Chapter 5

Second post-Newtonian gravitational radiation reaction for two-body systems

5.1 Introduction

As discussed in earlier chapters, the detection of gravitational waves from inspiraling compact binaries require extremely high phasing accuracy. This implies very accurate description of the evolution of the binary orbit. The orbital evolution, in reality has to be determined by local equations of motion that include damping terms due to the emission of gravitational radiation to infinity. Obtaining from first principles, approximate solutions of Einstein equations that incorporate into the "near-zone" gravitational fields, the back reaction from radiation to infinity, is a complicated and subtle exercise, involving near-zone iteration of Einstein's field equations upto the order required. In the absence of this difficult computation that provides the complete solution, a related question that provides a partial answer is the following. To what extent do the expressions of energy and angular momentum fluxes and the assumption of energy and angular momentum balance constrain the equations of motion? As discussed in detail in the introduction such heuristic methods based on the balance procedures have proved useful in earlier instances. For instance, as in [105] one can derive the the radiation reaction terms in the equations of motion sufficient to balance the energy radiated to the far-zone assuming 'energy balance'. Schematically, the method is based on the following arguments. The far-zone energy flux at the Newtonian order reads $\left(\frac{dE}{dt}\right) = \mathrm{E} \sim \left(I_{ij}^{(3)}\right)^2$. Integrating twice by parts and moving the total time derivatives to the left hand side for a redefinition of E, one obtains $\mathrm{E} \sim \dot{I}_{ij}\, I_{ij}^{(5)} \sim m\, v^i \left(x^j\, I_{ij}^{(5)}\right)$. As $\mathrm{E} \sim m\,\mathbf{v}.\mathbf{a}$, one can read off the appropriate radiation reaction terms in the acceleration a, related to the fifth time derivative of the quadrupole moment. Recently Iyer and Will (IW) proposed a 'refined balance procedure' to obtain the reactive terms in the equations of motion for binaries in general orbits. This 'refined balance procedure' which is extendable to higher PN orders, depends on the balance of both energy and angular momentum [33, 34]. The procedure leads to general expressions for the radiation reaction acceleration and the part in the reactive acceleration which is not fixed by the procedure corresponds to a residual gauge freedom inherent in the method. They also obtained, for the first time 3.5 PN terms (1PN radiation reaction) in the equations of motion of a binary using the 1PN accurate radiation reaction tensor potential obtained by Blanchet [31, 32]. The consistency of the above result with the reactive acceleration to 1PN obtained using the refined balance procedure was also established in [34]. This provided a valuable non-trivial check on the validity of the 1PN reaction potentials and the overall consistency of the direct methods based on iteration of the near-field equations and heuristic methods based on energy and angular momentum balance.

In this chapter we deduce the gravitational radiation reaction to 2PN order beyond the quadrupole approximation – 4.5PN terms in the equation of motion – using the refined balance method of Iyer and Will. We employ for our calculation the instantaneous 2PN accurate expressions for the energy and angular momentum fluxes obtained in chapter 2. We explore critically the features of their construction and illustrate them by contrast to other possible variants. As in the earlier orders, there exist arbitrary terms in the 4.5PN reactive terms too, which along the lines of [33, 34], are shown to be associated with the possible residual 'gauge' choice at the 4.5PN order. The limiting cases of circular orbits and radial infall are also discussed. The equations of motion are valid for general binary orbits and for a class of coordinate gauges. We also show that the far-zone flux formulae and the balance equations admit more general solutions than the one explored by Iyer and Will if one relaxes the requirement that the reactive acceleration be a power series in the individual masses of the binary or, equivalently, that it be nonlinear in the total mass. Most of the results presented in this chapter have been published in Ref. [45].

To summarize: Starting from 2PN accurate energy and angular momentum fluxes for compact binaries of arbitrary mass ratio moving in quasi-general orbits [44, 43], we obtain the 4.5PN reactive terms in the equations of motion by an extension of the IW method. Schematically, the equations of motion for spinless bodies of arbitrary mass ratio are

$$\mathbf{a} \equiv \frac{d^2\mathbf{x}}{\mathrm{d}t^2} \approx -\frac{\mathrm{G}\,m\mathbf{x}}{r^3}[1+O(\epsilon)+O(\epsilon^2)+O(\epsilon^{2.5})+O(\epsilon^3)+O(\epsilon^{3.5})+O(\epsilon^4)+O(\epsilon^{4.5})+\ldots], \tag{5.1}$$

where x and $r = |\mathbf{x}|$ denote the separation vector and distance between the bodies, and m $= m_1 + m_2$ denotes the total mass. The quantity ϵ is a small expansion parameter that satisfies $\epsilon \sim (v/c)^2 \sim \mathrm{G}m/(rc^2)$, where v and r are the orbital velocity and separation of the binary system. The symbols $O(\epsilon)$ and $O(\epsilon^2)$ represent post-Newtonian (PN), post-post-Newtonian (2PN) corrections and so on. Gravitational radiation reaction first appears at $O(\epsilon^{2.5})$ beyond Newtonian gravitation, or at 2.5PN order. We call this the "Newtonian" radiation reaction. "Post-Newtonian" radiation reaction terms, at $O(\epsilon^{3.5})$, were obtained by Iyer and Will [33, 34] and Blanchet [31, 32]. Here we obtain the 2PN radiation reaction, at $O(\epsilon^{4.5})$.

In the present chapter, for the ease of presentation we will work with the geometrical units: G = c = 1. This chapter is organized as follow. In section 5.2, we describe the Iyer-Will (IW) method to obtain the 2PN reactive terms. Section 5.3

examines the question of redundant equations and explores 'variants' of the original IW scheme that differ in their choice of the ambiguities in energy and angular momentum. Section 5.4 discusses the question of the undetermined parameters and arbitrariness in the choice of the gauge, in particular at 4.5PN order. Section 5.5 is devoted to the particular cases of quasi-circular orbits and head-on infall. In Section 5.6 for mathematical completeness, we prove that the far-zone flux formulae and the balance equations admit more general solutions if one relaxes the requirement that the reactive acceleration be a power series in individual masses m_1 and m_2. Section 5.7 contains some concluding remarks.

5.2 The IW method for reactive terms in the equations of motion

5.2.1 The Procedure

We consider only two-body systems containing objects which are sufficiently small that finite-size effects, such as spin-orbit, spin-spin, or tidal interactions can be ignored. The dynamics of such systems is well studied and the two-body equations of motion – conveniently cast into a relative one-body equation of motion – is given by :

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}^{(1)}_{\rm PN} + \mathbf{a}^{(2)}_{\rm 2PN} + \mathbf{a}^{(2.5)}_{\rm RR} + \mathbf{a}^{(3)}_{\rm 3PN} + \mathbf{a}^{(3.5)}_{\rm 1RR} + \mathbf{a}^{(4)}_{\rm 4PN} + \mathbf{a}^{(4)}_{\rm Tail} + +\mathbf{a}^{(4.5)}_{\rm 2RR} + O(\epsilon^5)\,, \quad (5.2)$$

where the subscripts denote the nature of the term, post-Newtonian (PN), post-post-Newtonian (2PN), Newtonian radiation reaction (RR), post-Newtonian radiation reaction (1RR), 2PN radiation reaction (2RR), tail radiation reaction and so on; and the superscripts denote the order in ϵ. For our purpose we need to know explicitly the acceleration terms through 2PN order and these are available in [104, 38, 130]. We have already displayed them as Eqs.(2.23) and reproduce these here with $G = c = 1$,

$$
\begin{aligned}
\mathbf{a}_N = \quad & -\frac{m}{r^2}\mathbf{n}\,, && (5.3a)\\
\mathbf{a}^{(1)}_{\rm PN} = \quad & -\frac{m}{r^2}\left\{\mathbf{n}\left[-2(2+\eta)\frac{m}{r}+(1+3\eta)v^2-\frac{3}{2}\eta\dot{r}^2\right]-2(2-\eta)\dot{r}\mathbf{v}\right\}, && (5.3b)\\
\mathbf{a}^{(2)}_{\rm 2PN} = \quad & -\frac{m}{r^2}\left\{\mathbf{n}\left[\frac{3}{4}(12+29\eta)\left(\frac{m}{r}\right)^2+\eta(3-4\eta)v^4+\frac{15}{8}\eta(1-3\eta)\dot{r}^4\right.\right.\\
& \left.-\frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2-\frac{1}{2}\eta(13-4\eta)\frac{m}{r}v^2-(2+25\eta+2\eta^2)\frac{m}{r}\dot{r}^2\right]\\
& \left.-\frac{1}{2}\dot{r}\mathbf{v}\left[\eta(15+4\eta)v^2-(4+41\eta+8\eta^2)\frac{m}{r}-3\eta(3+2\eta)\dot{r}^2\right]\right\}, && (5.3c)
\end{aligned}
$$

where $\mu \equiv m_1 m_2/m$ is the reduced mass, with $\eta = \mu/m$, and $\mathrm{n} = \mathbf{x}/r$. The n.5PN reactive accelerations are determined by following the 'What else can it be?' procedure employed in IW which we summarize here. One writes down a general form for the Newtonian ($\epsilon^{2.5}$), 1PN ($\epsilon^{3.5}$) and 2PN ($\epsilon^{4.5}$) radiation-reaction terms in the equations of motion for two bodies, ignoring tidal and spin effects. For the relative acceleration $\mathrm{a} \equiv \mathbf{a}_1 - \mathbf{a}_2$, one assumes the provisional form

$$
\mathbf{a} = -\frac{8}{5}\eta(m/r^2)(m/r)[-(A_{2.5}+A_{3.5}+A_{4.5})\dot{r}\mathbf{n}+(B_{2.5}+B_{3.5}+B_{4.5})\mathbf{v}]\,. \qquad (5.4)
$$

The form of Eq.(5.4) is dictated by the fact that it must be a correction to the Newtonian acceleration (*i.e.,* be proportional to m/r^2), must vanish in the test body limit when gravitational radiation vanishes (*i.e.,* be proportional to η), must be dissipative, or odd in velocities (*i.e.,* contain the factors $\dot{r}$, n and v linearly) and finally, must be related to the emission of gravitational radiation or be nonlinear in Newton's constant G (*i.e.,* contain another factor m/r). The last condition may be more precisely stated by requiring that the reactive acceleration be a power series in the individual masses m_1 and m_2 [150]. For spinless, structureless bodies, the acceleration must lie in the orbital plane (*i.e.,* depend only on the vectors n and v). The prefactor 8/5 is chosen for convenience. To make the leading term of $O(\epsilon^{2.5})$ beyond Newtonian order, $A_{2.5}$ and $B_{2.5}$ must be of $O(\epsilon)$. For this structureless two-body system the only variables in the problem of this order are v^2, m/r, and $\dot{r}^2$. Thus

$A_{2.5}$ and $B_{2.5}$ can each be a linear combination of these three terms; to those terms we assign six "Newtonian radiation reaction" parameters. Proceeding similarly, $A_{3.5}$ and $B_{3.5}$ must be of O(c^2), hence must each be a linear combination of the six terms v^4, v^2m/r, $v^2\dot{r}^2$, $\dot{r}^2m/r$, $\dot{r}^4$, and $(m/r)^2$. To these we assign 12 "1PN R R parameters. And finally, $A_{4.5}$ and $B_{4.5}$ must be of O(c^3), each a linear combination of the 10 terms v^6, $v^4\dot{r}^2$, v^4m/r, $v^2\dot{r}^4$, $v^2(m/r)^2$, $v^2\dot{r}^2(m/r)$, $\dot{r}^6$, $\dot{r}^4(m/r)$, $\dot{r}^2(m/r)^2$ and $(m/r)^3$ to which we assign 20 "2PN R R parameters. The 6 Newtonian RR and 12 post-Newtonian RR parameters were first determined in IW [33, 34]. This solution has been checked and reproduced in the preliminary part of this investigation and constitutes an input to supplement the conservative acceleration terms in Eq.(5.4) for the present study. Our aim is to evaluate these 20 parameters appearing in $A_{4.5}$ and $B_{4.5}$ that will determine the 2PN radiation reaction. It is worth pointing out that in the calculation we are setting up, the terms in the equations of motion of O(c^3) and $O(\epsilon^4)$ beyond Newtonian order do not play any role. The former is non-dissipative but not yet computed; the latter on the other hand includes dissipative parts due to the 'tail' effects [151, 152, 56, 128] which have been separately balanced by the tail luminosity in the works of Blanchet and Damour [151, 32]. However all the radiation reaction results will remain as 'partial results' in the saga of equations of motion until a complete treatment à la Chandrasekhar [100] and Damour [93] is available through 3PN order and later through 4PN order.

Through 2PN order, the equations of motion can be derived from a generalized Lagrangian that depends not only on positions and velocities but also on accelerations. To this order, that is in the absence of radiation reaction, the Lagrangian leads to a conserved energy and angular momentum given by [104, 38, 24]

$$E = E_N + E_{\rm PN} + E_{\rm 2PN}\,, \tag{5.5a}$$

$$\mathbf{J} = \mathbf{J}_N + \mathbf{J}_{\rm PN} + \mathbf{J}_{\rm 2PN}\,, \tag{5.5b}$$

where

$$
\begin{aligned}
E_N &= \mu\left(\frac{1}{2}v^2 - \frac{m}{r}\right), && (5.6a)\\
E_{\mathrm{PN}} &= \mu\left\{\frac{3}{8}(1-3\eta)v^4 + \frac{1}{2}(3+\eta)v^2\frac{m}{r} + \frac{1}{2}\eta\frac{m}{r}\dot{r}^2 + \frac{1}{2}\left(\frac{m}{r}\right)^2\right\}, && (5.6b)\\
E_{2\mathrm{PN}} &= \mu\Big\{\frac{5}{16}(1-7\eta+13\eta^2)v^6 + \frac{1}{8}(21-23\eta-27\eta^2)\frac{m}{r}v^4 \\
&\quad + \frac{1}{4}\eta(1-15\eta)\frac{m}{r}v^2\dot{r}^2 - \frac{3}{8}\eta(1-3\eta)\frac{m}{r}\dot{r}^4 - \frac{1}{4}(2+15\eta)\left(\frac{m}{r}\right)^3 \\
&\quad + \frac{1}{8}(14-55\eta+4\eta^2)\left(\frac{m}{r}\right)^2 v^2 + \frac{1}{8}(4+69\eta+12\eta^2)\left(\frac{m}{r}\right)^2\dot{r}^2\Big\}, \\
& && (5.6c)\\
\mathbf{J}_N &= \mathbf{L}_N, && (5.6d)\\
\mathbf{J}_{\mathrm{PN}} &= \mathbf{L}_N\left\{\frac{1}{2}v^2(1-3\eta) + (3+\eta)\frac{m}{r}\right\}, && (5.6e)\\
\mathbf{J}_{2\mathrm{PN}} &= \mathbf{L}_N\Big\{\frac{1}{2}(7-10\eta-9\eta^2)\frac{m}{r}v^2 - \frac{1}{2}\eta(2+5\eta)\frac{m}{r}\dot{r}^2 \\
&\quad + \frac{1}{4}(14-41\eta+4\eta^2)\left(\frac{m}{r}\right)^2 + \frac{3}{8}(1-7\eta+13\eta^2)v^4\Big\}, && (5.6f)
\end{aligned}
$$

and where $\mathbf{L}_N \equiv \mu\mathbf{x} \times \mathrm{v}$.

