Chapter V

SEMICLASSICAL DECAY OF THE KALUZA-KLEIN VACUUM IN HIGHER ORDER GRAVITY

In this chapter, we are going to extend the ideas and techniques stated in chapter III to the vacuum state coming from theories involving higher order terms. The starting point is the classical D-dimensional Einstein action modified by the Gauss-Bonnet combination of higher order terms given by Eq.(I.26).

As has been discussed in chapter III, a very much related question that one has to address in this regard is of the validity of the Positive Energy Theorem (PET) in this theory. There are exactly two distinct considerations :

(i) effect of the presence of the extra higher order terms,

(ii) the topology of the gravitating system.

The first consideration in an almost similar situation (i.e. $R + R^2$ gravity in four dimensions) was investigated by Strominger (1984) who proved PET in that theory. Also. flat space was shown to be the unique topologically Minkowskian stationary point of energy.

It is not too obvious that the PET will remain to be valid in the Einstein- Gauss-Bonnet theory or the Lovelock gravity as a whole. In field equations (1.27) if one identifies the contribution of the Gauss-Bonnet term with some stress energy tensor, $S_{\mu\nu}$ (although it is not a very comfortable idea), one may see that S_{00} is of indefinite sign. So, although the energy consideration is dependent on the dynamics at large distances which is mainly determined by the Einstein term, a negative sign of S_{00} (for $a > 0$) may make the total energy negative.

To see whether Witten-type arguments can be applied in this case, one has to take into account supersymmetric considerations. It seems plausible that the'model incorporating

Gauss-Bonnet term has a supersymmetric extension at least for $\alpha > 0$ [Deser, 1986]. It was also pointed out by Boulware and Deser (1985) that, due to this reason, the global energy will be positive only for the asymptotically Minkowskian flat solutions (a consideration also used in the proof of the theorem in four dimensions, see sec. III.B), but not for those which asymptotically approach de Sitter or anti de-Sitter spacetime.

As far as the consideration (ii) regarding the topology of the system is concerned, all arguments presented in sec. III(B) can be extended to this case. Here, we show that the theorem will not be valid for a multiply connected topology of the initial value hypersurface.

We consider $M^4 \times S^1$ to be the flat spacetime solution of such a theory. We are interested in the case of pure gravity without matter fields and, therefore, we study the source free Einstein-Gauss-Bonnet field equations given by Eq.(I.28) [Bhawal and Mani, 19921.

Due to such a choice of the topology of the ground state, the 5-dimensional gravitational constant is taken to be $G_5 = 2\pi G R_0$, where R_0 is the radius of the fifth dimension. So, by definition, $\kappa = 32\pi^2 G R_0$.

Now, to look for a semiclassical instability of this vacuum state, we have to search for an instanton-like 'bounce' solution of the classical Euclidean field equations of the higher order theory, Such a solution should be asymptotic to the flat infinite Euclidean spacetime

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2 + d\chi^2 \qquad (V.1)$$

By introducing polar coordinates $(\rho, \psi, \theta, \phi)$ in the four noncompact dimensions x, y, z, t, we can rewrite this as

$$ds^2 = d\rho^2 + \rho^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)] + d\chi^2 \qquad (V.2)$$

Also, this solution will represent an instability of the vacuum, if there exists negative action modes in small fluctuations around it. To look for such a solution, we can use the five-dimensional Boulware-Deser metric [Eq.II.1 & 5]. After analytically continuing that metric to the Euclidean time, the solution can be written as (in dimensionless variables):

$$ds^2 = P^{-1}d\rho^2 + \rho^2[d\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\phi^2)] + Pd\chi^2 \qquad (V.3)$$

$$P = 1 + \frac{\rho^2}{2a}[1 - (1 + \frac{8am}{\rho^4})^{1/2}], \qquad (V.4)$$

where a $= 32\pi\alpha$ (see Eq.II.4). The quantity m is an arbitrary parameter of dimension [L^2]. This solution satisfies the Euclidean field equations for any value of m, except at $\rho = \rho_i = \sqrt{2m-a}$, where P becomes zero. To study the behaviour at ρ_i, we make a coordinate transformation $\rho = \rho_i + X^2$, where X^2 is very small. Then as $X^2 \rightarrow 0$, the $\rho - \chi$ subspace will behave as

$$\frac{2(2m+a)}{\sqrt{2m-a}}\left[d\lambda^2 + \lambda^2\frac{2m-a}{(2m+a)^2}d\chi^2\right] \qquad (V.5)$$

where we have neglected the higher order terms of (λ^2) in $g_{\chi\chi}$.

