Chapter I

INTRODUCTION

The fact that General Relativity (GR) is the correct classical theory of gravity has been well established by sufficient experimental evidence. It is believed that this theory is the low frequency limit of a quantum theory of gravity which is yet to be developed. In the last few years the search for a consistent quantum theory of gravity and the quest for a unification of gravity with other forces have both led to a renewed interest in theories with extra spatial dimensions incorporated in them.

In this chapter, we first establish the motivation for studying higher dimensional Gravity and present a brief chronology and review of the works done in this line. We then go on to introduce higher order terms that may be naturally incorporated in the Lagrangian of such theories. In the third section, we present an analysis based on the existing literature of these higher order terms in the language of Differential Forms. The last section is a brief review of the features of some solutions of these theories. This chapter, as a whole, will provide the essential groundwork for the topics to be discussed in the rest of the chapters.

I.(A) Higher Dimensional Gravity

When Einstein first talked of a new kind of Physics in a 4-dimensional manifold unifying space and time, a large cross section of the society including physicists as well as nonspecialists raised eyebrows either in disbelief or in bewilderment. The 3-dimensional realm of the Newtonian Mechanics was an adequate framework to common human mind for studying phenomena encountered in its daily life. The special theory of relativity was revolutionary in the sense that it first disabused people of the idea that Physics is just the study of objects in their familiar 3-dimensional world where time acts just as a parameter.

A beautiful equation like $E = mc^2$ was an outcome of this kind of Physics in 4-manifold and could never be conceived through the Newtonian Mechanics. The real implication of this formula could be convincingly verified in different processes in nuclear and particle Physics.

Once this breakthrough was achieved and the psychological stumbling blocks got removed, there was no bar in extending the limit of imagination to higher ($D > 4$) dimensional manifolds by incorporating extra spatial dimensions *. In fact, in any of the basic laws of nature (e.g. Newton's laws of motion, the Lagrangian and Hamiltonian formalisms, principles of the special relativity, Principle of equivalence, Principle of general covariance, the geodesic principle, the quantum mechanics principles), neither the statement of the principle nor the mathematical machineries were ever restricted to three dimensions. The general hope was that such an exercise might as well be successful in giving us some distinctive results which can again be verified by either experiments or the observation of the universe.

T. Kaluza(1921) and O. Klein(1926) [see Appelquist et al.,1987, for the original papers and English translation thereof] suggested that gravitation and electromagnetism could be unified in a theory of 5-dimensional Riemannian Geometry. Over the years, however, such an idea became decrepit because of its failure to provide some unique verifiable prediction: instead, it yielded some unphysical results when combined with quantum theory [Bailin and love,1987].

However, with the growing importance of gauge invariance as a major guiding principle for the formulation of physical laws, the urgency of unifying gauge fields with General Relativity (GR) began to be strongly felt. So, in the 70's this approach was resurrected again with much more vigour [Scherk and Schwarz, 1975, Cho, 1975, Cremmer and Scherk, 1976].

In contrast to the classical literature, this modern approach [Witten, 1981b, Salam

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* Theories with additional time-like dimensions appear to be plagued with ghosts, see e.g. Duff, Nilsson, and Pope (1986)

and Stathdee, 1982; For a concise review, see Bailin and Love, 1987] established the idea that the extra dimensions should be regarded as true, physical dimensions on par with the four observed dimensions and not be treated as a mere mathematical device. In this line of attack the gauge invariance assumes the same status as spacetime invariance and internal symmetries originate in the spacetime symmetries associated with the extra dimensions. In this framework, therefore, it is essential that at every stage in the derivation of the effective 4-dimensional field theory, one maintains consistency with the higher dimensional field equations.