Through 2PN order, *the orbital energy and angular momentum per unit reduced mass,* $E \equiv E/\mu = \frac{1}{2}v^2 - m/r + \mathrm{O}(\mathrm{e}^2) + O(\epsilon^3)$, $\tilde{\mathbf{J}} = x \times \mathbf{v}[1 + O(\epsilon) + \mathrm{O}(\mathrm{e}^2)]$, are constant, and correspond to asymptotically measured quantities. However, the radiation reaction terms lead to non-vanishing expressions for $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ containing the 20 undetermined parameters. Following IW, starting from the 2PN-conserved expressions for E and $\tilde{\mathbf{J}}$ we calculate $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ using the 2PN two-body equations of motion [104, 38, 130] supplemented by the radiation-reaction terms of Eq.(5.4). In the balance approach, this time variation of the 'conserved' quantities is equated to the negative of the flux of energy and angular momentum carried by the gravitational waves to the far-zone. Thus in addition to the EOM and conserved quantities we need the 2PN accurate expressions for the far-zone fluxes of energy and angular momentum for a system of two particles moving on general quasi-general orbits. We have computed the instantaneous 2PN corrections to the far-zone fluxes in chapter 2 using the BDI formalism. We quote below the Eqs.(2.57) and (2.61) for the 2PN accurate instantaneous contributions to the far-zone fluxes but with $G = c = 1$ and *per unit reduced mass:*

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\rm far-zone} = \dot{\mathcal{E}}_N + \dot{\mathcal{E}}_{\rm 1PN} + \dot{\mathcal{E}}_{\rm 1.5PN} + \dot{\mathcal{E}}_{\rm 2PN}\,, \tag{5.7a}$$

$$\left(\frac{d\mathcal{J}}{dt}\right)_{\rm far-zone} = \tilde{\mathbf{L}}_N\left[\dot{\mathcal{J}}_N + \mathcal{J}_{\rm 1PN} + \mathcal{J}_{\rm 1.5PN} + \dot{\mathcal{J}}_{\rm 2PN}\right]\,, \tag{5.7b}$$

where

$$\dot{\mathcal{E}}_N = \frac{8}{5}\eta\frac{m^2}{r^3}\frac{m}{r}\left(4v^2 - \frac{11}{3}\dot{r}^2\right)\,, \tag{5.8a}$$

$$\begin{aligned}\dot{\mathcal{E}}_{\rm 1PN} = \frac{8}{5}\eta\frac{m^2}{r^3}\frac{m}{r}\Bigg[&\frac{1}{84}(785 - 852\eta)v^4 - \frac{1}{42}(1487 - 1392\eta)v^2\dot{r}^2\\ &-\frac{40}{21}(17 - \eta)v^2\frac{m}{r} + \frac{1}{28}(687 - 620\eta)\dot{r}^4\\ &+\frac{2}{21}(367 - 15\eta)\dot{r}^2\frac{m}{r} + \frac{4}{21}(1 - 4\eta)\left(\frac{m}{r}\right)^2\Bigg]\,,\end{aligned} \tag{5.8b}$$

$$\begin{aligned}\dot{\mathcal{E}}_{\rm 2PN} = \frac{8}{5}\eta\frac{m^2}{r^3}\frac{m}{r}\Bigg[&\frac{1}{126}(1692 - 5497\eta + 4430\eta^2)v^6\\ &-\frac{1}{42}(1719 - 10278\eta + 6292\eta^2)v^4\dot{r}^2\\ &-\frac{1}{63}(4446 - 5237\eta + 1393\eta^2)v^4\frac{m}{r}\\ &+\frac{1}{42}(2018 - 15207\eta + 7572\eta^2)v^2\dot{r}^4\\ &+\frac{1}{21}(4987 - 8513\eta + 2165\eta^2)v^2\dot{r}^2\frac{m}{r}\\ &+\frac{1}{2268}(281473 + 81828\eta + 4368\eta^2)v^2\left(\frac{m}{r}\right)^2\\ &-\frac{1}{126}(2501 - 20234\eta + 8404\eta^2)\dot{r}^6\\ &-\frac{1}{189}(33510 - 60971\eta + 14290\eta^2)\dot{r}^4\frac{m}{r}\\ &-\frac{1}{756}(106319 + 9798\eta + 5376\eta^2)\dot{r}^2\left(\frac{m}{r}\right)^2\\ &-\frac{2}{189}(253 - 1026\eta + 56\eta^2)\left(\frac{m}{r}\right)^3\Bigg]\,,\end{aligned} \tag{5.8c}$$

$$\dot{\mathcal{J}}_N = \frac{8}{5}\eta\frac{m}{r^2}\frac{m}{r}\left(2v^2 - 3\dot{r}^2 + 2\frac{m}{r}\right)\,, \tag{5.8d}$$

$$\dot{\mathcal{J}}_{\rm 1PN} = \frac{8}{5}\eta\frac{m}{r^2}\frac{m}{r}\Bigg[\frac{1}{84}(307 - 548\eta)v^4 - \frac{1}{14}(74 - 277\eta)v^2\dot{r}^2$$

$$
\begin{aligned}
& -\frac{1}{21}(58+95\eta)v^2\frac{m}{r}+\frac{1}{28}(95-360\eta)\dot{r}^4 \\
& +\frac{1}{42}(372+197\eta)\dot{r}^2\frac{m}{r}-\frac{1}{42}(745-2\eta)\left(\frac{m}{r}\right)^2\Big], \qquad (5.8e)\\
\dot{\mathcal{J}}_{2\mathrm{PN}} = {} & \frac{8}{5}\eta\frac{m}{r^2}\frac{m}{r}\Big[\frac{1}{504}(2665-12355\eta+12894\eta^2)v^6 \\
& -\frac{1}{168}(2246-12653\eta+15637\eta^2)v^4\dot{r}^2 \\
& +\frac{1}{504}(165-491\eta+4022\eta^2)v^4\frac{m}{r} \\
& +\frac{1}{168}(3575-16805\eta+15680\eta^2)v^2\dot{r}^4 \\
& +\frac{1}{504}(21853-21603\eta+2551\eta^2)v^2\dot{r}^2\frac{m}{r} \\
& -\frac{1}{252}(10651-10179\eta+3428\eta^2)v^2\left(\frac{m}{r}\right)^2 \\
& -\frac{5}{18}(39-163\eta+97\eta^2)\dot{r}^6 \\
& -\frac{1}{504}(22312-41398\eta+9695\eta^2)\dot{r}^4\frac{m}{r} \\
& +\frac{1}{252}(8436-25102\eta+4587\eta^2)\dot{r}^2\left(\frac{m}{r}\right)^2 \\
& +\frac{1}{2268}(170362+70461\eta+1386\eta^2)\left(\frac{m}{r}\right)^3\Big]. \qquad (5.8f)
\end{aligned}
$$

In the above expressions $\tilde{\mathbf{L}}_N = \mathbf{L}_N/\mu$ and the tail terms are not listed. It is important to emphasize that the 'tail' contribution to the reaction force is such that the balance equation for energy is verified for the tail luminosity [151, 32]. This corresponds to the 'tail' acceleration at **4PN.** With this part independently accounted for, in our analysis we focus on the 'instantaneous' terms without loss of generality. It is worth recalling that the 'balance' one sets up in the above treatment is always modulo total time derivatives of the variables involved. This is crucial to realize and in IW this was systematically accounted for by noting that at orders of approximation beyond those at which they are strictly conserved (and thus well-defined), E and $\tilde{\mathbf{J}}$ are ambiguous upto such terms. Consequently, we have the freedom to add to $\tilde{E}$ and $\tilde{\mathbf{J}}$ arbitrary terms of order $\epsilon^{2.5}$, $\epsilon^{3.5}$, and $\epsilon^{4.5}$ beyond the Newtonian expressions without affecting their conservation at **2PN** order. There are three such terms of the appropriate general form at $O(\epsilon^{2.5})$ in each of $\tilde{E}$ and $\tilde{\mathbf{J}}$, respectively, 6 each at $O(\epsilon^{3.5})$ and 10 each at $O(\epsilon^{4.5})$, resulting in 6 additional Newtonian RR parameters, 12 additional 1PN RR parameters and 20 additional 2PN RR parameters, respectively. As discussed in detail in the following section, these numbers are very much tied up with the 'functional form' we assume for the ambiguous terms and in this section we follow IW in close detail. Equating time derivatives of the resulting generalized energy and angular momentum expressions $\tilde{E}^*$ and $\tilde{\mathbf{J}}^*$ (rather than only the conserved expressions) to the negative of the far-zone flux formulae and comparing them term by term one seeks to determine the extent to which one can deduce the 4.5PN reactive acceleration terms by the refined balance approach.

5.2.2 The 2PN RR computation and results

The above procedure is implemented order by order. All the computations were done with MAPLE [134] and independently checked by MATHEMATICA [153]. At the leading order, when the flux is given by the quadrupole equation, one deduces the 'Newtonian RR' or 2.5PN term in the acceleration. In this case, in addition to the 6 unknowns in the reactive acceleration, one has 3 unknowns each for the possible 2.5PN ambiguities in the E^* and $\tilde{\mathbf{J}}^*$. As demonstrated in IW, the balance equations yield 12 constraints on these 12 Newtonian RR parameters. Of the 12 constraints, only 10 are linearly independent, and thus finally one obtains 10 linear inhomogeneous equations for 12 Newtonian radiation reaction variables. Solving these equations one obtains explicit forms for $A_{2.5}$, $B_{2.5}$ and $\tilde{E}_{2.5}$, $\tilde{J}_{2.5}$ in terms of two 2.5PN arbitrary parameters. To get the 3.5PN reactive terms, one adopts the above solution and extends the calculation to $O(\epsilon^{3.5})$ after introducing $\tilde{E}_{3.5}$ and $\tilde{J}_{3.5}$ with 12 additional 1PN RR parameters. At 3.5PN there are 20 constraints on the 24 post-Newtonian radiation reaction parameters; of the 20 only 18 are linearly independent; the solution to this system yields explicit forms for $A_{3.5}$, $B_{3.5}$ and $\tilde{E}_{3.5}$, $\tilde{J}_{3.5}$ in terms of six 3.5PN arbitrary parameters. Since we need these results for the present computation, we reproduce them from IW [154].

$$A_{2.5} = \quad 3(1+\alpha_3)v^2 + \frac{1}{3}(23 + 6\beta_2 - 9\alpha_3)\frac{m}{r} - 5\alpha_3\dot{r}^2\,, \tag{5.9a}$$

$$B_{2.5} = \quad (2+\beta_2)v^2 + (2-\beta_2)\frac{m}{r} - 3(1+\beta_2)\dot{r}^2\,, \tag{5.9b}$$

$$A_{3.5} = \quad f_1 v^4 + f_2 v^2\frac{m}{r} + f_3 v^2\dot{r}^2 + f_4\dot{r}^2\frac{m}{r} + f_5\dot{r}^4 + f_6\left(\frac{m}{r}\right)^2\,, \tag{5.9c}$$

$$B_{3.5} = \quad g_1 v^4 + g_2 v^2\frac{m}{r} + g_3 v^2\dot{r}^2 + g_4\dot{r}^2\frac{m}{r} + g_5\dot{r}^4 + g_6\left(\frac{m}{r}\right)^2\,, \tag{5.9d}$$

where

$$f_1 = \quad \frac{1}{28}(117 + 132\eta) - \frac{3}{2}\alpha_3(1-3\eta) + 3\xi_2 - 3\rho_5\,, \tag{5.10a}$$

$$f_2 = \quad -\frac{1}{42}(297 - 310\eta) - 3\beta_2(1-4\eta) - \frac{3}{2}\alpha_3(7+13\eta)$$

$$-2\xi_1 - 3\xi_2 + 3\xi_5 + 3\rho_5\,, \tag{5.10b}$$

$$f_3 = \quad \frac{5}{28}(19 - 72\eta) + \frac{5}{2}\alpha_3(1-3\eta) - 5\xi_2 + 5\xi_4 + 5\rho_5\,, \tag{5.10c}$$

$$f_4 = \quad -\frac{1}{28}(687 - 368\eta) - 6\beta_2\eta + \frac{1}{2}\alpha_3(54+17\eta) - 2\xi_2 - 5\xi_4 - 6\xi_5\,, \tag{5.10d}$$

$$f_5 = \quad -7\xi_4\,, \tag{5.10e}$$

$$f_6 = \quad -\frac{1}{21}(1533 + 498\eta) - \beta_2(14+9\eta) + 3\alpha_3(7+4\eta) - 2\xi_3 - 3\xi_5\,, \tag{5.10f}$$

$$g_1 = \quad -3(1-3\eta) - \frac{3}{2}\beta_2(1-3\eta) - \xi_1\,, \tag{5.10g}$$

$$g_2 = \quad -\frac{1}{84}(139 + 768\eta) - \frac{1}{2}\beta_2(5+17\eta) + \xi_1 - \xi_3\,, \tag{5.10h}$$

$$g_3 = \quad \frac{1}{28}(369 - 624\eta) + \frac{3}{2}(3\beta_2 + 2\alpha_3)(1-3\eta) + 3\xi_1 - 3\rho_5\,, \tag{5.10i}$$

$$g_4 = \quad \frac{1}{42}(295 - 335\eta) + \frac{1}{2}\beta_2(38 - 11\eta) - 3\alpha_3(1-3\eta) + 2\xi_1 + 4\xi_3 + 3\rho_5\,, \tag{5.10j}$$

$$g_5 = \quad \frac{5}{28}(19 - 72\eta) - 5\alpha_3(1 - 37) + 5\rho_5\,, \tag{5.10k}$$

$$g_6 = \quad -\frac{1}{21}(634 - 667) + \beta_2(7 + 3\eta) + \xi_3\,. \tag{5.101}$$

The quantities α_3, β_2, ξ_1, J2, ξ_3, ξ_4, ξ_5 and ρ_5 are parameters that represent the unconstrained degrees of freedom that correspond to gauge transformations. In addition to the reactive terms listed above, one of the coefficients that determine the 2.5PN ambiguity in $\tilde{E}$ and $\tilde{\mathbf{J}}$ and three of the coefficients that determine the corresponding 3.5PN ambiguity are nonvanishing. We list these also since they are needed for setting up the 4.5PN computation:

$$\begin{aligned}
\alpha_1 &= -(2+\beta_2) \, , && (5.11a)\\
\xi_6 &= -\frac{4}{21}(1-4\eta) \, , && (5.11b)\\
\rho_3 &= \xi_1 + \frac{1}{84}(307-548\eta) \, , && (5.11c)\\
\rho_6 &= \xi_3 - \frac{1}{42}(271-214\eta) \, . && (5.11d)
\end{aligned}$$

We now adopt the 2.5PN and 3.5PN solutions given by Eqs.(5.9), (5.10) and (5.11). Following the IW strategy, we assume the 4.5PN terms in the equations of motion to be of the form

$$\begin{aligned}
A_{4.5} &= h_1 v^6 + h_2 v^4 \dot{r}^2 + h_3 v^4 \frac{m}{r} + h_4 v^2 \dot{r}^4 + h_5 v^2 \left(\frac{m}{r}\right)^2 \\
&\quad + h_6 v^2 \dot{r}^2 \frac{m}{r} + h_7 \dot{r}^6 + h_8 \dot{r}^4 \frac{m}{r} + h_9 \dot{r}^2 \left(\frac{m}{r}\right)^2 + h_{10} \left(\frac{m}{r}\right)^3 , && (5.12a)\\
B_{4.5} &= k_1 v^6 + k_2 v^4 \dot{r}^2 + k_3 v^4 \frac{m}{r} + k_4 v^2 \dot{r}^4 + k_5 v^2 \left(\frac{m}{r}\right)^2 \\
&\quad + k_6 v^2 \dot{r}^2 \frac{m}{r} + k_7 \dot{r}^6 + k_8 \dot{r}^4 \frac{m}{r} + k_9 \dot{r}^2 \left(\frac{m}{r}\right)^2 + k_{10} \left(\frac{m}{r}\right)^3 . && (5.12b)
\end{aligned}$$

