,11-338\,8\eta)e_r^4\Big] \\
& +\zeta^2\Big[\frac{1}{201\,6}\Big((174\,096\,16+170\,583\,84\eta+491\,904\eta^2) \\
& -(120\,536\,4-397\,143\,72\eta-760\,788\eta^2)e_r^2 \\
& -(150\,068\,86-224\,584\,2\eta-560\,952\eta^2)e_r^4 \\
& +(384\,043\,5-619\,614\eta+914\,76\eta^2)e_r^6\Big) \\
& -\frac{3}{2}(1-e_r^2)^6\,(304+121e_r^2)\,(5-2\eta)\Big]\Big\}\,. && (3.28)
\end{aligned}
$$

To 1PN accuracy we recover the results of [127].

3.6 Limits

We observe that in the test particle limit ($\eta \to 0$) and for small radial eccentricities, Eqs.(3.21) and (3.22) become

$$
\begin{aligned}
< \frac{d\mathcal{E}}{dt} >_{\eta=0} \;=\; & \frac{32}{5}\,\frac{c^5}{G}\,\frac{\mu^2}{m^2}\,\zeta^5\Big\{1-\frac{292\,7}{336}\zeta+\frac{282\,043}{907\,2}\zeta^2 \\
& +\Big[\frac{157}{24}-\frac{639\,7}{84}\zeta+\frac{273\,523}{864}\zeta^2\Big]e_r^2\Big\}\,, && (3.29a)\\
< \frac{d\mathcal{J}}{dt} >_{\eta=0} \;=\; & \frac{32}{5}\,\frac{\mu^2}{m}\,c^2\,\zeta^{\frac{7}{2}}\Big\{1-\frac{2423}{336}\,\zeta+\frac{340\,607}{181\,44}\,\zeta^2 \\
& +\Big[\frac{23}{8}-\frac{947\,9}{336}\,\zeta+\frac{101\,464\,7}{181\,44}\,\zeta^2\Big]e_r^2\Big\}\,. && (3.29b)
\end{aligned}
$$

Such expressions for average energy and angular momentum fluxes for a test particle moving in a slightly eccentric orbit around a Schwarzschild black hole have been obtained by Tagoshi [61], using the black hole perturbation methods: Eqs.(4.9) and (4.12) of [61] (with $q=0$). They are given by

$$
< \frac{d\mathcal{E}}{dt} > \;=\; \frac{32}{5}\,\frac{\mu^2}{G\,m^2\,c^5}\,v^{10}\Big\{1-\frac{124\,7}{336}\,\frac{v^2}{c^2}-\frac{447\,11}{907\,2}\,\frac{v^4}{c^4}
$$

$$+\left[\frac{37}{24}-\frac{65}{21}\frac{v^2}{c^2}-\frac{465\,337}{907\,2}\frac{v^4}{c^4}\right]e^2\Big\}\,, \tag{3.30a}$$

$$<\frac{d\mathcal{J}}{dt}> \;=\; \frac{32}{5}\frac{\mu^2}{m\,c^5}v^7\Big\{1-\frac{124\,7}{336}\frac{v^2}{c^2}-\frac{447\,11}{907\,2}\frac{v^4}{c^4}$$

$$+\left[-\frac{5}{8}+\frac{749}{96}\frac{v^2}{c^2}-\frac{232\,181}{604\,8}\frac{v^4}{c^4}\right]e^2\Big\}\,, \tag{3.30b}$$

where v and e refer to the radial velocity and the eccentricity in Schwarzschild coordinates. Eqs.(3.29) and (3.30) are consistent, if the ADM variables a, and e, are related to the Schwarzschild variables v and e by

$$\frac{\mathrm{Gm}}{a_r} \;=\; v^2\left\{1+\frac{v^2}{c^2}+\frac{5}{4}\frac{v^4}{c^4}-\left[1+\text{[illegible]}-\frac{9}{2}\frac{v^4}{c^4}\right]e^{2)}\right.\,, \tag{3.31a}$$

$$e_r^2 \;=\; e^2\left\{1+2\frac{v^2}{c^2}+4\frac{v^4}{c^4}\right\}. \tag{3.31b}$$

As stressed by Tagoshi, the fluxes reveal the more familiar coefficients in terms of a parameter v', related to the angular frequency in the ϕ coordinates rather than v, which is adapted to the radial coordinate r. For slightly eccentric orbits, v and v' are related by

$$v = v'\left\{1+\frac{1}{2}\left[1-3\frac{v'^2}{c^2}-12\frac{v'^4}{c^4}\right]e^2\right\}. \tag{3.32}$$

In terms of v' the far-zone fluxes for a test particle in Schwarzschild geometry, Eqs.(3.30) may be written as

$$<\frac{d\mathcal{E}}{dt}> \;=\; \frac{32}{5}\frac{\mu^2}{\mathrm{Gm}^2\mathrm{c}^5}v'^{10}\Big\{1-\frac{124\,7}{336}\frac{v'^2}{c^2}-\frac{447\,11}{9072}\frac{v'^4}{\mathrm{c}^4}$$

$$+e^2\left[\frac{157}{24}-\frac{678\,1}{168}\frac{v'^2}{\mathrm{c}^2}-\frac{149\,29}{189}\frac{v'^4}{c^4}\right]\Big\}\,, \tag{3.33a}$$

$$<\frac{d\mathcal{J}}{dt}> \;=\; \frac{32}{5}\frac{\mu^2}{m\,c^5}v'^7\Big\{1-\frac{124\,7}{336}\frac{v'^2}{\mathrm{c}^2}-\frac{447\,11}{9072}\frac{v'^4}{\mathrm{c}^4}$$

$$+e^2\left[\frac{23}{8}-\frac{325\,9}{168}\frac{v'^2}{\mathrm{c}^2}-\frac{104\,134\,9}{18144}\frac{v'^4}{c^4}\right]\Big\}. \tag{3.33b}$$