The sizes of the extra dimensions are free parameters of the model, ones that are not determined even when we specify all parameters of the Lagrangian. However, establishing a relationship among the coupling constants, gravitational constant G and the size of the extra dimensions (each coupling is given by the ratio of $2\pi(16\pi G)^{1/2}$ to an arbitrary root-mean-square circumference), Weinberg (1983) and Candelas and Weinberg (1984) suggested that the size of the extra dimensions are at most a few orders of magnitude greater than the Planck length ($\sim 10^{-33}$cm.). Since the present day accelerators can probe matter at 10^{-16} cm. only, resolving the extra dimensions at currently available energies is out of question. But this may not have been always so. If we go back in time, according to the standard cosmological model, there must have been a time when the visible universe was of a comparable size to that of the internal space. The present difference between the four observed dimensions and the extra microscopic ones could arise from a spontaneous breakdown of the vacuum symmetry i.e.'spontaneous compactification' of the extra dimensions.

The idea of extra hidden dimensions also stimulated much work in supersymmetry theory. This idea permits a simple derivation of the $SO(8)$ supergravity Lagrangian by an appropriate dimensional reduction of the N = 1 supergravity Lagrangian in eleven dimensions [Cremmer and Julia, 1979].

The foundation of the superstring theory has also been built on spacetime of more than four dimensions. In this theory, the dimension of the Minkowskian spacetime in which the first quantization can be perturbatively done turns out to be $D = 10$, while for the old bosonic string, this same number is $D = 26$ [Green et al., 1987].

The fact that higher dimensions naturally arise in different unification theories involving gravitation provides the main motivation for studying higher dimensional gravity. However, we should remember that there remains a more subtle aim of such a study. This direction of investigation concentrates on attempting to explain why the observed space is so specific to choose three dimensions, although almost always there exists a greater generality in the statements and mathematics describing the laws of nature. Such a question should also be addressed to all the unification schemes since those also involve the 'specificity' of choosing a particular number of dimensions.

I.(B) Higher order Terms

Einstein's gravitational tensor (together with the cosmological term) was, in any dimension, the only symmetric and conserved tensor depending only on the metric and its first and second derivatives, with a linear dependence on the latter. He was able to deduce the simple Ricci scalar Lagrangian only by making certain simplifying assumptions [Einstein, 1916]. The gravity Lagrangian could, in fact, contain an arbitrary number of terms consisting of the invariants which can be constructed from powers of the Riemann curvature tensor. It is hard to argue on experimental ground that such additional terms should not be present since, in all practical situations, the curvature is very small. It was Weyl(1919) who first introduced such terms in his affine theory which claimed to unify gravity and electromagnetism.

It is false to assume that adding a higher order correction term with a small coefficient will only perturb the original theory. The presence of an unconstrained higher order term. no matter how small it may naively appear, may make the new theory dramatically different from the original.

The classical Einstein theory should be the low energy limit of a quantum theory of gravity. It was suggested that the action for a quantum theory of gravity should contain some nonminimal functionals of the metric tensor which involve more than two derivatives. The action gets modified by higher order interactions in any attempt to perturbatively quantize gravity as a field theory [Grisaru et *al*, 1976; Deser, Kay, Stelle,1977; Goroff and Sagnotti, 1986]. Gravitational actions which include terms quadratic in curvature tensor are renormalizable [Stelle, 1977; Birrel and Davies, 1982].

It is hoped that the full low energy theory will solve the problem of singularities in GR. However, in the absence of the knowledge of the details of such a theory, attempts have been made to gain further insight by studying models which include only the leading order corrections.

From time to time people studied different objects and issues arising in ordinary GR in the context of four dimensional theories involving higher powers of the curvature tensor (e.g., $R + R^2$ theory etc.). The associated field equations are typically of fourth order in the derivatives and are exceedingly nonlinear [For a review of such theories. see Boulware, Strominger, Tomboulis, 1984].

In cosmology, such terms were introduced first by Starobinsky (1980.1983) with an aim to avoid the initial singularity. It was found that such models may lead to inflationary expansion driven only by gravity.

Recent developments in the superstring approach to the unification of all forces have also provided concrete suggestions for higher order corrections to the Einstein action. In their field theoretic limit [Scherk and Schwarz, 1974], string theories give rise to effective models of gravity in higher dimensions which involve higher powers of Riemann curvature. Of these, the quadratic term is of particular importance because it is the leading one and can affect the gravitational excitation spectrum near the flat space.