We also assume for the ambiguity in $\tilde{E}_{4.5}$ and $\tilde{J}_{4.5}$ the restrictions and functional forms adopted in IW and also require that $\tilde{\mathbf{J}}$ remain a pseudo-vector. The 'generalized' "energy" and "angular momentum" through 4.5PN are thus given as sums of the conserved parts Eqs.(5.6), the 'most general' 2.5PN and 3.5PN contributions – *i.e.,* with coefficients determined by the Newtonian RR and 1PN RR calculations, and arbitrary 4.5PN terms. We use E_* and $\tilde{\mathbf{J}}^*$ to distinguish these quantities from the conserved energy and angular momentum. We get (per unit reduced mass)

$$\begin{aligned}
\tilde{E}^* \equiv\; & \tilde{E}_N + \tilde{E}_{\rm PN} + \tilde{E}_{\rm 2PN} + \tilde{E}_{2.5} + \tilde{E}_{3.5} + \tilde{E}_{4.5} \\
=\; & \tilde{E}_N + \tilde{E}_{\rm PN} + \tilde{E}_{\rm 2PN} + \frac{8}{5}\eta\left(\frac{m}{r}\right)^2 \dot{r}[(2+\beta_2)v^2 - \alpha_3 \dot{r}^2] \\
& -\frac{8}{5}\eta\left(\frac{m}{r}\right)^2 \dot{r}\left[\xi_1 v^4 + \xi_2 v^2 \dot{r}^2 + \xi_3 v^2 \frac{m}{r} + \xi_4 \dot{r}^4 + \xi_5 \dot{r}^2 \frac{m}{r} - \frac{4}{21}(1-4\eta)\left(\frac{m}{r}\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
& -\frac{8}{5}\eta\left(\frac{m}{r}\right)^2\dot{r}\left[\psi_1 v^6+\psi_2 v^4\dot{r}^2+\psi_3 v^4\frac{m}{r}+\psi_4 v^2\dot{r}^4+\psi_5 v^2\left(\frac{m}{r}\right)^2\right.\\
& \left.+\psi_6 v^2\dot{r}^2\frac{m}{r}+\psi_7\dot{r}^6+\psi_8\dot{r}^4\frac{m}{r}+\psi_9\dot{r}^2\left(\frac{m}{r}\right)^2+\psi_{10}\left(\frac{m}{r}\right)^3\right],\\
\tilde{\mathbf{J}}^* \equiv\ & \tilde{\mathbf{J}}_N+\tilde{\mathbf{J}}_{\rm PN}+\tilde{\mathbf{J}}_{\rm 2PN}+\tilde{\mathbf{J}}_{2.5}+\tilde{\mathbf{J}}_{3.5}+\tilde{\mathbf{J}}_{4.5}\\
=\ & \tilde{\mathbf{J}}_N+\tilde{\mathbf{J}}_{\rm PN}+\tilde{\mathbf{J}}_{\rm 2PN}+\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r}\dot{r}\left(\beta_2\frac{m}{r}\right)\\
& -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r}\dot{r}\left[\frac{1}{84}(307-548\eta+84\xi_1)v^2\frac{m}{r}+\rho_5\dot{r}^2\frac{m}{r}\right.\\
& \left.-\frac{1}{42}(271-214\eta-42\xi_3)\left(\frac{m}{r}\right)^2\right]\\
& -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r}\dot{r}\left[\chi_1 v^6+\chi_2 v^4\dot{r}^2+\chi_3 v^4\frac{m}{r}+\chi_4 v^2\dot{r}^4\right.\\
& +\chi_5 v^2\left(\frac{m}{r}\right)^2+\chi_6 v^2\dot{r}^2\frac{m}{r}+\chi_7\dot{r}^6\\
& \left.+\chi_8\dot{r}^4\frac{m}{r}+\chi_9\dot{r}^2\left(\frac{m}{r}\right)^2+\chi_{10}\left(\frac{m}{r}\right)^3\right],
\end{aligned}
\tag{5.13}
$$

We now compute the 4.5PN terms in $d\tilde{E}^*/dt$ and $d\tilde{\mathbf{J}}^*/dt$ using the identities

$$\frac{1}{2}\frac{dv^2}{dt} \equiv \mathbf{v}\cdot\mathbf{a}, \tag{5.14a}$$

$$\frac{d(\mathbf{x}\times\mathbf{v})}{dt} \equiv \mathbf{x}\times\mathbf{a}, \tag{5.14b}$$

$$\ddot{r} \equiv \frac{v^2+\mathbf{r}\cdot\mathbf{a}-\dot{r}^2}{r}, \tag{5.14c}$$

where a is given by Eqs.(5.2), (5.3), (5.4), (5.9), (5.10) and (5.12). To compute E* and $\tilde{\mathbf{J}}^*$ to $O(\epsilon^{4.5})$, one needs to evaluate ($\dot{\tilde{E}}_N$,$\dot{\tilde{\mathbf{J}}}_N$), ($\tilde{E}_{\rm 1PN}$, $\tilde{\mathbf{J}}_{\rm 1PN}$) and ($\tilde{E}_{\rm 2PN}$, $\tilde{\mathbf{J}}_{\rm 2PN}$) by using a to $O(\epsilon^{4.5})$, $O(\epsilon^{3.5})$ and $O(\epsilon^{2.5})$, respectively. On the other hand, for time derivatives of the 'ambiguity parts', ($\tilde{E}_{4.5}$, $\tilde{\mathbf{J}}_{4.5}$), ($\tilde{E}_{3.5}$, $\tilde{\mathbf{J}}_{3.5}$) and ($\tilde{E}_{2.5}$, $\tilde{\mathbf{J}}_{2.5}$), the relevant accelerations are the 'conservative' accelerations to order Newtonian, post-Newtonian and second post-Newtonian, respectively. Schematically, we get,

$$
\begin{aligned}
\frac{d\tilde{E}^*}{dt} =\ & -\frac{8}{15}\eta\frac{m^2}{r^3}\left[\frac{m}{r}\left(12v^2-11\dot{r}^2\right)+\frac{m}{r}\left\{\frac{1}{28}\left[(785-852\eta)v^4\right.\right.\right.\\
& +2(-1487+1392\eta)v^2\dot{r}^2+160(-17+\eta)\frac{m}{r}v^2\\
& +3(687-620\eta)\dot{r}^4+8(367-15\eta)\frac{m}{r}\dot{r}^2\\
& \left.\left.\left.+16(1-4\eta)\left(\frac{m}{r}\right)^2\right]\right\}+\sum_{i=1}^{15}\mathcal{R}_i^{[4.5]}\mathcal{Y}_i^{[4]}\right],
\end{aligned}
\tag{5.15a}
$$

$$\frac{d\tilde{\mathbf{J}}^*}{dt} = -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\left[\frac{m}{r}\left(2v^2+2\frac{m}{r}-3\dot{r}^2\right)+\frac{m}{r}\left\{\frac{1}{84}\left[(307-548\eta)v^4\right.\right.\right.$$
$$+6\left(-74+277\eta\right)v^2\dot{r}^2-4(58+95\eta)\frac{m}{r}v^2$$
$$+3(95-360\eta)\dot{r}^4+2(372+197\eta)\frac{m}{r}\dot{r}^2$$
$$\left.\left.\left.+2(-745+2\eta)\left(\frac{m}{r}\right)^2\right]\right\}+\sum_{i=1}^{15}\mathcal{S}_i^{[4.5]}\mathcal{Y}_i^{[4]}\right], \tag{5.15b}$$

where

$$\mathcal{Y}_i^{[4]}(i=1\ldots 15)=\left[\quad v^8, v^6\left(\frac{m}{r}\right), v^6\dot{r}^2, v^4\left(\frac{m}{r}\right)^2, v^4\dot{r}^4, v^4\left(\frac{m}{r}\right)\dot{r}^2,\right.$$
$$v^2\left(\frac{m}{r}\right)^3, v^2\dot{r}^6, v^2\left(\frac{m}{r}\right)^2\dot{r}^2, v^2\left(\frac{m}{r}\right)\dot{r}^4,$$
$$\left.\left(\frac{m}{r}\right)^4, \left(\frac{m}{r}\right)^3\dot{r}^2, \left(\frac{m}{r}\right)^2\dot{r}^4, \left(\frac{m}{r}\right)\dot{r}^6, \dot{r}^8\right], \tag{5.16}$$

and $\mathcal{R}_i^{[4.5]}$ and $\mathcal{S}_i^{[4.5]}$ consist of combinations of the parameters h_i and k_i from $A_{4.5}$ and $B_{4.5}$, ψ_i, χ_i combined with functions of η from $\tilde{E}_{4.5}$ and $\tilde{\mathbf{J}}_{4.5}$, $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \rho_5$ combined with functions of η from 1PN corrections of 3.5PN terms and α_3 and β_2 combined with functions of η from 2PN corrections of 2.5PN terms. We equate $d\tilde{E}^*/dt$ and $d\tilde{\mathbf{J}}^*/dt$ thus obtained to the negative of the 2PN far-zone fluxes given by Eqs.(5.8). This results in 30 constraints on the 40 parameters h_i, k_i, ψ_i and χ_i. Two of these constraints being redundant, of the 30 constraints only 28 are linearly independent. The system of 28 linear inhomogeneous equations for 40 variables is therefore upder-determined to the extent of 12 arbitrary parameters, and we choose these to be $\psi_1 \ldots \psi_9$, χ_6, χ_8 and χ_9. With this choice, the coefficients in Eq.(5.12) determining the 4.5PN reactive acceleration are given by

$$h_1 = -\frac{1}{168}(121-2278\eta+4012\eta^2)-\frac{3}{8}\alpha_3(1-9\eta+21\eta^2)-\frac{3}{2}(\xi_2-\rho_5)(1-3\eta)$$
$$+3\psi_2-3\chi_6\,, \tag{5.17a}$$

$$h_2 = \frac{5}{84}(329-1487\eta+1244\eta^2)+\frac{5}{8}\alpha_3(1-9\eta+21\eta^2)+\frac{5}{2}(\xi_2-\xi_4-\rho_5)(1-3\eta)$$
$$-5\psi_2+5\psi_4+5\chi_6-5\chi_8\,, \tag{5.17b}$$

$$h_3 = \frac{1}{504}(7692-87429\eta+11218\eta^2)+\frac{3}{8}\alpha_3(1-97\eta+25\eta^2)+\frac{1}{4}\beta_2(3-3\eta-19\eta^2)$$

$$+3\xi_1(1-4\eta)-\frac{3}{2}(\xi_2-\rho_5)(7+13\eta)-\frac{3}{2}\xi_5(1-3\eta)$$

$$-2\psi_1-3\psi_2+3\psi_6+3\chi_6-3\chi_9\,, \qquad (5.17c)$$

$$h_4 = -\frac{5}{18}(39-163\eta+97\eta^2)+\frac{7}{2}\xi_4(1-3\eta)-7\psi_4+7\psi_7+7\chi_8\,, \qquad (5.17d)$$

$$h_5 = -\frac{1}{252}(37089-64005\eta+11297\eta^2)+9\alpha_3(2+13\eta+2\eta^2)$$

$$+\frac{1}{4}\beta_2(48-121\eta-54\eta^2)+\xi_1(14+9\eta)+3(\xi_2-\rho_5)(7+4\eta)$$

$$+3\xi_3(1-4\eta)-\frac{3}{2}\xi_5(7+13\eta)-2\psi_3-3\psi_6+3\psi_9+3\chi_9\,, \qquad (5.17e)$$

$$h_6 = -\frac{1}{504}(45475-219535\eta+43121\eta^2)-\frac{1}{4}\alpha_3(14-403\eta+77\eta^2)$$

$$-\frac{3}{2}\beta_2\eta(7-13\eta)+6\eta\xi_1+\frac{1}{2}\xi_2(68-9\eta)-\frac{5}{2}\xi_4(7+13\eta)$$

$$+3\xi_5(1-3\eta)-\frac{1}{2}\rho_5(62+19\eta)-4\psi_2-5\psi_4-6\psi_6$$

$$+5\psi_8+2\chi_6+5\chi_8+6\chi_9\,, \qquad (5.17f)$$

$$h_7 = -9\psi_7\,, \qquad (5.17g)$$

$$h_8 = \frac{1}{252}(5002-36589\eta+4496\eta^2)-\frac{1}{8}\alpha_3\eta(233-63\eta)+\frac{33}{4}\beta_2\eta(1-3\eta)$$

$$+3\eta\xi_2+\frac{1}{2}\xi_4(82+23\eta)+5\eta\rho_5-2\psi_4-7\psi_7-8\psi_8\,, \qquad (5.17h)$$

$$h_9 = \frac{1}{756}(181371-342479\eta+42598\eta^2)-\frac{1}{2}\alpha_3(117+109\eta+6\eta^2)$$

$$-\frac{1}{4}\beta_2(28+245\eta+20\eta^2)+2\eta\xi_1+(2\xi_2+5\xi_4)(7+4\eta)+7\eta\xi_3$$

$$+\frac{1}{2}\xi_5(60+21\eta)+3\eta\rho_5-2\psi_6-5\psi_8-7\psi_9\,, \qquad (5.17i)$$

$$h_{10} = \frac{1}{756}(265265+262230\eta+15072\eta^2)-\frac{3}{4}\alpha_3(102+177\eta+16\eta^2)$$

$$+\frac{1}{4}\beta_2(200+325\eta+40\eta^2)+\xi_3(14+9\eta)+3\xi_5(7+4\eta)-2\psi_5-3\psi_9\,,$$

$$(5.17j)$$

$$k_1 = \frac{3}{8}(\beta_2+2)(1-\eta-11\eta^2)+\frac{3}{2}\xi_1(1-3\eta)-\psi_1\,, \qquad (5.17k)$$

$$k_2 = -\frac{1}{168}(499-2656\eta-146\eta^2)-\frac{3}{2}\alpha_3(1-3\eta-3\eta^2)-\frac{9}{8}\beta_2(1-\eta-11\eta^2)$$

$$-\frac{3}{2}(3\xi_1-2\xi_2-\rho_5)(1-3\eta)+3\psi_1-3\chi_6\,, \qquad (5.17l)$$

$$k_3 = \frac{1}{504}(81-9127\eta-14482\eta^2)-\frac{1}{8}\beta_2(3+121\eta+7\eta^2)+\frac{1}{2}\xi_1(5+17\eta)$$

$$
\begin{aligned}
& +\frac{3}{2}\xi_3(1-3\eta)+\psi_1-\psi_3\,, && (5.17\text{m})\\
k_4 = \quad & \frac{5}{84}(329-1487\eta+1244\eta^2)+\frac{5}{2}\alpha_3(1-3\eta-3\eta^2)\\
& -\frac{5}{2}(2\xi_2-2\xi_4+\rho_5)(1-3\eta)+5\chi_6-5\chi_8\,, && (5.17\text{n})\\
k_5 = \quad & -\frac{11}{252}(1107-805\eta-508\eta^2)+\frac{1}{4}\beta_2(16+255\eta+22\eta^2)-\xi_1(7+3\eta)\\
& +\frac{1}{2}\xi_3(5+17\eta)+\psi_3-\psi_5\,, && (5.17\text{o})\\
k_6 = \quad & \frac{1}{504}(1797+54816\eta-22463\eta^2)+\frac{3}{2}\alpha_3(1+3\eta+5\eta^2)\\
& -\frac{1}{4}\beta_2(42-485\eta+173\eta^2)-\frac{1}{2}\xi_1(56-49\eta)\\
& -3(\xi_2+2\xi_3-\xi_5)(1-3\eta)+\frac{3}{2}\rho_5(7+11\eta)\\
& +4\psi_1+4\psi_3+3\chi_6-3\chi_9\,, && (5.17\text{p})\\
k_7 = \quad & -\frac{5}{18}(39-163\eta+97\eta^2)-7\xi_4(1-3\eta)+7\chi_8\,, && (5.17\text{q})\\
k_8 = \quad & -\frac{1}{504}(39808-92788\eta+24563\eta^2)+\frac{1}{2}\alpha_3(14-105\eta+59\eta^2)\\
& -\frac{3}{8}\beta_2\eta(69+13\eta)-3\eta\xi_1-(2\xi_2+5\xi_4+6\xi_5)(1-3\eta)\\
& -\frac{1}{2}\rho_5(62+3\eta)+2\chi_6+5\chi_8+6\chi_9\,, && (5.17\text{r})\\
k_9 = \quad & \frac{1}{252}(8319-7683\eta+11809\eta^2)+3\alpha_3(3-13\eta-\eta^2)\\
& -\frac{1}{4}\beta_2(194+215\eta+24\eta^2)-(2\xi_1+3\rho_5)(7+3\eta)-\frac{1}{2}\xi_3(44-9\eta)\\
& -3\xi_5(1-3\eta)+2\psi_3+5\psi_5+3\chi_9\,, && (5.17\text{s})\\
k_{10} = \quad & \frac{1}{2268}(425413+111636\eta-6912\eta^2)-\frac{1}{2}\beta_2(53+103\eta+4\eta^2)\\
& -\xi_3(7+3\eta)+\psi_5\,. && (5.17\text{t})
\end{aligned}
$$