The expression within the bracket can be compared with the standard line element for the metric of the plane in polar coordinates. Therefore? this will describe a nonsingular space, if χ is a periodic variable with periodicity $2\pi(2m + a)/\sqrt{2m-a}$. So, we obtain

$$\frac{2m+a}{\sqrt{2m-a}} = R_0 \qquad (V.6).$$

We get two solutions for m

$$m_\pm = \frac{1}{2}\left[\frac{R_0^2}{2} - a \pm \frac{R_0}{2}\sqrt{R_0^2 - 8a}\right] \qquad (V.7)$$

Note that $R_0^2 \geq 8a$. Also, now ρ can run between $\rho_i \leq \rho < \infty$. The value of ρ_i in the two cases will be different

$$\rho_{i\pm} = \left[\frac{R_0^2}{2} - 2a \pm \frac{R_0}{2}\sqrt{R^2 - 8a}\right]^{1/2}. \qquad (V.8)$$

Unfortunately, it is not easy to arrive at a conclusion regarding the presence of negative action modes in small fluctuations around these solutions, since the perturbation equations turn out to be extremely complex. The general procedure is as follows.

Let us consider a perturbation, h„ of a metric $g_{\mu\nu}$ which satisfies the Euclidean vacuum field equations corresponding to Eq.(I.27)

$$G_{\mu\nu}(g_{\alpha\beta}) = \alpha\kappa\, S_{\mu\nu}(g_{\alpha\beta}), \qquad (V.9)$$

where $g_{\alpha\beta}$ is with Euclidean signature. The perturbed metric ($g_{\mu\nu}$ + h„) also should satisfy the field equations and so

$$G_{\mu\nu}(g_{\alpha\beta}+h_{\alpha\beta})-\alpha\kappa\, S_{\mu\nu}(g_{\alpha\beta}+h_{\alpha\beta}) = G_{\mu\nu}(g_{\alpha\beta})+\delta G_{\mu\nu}(h_{\alpha\beta})-\alpha\kappa\left[S_{\mu\nu}(g_{\alpha\beta})+\delta S_{\mu\nu}(h_{\alpha\beta})\right]$$

$$\text{So,} \qquad \delta G_{\mu\nu}(h_{\alpha\beta}) = \alpha\kappa\, \delta S_{\mu\nu}(h_{\alpha\beta}). \qquad (V.10)$$

One may choose to work in a tracefree transverse gauge for $h_{\mu\nu}$, so that

$$g^{\mu\nu}h_{\mu\nu} = 0 \qquad (V.11)$$

$$D_{\mu}h^{\mu\nu} = 0. \qquad (V.12)$$

The most general traceless metric perturbation of Eq.(V.3) may be given as

$$\text{h„} \;= A(\rho)P^{-1}d\rho^2 - \frac{1}{3}[A(\rho)+B(\rho)]\rho^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)] + B(\rho)P\, d\chi^2. \quad (V.13)$$

This preserves both the rotational and time symmetry of the metric.

One may use the p–component of Eq.(V.12) to derive a relationship between A and B. The other components of this equation will be trivially satisfied. The next task is to solve the appropriate eigenvalue equation corresponding to Eq.(V.10). This equation which essentially comes out of the second variation of the action turns out to be a very complicated one involving various combinations of higher order terms [To get an idea of how it may look like, one may see Wiltshire (1988) where the Lorentzian version of the same perturbation equation has been explicitly written by choosing a de Donder gauge $(h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}h^{\lambda}{}_{\lambda})_{;\mu} = 0$. The situation will not improve much in a transverse traceless gauge]. It is almost impossible to solve such an equation except probably by the use of a powerful computer symbolic manipulation program.

Since the m_+ solution approaches the 'bounce' solution given by Witten [eq.III.10] in the $a \to 0$ limit, we expect a negative action mode to be present in this case. It is difficult to guess any result on the m_- solution. But it is quite possible that the actual higher order perturbation equation in each case may contain more than one negative eigenvalue.