However, if like the string itself, its slope expansion is to be ghost free, the quadratic term, if any, must be the Gauss-Bonnet (GB) combination [Zwiebach, 1985]

$$(\mathrm{GB}) \equiv R_{\alpha\beta\gamma\delta}{}^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2. \qquad (I.1)$$

Such an addition would not modify the propagator because now if one expands the action about Minkowski space, the terms quadratic in the gravitational field combine to a total derivative and integrate to zero. In four dimensions the combination multiplied by $\sqrt{-g}$ is a total derivative and proportional to the Euler topological invariant.

Therefore, the Gauss-Bonnet combination can act as the leading correction to Einstein theory in the low frequency effective field theory of the string only for dimensions greater than four.

The presence of these higher order terms can also be understood from the point of view of Lovelock's theorem. It is interesting to note that Lovelock [1971,72] was also led to this action by a different route.

Within the realm of classical gravity, Lovelock tried to obtain the most general second rank tensor in arbitrary dimensions, which is (i) symmetric, (ii) depends on the metric and its derivatives upto second order and (iii) divergence free. He relaxed the requirement that the tensor be linear in second derivatives of the metric. However, it turns out that in 4-dimensions, this follows naturally from the above assumptions and nonlinear terms arise only in higher dimensions.

The Lovelock theorem states that in D-dimensions, the number of such independent tensors is m $= D/2$ for D even and m $= (D+1)/2$ for D odd. The most general metric Lagrangian is given by a finite sum of the dimensionally extended Euler densities (to be explained in detail in section 1.3)

$$A_\mu{}^\nu = \sum_{p=1}^{m-1} a_p A^{(p)}{}_\mu{}^\nu + a\delta_\mu{}^\nu \qquad (I.2a)$$

$$A^{(p)}{}_\mu{}^\nu = \delta^{\nu\nu_1\cdots\nu_{2p}}_{\mu\mu_1\cdots\mu_{2p}} R_{\nu_1\nu_2}{}^{\mu_1\mu_2}\cdots R_{\nu_{2p-1}\nu_{2p}}{}^{\mu_{2p-1}\mu_{2p}} \qquad (I.2b)$$

where $a_p(p = 1,\cdots, m-1)$ and '**a**' are arbitrary constants. The generalised Kronecker delta symbol is given as

$$\delta^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_N} = \delta^{[\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_N]}_{\mu_N]}. \qquad (I.2c)$$

In 10 dimensions, for example, we would expect terms in the action of up to quartic order in the curvature. The quantity 'a' is equivalent to the cosmological constant. The lowest order term (p = 1) in the summation is identical with the Einstein tensor, whereas the (p = 2) term corresponds to the leading quadratic curvature correction or the Gauss-Bonnet combination.

Again, the field equations have the anomalous property that in D > 4, the tensor A_{ν}' is nonlinear in the second order derivatives of $g_{\mu\nu}$ and differs from Einstein tensor only if the spacetime has more than four dimensions. Therefore, it yields the most natural generalisation of GR in higher dimensional spacetimes.

A **misnomer :** Before we end this section, we would like to point out that the Lovelock gravity is frequently referred to in the literature as a higher derivative theory. It is a misnomer to call such theories to be 'higher derivative' ones since the Lagrangian, just like the Einstein-Hilbert gravity, does not contain more than the second derivative of the metric. 'Higher Order Gravity' is an appropriate name for such theories.

I.(C) Higher Order Lagrangian In The Language Of Differential Forms

In this section we present an analysis of higher order terms arising in the Lagrangian of higher dimensional gravity by making use of the calculus of differential forms. This presentation is based on the existing literature on this subject [Zumino, 1986; Teitelboim and Zanelli, 1987]. Our aim is to systematize the whole procedure and present the same in a coherent way.