At the 4.5PN order, 4 parameters determining $\tilde{E}_{4.5}$ and $\tilde{\mathbf{J}}_{4.5}$ are non-vanishing and are given by

$$
\begin{aligned}
\psi_{10} &= \frac{1}{189}\left(362-1548\eta+400\eta^2\right)\,,\\
\chi_3 &= \psi_1+\frac{1}{504}\left(2665-12355\eta+12894\eta^2\right)\,,\\
\chi_5 &= \psi_3+\frac{7}{2}\beta_2\eta-\frac{1}{126}\left(524-4483\eta+3675\eta^2\right)\,,
\end{aligned}
$$

$$\chi_{10} = \psi_5 - \frac{7}{2}\beta_2\eta + \frac{1}{252}\left(775 - 3939\eta + 2942\eta^2\right) . \tag{5.18}$$

A final minor remark is with regard to the two possible ways one may implement the requirement that the ambiguity in $\tilde{\mathbf{J}}^*$ be a pseudovector. One may either choose it proportional to $\tilde{\mathbf{L}}_N$ as in the treatment above or to the conserved angular momentum $\tilde{\mathbf{J}}$. At 2.5PN order both choices are identical. At the 3.5PN order, the two choices lead to an identical system of linear equations barring a translation in the values of ρ_3 and ρ_6 by an amount given by the coefficients of v^2 and m/r in $\mathbf{J}_{1PN}$:

$$\begin{aligned} \rho_3 \rightarrow \bar{\rho_3} &= \rho_3 + \frac{1}{2}(1 - 3\eta)\beta_2 , \\ \rho_6 \rightarrow \bar{\rho_6} &= \rho_6 + (3 + \eta)\beta_2 . \end{aligned} \tag{5.19}$$

Since ρ_3 and ρ_6 are not among the arbitrary parameters determining the solution, the solution determining the reactive terms and ξ_6 is unchanged. Only the expressions for ρ_3 and ρ_6 are changed to

$$\begin{aligned} \bar{\rho_3} &= \xi_1 + \frac{1}{84}(307 - 548\eta) + \frac{1}{2}(1 - 3\eta)\beta_2 , \\ \bar{\rho_6} &= \xi_3 - \frac{1}{42}(271 - 214\eta) + (3 + \eta)\beta_2 . \end{aligned} \tag{5.20}$$

At 4.5PN order, however, the situation is different. Indeed, as before, the two choices lead to an identical system of linear equations barring a translation in the values of the five parameters χ_3, χ_5, χ_6, χ_9 and χ_{10}.

$$\begin{aligned} \chi_3 \rightarrow \bar{\chi_3} &= \chi_3 + \frac{1}{8}\left(1 - 9\eta + 21\eta^2\right)\beta_2 - \frac{1}{2}(1 - 3\eta)\xi_1 \\ &\quad - \frac{1}{168}\left(307 - 1469\eta + 1644\eta^2\right) , \\ \chi_5 \rightarrow \bar{\chi_5} &= \chi_5 + \frac{1}{2}\left(1 + 6\eta - 3\eta^2\right)\beta_2 - (3 + \eta)\xi_1 - \frac{1}{2}(1 - 3\eta)\xi_3 \\ &\quad - \frac{1}{42}\left(325 - 155\eta - 595\eta^2\right) , \\ \chi_6 \rightarrow \bar{\chi_6} &= \chi_6 - \frac{1}{2}(1 - 3\eta)\rho_5 , \\ \chi_9 \rightarrow \bar{\chi_9} &= \chi_9 - \frac{1}{2}(2 + 5\eta)\eta\beta_2 - (3 + \eta)\rho_5 , \\ \chi_{10} \rightarrow \bar{\chi_{10}} &= \chi_{10} - \frac{1}{4}(22 + 65\eta)\beta_2 - (3 + \eta)\xi_3 \end{aligned}$$

$$+\frac{1}{294}\left(5691 - 2597\eta - 1498\eta^2\right) \tag{5.21}$$

Consequently, in terms of the above 'shifted' variables, the solutions for the reactive accelerations are identical. As χ_6 and χ_9 are among the *independent* parameters that determine the reactive acceleration, in terms of χ_6 and χ_9 the two choices yield equivalent but different looking solutions for the 4.5PN reactive terms in the equations of motion.

Of the two choices, the second choice is more convenient for calculations by hand since $d\mathbf{J}/dt = 0$ to $O(\epsilon^2)$, but has no special advantage when the calculation is done on a computer.

5.3 Redundant equations and related variant schemes

It was noticed in IW that both at the 2.5PN and at the 3.5PN order, the 'balance procedure' leads to two redundant constraint equations [34]. Here, at 4.5PN order, we once again obtain two redundant constraint equations. In this section, we examine critically the origin of these redundant equations.

In implementing the 'refined balance procedure' for the general orbits, IW [34] balance the 'energy flux' and 'angular momentum flux'[7] completely independently of each other. However, for circular orbits, these fluxes are not independent but related [140] via:

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{far-zone}} = v^2\dot{\mathcal{J}}$$

where $\mathcal{J}$ is defined by the equation

$$\left(\frac{d\mathbf{J}}{dt}\right)_{\text{far-zone}} = \mathbf{L}_N\dot{\mathcal{J}}$$

The general balance should reflect this limit and we find that for Newtonian RR a linear combination of the 6 equations representing energy balance and another linear combination of the 6 equations representing angular momentum balance are indeed identical and given by:

$$e_1 + e_2 - 4 = 0\,. \tag{5.22}$$

Similarly at 3.5PN we have

$$g_1 + g_2 + g_6 - (3-\eta)\,\beta_2 + \frac{1}{84}\,(2927 - 252\eta) = 0\,, \tag{5.23}$$

and finally at 4.5PN order the 'degenerate' equation is

$$\begin{aligned} k_1 + k_3 + k_5 + k_{10} + (3-\eta)(\xi_1+\xi_3) + \frac{1}{4}(90 + 13\eta + 6\eta^2)\beta_2 & \\ -\frac{1}{4536}(635771 + 297117\eta - 81000\eta^2) = & \quad 0\,. \end{aligned} \tag{5.24}$$

Thus we can trace the existence of one of the redundant equations in the IW procedure to the fact that for circular orbits the energy and angular momentum fluxes are not independent but proportional to each other.

The mystery of the other redundant equation was not so easy to resolve but after a careful examination of the system of equations and 'experiments' in modifying the system, we could finally track it back to its source. The observation that this redundant equation relates the coefficients of the polynomial representing the ambiguity in $\tilde{\mathbf{J}}$ led us to examine the functional form that IW proposed as the starting ansatz for the calculation. **A** comparison of the functional forms for the ambiguity in E and $\tilde{\mathbf{J}}$ Eqs.(5.13) reveals that indeed IW assume a more general possibility for $\tilde{\mathbf{J}}$ than required. The ambiguity in angular momentum leads to terms more general than required by the far-zone flux formula and time derivative of the leading term using the reactive acceleration. The absence of such terms in the far-zone flux then yields only the trivial solution for these additional variables in $\tilde{\mathbf{J}}$, and the second redundant equation is just a homogeneous linear combination of these trivial solutions. Thus the second redundant equation in the IW scheme is due to the fact that the IW scheme – extended here to 4.5PN order – is not a 'minimal' one.

To verify this 'conjecture' we experimented with alternatives for the functional form that one assumes as the starting expression for the ambiguity in $\tilde{E}$ and $\tilde{\mathbf{J}}$ – the 2.5PN, 3.5PN and 4.5PN order terms. In the first instance, we replace the IW scheme – labelled for clarity of reference by IW21 – by the 'minimal' variant in Eq.(5.13) – labelled by IW22. The notation IW21 indicates e.g., that $(m/r)^2$ is pulled out in $\tilde{E}$ while only $(m/r)^1$ is pulled out in $\tilde{\mathbf{J}}$. As explained above, the minimal choice for $\tilde{\mathbf{J}}^*$ is obtained by pulling out the factor $(8/5)\eta\tilde{\mathbf{L}}_N(m/r)^2\dot{r}$ from arbitrary terms in $\tilde{\mathbf{J}}^*$, rather than the factor $(8/5)\eta\tilde{\mathbf{L}}_N(m/r)\dot{r}$ as in the IW scheme for $\tilde{\mathbf{J}}^*$. This reduces by one the order of the polynomial in v^2, $\dot{r}^2$, and m/r that constitutes the arbitrariness, and consequently implies a reduction in the number of variables that characterize the ambiguity in $\tilde{\mathbf{J}}$ to one for $\tilde{\mathbf{J}}_{2.5}$, three in $\tilde{\mathbf{J}}_{3.5}$ and six in $\tilde{\mathbf{J}}_{4.5}$. Thus in the IW22 scheme, at the 2.5PN level we have 6 variables in the reactive acceleration, 3 variables determining the energy ambiguity $\tilde{E}_{2.5}$ and 1 variable determining the ambiguity in $\tilde{\mathbf{J}}_{2.5}$ *i.e.,* 10 variables in all. The balance equations lead to 9 equations – 6 from energy and 3 from angular momentum – of which 8 are linearly independent. In other words, there is only one redundant equation. The linear system of 8 equations for 10 variables is then the same as before and leads to the IW21 solution in terms of 2 arbitrary parameters. (The two extra variables in IW21 are identically zero.) Similarly, at the 3.5PN level we have 12 variables in the reactive acceleration, 6 variables determining the energy ambiguity $\tilde{E}_{3.5}$ and 3 variables determining the ambiguity in $\tilde{\mathbf{J}}_{3.5}$, *i.e.,* 21 variables in all. The balance equations lead to 16 equations – 10 from energy and 6 from angular momentum – of which 15 are linearly independent, leaving only one redundant equation. The linear system of 15 equations for 21 variables is then the same as before and leads to the IW21 solution in terms of 6 arbitrary parameters. (The three extra variables in IW21 are identically zero.) Finally, at the 4.5PN level, we have 20 variables in the reactive acceleration, 10 variables determining the energy ambiguity $\tilde{E}_{4.5}$ and 6 variables determining the ambiguity in $\tilde{\mathbf{J}}_{4.5}$, *i.e.,* 36 variables in all. The balance equations lead to 25 – 15 from energy and 10 from angular momentum – equations of which 24 are linearly independent, again leaving *only one redundant equation.* The linear system of 24 equations for 36 variables is the same as before and leads to the solution obtained in the previous section in terms of 12 arbitrary parameters. (The four extra variables in the IW21 scheme are identically zero.) The IW22 (minimal) scheme thus confirms the conjecture that the occurrence of the second redundant equation is special to the IW scheme (IW21) and is related to the choice they make for the functional form of the $\tilde{\mathbf{J}}$ ambiguity by pulling out only one factor of nonlinearity m/r rather than its square – the minimal choice. To double check the above explanation, we performed another experiment by examining a variant that would generate an increased number of redundant or degenerate equations. This scheme denoted by IW11 differs from IW21 in that the ambiguity in $\tilde{\mathbf{E}}^*$ is assumed to have $(8/5)\eta(m/r)\dot{r}$ as the common factor, *i.e.,* by pulling out only one order of nonlinearity m/r rather than its square as in IW21; the polynomial representing the ambiguity in E is consequently of *one order more* than in IW21. In this case, at 2.5PN order one has $6+6+3=15$ variables and $10+6=16$ equations of which *3* are *redundant.* The 13 equations for 15 variables thus yield the required solution in terms of 2 arbitrary parameters and similarly for higher orders. One may also explore the most general of choices in which only $(8/5)\eta$ is pulled outside and the ambiguity is the highest order polynomial consistent with the order of the approximation. We studied one such scheme (IWOO) in the Newtonian RR case. For convenience, the various experiments are summarized in Table 5.1.

To conclude: at 2.5PN, 3.5PN and 4.5PN orders all variants of IW examined in this subsection with different forms of the ambiguities in E and $\tilde{\mathbf{J}}$ – minimal (IW22) or IW11 – lead to identical reactive accelerations including their gauge arbitrariness.

At this juncture one may wonder about the issues of the 'uniqueness' and 'am-

Table 5.1: Comparison of four Alternative Schemes : IW21, IW22 (Minimal), IW11 and IW00. N denotes the order of approximation, NV the number of variables, NC the number of constraints coming from balance equations, ND the number of degenerate equations, NI the number of independent equations and NA the number of arbitrary parameters determining the solution. In the NV column, a + b + c means a variables of reactive acceleration, b in energy ambiguity and c in angular momentum ambiguity.

N	NV	NC	ND	NI	NA
IW21: IW Scheme					
2.5PN	6+3+3	12	2	10	2
3.5PN	12+6+6	20	2	18	6
4.5PN	20+10+10	30	2	28	12
IW22: Minimal Scheme					
2.5PN	6+3 +1	9	1	8	2
3.5PN	12+6+3	16	1	15	6
4.5PN	20+10+6	25	1	24	12
IW11 Scheme					
2.5PN	6+6+3	16	3	13	2
3.5PN	12+10+6	25	3	22	6
4.5PN	20+15+10	36	3	33	12
IW00 Scheme					
2.5PN	6+10+6	25	5	20	2

biguities' of the schemes discussed earlier. In this regard, we would like to make the following general remarks. For general orbits, in addition to the balance of energy one must take into account the balance of angular momentum. Thus, schemes involving only energy balance are not relevant except in special cases like 'circular orbits' and 'radial infall' (see section 5.5). Can one have schemes where one implements both energy and angular momentum balance but does *not* take into account the possible ambiguities in E and $\tilde{\mathbf{J}}$? One can show that even at the 2.5PN level this system of equations is inconsistent! Further, is the ambiguity necessary *both* in E and $\tilde{\mathbf{J}}$? If one examines a scheme with both energy and angular momentum balance taking account of the ambiguity *only* in E one does obtain a consistent solution upto 4.5PN order but with only half the number of arbitrary parameters as in the IW scheme. The reduced 'gauge' freedom is not adequate to treat as special cases the

Burke-Thorne gauge at the 2.5PN level or the Blanchet choice at the 3.5PN level. And finally, in a scheme with both energy and angular momentum balance taking account of the ambiguity only in $\tilde{\mathbf{J}}$ one obtains a consistent solution at 2.5PN order containing no arbitrary parameters at all. No solution is possible at higher orders.

On general considerations, the reactive acceleration should be a power series in the individual masses m_1 and m_2 or equivalently, it should be nonlinear in the total mass m as assumed in earlier sections. It is interesting to investigate whether the functional forms of the far-zone fluxes and the balance procedure necessarily lead to such 'physical' solutions alone or whether they are consistent with more general possibilities. In section 5.6, for mathematical completeness [155] we investigate this question in detail and prove that the flux formulas and balance equations do not constrain the reactive acceleration to their 'physical' forms alone but allow for a more general form for the reactive acceleration.

5.4 Arbitrariness in reactive terms and gauge choice

It is well known that the formulas for the energy and angular momentum fluxes in the far-zone are gauge invariant, *i.e.*, independent of the changes in the coordinate system that leave the spacetime asymptotically flat. On the other hand, the expressions for the reactive force are 'gauge dependent' and consequently e.g., the Chandrasekhar form is different from the Burke-Thorne or Damour-Deruelle forms. In IW it was shown that the Burke-Thorne gauge corresponds to the values $\beta_2 = 4$ and $\alpha_3 = 5$, while the Damour-Deruelle choice corresponds to $\beta_2 = -1$ and $\alpha_3 = 0$. It can also be shown that the ADM choice corresponds to $\beta_2 = \frac{5}{3}$ and $\alpha_3 = 3$ [156]. It was further shown that the reactive acceleration implied by Blanchet's first principles determination of the 1PN radiation reaction indeed corresponds to a particular choice of the arbitrary parameters in the IW solution. One of the satisfactory aspects of IW was the demonstration that the part of the reactive acceleration not determined by the balance requirement was precisely related to the possible ambiguity in the choice of the gauge at that order. (The flux is equal to the time variation of the conserved quantities only upto total time derivatives; this ambiguity may be absorbed in a 'change' in the relative separation vector as discussed below.)