However, as pointed out by Witten (1982), the much simpler way to see that the bounce solutions (V.3) actually describe the instability, is to perform a suitable analytical continuation of these solutions from the Euclidean to the Minkowskian space. If a real Euclidean solution remains to be a real valued Minkowski solution after the analytical continuation, it should describe the instability. This argument will also be confirmed by the fact that both the Minkowski solutions have zero energy and, therefore, represent the nonuniqueness of the assumed ground state, thus violating the positive energy theorem.

Therefore, performing the transformation $\psi \to \frac{\pi}{2} + i\tau$ on the metrics [Eq.V.3], we write the Minkowskian signature solutions as

$$ds^2 = -\rho^2 d\tau^2 + P^{-1} d\rho^2 + \rho^2 \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2) + P d\chi^2. \qquad (V.14)$$

As discussed in sec.III.(B), these represent the alternative spacetimes into which the assumed ground state decays. We verify below that the energy of these spacetimes is zero.

We can easily see that the presence of higher order terms will not create any problem in defining the energy integral for such a system. Because the dynamics of large distances are governed by the lower order Einstein term, the conserved energy of an asymptotically flat spacetime can be given by the usual ADM expression. For a quasi-Minkowskian spacetime, the metric $g_{\mu\nu}$ can be splitted up into its asymptotic value $\eta_{\mu\nu}$ and a deviation $h_{\mu\nu}$: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Then we can rewrite Einstein's equations as

$$G^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^L = -\frac{\kappa}{2}(T_{\mu\nu} + \tau_{\mu\nu}), \qquad (V.15)$$

where the superscript L represents the linear part of the corresponding quantities. $\tau_{\mu\nu}$

includes all the terms of $G_{\mu\nu}$, nonlinear in h.

$$\begin{aligned}\tau_{\mu\nu} =& \frac{2}{\kappa}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) - R^L_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^L \\ &+ 2\alpha[2RR_{\mu\nu} - 4R_{\mu\alpha}R^{\alpha}{}_{\nu} - 4R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} + 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} \\ &- \frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)]\end{aligned} \tag{V.16}$$

Then one can proceed in the standard way [Weinberg, 1972] to define the energy of such a spacetime geometry. For our purpose, we are using the energy integral defined by Deser and Soldate, 1988, for a geometry with a compactified dimension :

$$E = \frac{1}{16\pi G_5}\int_0^{2\pi R_0} dx^5 \int d^2S^k[h_{kj,j} - h_{jj,k} - h_{55,k}], \tag{V.17}$$

Now we can see that only the terms in the metric of order $1/\rho$ are relevant. However, since P contains no term of order $1/\rho$, the energy integrals for these spacetime geometries will give zero energy. These two solutions are, therefore, two counter-examples of the uniqueness of the $M^4 \times S^1$ ground state in Einstein-Gauss-Bonnet theory.

Both the solutions will have the same behaviour as that of ordinary Witten bubble, but their initial radii will be different : ρ_i+ and ρ_i-. Any of the two solutions will represent a perfectly reflecting expanding bubble of area $4\pi\rho_i^2\cosh^2\tau$ and at any time t, its radius will be $\rho(t) = \sqrt{\rho_i^2 + t^2}$. This corresponds to a distorted Minkowskian space in which the interior of the hyperboloid $x^2 - t^2 < \rho_i^2$ has been deleted [refer to Fig.9 by replacing r $\rightarrow$ p and $R_0 \rightarrow \rho_i$].

We can study the behaviour of the solutions in the limit $a \rightarrow 0$. In that limit, two solutions of m will behave as

$$m_+ = \frac{R_0^2}{2} - \frac{3}{2}a \qquad m_- = \frac{a}{2} \tag{V.18}$$

When a=0, the solutions are given by the values of P as.

$$ \tag{V.19}$$

So, the '+' solution, in the $\alpha \rightarrow 0$ limit, approaches the Witten bubble solution. On the other hand, the second solution, in that limit, approaches the assumed vacuum state. We can say that the '+' solution is actually the modified Witten bubble solution, whereas, the '$-$'solution is an entirely new one, but with the same physical properties. One should note the significance of the higher order terms here. Higher order corrections not only modifies the Witten bubble solution, but also provides a new solution. Only because of the presence of nonzero string parameter (and consequently higher order terms), the second solution comes into being to be interpreted as an alternative decay solution of the Kaluza-Klein vacuum.