We consider a D-dimensional spacetime with a metric g of signature $(-++\cdots+)$. Let e^A, $A = 1, \ldots D$ denote the orthogonal coframe (vielbein 1-forms).

$$g = \eta_{AB}\, e^A \otimes e^B \qquad \eta \equiv \mathrm{diag}(-1, 1, \ldots, 1) \qquad (I.3)$$

Let us introduce the differential forms

$$\epsilon_{A_1\cdots A_m} = \frac{1}{(D-m)!} \epsilon_{A_1\cdots A_m A_{m+1}\cdots A_D}\ e^{A_{m+1}} \wedge \cdots \wedge e^{A_N}, \qquad (I.4)$$

where $\epsilon_{A_1\cdots A_D}$ is totally antisymmetric with $\epsilon_{1\cdots D}=1$. They satisfy

$$\begin{aligned} e^B \wedge \epsilon_{A_1\cdots A_m} = \delta^B_{A_M}\ \epsilon_{A_1\cdots A_{m-1}} - \delta^B_{A_{m-1}}\ \epsilon_{A_1\cdots A_{m-2}A_m} \\ + \cdots + (-1)^{m-1}\ \epsilon_{A_2\cdots A_m} \end{aligned} . \qquad (I.5)$$

Let $w^A{}_B$ be a spin connection one-form compatible with the metric g. The torsion and curvature 2-forms are defined respectively as

$$T^A = de^A + \omega^A{}_B \wedge e^B = \mathcal{D}e^A \qquad (I.6)$$

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = \frac{1}{2} R^A{}_{BCD} e^C \wedge e^D = \frac{1}{2} R^A{}_{B\mu\nu} dx^\mu dx^\nu \qquad (I.7)$$

$$\text{Also,} \quad R_{AB} = -R_{BA} \qquad (I.8)$$

From now onwards, we stop indicating explicitly the wedge product sign.

The torsion and curvature 2-forms satisfy the Bianchi identities

$$dT^A + \omega^A{}_B T^B = R^A{}_B e^B = \mathcal{D}T^A \qquad (I.9)$$

$$(dR + \omega R - R\omega)^A{}_B = (\mathcal{D}R)^A{}_B = 0 \qquad (I.10)$$

A small variation δw of the connection form induces a variation of R given by

$$\delta R = \mathcal{D}\delta\omega = d\delta\omega + \omega\delta\omega + \delta\omega\ \ \omega \qquad (I.11)$$

The Lagrangian can be considered to be a linear combination of D-forms (in D dimensions) which, integrated over the manifold, gives the action. A particularly interesting class of D-forms invariant under local lorentz transformation are given by

$$L_{K,D-2K} = R^{A_1B_1} R^{A_2B_2} \cdots R^{A_KB_K} e^{A_{2K+1}} \cdots e^{A_D}\ \epsilon_{A_1B_1\cdots A_KB_KA_{2K+1}\cdots A_D}, \qquad (I.12)$$

where $0 \le K \le D/2$. Therefore, in a given dimension, the number of Lagrangians considered here is finite.

Rewriting this expression in terms of the 4-index Riemann curvature tensor,

$$
\begin{aligned}
L_{K,D-2K} =& \frac{1}{2^K} R^{A_1 B_1}{}_{M_1 N_1} \cdots R^{A_K B_K}{}_{M_K N_K} e^{M_1} e^{N_1} \cdots e^{M_K} e^{N_K} e^{A_{2K+1}} \\
& \cdots e^{A_D} \in_{A_1 B_1 \cdots A_K B_K A_{2K+1} \cdots A_D} \\
=& \frac{(D-2K)!}{2^K} e R^{A_1 B_1}{}_{M_1 N_1} \cdots R^{A_K B_K}{}_{M_K N_K} \delta^{M_1 N_1 \cdots M_K N_K}_{A_1 B_1 \cdots A_K B_K} d^D x \qquad (I.13)
\end{aligned}
$$

where $e = \det e^A,$.

When $D(=2E)$ is even, the particular 2E-form corresponding to $K = E$ will be given by

$$L_{E,0} = R^{A_1 B_1} \cdots R^{A_E B_E} \in_{A_1 B_1 \cdots A_E B_E} \qquad (1.14)$$

This is proportional to the Euler invariant. For obvious reasons, Euler invariants corresponding to odd dimensions do not exist at all. Now, rewriting Eq.(I.15) in terms of 4-index Riemann curvature tensor,

$$L_{E,0} = \frac{1}{2^E} e R^{A_1 B_1}{}_{M_1 N_1} \cdots R^{A_E B_E}{}_{M_E N_E} \delta^{M_1 N_1 \cdots M_E N_E}_{A_1 B_1 \cdots A_E B_E} d^{2E} x \qquad (I15)$$

So, barring a multiplying constant, both $L_{K,D-2K}$ and $L_{E,0}$ will generate the same expression, if $K = E$, except that the indices will run over different number of dimensions. In the first case, number of dimensions may be any $D > 2K$. In the latter case, it is equal to $2K$.