Following IW, we seek to establish the correspondence between the arbitrary parameters contained in the radiation reaction terms and the residual gauge freedom in the construction. The residual gauge freedom arises from the fact that the far-zone fluxes Eqs.(5.7), (5.8) are independent of changes in the coordinate system that leave the spacetime asymptotically flat. These coordinate changes will induce a change in x which is the difference between the centers of mass of the two bodies $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ at coordinate time t. Following IW, we choose the transformation to be of the form x $\rightarrow \mathbf{x}' =$ x $+\ \delta\mathbf{x}$, where Sx can depend only on the two vectors x and v,

$$\delta\mathbf{x} = (f2.5 + f3.5 + f_{4.5})\dot{r}\mathbf{x} + (92.5 + 93.5 + g_{4.5})r\mathbf{v}\,. \tag{5.25}$$

In order that $\delta\mathbf{x}/\mathbf{x}$ be $O(\epsilon^{2.5})$, $O(\epsilon^{3.5})$ and $O(\epsilon^{4.5})$, $f_{2.5}$ and $g_{2.5}$ must be $O(\epsilon^2)$, $f_{3.5}$ and $g_{3.5}$ must be $0(c^3)$ and $f_{4.5}$ and $g_{4.5}$ must be $O(c^4)$. As for the other variables, the f's and g's will also be polynomials in the variables m/r, v^2 and $\dot{r}^2$. As pointed out in [34], we do not independently take into account changes in the coordinate time t since the v-dependent term in $\delta\mathbf{x}$ includes this contribution via $\mathbf{x}(\mathbf{t} + \mathrm{St}) \sim \mathbf{x}(\mathbf{t}) + \mathbf{v}\delta\mathbf{t}$.

In [34] it was proved that to cancel the dependence on the two 2.5PN arbitrary parameters and the six 3.5PN arbitrary parameters, Sx should be chosen such that

$$
\begin{aligned}
f_{2.5} &= \frac{8}{15}\eta\left(\frac{m}{r}\right)^2\alpha_3\,, && (5.26a)\\
g_{2.5} &= \frac{8}{15}\eta\left(\frac{m}{r}\right)^2(2\alpha_3 - 3\beta_2)\,, && (5.26b)\\
f_{3.5} &= \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\left[P_{21}v^2 + P_{22}\left(\frac{m}{r}\right) + P_{23}\dot{r}^2\right], && (5.26c)\\
g_{3.5} &= \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\left[Q_{21}v^2 + Q_{22}\left(\frac{m}{r}\right) + Q_{23}\dot{r}^2\right], && (5.26d)
\end{aligned}
$$

where P_{ab}'s and Q_{ab}'s are given by

$$
\begin{aligned}
P_{21} &= \frac{1}{3}\left[\xi_2+\frac{2}{5}\xi_4-\rho_5-\frac{1}{2}\alpha_3(1-3\eta)\right], && (5.27a)\\
P_{22} &= -\frac{1}{6}\left[\xi_2+\xi_4-\frac{3}{2}\xi_5-\rho_5-\frac{3}{2}\beta_2\eta+\frac{1}{2}\alpha_3(4+11\eta)\right], && (5.27b)\\
P_{23} &= \frac{1}{5}\xi_4, && (5.27c)\\
Q_{21} &= \left[\xi_1+\frac{2}{3}\xi_2+\frac{8}{15}\xi_4+\frac{1}{2}(3\beta_2-2\alpha_3)(1-3\eta)\right], && (5.27d)\\
Q_{22} &= -\frac{1}{6}\left[6\xi_1+5\xi_2-3\xi_3+5\xi_4-\frac{3}{2}\xi_5+\rho_5-\frac{63}{2}\beta_2\eta-\frac{1}{2}\alpha_3(4-55\eta)\right], && (5.27e)\\
Q_{23} &= \frac{1}{3}\left[\frac{2}{5}\xi_4+\rho_5-\alpha_3(1-3\eta)\right]. && (5.27f)
\end{aligned}
$$

We provisionally choose the 4.5PN part of $\mathbf{S}\mathbf{x}$ to be of the form

$$
\begin{aligned}
f_{4.5} &= \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\left[P_{41}v^4+P_{42}v^2\frac{m}{r}+P_{43}v^2\dot{r}^2+P_{44}\left(\frac{m}{r}\right)^2\right.\\
&\quad\left.+P_{45}\left(\frac{m}{r}\right)\dot{r}^2+P_{46}\dot{r}^4\right], && (5.28a)\\
g_{4.5} &= \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\left[Q_{41}v^4+Q_{42}v^2\frac{m}{r}+Q_{43}v^2\dot{r}^2+Q_{44}\left(\frac{m}{r}\right)^2\right.\\
&\quad\left.+Q_{45}\left(\frac{m}{r}\right)\dot{r}^2+Q_{46}\dot{r}^4\right]. && (5.28b)
\end{aligned}
$$

The change in the 2PN equations of motion Eqs.(5.3) produced by this change of variable Eq.(5.25) can be determined using the known form of $\delta\mathbf{x}$ upto 3.5PN order Eqs.(5.26), (5.27), the provisional form chosen above for the 4.5PN terms Eq.(5.28) and the transformations given below:

$$
\begin{aligned}
\mathbf{x}\to\mathbf{x}' &= \mathbf{x}+\delta\mathbf{x},\\
\mathbf{v}\to\mathbf{v}' &= \mathbf{v}+\delta\mathbf{v}=\frac{d\mathbf{x}}{dt}+\frac{d\delta\mathbf{x}}{dt},\\
r\to r' &= r\left[1+\frac{\mathbf{n}\cdot\delta\mathbf{x}}{r}\right],\\
\frac{\mathbf{x}'}{r'^p} &= \frac{\mathbf{x}}{r^p}+\frac{\delta\mathbf{x}}{r^p}-\frac{p\mathbf{n}}{r^p}(\mathbf{n}\cdot\delta\mathbf{x}),\\
v^2\to v'^2 &= v^2+\left[2\mathbf{v}\cdot\frac{d\delta\mathbf{x}}{dt}\right],\\
\dot{r}\to\dot{r}' &= \frac{1}{r}\left[r\dot{r}+\delta\mathbf{x}\cdot\mathbf{v}+\mathbf{x}\cdot\frac{d\delta\mathbf{x}}{dt}-(\mathbf{n}\cdot\delta\mathbf{x})\dot{r}\right]. && (5.29)
\end{aligned}
$$

The gauge change generates reactive terms and the requirement that this change should cancel the dependence of the radiation-reaction terms on arbitrary parameters dictates that

$$P_{41} = -\frac{1}{24}\alpha_3(1-9\eta+21\eta^2) - \frac{1}{30}(5\xi_2+2\xi_4-5\rho_5)(1-3\eta) + \frac{1}{3}\psi_2 + \frac{2}{15}\psi_4 + \frac{8}{105}\psi_7 - \frac{1}{3}\chi_6 - \frac{2}{15}\chi_8 \tag{5.30a}$$

$$P_{42} = -\frac{1}{6}\alpha_3(3+\eta^2) + \frac{3}{8}\beta_2\eta - \frac{1}{4}\xi_1\eta + \frac{1}{12}\xi_2(3-23\eta) + \frac{1}{60}\xi_4(19-77\eta) - \frac{1}{8}\xi_5(1-3\eta) - \frac{1}{12}\rho_5(3-22\eta) - \frac{1}{3}\psi_2 - \frac{4}{15}\psi_4 + \frac{1}{4}\psi_6 - \frac{1}{5}\psi_7 + \frac{1}{12}\psi_8 + \frac{1}{3}\chi_6 + \frac{4}{15}\chi_8 - \frac{1}{4}\chi_9 \tag{5.30b}$$

$$P_{43} = -\frac{1}{10}\xi_4(1-3\eta) + \frac{1}{5}\psi_4 + \frac{4}{35}\psi_7 - \frac{1}{5}\chi_8 \tag{5.30c}$$

$$P_{44} = \frac{1}{30}\alpha_3(13+12\eta+16\eta^2) + \frac{1}{5}\beta_2(1+12\eta-2\eta^2) + \frac{1}{10}(\xi_1-2\xi_3)\eta + \frac{1}{30}(\xi_2+\xi_4)(9+31\eta) - \frac{1}{20}\xi_5(7+13\eta) - \frac{1}{30}\rho_5(9+28\eta) + \frac{2}{15}\psi_2 + \frac{2}{15}\psi_4 - \frac{1}{10}\psi_6 + \frac{2}{15}\psi_7 - \frac{1}{10}\psi_8 + \frac{1}{5}\psi_9 - \frac{2}{15}\chi_6 - \frac{2}{15}\chi_8 + \frac{1}{10}\chi_9 \tag{5.30d}$$

$$P_{45} = \frac{1}{12}\alpha_3\eta^2 - \frac{1}{4}\beta_2\eta(1-3\eta) + \frac{1}{6}\xi_2\eta - \frac{1}{15}\xi_4(1+7\eta) - \frac{1}{3}\rho_5\eta - \frac{1}{15}\psi_4 - \frac{2}{15}\psi_7 + \frac{1}{6}\psi_8 + \frac{1}{15}\chi_8 \tag{5.30e}$$

$$P_{46} = \frac{1}{7}\psi_7 \tag{5.30f}$$

$$Q_{41} = \frac{1}{8}(2\alpha_3-3\beta_2)(1-\eta-11\eta^2) - \frac{1}{10}(15\xi_1+10\xi_2+8\xi_4)(1-3\eta) + \psi_1 + \frac{2}{3}\psi_2 + \frac{8}{15}\psi_4 + \frac{16}{35}\psi_7 \tag{5.30g}$$

$$Q_{42} = -\frac{1}{24}\alpha_3(108-331\eta+197\eta^2) + \frac{1}{8}\beta_2(48-121\eta+63\eta^2) + \frac{1}{2}\xi_1(9-28\eta) + \frac{1}{12}\xi_2(49-142\eta) - \frac{1}{8}(6\xi_3+3\xi_5-2\rho_5)(1-3\eta) + \frac{1}{60}\xi_4(231-653\eta) - 2\psi_1 - \frac{5}{3}\psi_2 + \frac{1}{2}\psi_3 - \frac{47}{30}\psi_4 + \frac{1}{4}\psi_6 - \frac{22}{15}\psi_7 + \frac{1}{6}\psi_8 - \frac{1}{6}\chi_6 - \frac{1}{10}\chi_8 \tag{5.30h}$$

$$Q_{43} = \frac{1}{6}\alpha_3(1-3\eta-3\eta^2) - \frac{1}{6}(2\xi_2+2\xi_4+\rho_5)(1-3\eta) + \frac{2}{15}\psi_4 + \frac{16}{105}\psi_7 + \frac{1}{3}\chi_6 + \frac{2}{15}\chi_8 \tag{5.30i}$$

$$Q_{44} = \frac{1}{30}\alpha_3(32+73\eta+254\eta^2) - \frac{1}{30}\beta_2(51+157\eta+258\eta^2) - \frac{1}{30}\xi_1(10-307\eta)$$

$$
\begin{aligned}
& -\frac{1}{30}\xi_2(19-279\eta)-\frac{1}{30}\xi_3(15+59\eta)-\frac{1}{30}\xi_4(19-279\eta)-\frac{1}{60}\xi_5(9+91\eta) \\
& +\frac{1}{30}\rho_5(9+28\eta)+\frac{4}{3}\psi_1+\frac{6}{5}\psi_2-\frac{1}{3}\psi_3+\frac{6}{5}\psi_4+\frac{1}{3}\psi_5-\frac{7}{30}\psi_6 \\
& +\frac{6}{5}\psi_7-\frac{7}{30}\psi_8+\frac{2}{15}\psi_9+\frac{2}{15}\chi_6+\frac{2}{15}\chi_8-\frac{1}{10}\chi_9 \qquad (5.30j) \\
Q_{45} = & -\frac{1}{24}\alpha_3(24-29\eta-91\eta^2)-\frac{33}{8}\beta_2\eta^2+\frac{3}{4}\xi_1\eta+\frac{1}{12}\xi_2(2+\eta)+\frac{1}{15}\xi_4(6-13\eta) \\
& -\frac{1}{4}\xi_5(1-3\eta)-\frac{3}{4}\rho_5\eta-\frac{1}{10}\psi_4-\frac{1}{5}\psi_7+\frac{1}{12}\psi_8-\frac{1}{6}\chi_6-\frac{7}{30}\chi_8+\frac{1}{4}\chi_9 \qquad (5.30k) \\
Q_{46} = & -\frac{1}{5}\xi_4(1-3\eta)+\frac{2}{35}\psi_7+\frac{1}{5}\chi_8\,. \qquad (5.30l)
\end{aligned}
$$

The above computation shows that as at the 3.5PN order the (12 parameter) arbitrariness in the 4.5PN radiation reaction formulas reflects the residual freedom that is available to one in the choice of a 4.5PN accurate 'gauge'. Every particular 4.5PN accurate radiation reaction formula should correspond to a particular choice of these 12 parameters.

5.5 Particular cases: quasi-circular orbits and head-on infall

In this section we specialise our solutions valid for general orbits to the particular case of quasi-circular orbits and. radial infall and verify that they indeed reproduce the simpler reactive solutions one would obtain if one formulated the problem ab *initio* appropriate to these two special cases. We first consider the quasi-circular limit that is of immediate relevance to sources for the ground based interferometric gravitational wave detectors. In this particular case, the reactive acceleration may be deduced using *only* the energy balance. Using the reactive acceleration we compute the 4.5PN contribution to $\dot{r}$ and w. We also discuss the complementary case of the radial infall of two compact objects of arbitrary mass ratio and determine the 4.5PN contribution to the radial infall velocity for the two special cases: radial infall from infinity and radial infall with finite initial separation.

5.5.1 Quasi-circular inspiral

Using our general reactive solution we can compute the physically relevant quantities $\dot{r}$ and $\dot{\omega}$ for quasi-circular inspiral, where r and *w* are the orbital separation and the orbital angular frequency in harmonic coordinates, respectively. As would be expected, these results are independent of the arbitrary parameters that are present in the reactive solution. We obtain the radiation reaction contribution to a upto 4.5PN for quasi-circular inspiral by setting $\dot{r} = 0 + O(\epsilon^{2.5})$ and using

$$v^2 = \frac{m}{r}\left[1 - (3-\eta)\frac{m}{r} + \left(6 + \frac{41\eta}{4} + \eta^2\right)\left(\frac{m}{r}\right)^2\right], \tag{5.31}$$

in Eqs.(5.4), (5.9), (5.12) and (5.17). We get

$$\begin{aligned} \mathbf{a}_{RR} &= -\frac{32\eta m^3 \mathbf{v}}{5r^4}\left[1 - \left(\frac{3431}{336} - \frac{5}{4}\right)\frac{m}{r}\right. \\ &\quad \left. + \left(\frac{794369}{18144} + \frac{26095}{2016}\eta - \frac{7}{4}\eta^2\right)\left(\frac{m}{r}\right)^2\right]. \end{aligned} \tag{5.32}$$

It is worth noting that for quasi-circular inspiral the energy flux determines the reactive acceleration without any gauge ambiguity. All the arbitrary terms in energy are proportional to $\dot{r}$ and hence play no role in this instance. Inverting Eq.(5.31), we get

$$\frac{m}{r} = u^2\left[1 + (3-\eta)v^2 + \frac{1}{4}(48 - 89\eta + 4\eta^2)v^4\right]. \tag{5.33}$$

Differentiating Eq.(5.33) w.r.t t and noting that the a that appears is the total acceleration (conservative + reactive) we get, after some rearrangement

$$\begin{aligned} r &= -\frac{64}{5}\eta\left(\frac{m}{r}\right)^3\left[1 - \left(\frac{1751}{336} + \frac{7\eta}{4}\right)\frac{m}{r}\right. \\ &\quad \left. + \left(\frac{303455}{18144} + \frac{40981\eta}{2016} + \frac{\eta^2}{2}\right)\left(\frac{m}{r}\right)^2\right]. \end{aligned} \tag{5.34}$$

Using Eq.(5.34) and the expression for angular velocity ($w \equiv v/r$)

$$\omega^2 = \frac{m}{r^3}\left[1 - (3-\eta)\frac{m}{r} + (6 + \frac{41\eta}{4} + \eta^2)\left(\frac{m}{r}\right)^2\right], \tag{5.35}$$

we may express w as

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\eta(m\omega)^{\frac{5}{3}}\left[1-(m\omega)^{\frac{2}{3}}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\right.$$
$$\left.+\left(\frac{34103}{18144}+\frac{13661}{2016}\eta+\frac{59}{18}\eta^2\right)(m\omega)^{\frac{4}{3}}\right]. \quad (5.36)$$

The results Eqs.(5.34) and (5.36) are in agreement with [18] as expected and required, suggesting that the reactive terms obtained here could be used to evolve orbits in the more general case also [157] .