In this form at the Newtonian order, one recovers the results of Peters and Mathews [144]. The quantities a, and e, in the ADM coordinates are related to v' and e by the following relations

$$\frac{Gm}{a_r} \;=\; v'^2\left\{1+\frac{1}{c^2}(1-2e^2)\,v'^2+\frac{1}{4\,c^4}\left(5-39e^2\right)\,v'^4\right\}, \tag{3.34a}$$

$$e_r^2 = e^2\left\{1 + 2\,\frac{v'^2}{c^2} + 4\frac{v'^4}{c^4}\right\} \tag{3.34b}$$

The above relations may be rewritten, in terms of the conserved energy E using [145]

$$v^2 = -2E\left\{1 + e^2 - \frac{E}{2\,c^2}\left(3 - e^2\right) + \frac{E^2}{c^4}\left(18 + 4\,e^2\right)\right\}, \tag{3.35a}$$
$$v'^2 = -2\,E\left\{1 - \frac{E}{2\,c^2}\left(3 + 8\,e^2\right) + \frac{E^2}{c^4}\left(18 + 52\,e^2\right)\right\}. \tag{3.35b}$$

We obtain

$$\frac{G\,m}{a_r} = v'^2\left\{1 - \frac{E}{c^2}(2 - 4e^2) + \frac{E}{c^4}\,(8 - 37e^2)\right\}\ , \tag{3.36a}$$
$$e_r^2 = e^2\left\{1 - 4\frac{E}{c^2} + 22\frac{E^2}{c^4}\right\}, \tag{3.36b}$$

which are the generalizations of similar 1PN relations in [127]. For the special case of circular orbits Eq.(3.27) for $< da_r/dt >$, takes the simple form

$$< \frac{da_r}{dt} >= -\frac{64}{5}\,\zeta^3\,\eta\,c\left\{1 - \zeta\left[\frac{1751}{336} + \frac{7}{4}\eta\right] + \zeta^2\left[\frac{294\,383}{181\,44} + \frac{263\,65}{201\,6}\eta + \frac{1}{2}\eta^2\right]\right\}. \tag{3.37}$$

Eq.(3.37) is consistent with the expression for $\dot{r}$ given in [45], after taking due account of the coordinate transformations required to relate the ADM and the harmonic gauges for the circular orbits.

3.7 Conclusions

In this chapter employing the 2PN accurate expressions for the instantaneous far-zone energy and angular momentum fluxes for general orbits and the 2PN accurate generalized quasi-Keplerian representation for elliptic orbits, we have computed the instantaneous 2PN contributions to $< d\mathcal{E}/dt >$ and $< d\mathcal{J}/dt >$, the far-zone fluxes averaged over one orbital timescale in the ADM coordinates. Using the averaged far-zone fluxes and a heuristic argument based on energy and angular momentum balance to the 2PN order, we compute the evolution of the orbital elements of the generalized quasi-Keplerian representation, in particular the 2PN contributions to $\dot{P}$, $\dot{e}_r$ and a',. It should be noted that in a similar manner, it is possible to obtain the orbital evolution for all the other parameters of the generalized quasi-Keplerian representation. The method employed to compute $< d\mathcal{E}/dt >$ and $< d\mathcal{J}/dt >$ could also be adapted to the case of hyperbolic orbits to generalize the work of Simone, Poisson and Will on the head-on collision [139].

As mentioned earlier, Blanchet and Schafer have obtained the 1PN and the 1.5PN corrections to $\dot{P}$, the rate of decay of the orbital period $\mathbf{P}$ [126, 128]. They have shown that for the binary pulsar PSR 1913+16, the relative 1PN and 1.5PN corrections are numerically equal to $+2.15 \times 10^{-5}$ and $+1.65 \times 10^{-7}$ respectively. These are unfortunately far below the present accuracy in the measurements of $\mathbf{P}$ for 1913+16. Therefore 2PN corrections to the decay of the orbital elements of the binary may not be useful for the timing observations of the known relativistic binary pulsars. However these expressions may be useful for the construction of 'ready to use' search templates needed to detect gravitational radiation from inspiraling compact binaries in quasi-elliptical orbits. The construction of 'ready to use' search templates (waveforms) for gravitational radiation from inspiraling binaries can be decomposed into two different parts. These two distinct parts are referred as the 'wave generation problem' and the 'radiation reaction problem' [2]. The wave generation problem deals with the computation of the gravitational wave polarizations at the leading order in $1/R$, when the orbital phase and other parameters of the binary orbit take some specific values. The radiation reaction problem consists in determining the evolution of the orbital phase and other orbital elements as a function of time under the effects of gravitational radiation reaction forces. The expressions derived in this chapter will be useful for the evolution of the orbital elements in the 'ready to use' search waveforms. In the next chapter, we will tackle the 'wave generation problem' and will obtain all the instantaneous 2PN corrections to the 'plus' and 'cross' gravitational wave polarizations for inspiraling binaries of arbitrary mass ratio, moving in elliptical orbits.