In essence, all the D-forms $L_{K,D-2K}$ can be interpreted as the extension to a higher number of dimensions D of the Euler number $L_{K,0}$ corresponding to dimensions $2K$. We may, therefore, represent the total Lagrangian as a linear combination of dimensionally extended Euler invariants :

$$\mathcal{L} = \sum_{K=0}^{p} \lambda_K L_K, \qquad \text{where} \quad p \le p_{\max} \qquad (I.16a)$$

λ_K are constants

$$\text{and} \quad p_{\max} = \begin{cases} D/2 & \text{when D is even} \\ (D-1)/2 & \text{when D is odd.} \end{cases} \qquad (I.16b)$$

$$L_K \equiv L_{K,0}$$
$$= R^{A_1 B_1} \cdots R^{A_K B_K} \in_{A_1 B_1 \cdots A_K B_K}, \qquad 2K \leq D. \qquad (I.17)$$

In particular, we have

$$L_0 = \in = \text{volume form},$$
$$L_1 = R^{AB} \in_{AB} = \text{R} \in,$$
$$L_2 = R^{AB} R^{CD} \in_{ABCD} = (R_{ABCD} R^{ABCD} - 4 R_{AB} R^{AB} + R^2) \in \qquad (1.18)$$
$$L_3 = R^{AB} R^{CD} R^{EF} \in_{ABCDEF}$$
$$\ldots \text{etc.}$$

So, the first term in L gives a cosmological constant, the second is proportional to the Einstein-Hilbert Lagrangian, the third is the Gauss-Bonnet combination. In D $= 2K$ dimensions, L_K is proportional to the Euler form which gives rise to the Euler (characteristic) class. In that case, there is no contribution to the field equations, since the entire expression ($\sqrt{-g} L_K$) is a total derivative. This will have contribution only in a theory with D $> 2K$. For example, in four dimensions, the Gauss-Bonnet combination will not have any contribution to the field equations and this fact leads to the Bach-Lanczos identity [Lanczos, 1938]

$$C_{\alpha\mu\nu\lambda} C_\beta{}^{\mu\nu\lambda} = \frac{1}{4} g_{\alpha\beta} C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}, \qquad (I.19)$$

where $C_{\mu\nu\lambda\sigma}$ is the Weyl tensor.

Now, let us consider an infinitesimal variation of the connection and vielbein forms. The corresponding variation of L_K is $\delta_e L_K + \delta_\omega L_K$.

For $1 \leq K \leq D/2$, variation of $w^A{}_B$ yields (using Eq.I.12)

$$\delta_\omega L_K = K(\mathcal{D} \delta\omega^{A_1 B_1}) R^{A_2 B_2} \cdots R^{A_K B_K} \in_{A_1 B_1 \cdots A_K B_K} . \qquad (I.20)$$

Now, using Eq.(I.11),

$$d(\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}\in_{A_1B_1\cdots A_KB_K})$$
$$=(\mathcal{D}\delta\omega^{A_1B_1})R^{A_2B_2}\in_{A_1B_1\cdots A_KB_K}+\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}\mathcal{D}\in_{A_1B_1\cdots A_KB_K} \quad (I.21)$$

So, $$\delta_\omega L_K = K\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}(\mathcal{D}\in_{A_1B_1\cdots A_KB_K}) + \text{exact form} \quad (1.22)$$

If $D = 2K$, the above variation is in exact form. For $D > 2K$,

$$\delta_\omega L_K = K\delta\omega^{A_1B_1}R^{A_2B_2}\dots R^{A_KB_K}T^C\in_{A_1B_1\cdots A_KB_KC} + \text{exact form} \quad (1.23)$$