5.5.2 Head-on infall

Recently Simone, Poisson and Will [139] have obtained to 2PN accuracy the gravitational wave energy flux produced during head-on infall and starting from these formulas one can deduce ab *initio* the reactive acceleration in this limit adapting IW to the radial infall case. As required, these results match exactly with expressions obtained by applying radial infall limits to the general orbit solutions and we summarize the relevant formulas in this limit in what follows. Equations representing the head-on infall can be obtained from the general orbit expressions by imposing the restrictions, x = zn, v = $\dot{z}\hat{\mathbf{n}}$, $r = z$ and v = $\dot{r} = \dot{z}$. For radial infall the conserved energy Eq.(5.6) to 2PN order then becomes

$$E(z) = \mu\left\{\frac{\dot{z}^2}{2}-\gamma+\frac{3(1-3\eta)\dot{z}^4}{8}+\frac{(3+2\eta)\gamma\dot{z}^2}{2}+\frac{\gamma^2}{2}\right.$$
$$+\frac{5(1-7\eta+13\eta^2)\dot{z}^6}{16}+\frac{3(7-8\eta-16\eta^2)\gamma\dot{z}^4}{8}+$$
$$\left.\frac{(9+7\eta+8\eta^2)\gamma^2\dot{z}^2}{4}-\frac{(2+15\eta)\gamma^3}{4}\right\}, \quad (5.37)$$

where $\gamma = m/z$. Unlike the quasi-circular inspiral, for head-on infall we can distinguish between two different cases. Following [139] we denote them by (A) and (B), respectively, and list the expressions relevant for our computations. In case (A), the radial infall proceeds from rest at infinite initial separation, $E(z) = E(\infty) = 0$, and inverting Eq.(5.37) we get

$$\dot{z} = -\left\{\frac{2m}{z}\left[1 - 5\gamma\left(1 - \frac{\eta}{2}\right) + \gamma^2\left(13 - \frac{81\eta}{4} + 5\eta^2\right)\right]\right\}^{1/2}. \tag{5.38}$$

In case (B), the radial infall proceeds from rest at finite initial separation z_0, which implies

$$E(z) = E(z_0) = -\mu\left\{\gamma_0 - \frac{\gamma_0^2}{2} + \frac{\gamma_0^3}{2}\left(1 + \frac{15\eta}{2}\right)\right\}. \tag{5.39}$$

We obtain as in case (A), an expression for $\dot{z}$ given by

$$\begin{aligned}
\dot{z} \;=\; & -\left\{2(\gamma - \mathrm{yo})\left[1 - 5\gamma\left(1 - \frac{\eta}{2}\right) + \gamma_0\left(1 - \frac{9\eta}{2}\right)\right.\right.\\
& +\gamma^2\left(13 - \frac{81\eta}{4} + 5\eta^2\right) - \gamma\gamma_0\left(5 - \frac{173\eta}{4} + 13\eta^2\right)\\
& \left.\left.+\gamma_0^2\left(1 - \frac{5\eta}{4} + 8\eta^2\right)\right]\right\}^{\frac{1}{2}},
\end{aligned} \tag{5.40}$$

where $\gamma_0 = m/z_0$. We first compute the 4.5PN contribution to $\ddot{z}$ for case (B), the radial infall from finite initial separation. We use the radial infall restriction along with Eq.(5.40) in Eqs.(5.4), (5.9), (5.12) and (5.17) to obtain 4.5PN terms in $\ddot{z}$ as

$$\begin{aligned}
\ddot{z} \;=\; & \frac{8\eta\gamma^3}{5m}(2\gamma - 2\gamma_0)^{\frac{1}{2}}\left\{\frac{1}{3}(-41 + 21\zeta_1)\gamma + (8 - 4\zeta_1)\gamma_0\right.\\
& +\left[\left(\frac{1}{84}(18054 - 13231\eta) - \frac{1}{4}(438 - 331\eta)\zeta_1 + 18\zeta_2 + 9\zeta_3\right)\gamma^2\right.\\
& +\left(-\frac{1}{28}(5510 - 8849\eta) + \frac{1}{4}(402 - 643\eta)\zeta_1 - 26\zeta_2 - 6\zeta_3\right)\gamma\gamma_0\\
& \left.+\left(36 - 126\eta - (18 - 63\eta)\zeta_1 + 8\zeta_2\right)\gamma_0^2\right]\\
& +\left[\left(-\frac{1}{18144}(30549820 - 54233376\eta + 15776427\eta^2)\right.\right.\\
& +\frac{1}{32}(27156 - 49816\eta + 15057\eta^2)\zeta_1\\
& \left.-\frac{1}{2}(766 - 527\eta)\zeta_2 - \frac{1}{4}(546 - 417\eta)\zeta_3 + 22\zeta_4 + 44\zeta_5 + 11\zeta_6\right)\gamma^3\\
& +\left(\frac{1}{3024}(6314916 - 20766190\eta + 8663249\eta^2)\right.\\
& -\frac{1}{16}(17052 - 56198\eta + 23811\eta^2)\zeta_1\\
& \left.+(680 - 759\eta)\zeta_2 + \frac{1}{4}(546 - 855\eta)\zeta_3 - 34\zeta_4 - 104\zeta_5 - 8\zeta_6\right)\gamma^2\gamma_0\\
& +\left(-\frac{1}{2016}(1521308 - 7938232\eta + 5800187\eta^2)\right.
\end{aligned}$$

$$+\frac{1}{32}(12372-64104\eta+46641\eta^2)\zeta_1$$
$$-\frac{1}{2}(682-1315\eta)\zeta_2-\frac{1}{2}(54-189\eta)\zeta_3+12\zeta_4+76\zeta_5\Big)\gamma\gamma_0^2$$
$$+\Big(\frac{1}{8}(348-2016\eta+3339\eta^2)(2-\zeta_1)+(44-162\eta)\zeta_2-16\zeta_5\Big)\gamma_0^3\Big]\Big\}. \tag{5.41}$$

To obtain the 2PN reactive terms for case (A), the radial infall from infinity, we use in Eqs.(5.4), (5.9), (5.12) and (5.17) the radial infall restriction and Eq.(5.38). The expression thus obtained is the same as obtained by putting $\gamma_0 = 0$ in Eq.(5.41). The ζ 's in Eq.(5.41) are given by

$$\begin{aligned}
\zeta_1 &= \alpha_3-\beta_2\,,\\
\zeta_2 &= \xi_1+\xi_2+\xi_4\,,\\
\zeta_3 &= \xi_3+\xi_5\,,\\
\zeta_4 &= \psi_3+\psi_6+\psi_8\,,\\
\zeta_5 &= \psi_1+\psi_2+\psi_4+\psi_7\,,\\
\zeta_6 &= \psi_5+\psi_9\,.
\end{aligned} \tag{5.42}$$

We have also computed the 2PN reactive terms for cases (A) and (B) ab *initio* using the IW method adapted to radial infall. In this case, only energy balance is needed as J = 0 for head-on infall. The result thus obtained is in agreement with Eq.(5.41). Eq.(5.41) may be integrated straightforwardly to obtain the 4.5PN contribution to $\dot{z}^2$ in case (B) and it yields

$$\begin{aligned}
\dot{z}^2 = \frac{16(2\gamma-2\gamma_0)^{\frac{3}{2}}\eta}{5}\Big\{&\frac{1}{21}(41-21\zeta_1)\gamma^2-\frac{4}{105}\gamma\gamma_0-\frac{8}{315}\gamma_0^2\\
&+\Big[\Big(\frac{1}{756}(-18054+13231\eta)+\frac{1}{36}(438-331\eta)\zeta_1-2\zeta_2-\zeta_3\Big)\gamma^3\\
&+\Big(\frac{1}{252}(1926-7597\eta)+\frac{1}{168}(660-2534\eta)\zeta_1+2\zeta_2\Big)\gamma_0\gamma^2\\
&+\Big(\frac{1}{315}(-342+341\eta)+\frac{1}{525}(240-280\eta)\zeta_1\Big)\gamma_0^2\gamma\\
&+\Big(\frac{1}{945}(-684+682\eta)+\frac{1}{4725}(1440-1680\eta)\zeta_1\Big)\gamma_0^3\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\Big[\Big(\frac{1091065}{7128}-\frac{564931\eta}{2079}+\frac{5258809\eta^2}{66528}\\
&-\frac{1}{352}(27156-49816\eta+15057\eta^2)\zeta_1\\
&+\frac{1}{22}(766-527\eta)\zeta_2+\frac{1}{44}(546-417\eta)\zeta_3-2\zeta_4-4\zeta_5-\zeta_6\Big)\gamma^4\\
&+\Big(-\frac{21548237}{224532}+\frac{26019487\eta}{49896}-\frac{2750389\eta^2}{11088}\\
&+\frac{1}{528}(26316-139638\eta+67231\eta^2)\zeta_1\\
&-\frac{1}{99}(4416-6241\eta)\zeta_2-\frac{1}{132}(546-2023\eta)\zeta_3+2\zeta_4+8\zeta_5\Big)\gamma_0\gamma^3\\
&+\Big(\frac{3823453}{149688}-\frac{1681430\eta}{14553}+\frac{4399627\eta^2}{22176}\\
&-\frac{1}{2464}(30828-146592\eta+244127\eta^2)\zeta_1\\
&+\frac{1}{5082}(53262-202741\eta)\zeta_2+\frac{1}{77}(24-28\eta)\zeta_3-4\zeta_5\Big)\gamma_0^2\gamma^2\\
&+\Big(\frac{567739}{187110}+\frac{608992\eta}{72765}-\frac{228227\eta^2}{27720}-\frac{1}{385}(504+1080\eta-1622\eta^2)\zeta_1\\
&-\frac{1}{2541}(1056-1232\eta^2)\zeta_2+\frac{1}{385}(96-112\eta)\zeta_3\Big)\gamma_0^3\gamma\\
&+\Big(\frac{567739}{280665}+\frac{1217984\eta}{218295}-\frac{228227\eta^2}{41580}-\frac{1}{1155}(1008+2160\eta-3244\eta^2)\zeta_1\\
&-\frac{1}{22869}(6336-7392\eta)\zeta_2\\
&+\frac{1}{63525}(10560-12320\eta)\zeta_3\Big)\gamma_0^4\Big]\Big\}\,. \qquad (5.43)
\end{aligned}
$$

We obtain the 4.5PN contribution to $\dot{z}^2$ for case **(A)** by putting $\gamma_0 = 0$ in Eq.(5.43). Unlike in the case of quasi-circular **inspiral** the expressions in the head-on or radial **infall** cases are dependent on the choice of arbitrary variables or the choice of 'gauge'.

5.6 The general solution to the balance method

5.6.1 The 2.5PN reactive solution

It should be noted that all the discussion in section **5.3** follows only after one has *assumed* a functional form for the reactive acceleration – in particular, the intuitive requirement that it be nonlinear, *i.e.*, contain an overall factor of m/r. It is pertinent to ask whether more general possibilities obtain, consistent with the far-zone fluxes, if one relaxes this requirement. We have explored this question in detail at the 2.5PN level and we summarize the results in what follows. In this instance the reactive acceleration is assumed to be:

$$\begin{aligned}
\mathbf{a} &= -\frac{8}{5}\eta\left(\frac{m}{r^2}\right)\left[-(\mathcal{A}_{2.5})\dot{r}\mathbf{n} + (\mathcal{B}_{2.5})\mathbf{v}\right], \\
\mathcal{A}_{2.5} &= a_1' v^4 + a_2' v^2\frac{m}{r} + a_3' v^2\dot{r}^2 + a_4'\left(\frac{m}{r}\right)^2 + a_5'\left(\frac{m}{r}\right)\dot{r}^2 + a_6'\dot{r}^4, \\
\mathcal{B}_{2.5} &= b_1' v^4 + b_2' v^2\frac{m}{r} + b_3' v^2\dot{r}^2 + b_4'\left(\frac{m}{r}\right)^2 + b_5'\left(\frac{m}{r}\right)\dot{r}^2 + b_6'\dot{r}^4, \qquad (5.44)
\end{aligned}$$

i.e., it is determined by 12 reactive coefficients instead of the earlier 6. Recall that the nomenclature IW22, IW21 and IW11 refers to the functional forms chosen for the ambiguity in energy and angular momentum and we introduce similar notation EJ22, EJ21 and EJ11, respectively, in this section. where the acceleration has a more general form as given by Eq.(5.44). With this form of the reactive acceleration, however, one gets e.g., in the EJ21 scheme at 2.5PN

$$\begin{aligned}
\tilde{E}^* &\equiv \tilde{E}_N + \tilde{E}_{2.5} \\
&= \tilde{E}_N - \frac{8}{5}\eta\left(\frac{m}{r}\right)^2 \dot{r}(\alpha_1 v^2 + \alpha_2\frac{m}{r} + \alpha_3\dot{r}^2), \qquad (5.45a) \\
\tilde{\mathbf{J}}^* &\equiv \tilde{\mathbf{L}}_N + \tilde{\mathbf{J}}_{2.5} \\
&= \tilde{\mathbf{L}}_N + \frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r}\dot{r}(\beta_1 v^2 + \beta_2\frac{m}{r} + \beta_3\dot{r}^2). \qquad (5.45b)
\end{aligned}$$

The derivatives of E* and $\tilde{\mathbf{J}}^*$ with the new form of the reactive acceleration are given by

$$\begin{aligned}
\frac{d\tilde{E}^*}{dt} = &-\frac{8}{5}\eta\frac{m}{r^2}\Big[(b_1')v^6 + (b_2' + \alpha_1)\frac{m}{r}v^4 + (-a_1' + b_3')\dot{r}^2 v^4 + (b_4' - \alpha_1 + \alpha_2)\left(\frac{m}{r}\right)^2 v^2 \\
&+(-a_3' + b_6')\dot{r}^4 v^2 + (-a_2' + b_5' - 3\alpha_1 + 3\alpha_3)\left(\frac{m}{r}\right)\dot{r}^2 v^2 - \alpha_2\left(\frac{m}{r}\right)^3 \\
&-(a_4' + 2\alpha_1 + 4\alpha_2 + 3\alpha_3)\left(\frac{m}{r}\right)^2\dot{r}^2 - (a_5' + 5\alpha_3)\left(\frac{m}{r}\right)\dot{r}^4 - a_6'\dot{r}^6\Big], \qquad (5.46a)
\end{aligned}$$

$$\frac{d\tilde{\mathbf{J}}^*}{dt} = -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\left(\frac{m}{r^2}\right)\Big[(b_1' - \beta_1)v^4 + (b_2' + \beta_1 - \beta_2)\left(\frac{m}{r}\right)v^2 + (b_3' + 2\beta_1 - 3\beta_3)\dot{r}^2 v^2$$

$$+(b'_4+\beta_2)\left(\frac{m}{r}\right)^2+(b'_5+2\beta_1+3\beta_2+3\beta_3)\left(\frac{m}{r}\right)\dot{r}^2+(b'_6+4\beta_3)\dot{r}^4\Bigg]\,. \tag{5.46b}$$

Using Eqs.(5.45) and (5.46) one can understand the counts of the various variables summarized in Table 5.2.