It is obvious from Eq.(V.8) that the initial radius ρ_i of the bubble represented by '+' solution will always be greater than or equal to $R_0/2$. But that for the '$-$' solution will always be less than or equal to $R_0/2$. As $a \rightarrow 0$, ρ_{i-} goes as $\sqrt{a}$ and eventually becomes zero for $a = 0$, which represents the Kaluza-Klein ground state.

Another interesting feature in our calculation is the relationship: $R_0^2 \geq 8a$. One can interpret this in two ways. One can say that the presence of nonzero string parameter has set a lower limit to the radius of the fifth dimension, although the upper limit is not fixed. On the other hand, as has already been pointed out, in ordinary Einstein action, the radius of the fifth dimension is undetermined. This can be determined only by including quantum corrections. Therefore, if one wishes to keep R_0 theoretically unrestricted, one can say that for any determined value of R_0, the string parameter will have an upper limit.

To probe into the geometry of these spacetimes, one may wish to study the behaviour of geodesics and scalar waves in these spacetimes. The nature of time-like and null geodesics in the Witten bubble solution was studied by Brill and Matlin (1989). We can easily see that there will not be any qualitative difference between that case and present ones, since respective 'P' in different cases is always positive and zero only at p $= \rho_i$. The time-like geodesics will execute oscillating behaviour in ρ, with turning points at ρ_i and at $k = \sqrt{k_\tau^2 - k_\phi^2 - k_\chi^2}$, where k_τ, k,, k_χ are constants of motion in the corresponding directions. On the other hand, if a massless particle is directed toward the bubble, it will be reflected only once and then will move away at the speed of light.

For studying scalar waves in these spacetimes, the mathematical formalism developed in the work of Bhawal and Vishveshwara, 1990, on Witten bubble may be used here in a straightforward manner. Only the radial solutions in the present cases will differ from the radial solution given by them. But, as discussed in the appendix of their paper, appropriate coordinate transformations can bring the radial equations in the Schrodinger's form which can be studied. However, that will not provide any result qualitatively different from the Witten bubble case.

A few points discussed here will be elaborated in the next chapter.

Chapter VI

EPILOGUE

We discussed our motivation and summarized our successes and failures in various problems in the previous chapters. Here, we attempt to interrelate our failures and propose some logical extensions and improvements thereof. We indicate some important works done by others to shed some light on some unexplored prospects and problems in higher order gravity.

Let us start with black holes. One solution that holds the string of most of the works done by us is the Boulware–Deser black hole (BDBH). There is a series of other more generalised black hole solutions in the general second order Lovelock gravity [see sec.I(D)]. However, there still remains some more unsettled profound issues related to black holes in quadratic gravity.

We do not yet know the status of the singularity theorems in these theories, since the condition of the validity of these theorems may get violated by the higher order Lagrangians. Questions related to the different energy conditions (weak, strong, dominant) are also to be revisited, since they play an important role in both the classical and semi-classical aspects of various models. Since the vacuum field equations add up extra higher order terms with an indefinite sign, the above issues affect the status of the positive energy theorem which we discussed in chapter V.

One also has to explore carefully and in details the intricate issues related to the uniqueness theorem, quantum coherence problem arising out of the evaporation of the black holes and the back reaction problem in this context.

Another important problem that remains to be unsolved is the investigation of the classical stability (gravitational) of these black holes. This can be checked by performing similar analyses done in the case of four dimensional Lorentzian Schwarzschild solution by several authors [Regge and Wheeler, 1957; Vishveshwara, 1970; Edelstein and Vishveshwara, 1970]. The actual calculation, however, faces severe problems by the fact that the perturbation equations are extremely complex. One needs to develop a powerful computer symbolic manipulation program to solve these problems. It has, however, been argued by Boulware and Deser (1985) that the asymptotically de Sitter branch of these solutions ($M > 0$, $\alpha > 0$) is unstable.