If we restrict our considerations to the pure metric theory where $T^C = 0$, we see that that the variation of the connection does not contribute to the field equations : (if D $> 2K$)

$$\sum_{K=0}^{p}\lambda_K\delta_e L_K = 0 \quad (I.24a)$$

where $$\delta_e L_K = \delta e^C R^{A_1B_1}\dots R^{A_KB_K}\in_{A_1B_1\cdots A_KB_KC} \quad (I.24b)$$

In particular, one gets

$$\begin{aligned}
\delta_e L_0 &= \delta e^A\in_A\\
\delta_e L_1 &= \delta e^A(Rg_{AB}-2R_{BA})\in^B\\
\delta_e L_2 &= \delta e^E\big[(R_{ABCD}R^{CDAB}-4R_{AB}R^{BA}+R^2)g_{EF}\\
&\quad +4(R_{FCAB}R^{ABC}{}_E+2R_{FAEB}R^{BA}+2R_{FC}R^C{}_E-RR_{FE})\big]\in^F\\
&\quad \dots\text{etc.}
\end{aligned} \quad (I.25)$$

All these terms contribute higher order corrections to the field equations of these theories.

I.(D) Solutions of Higher Dimensional and Higher Order Gravity

Many known objects which naturally arise in ordinary GR have been studied in the context of higher dimensional theories with or without higher order terms. In the former case, the interest has been mainly centered on the simplest version of higher order gravity, i.e. the Einstein-Gauss-Bonnet (EGB) theory in which the Lagrangian is a sum of the Einstein-Hilbert and the Gauss-Bonnet terms. The action can thus be written as

$$I = \int d^D x\sqrt{-g}[R/\kappa + \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)], \qquad (I.26)$$

$$\text{where} \qquad \kappa = 4\mathbf{G}\int d\Omega_n, \qquad n = D-2.$$

$\int d\Omega_n$ is the area of a unit n-sphere and α is the string slope parameter with magnitude of the order of the square of the Planck length, and is positive as long as we consider EGB theory to be the low frequency limit of Superstring theory. However, if the EGB theory is assumed as a theory in its own right (i.e. Lovelock gravity), there is no restriction on the magnitude or the sign of α and it is viewed as just a coupling coefficient of higher order terms. Both the Newton's constant **G** and κ have dimensions $L^{(D-3)}/M$.

The field equations that follow from such an action are given as

$$\begin{aligned} 0 =& G_{\mu\nu}/\kappa - \alpha S_{\mu\nu} \\ =& \frac{1}{\kappa}\big[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big] - \alpha\big[\frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) \\ & - 2RR_{\mu\nu} + 4R_{\mu\alpha}R^{\alpha}{}_{\nu} + 4R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} - 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}\big]. \qquad (I.27) \end{aligned}$$

An action like (1.26) is of independent interest since this allows spontaneous compactification [Müller-Hoissen, 1985; Mignemi, 1986]. The multidimensional solutions which have been studied so far may be categorised into two classes:

(a) WITHOUT COMPACTIFIED DIMENSIONS :

In these cases, all dimensions are considered to be on equal footing. The generalised Einstein equations have as 'ground states' the maximally symmetric solutions– Minkowski space, de Sitter space and Anti de Sitter space. So, the solutions asymptotically approach either the Minkowski space (M^4) if the spacetime becomes flat at infinity, or D-dimensional de Sitter or Anti de Sitter space if the spacetime remains curved at infinity. Although these solutions are irrelevent to present day low energy physics, but the study of such solutions may give us important insight into the concept of the existence of higher dimensions and may answer some questions related to the 'specificity' of the number of dimensions discussed in section I.(A).

Here we are mentioning various works done in this line :

(i) Black Holes : In ordinary (without higher order terms) higher dimensional gravity. both static and stationary solutions (charged or uncharged) have been studied [see Tangherlini,1963, Myers and Perry,1986, Dianyan,1988]. In higher order gravity, so far, only static spherically symmetric black holes seem to have been investigated [see Boulware and Deser,1985, Wheeler,1986, Myers,1987, Myers and Simon, 1988, Wiltshire,1988, Callan, Myers and Perry,1989]. Wiltshire (1986) did the electromagnetic extension of such solutions. The assumed topology for such solutions is $R^2 \times S^{D-2}$ and spherical symmetry in all the $(D-1)$ spatial dimensions.