Table 5.2: Comparison of the four Alternative Schemes: EJ21, EJ22, EJ11, and EJOO at 2.5PN level. The notation is as in Table 5.1. In the NC column, a + b indicates that a constraints arise from energy balance and b from angular momentum balance.

Scheme	NV	NC	ND	NI	NA
EJ22	12+3+1	10+6	2	14	2
EJ21	12+3 +3	10+6	1	15	3
EJ11	12+6+3	10+6	1	15	6
EJOO	12+10+6	15+10	3	22	6

One can explain the new counts for the arbitrary parameters by comparing e.g., the EJ21 scheme with a general form for the reactive acceleration as in this section with the IW21 scheme with the restricted form for reactive acceleration as in section 5.3. One has 6 extra variables and 4 extra equations. However one gains an extra equation because one of the degeneracies is lifted. The resulting 5 equations for 6 variables lead to an extra arbitrary parameter resulting in a 3 parameter solution in this instance. All the other entries in Table 5.2 can be similarly understood by comparison of Tables 5.1 and 5.2.

The reactive solution resulting from the EJ22 scheme in this instance is exactly the same as the IW21 reactive solution discussed earlier. From the EJ21 scheme one obtains a solution with three arbitrary parameters given by

$$a'_1=3\beta_3\,,\quad a'_2=3(1+\alpha_3-\beta_3)\,,\quad a'_3=-4\beta_3\,,$$

$$a'_4=23/3-3\alpha_3+2\beta_2\,,a'_5=-5\alpha_3\,,\quad a'_6=0\,, \tag{5.47a}$$

$$b'_1=0\,,\quad b'_2=2+\beta_2\,,\quad b'_3=3\beta_3\,,\quad b'_4=2-\beta_2\,,$$

$$b'_5=-3(1+\beta_2+\beta_3)\,,\quad b'_6=-4\beta_3\,. \tag{5.47b}$$

This construction can be generalized to 3.5PN and 4.5PN orders in which cases the number of arbitrary parameters are 8 and 15, respectively. The EJ11 and EJOO schemes on the other hand lead to a solution with six arbitrary parameters at the 2.5PN level. However, not all these solutions are similar in regard to the possibility of gauging away all the arbitrary parameters they contain.

5.6.2 The 2.5PN gauge arbitrariness

We have also investigated the question whether all the extra arbitrary parameters appearing in schemes with the general form of reactive acceleration (See Table 5.2.) can be gauged away? We find that at 2.5PN order, though this is possible with the 3 parameters of the EJ21 scheme, it is not true for the 6 arbitrary parameters in the EJ1l and EJOO schemes. For this reason the EJ1l and EJOO schemes are not satisfactory and we discuss them no further. We present here for the EJ21 scheme details of the gauge calculation at 2.5PN order. We choose **Sx** to be

$$\delta\mathbf{x} = \frac{8\eta}{5}(\frac{m}{r})\,(f'_{2.5}\dot{r}\mathbf{x} + g'_{2.5}r\mathbf{v})\ , \tag{5.48}$$

where $f'_{2.5}$ and $g'_{2.5}$ are given by

$$\begin{aligned} f'_{2.5} &= P'_{01}(\frac{m}{r}) + P'_{02}v^2 + P'_{03}\dot{r}^2\,, \\ g'_{2.5} &= Q'_{01}(\frac{m}{r}) + Q'_{02}v^2 + Q'_{03}\dot{r}^2\,. \end{aligned} \tag{5.49}$$

For the reactive acceleration given by Eqs.(5.44) and (5.47) we obtain

$$P'_{01} = \frac{1}{3}(\alpha_3 - \beta_3)\,, \tag{5.50a}$$

$$P'_{02} = \frac{1}{2}\beta_3\,, \tag{5.50b}$$

$$P'_{03} = 0\,, \tag{5.50c}$$

$$Q'_{01} = \frac{1}{3}(2\alpha_3 - 3\beta_2 + \beta_3)\,, \tag{5.50d}$$

$$Q'_{02} = 0\,, \tag{5.50e}$$

$$Q'_{03} = -\frac{1}{2}\beta_3\,. \tag{5.50f}$$

The EJ21 scheme leads to a more general solution to the balance equations, and as in IW all the arbitrary parameters that appear in its solution can be associated with a residual choice of gauge. It has been explored in detail upto 4.5PN and the results are summarized below. We list the new general reactive solutions and the corresponding gauge transformations for the arbitrary parameters they contain. For brevity, the solutions are presented in the form : 'New solution' = 'Old solution' + 'Difference'.

5.6.3 The 3.5PN and 4.5PN reactive solutions

The reactive acceleration is assumed to have the following general form

$$\mathbf{a} = -\frac{8}{5}\eta\frac{m}{r^2}[-(\mathcal{A}_{2.5}+\mathcal{A}_{3.5}+\mathcal{A}_{4.5})\dot{r}\mathbf{n}+(\mathcal{B}_{2.5}+\mathcal{B}_{3.5}+\mathcal{B}_{2.5})\mathbf{v}]\,, \tag{5.51}$$

with $\mathcal{A}_{2.5}$ and $\mathcal{B}_{2.5}$ given in Eqs.(5.44) and (5.47) and $\mathcal{A}_{3.5}$, $\mathcal{B}_{3.5}$, d4.5 and $\mathcal{B}_{4.5}$ given by

$$\begin{aligned}
\mathcal{A}_{3.5} &= f'_1 v^6 + f'_2 v^4\frac{m}{r} + f'_3 v^4\dot{r}^2 + f'_4 v^2\dot{r}^2\frac{m}{r} + f'_5 v^2\dot{r}^4 \\
&\quad + f'_6 v^2\left(\frac{m}{r}\right)^2 + f'_7\frac{m}{r}\dot{r}^4 + f'_8\left(\frac{m}{r}\right)^2\dot{r}^2 + f'_9\left(\frac{m}{r}\right)^3 + f'_{10}\dot{r}^6\,, && (5.52a)\\
\mathcal{B}_{3.5} &= g'_1 v^6 + g'_2 v^4\frac{m}{r} + g'_3 v^4\dot{r}^2 + g'_4 v^2\dot{r}^2\frac{m}{r} + g'_5 v^2\dot{r}^4 \\
&\quad + g'_6 v^2\left(\frac{m}{r}\right)^2 + g'_7\frac{m}{r}\dot{r}^4 + g'_8\left(\frac{m}{r}\right)^2\dot{r}^2 + g'_9\left(\frac{m}{r}\right)^3 + g'_{10}\dot{r}^6\,, && (5.52b)\\
\mathcal{A}_{4.5} &= h'_1 v^8 + h'_2 v^6\dot{r}^2 + h'_3 v^6\frac{m}{r} + h'_4 v^4\dot{r}^4 + h'_5 v^4\left(\frac{m}{r}\right)^2 \\
&\quad + h'_6 v^4\dot{r}^2\frac{m}{r} + h'_7 v^2\dot{r}^6 + h'_8 v^2\dot{r}^4\frac{m}{r} + h'_9 v^2\dot{r}^2\left(\frac{m}{r}\right)^2 + h'_{10} v^2\left(\frac{m}{r}\right)^3 \\
&\quad + h'_{11}\left(\frac{m}{r}\right)^4 + h'_{12}\dot{r}^2\left(\frac{m}{r}\right)^3 + h'_{13}\dot{r}^4\left(\frac{m}{r}\right)^2 + h'_{14}\dot{r}^6\frac{m}{r} + h'_{15}\dot{r}^8\,, && (5.52c)\\
\mathcal{B}_{4.5} &= k'_1 v^8 + k'_2 v^6\dot{r}^2 + k'_3 v^6\frac{m}{r} + k'_4 v^4\dot{r}^4 + k'_5 v^4\left(\frac{m}{r}\right)^2 \\
&\quad + k'_6 v^4\dot{r}^2\frac{m}{r} + k'_7 v^2\dot{r}^6 + k'_8 v^2\dot{r}^4\frac{m}{r} + k'_9 v^2\dot{r}^2\left(\frac{m}{r}\right)^2 + k'_{10} v^2\left(\frac{m}{r}\right)^3 \\
&\quad + k'_{11}\left(\frac{m}{r}\right)^4 + k'_{12}\dot{r}^2\left(\frac{m}{r}\right)^3 + k'_{13}\dot{r}^4\left(\frac{m}{r}\right)^2 + k'_{14}\dot{r}^6\frac{m}{r} + k'_{15}\dot{r}^8\,. && (5.52d)
\end{aligned}$$

With this form of the acceleration we have at 3.5PN

$$\frac{d\tilde{E}^*}{dt} = -\frac{8}{15}\eta\frac{m}{r^2}\Big[(\frac{m}{r})^2\Big(12v^2 - 11\dot{r}^2\Big) + \sum_{i=1}^{15}\mathcal{R}'^{[3.5]}_i\mathcal{Y}^{[4]}_i\Big], \tag{5.53a}$$

$$\frac{d\tilde{\mathbf{J}}^*}{dt} = -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\Big[\frac{m}{r}\Big(2v^2 + 2\frac{m}{r} - 3\dot{r}^2\Big) + \sum_{i=1}^{10}\mathcal{S}'^{[3.5]}_i\mathcal{Y}^{[3]}_i\Big], \tag{5.53b}$$

where $\mathcal{Y}^{[4]}_i$ is given by Eqs.(5.16),

$$\begin{aligned}\mathcal{Y}^{[3]}_i(i = 1\ldots 10) = & \Big[v^6, v^4\frac{m}{r}, v^4\dot{r}^2, v^2\left(\frac{m}{r}\right)^2, v^2\frac{m}{r}\dot{r}^2, v^2\dot{r}^4, \left(\frac{m}{r}\right)^3, \\ & \left(\frac{m}{r}\right)^2\dot{r}^2, \frac{m}{r}\dot{r}^4, \dot{r}^6\Big],\end{aligned} \tag{5.54}$$

and $\mathcal{R}'^{[3.5]}_i$, $\mathcal{S}'^{[3.5]}_i$ consist of corresponding linear combinations of the parameters involved. Repeating the procedure explained in the text, the 3.5PN reactive solution obtained is:

$$f'_1 = -\frac{3}{2}(1-3\eta)\beta_3 - 3\rho_2\,, \tag{5.55a}$$

$$f'_2 = f_1 - \frac{1}{2}(21+39\eta)\beta_3 + 3\rho_2\,, \tag{5.55b}$$

$$f'_3 = 2(1-3\eta)\beta_3 + 4\rho_2 - 5\rho_4\,, \tag{5.55c}$$

$$f'_4 = f_3 + \frac{1}{2}(56+15\eta)\beta_3 + 2\rho_2 + 5\rho_4\,, \tag{5.55d}$$

$$f'_5 = 6\rho_4\,, \tag{5.55e}$$

$$f'_6 = f_2 + (21+12\eta)\beta_3\,, \tag{5.55f}$$

$$f'_7 = f_5 - 4\eta\beta_3\,, \tag{5.55g}$$

$$f'_8 = f_4 - 3\eta\beta_3\,, \tag{5.55h}$$

$$f'_9 = f_6\,, \tag{5.55i}$$

$$f'_{10} = 0\,, \tag{5.55j}$$

$$g'_1 = 0\,, \tag{5.55k}$$

$$g'_2 = g_1\,, \tag{5.55l}$$

$$g'_3 = -\frac{3}{2}(1-3\eta)\beta_2 - 3\rho_2\,, \tag{5.55m}$$

$$g'_4 = g_3 - \frac{1}{2}(21+33\eta)\beta_3 + 3\rho_2\,, \tag{5.55n}$$

$$g'_5 = 2(1-3\eta)\beta_3 + 4\rho_2 - 5\rho_4\,, \tag{5.55o}$$

$$g'_6 = g_2\,, \tag{5.55p}$$

$$g'_7 = g_5 + \frac{1}{2}(56+\eta)\beta_3 + 2\rho_2 + 5\rho_4\,, \tag{5.55q}$$

$$g'_8 = g_4 + (21+9\eta)\beta_3\,, \tag{5.55r}$$

$$g'_9 = g_6\,, \tag{5.55s}$$

$$g'_{10} = 6\rho_4\,, \tag{5.55t}$$

where f_i, g_i are given by Eqs.(5.10). The solution corresponding to Eqs.(5.11) remains identical.

Similarly at 4.5PN we have

$$\begin{aligned}
\frac{d\tilde{E}^*}{dt} &= -\frac{8}{15}\eta\frac{m}{r^2}\Bigg[\left(\frac{m}{r}\right)^2\left(12v^2 - 11\dot{r}^2\right) + \left(\frac{m}{r}\right)^2\Bigg\{\frac{1}{28}\Big[(785-852\eta)v^4 \\
&\quad +2(-1487+1392\eta)v^2\dot{r}^2 + 160(-17+\eta)\frac{m}{r}v^2 + 3(687-620\eta)\dot{r}^4 \\
&\quad +8(367-15\eta)\frac{m}{r}\dot{r}^2 + 16(1-4\eta)\left(\frac{m}{r}\right)^2\Big]\Bigg\} + \sum_{i=1}^{21}\mathcal{R}'^{[4.5]}_i\mathcal{Y}^{[5]}_i\Bigg]\,, \qquad (5.56a) \\
\frac{d\tilde{\mathbf{J}}^*}{dt} &= -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\Bigg[\frac{m}{r}\left(2v^2 + 2\frac{m}{r} - 3\dot{r}^2\right) + \frac{m}{r}\Bigg\{\frac{1}{84}\Big[(307-548\eta)v^4 \\
&\quad +6\left(-74+277\eta\right)v^2\dot{r}^2 - 4(58+95\eta)\frac{m}{r}v^2 + 3(95-360\eta)\dot{r}^4 \\
&\quad +2(372+197\eta)\frac{m}{r}\dot{r}^2 + 2(-745+2\eta)\left(\frac{m}{r}\right)^2\Big]\Bigg\} + \sum_{i=1}^{15}\mathcal{S}'^{[4.5]}_i\mathcal{Y}^{[4]}_i\Bigg]\,,
\end{aligned}$$

(5.56b)

where

$$\begin{aligned}
\mathcal{Y}^{[5]}_i(i=1\ldots 21) &= \Bigg[v^{10}, v^8\frac{m}{r}, v^8\dot{r}^2, v^6\left(\frac{m}{r}\right)^2, v^6\frac{m}{r}\dot{r}^2, v^6\dot{r}^4, v^4\left(\frac{m}{r}\right)^3, v^4\left(\frac{m}{r}\right)^2\dot{r}^2, \\
&\quad v^4\frac{m}{r}\dot{r}^4, v^4\dot{r}^6, v^2\left(\frac{m}{r}\right)^4, v^2\left(\frac{m}{r}\right)^3\dot{r}^2, v^2\left(\frac{m}{r}\right)^2\dot{r}^4, \\
&\quad v^2\frac{m}{r}\dot{r}^6, v^2\dot{r}^8, \left(\frac{m}{r}\right)^5, \left(\frac{m}{r}\right)^4\dot{r}^2, \left(\frac{m}{r}\right)^3\dot{r}^4, \left(\frac{m}{r}\right)^2\dot{r}^6, \frac{m}{r}\dot{r}^8, \dot{r}^{10}\Bigg]\,,
\end{aligned}$$