The same problem has thwarted our attempt to find negative modes in small oscillations around the 'bounce' solutions obtained by us in chapter V, which may interpolate between the $M^4 \times S^1$ ground state and alternative zero energy bubble solutions in Einstein-Gauss-Bonnet (EGB) theory.

One may guess from the structure of the higher order perturbation equation that there may exist some possibilty of obtaining more than one negative eigenvalue. In quantum field theory, it was pointed out and proved by Coleman (1988) that, in all cases of the decay of a metastable state by quantum tunneling, the second variation derivative of the Euclidean action at the bounce has one and only one negative eigenvalue. However, the same problem has not yet been investigated in the case of an Euclidean Lagrangian with higher order or higher derivative terms.

The 'bounce' solution obtained by Witten (1982) representing the decay of $M^4 \times S^1$ in ordinary GR possesses only one negative eigenvalue in the functional determinant for small oscillations around it. This result actually stems out from the fact that there also exist only one negative eigenvalue in the Lichnerowitz equation representing the perturbation of the Euclidean Schwarzschild solution which Witten (1982) used.

So, in our context, it will be very interesting to know whether the perturbation around the BDBH solution possesses more than one negative eigenvalue. The comparison of this result with that in a higher order or higher derivative extension of the field theoretic analysis by Coleman (1988) may give us good insight into the nature of the vacuum decay in these theories. The implication of the extra negative modes may then be studied.

In this context, we would also like to point out a related problem which has not attracted much attention. Although the study of scalar perturbations in higher dimensional spacetimes is relatively straightforward the case of 'higher spin' perturbations is technically more complicated and should prove to be of mathematical interest at least.

Now, we would like to mention two significant related works done by others, which may have far-reaching consequences on the future of Lovelock gravity.

An interesting study related to the classical stability of the EGB theory and, in general, the Lovelock gravity with compactified higher dimensions has been reported by Sokolowski et *al* (1991). They showed that the presence of dilaton (and of other geometric scalar fields) may render the possible ground state solutions of the reduced theory (i.e. the Minkowski and anti-de-Sitter spaces) unstable against perturbation of the scalar field. They have argued that the Gauss-Bonnet combination should, therefore, be discarded because this poses a serious problem which, unlike in the case of higher dimensional Einstein gravity, cannot be removed by field redefinitions.

Another interesting and somewhat controversial point has been raised in a series of papers by Simon (1990,91,92). He pointed out that if the second order terms are thought to be semiclassical perturbation corrections (of order $\hbar$), some new nonperturbative solutions may arise in this theory [note that one of our 'bounce' solutions, m_- in Eq.V.7 falls in this category]. But unlike the effective action and the field equations which generate them, most of these new solutions do not satisfy the initial perturbative ansatz, i.e. they are not perturbatively expandable in $\hbar$. The anti-de-Sitter branch of the solutions given by Boulware and Deser (1985) has this property.

Simon's argument is that the most self-consistent approach would be to discard these nonperturbative solutions because semiclassical gravity is only expected to approximate a perturbative expansion of the full theory and so, these 'pseudo-solutions' will fail to give any insight into the nonperturbative features of the full theory of quantum gravity. By these arguments, he showed that flat space is perturbatively stable to first order in

$\hbar$ against quantum fluctuations in semiclassical approximations to quantum gravity, although the past predictions had gone to the contrary [Hartle and Horowitz, 1981]. Similar arguments rule out Starobinsky (1980) inflation (de Sitter solutions driven only by higher order curvature terms).

All these points are to be cautiously studied before we arrive at any final conclusion. We would like to point out, while drawing an end to this thesis, that our physical intuition always suffers a drawback gaining its experience mostly from the ordinary theories. Higher order or higher derivative theories have never been extensively studied despite the fact that these may naturally arise in different branches of Physics [e.g., the relativistic model of the classsical radiating electron given by Dirac(1938)]. We should be careful enough before making any statement or jumping into any conclusion regarding any problem in these theories.

The not–so–happy marriage of gravity with quantum theory gave *a* 'natural' birth to the twins — higher dimensional and higher order gravity. They are here to stay and grow up and only God knows, when they will find themselves at the limiting end of the complete thory of quantum gravity and will come to know what God only knows.