(5.57)

and $\mathcal{Y}^{[4]}_i$ is given by Eq.(5.16). Here $\mathcal{R}'^{[4.5]}_i$,$\mathcal{S}'^{[4.5]}_i$ consist of linear combinations of the parameters involved. The 4.5PN reactive solution reads as:

$$
\begin{aligned}
h'_1 &= -\frac{1}{8}(3-27\eta+63\eta^2)\beta_3+\frac{3}{2}(1-3\eta)\rho_2-3\chi_2\,, && (5.58a)\\
h'_2 &= \frac{1}{2}(1-9\eta+21\eta^2)\beta_3-2(1-3\eta)\rho_2\\
&\quad+\frac{5}{2}(1-3\eta)\rho_4+4\chi_2-5\chi_4\,, && (5.58b)\\
h'_3 &= h_1+\frac{1}{8}(3-207\eta+75\eta^2)\beta_3+\frac{1}{2}(21+39\eta)\rho_2+3\chi_2\,, && (5.58c)\\
h'_4 &= -3(1-3\eta)\rho_4+6\chi_4-7\chi_7\,, && (5.58d)\\
h'_5 &= h_3+(18+96\eta+18\eta^2)\beta_3-(21+12\eta)\rho_2\,, && (5.58e)\\
h'_6 &= h_2-\frac{1}{4}(24-397\eta+95\eta^2)\beta_3-\frac{1}{2}(70-11\eta)\rho_2\\
&\quad+\frac{1}{2}(35+65\eta)\rho_4+4\chi_2+5\chi_4\,, && (5.58f)\\
h'_7 &= 8\chi_7\,, && (5.58g)\\
h'_8 &= h_4+\frac{1}{8}(-353+195\eta)\eta\beta_3+\eta\rho_2\\
&\quad-\frac{1}{2}(84+25\eta)\rho_4+2\chi_4+7\chi_7\,, && (5.58h)\\
h'_9 &= h_6-\frac{1}{4}(260+119\eta+30\eta^2)\beta_3-(14+5\eta)\rho_2-(35+20\eta)\rho_4\,, && (5.58i)\\
h'_{10} &= h_5-\frac{1}{4}(306+489\eta+48\eta^2)\beta_3\,, && (5.58j)\\
h'_{11} &= h_{10}\,, && (5.58k)\\
h'_{12} &= h_9-\frac{1}{4}(12+87\eta-24\eta^2)\beta_3\,, && (5.58l)\\
h'_{13} &= h_8-\frac{1}{2}(8+49\eta+34\eta^2)\beta_3+2\eta\rho_2+5\eta\rho_4\,, && (5.58m)\\
h'_{14} &= h_7+6\eta[(1-3\eta)\beta_3+\rho_4]\,, && (5.58n)\\
h'_{15} &= 0\,, && (5.58o)\\
& && (5.58p)
\end{aligned}
$$

$$
\begin{aligned}
k'_1 &= 0\,, && (5.59a)\\
k'_2 &= -\frac{1}{8}(3-27\eta+63\eta^2)\beta_3+\frac{3}{2}(1-3\eta)\rho_2-3\chi_2\,, && (5.59b)\\
k'_3 &= k_1\,, && (5.59c)\\
k'_4 &= \frac{1}{2}(1-9\eta+21\eta^2)\beta_3-2(1-3\eta)\rho_2+\frac{5}{2}(1-3\eta)\rho_4+4\chi_2-5\chi_4\,,
\end{aligned}
$$

(5.59d)

$$
\begin{aligned}
k'_5 &= k_3\,, && (5.59e)\\
k'_6 &= k_2 + \frac{3}{8}(1 - 81\eta + 13\eta^2)\beta_3 + \frac{1}{2}(21 + 33\eta)\rho_2 + 3\chi_2\,, && (5.59f)\\
k'_7 &= -3(1 - 3\eta)\rho_4 + 6\chi_4 - 7\chi_7\,, && (5.59g)\\
k'_8 &= k_4 - \frac{1}{4}(24 - 421\eta - \eta^2)\beta_3 - \frac{1}{2}(70 - 25\eta)\rho_2 \\
&\quad + \frac{1}{2}(35 + 55\eta)\rho_4 + 4\chi_2 + 5\chi_4\,, && (5.59h)\\
k'_9 &= k_6 + \frac{1}{4}(84 + 525\eta + 54\eta^2)\beta_3 - (21 + 9\eta)\rho_2\,, && (5.59i)\\
k'_{10} &= k_5\,, && (5.59j)\\
k'_{11} &= k_{10}\,, && (5.59k)\\
k'_{12} &= k_9 - \frac{1}{2}(159 + 288\eta + 12\eta^2)\beta_3\,, && (5.59l)\\
k'_{13} &= k_8 - \frac{1}{2}(144 + 179\eta + 40\eta^2)\beta_3 - (14 + 6\eta)\rho_2 - (35 + 15\eta)\rho_4\,, && (5.59m)\\
k'_{14} &= k_7 - \frac{1}{8}(317 + 105\eta)\eta\beta_3 - \frac{1}{2}(84 + 3\eta)\rho_4 - 3\eta\rho_2 + 2\chi_4 + 7\chi_7\,, && (5.59n)\\
k'_{15} &= 8\chi_7\,, && (5.59o)
\end{aligned}
$$

where h_i, k_i are given by Eqs.(5.17) of the text and Eqs.(5.18) remain the same.

5.6.4 The 3.5PN and the 4.5PN gauge arbitrariness

Finally it can be shown that all the arbitrary parameters in the reactive solution may be absorbed in a choice of 'gauge' of the form

$$
\delta\mathbf{x} = \frac{8}{5}\eta\frac{m}{r}(f'_{2.5} + f'_{3.5} + f'_{4.5})\dot{r}\mathbf{x} + (g'_{2.5} + g'_{3.5} + g'_{4.5})r\mathbf{v}\,, \qquad (5.60)
$$

where $f'_{2.5}$ and $g'_{2.5}$ are given by Eqs.(5.49), (5.50), while $f'_{3.5}$, $f'_{4.5}$, $g'_{3.5}$ and $g'_{4.5}$ have the form

$$
\begin{aligned}
f'_{3.5} &= \left[P'_{21}v^4 + P'_{22}v^2\frac{m}{r} + P'_{23}v^2\dot{r}^2 + P'_{24}\frac{m}{r}\dot{r}^2 + P'_{25}\left(\frac{m}{r}\right)^2 + P'_{26}\dot{r}^4\right],\\
g'_{3.5} &= \left[Q'_{21}v^4 + Q'_{22}v^2\frac{m}{r} + Q'_{23}v^2\dot{r}^2 + Q'_{24}\frac{m}{r}\dot{r}^2 + Q'_{25}\left(\frac{m}{r}\right)^2 + Q'_{26}\dot{r}^4\right],\\
f'_{4.5} &= \left[P'_{41}v^6 + P'_{42}\frac{m}{r}v^4 + P'_{43}v^4\dot{r}^2 + P'_{44}v^2\left(\frac{m}{r}\right)^2 + P'_{45}v^2\frac{m}{r}\dot{r}^2\right.
\end{aligned}
$$

$$+P'_{46}v^2\dot{r}^4 + P'_{47}\frac{m}{r}\dot{r}^4 + P'_{48}\left(\frac{m}{r}\right)^2\dot{r}^2 + P'_{49}\left(\frac{m}{r}\right)^3 + P'_{410}\dot{r}^6\Big],$$

$$g'_{4.5} = \Big[Q'_{41}v^6 + Q'_{42}\frac{m}{r}v^4 + Q'_{43}v^4\dot{r}^2 + Q'_{44}v^2\left(\frac{m}{r}\right)^2 + Q'_{45}v^2\frac{m}{r}\dot{r}^2$$

$$+Q'_{46}v^2\dot{r}^4 + Q'_{47}\frac{m}{r}\dot{r}^4 + Q'_{48}\left(\frac{m}{r}\right)^2\dot{r}^2 + Q'_{49}\left(\frac{m}{r}\right)^3 + Q'_{410}\dot{r}^6\Big]. \quad (5.61)$$

At 3.5PN we have

$$P'_{21} = -\frac{1}{4}(1-3\eta)\beta_3 - \frac{1}{4}(2\rho_2+\rho_4)\,, \quad (5.62a)$$

$$P'_{22} = P_{21} + \frac{1}{3}(3-10\eta)\beta_3 + \frac{1}{30}(20\rho_2+17\rho_4)\,, \quad (5.62b)$$

$$P'_{23} = -\frac{1}{4}\rho_4\,, \quad (5.62c)$$

$$P'_{24} = P_{23} + \frac{1}{10}(5\eta\beta_3+\rho_4)\,, \quad (5.62d)$$

$$P'_{25} = P_{22} + \frac{1}{12}(2+25\eta)\beta_3 - \frac{1}{3}(\rho_2+\rho_4)\,, \quad (5.62e)$$

$$P'_{26} = 0\,, \quad (5.62f)$$

$$Q'_{21} = 0\,, \quad (5.62g)$$

$$Q'_{22} = Q_{21} - \frac{1}{2}(1-3\eta)\beta_3 - \frac{1}{30}(10\rho_2+7\rho_4)\,, \quad (5.62h)$$

$$Q'_{23} = \frac{1}{4}(1-3\eta)\beta_3 + \frac{1}{4}(2\rho_2+\rho_4)\,, \quad (5.62i)$$

$$Q'_{24} = Q_{23} - \frac{1}{6}(3-8\eta)\beta_3 - \frac{1}{30}(10\rho_2+13\rho_4)\,, \quad (5.62j)$$

$$Q'_{25} = Q_{22} - \frac{1}{12}(2+25\eta)\beta_3 + \frac{1}{3}(\rho_2+\rho_4)\,, \quad (5.62k)$$

$$Q'_{26} = \frac{1}{4}\rho_4\,. \quad (5.62l)$$

Similarly at 4.5PN we have

$$P'_{41} = -\frac{1}{16}(1-9\eta+21\eta^2)\beta_3 + \frac{1}{4}(1-3\eta)\rho_2 + \frac{1}{8}(1-3\eta)\rho_4$$
$$-\frac{1}{24}(12\chi_2+6\chi_4+6\chi_7)\,, \quad (5.63a)$$

$$P'_{42} = P_{41} - \frac{1}{840}(1155-4817\eta+367\eta^2)\beta_3 - \frac{1}{60}(130-318\eta)\rho_2$$
$$-\frac{1}{30}(53-117\eta)\rho_4 + \frac{1}{105}(105\chi_2+84\chi_4+68\chi_7)\,, \quad (5.63b)$$

$$P'_{43} = \frac{1}{8}(1-3\eta)\rho_4 - \frac{1}{24}(6\chi_4+4\chi_7)\,, \quad (5.63c)$$

$$P'_{44} = P_{42} + \frac{1}{120}(420-1917\eta+967\eta^2)\beta_3 + \frac{1}{30}(55-228\eta)\rho_2$$

$$
\begin{aligned}
& +\frac{1}{120}(220-834\eta)\rho_4-\frac{1}{15}(15\chi_2+14\chi_4+13\chi_7)\,, && (5.63\text{d})\\
P'_{45} &= P_{43}-\frac{1}{140}\eta(47+48\eta)\beta_3-\frac{4}{5}\eta\rho_2 \\
& -\frac{1}{20}(8-17\eta)\rho_4+\frac{1}{105}(21\chi_4+32\chi_7)\,, && (5.63\text{e})\\
P'_{46} &= -\frac{1}{6}\chi_7\,, && (5.63\text{f})\\
P'_{47} &= P_{46}-\frac{27}{56}\eta(1-3\eta)\beta_3-\frac{1}{4}\eta\rho_4+\frac{1}{21}\chi_7\,, && (5.63\text{g})\\
P'_{48} &= P_{45}+\frac{1}{600}(300+4935\eta-1360\eta^2)\beta_3+\frac{1}{20}\eta(12\rho_2-\rho_4) \\
& -\frac{1}{15}(\chi_4+2\chi_7)\,, && (5.63\text{h})\\
P'_{49} &= P_{44}-\frac{1}{60}(92+121\eta+309\eta^2)\beta_3+\frac{1}{15}(1+52\eta)(\rho_2+\rho_4) \\
& +\frac{2}{5}(\chi_2+\chi_4+\chi_7)\,, && (5.63\text{i})\\
P'_{410} &= 0\,, && (5.63\text{j})\\
Q'_{41} &= 0\,, && (5.63\text{k})\\
Q'_{42} &= Q_{41}+\frac{1}{840}(105-659\eta-347\eta^2)\beta_3+\frac{1}{2}(1-3\eta)\rho_2+\frac{7}{20}(1-3\eta)\rho_4 \\
& -\frac{1}{30}(10\chi_2+7\chi_4)-\frac{19}{105}\chi_7\,, && (5.63\text{l})\\
Q'_{43} &= \frac{1}{16}(1-9\eta+21\eta^2)\beta_3-\frac{1}{8}(1-3\eta)(2\rho_2+\rho_4) \\
& +\frac{1}{24}(12\chi_2+6\chi_4+4\chi_7)\,, && (5.63\text{m})\\
Q'_{44} &= Q_{42}-\frac{1}{240}(420-1604\eta+1434\eta^2)\beta_3-\frac{1}{60}(80-301\eta)\rho_2 \\
& -\frac{1}{30}(40-131\eta)\rho_4+\frac{1}{15}(10\chi_2+9\chi_4+8\chi_7)\,, && (5.63\text{n})\\
Q'_{45} &= Q_{43}+\frac{1}{140}(175-639\eta+146\eta^2)\beta_3+\frac{1}{30}(50-99\eta)\rho_2 \\
& +\frac{1}{60}(94-183\eta)\rho_4-\frac{1}{21}(14\chi_2+14\chi_4+12\chi_7)\,, && (5.63\text{o})\\
Q'_{46} &= -\frac{1}{8}(1-3\eta)\rho_4+\frac{1}{24}(6\chi_4+4\chi_7)\,, && (5.63\text{p})\\
Q'_{47} &= Q_{46}+\frac{1}{280}\eta(121-363\eta)\beta_3+\frac{3}{10}\eta\rho_2+\frac{1}{20}(5-8\eta)\rho_4 \\
& -\frac{1}{10}\chi_4-\frac{26}{105}\chi_7\,, && (5.63\text{q})\\
Q'_{48} &= Q_{45}-\frac{1}{60}(135-64\eta-11\eta)\beta_3-\frac{1}{60}(30-119\eta)\rho_2-\frac{1}{30}(15-79\eta)\rho_4
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{15}(5\chi_2+6\chi_4+7\chi_7)\,, && (5.63\text{r})\\
Q'_{49} &= Q_{44}+\frac{1}{60}(92+121\eta+309\eta^2)\beta_3-\frac{1}{15}(1+52\eta)(\rho_2+\rho_4)\\
& -\frac{2}{5}(\chi_2+\chi_4+\chi_7)\,, && (5.63\text{s})\\
Q'_{410} &= \frac{1}{6}\chi_7\,. && (5.63\text{t})
\end{aligned}
$$

In the above, the P_{ab} and Q_{ab} are given by Eqs.(5.27) and (5.30) of the text.

To conclude: the far-zone flux formulas and the balance equations by themselves do not constrain the reactive acceleration to be a power series in m_1 and m_2, or equivalently nonlinear in the total mass m, as assumed in the initial sections of the present chapter, following IW. They are also consistent with the more general form of the reactive acceleration discussed in this section.

5.7 Concluding remarks

Starting from the 2PN accurate energy and angular momentum fluxes for structure-less non-spinning compact binaries of arbitrary mass ratio moving in quasi-general orbits we deduce the 4.5PN reactive terms in the equation of motion by an application of the IW method. The 4.5PN reactive terms are determined in terms of twelve arbitrary parameters which are associated with the possible residual choice of 'gauge' at this order. These general results could prove useful to studies of the evolution of the orbits. The limiting and complementary cases of circular orbits and head-on infall have also been examined.

We have systematically and critically explored different facets of the IW choice like the functional form of the reactive acceleration and provided a better understanding of the origin of redundant equations by studying variants obtained by modifying the functional forms of the ambiguities in E^* and $\tilde{\mathbf{J}}^*$. The main conclusions we arrive at by this analysis are

- In terms of the number of arbitrary parameters and the corresponding gauge transformations, the IW scheme exhibits remarkable stability for a variety of choices for the form of the ambiguity in energy and angular momentum. The different choices merely produce different numbers of degenerate equations. This indicates the essential validity and soundness of the scheme. These solutions are general enough to treat as special cases any particular solutions obtained from first principles in the future.

- Relaxing the requirement of nonlinearity in m or more precisely the power series behaviour in m_1 and m_2 permits mathematically more general solutions for the reactive accelerations involving more arbitrary parameters. Solutions more general than the ones discussed in the Section 5.6, e.g., a solution involving 6 parameters at the Newtonian level, cannot be gauged away either by gauge transformations of the form discussed by IW or by more general gauge transformations that differ in their powers of nonlinearity (m/r dependence). However, none of these solutions are of 'physical' interest to describe the gravitational radiation reaction of two-body systems.