(ii) Gravitational *Waves* and their propagation in higher order theory have been studied by Boulware and Deser (1985), Gibbons and Ruback (1986) and Tomimatsu and Ishihara (1987).

(iii) Euclidean *Wormholes* : Gonsalez-Diaz (1990) and Jianjun and Sicong (1991) studied Euclidean wormholes in the context of the EGB theory.

(iv) Lorentzian Wormholes : Bhawal and Kar (1992) first investigated the possibility of the existense of Lorentzian Wormhole solutions in EGB theory. They have shown that similar to the situation in four dimensional GR, the matter that threads the wormhole violates the Weak Energy Condition for positive values of a. For negative values of α, the condition may or may not be violated. They have also suggested the possible construction of a solution with matter satisfying Weak energy condition everywhere.

(b) WITH COMPACTIFIED EXTRA DIMENSIONS :

This class of solutions is based on the Kaluza-Klein view of the world geometry and more relevent to our present day low energy physics where the extra dimensions are unobservable. All cosmological solutions belong to this class. The main purpose for developing them is to find the reasons for a phase transition and the consequent dynamical reduction mechanism which may account for the huge discrepency of scales between the three observed dimensions and the extra microscopic ones. These models may also explore the possibility of solving horizon and flatness problems which arise in the standard cosmology. Thus, these also appear to be possible alternatives to the usual inflationary models. When the ordinary dimensions increase, the extra dimensions decrease and with them the mean volume. This corresponds to the increase in temperature which, in turn, may be interpreted to be an increase in the entropy of the universe.

Ordinary higher dimensional Kaluza-Klein cosmological models have been studied by several authors [see Sahdev, 1984 and references therein. Also see Appelquist et al.,1987 for a collection of papers]. In the context of higher order theories such models were studied by Shafi and Wetterich(1985), Madore(1985,1986), Yoshimura1986), Wheeler(1986), Ishihara(1986), Maeda(1986), Müller-Hoissen(1986), Henriques (1986)[but initial field equations(2.7) written in this paper are not correctly written], Deruelle and Madore(1987), Mukherjee and Paul(1990, Barrow and Cotsakis(1991) etc.

In the next chapter, we shall study and discuss different classical and semiclassical aspects of a simple model of this theory— a static, spherically symmetric solution— the Boulware Deser black hole.

Chapter II

THE BOULWARE-DESER BLACK HOLE

In this chapter, we concentrate our study on a static spherically symmetric black hole solution of the EGB theory given by Boulware and Deser (1985). This solution may be thought to be the extension of higher dimensional Schwarzschild black hole to higher order theory. Therefore, study of such a simple model may reveal the nature of physical processes involved in a higher order gravity model. We shall specifically study two cases, i.e. Geodesic motion and Hawking radiation, and try to see to what extent the extra dimensions and/or higher order terms affect these aspects.

The metric element of this exact solution of the field equations (1.27) is written as (in unit $c = 1$)

$$ds^2 = -P\,dt^2 + P^{-1}\,dr^2 + r^2\,d\Omega_n^2, \qquad (II.1)$$

where $$P = 1 + \frac{r^2}{2\bar{\alpha}\kappa}\left[1 - \left[1 + \frac{8GM\bar{\alpha}\kappa}{r^{n+1}}\right]^{1/2}\right], \qquad (II.2)$$

and $\bar{\alpha} = \alpha(n-1)(n-2), \qquad n = D - 2.$

$d\Omega_n^2$ is the surface element of a unit n-sphere :

$$d\Omega_n^2 = d\theta_n^2 + \sin^2\theta_n[\,d\theta_{n-1}^2 + \sin^2\theta_{n-1}[\,d\theta_{n-2}^2 + \cdots + \sin^2\theta_3(\,d\theta_2^2 + \sin^2\theta_2\,d\theta_1^2))\cdots]] \quad (II.3)$$

where $0 \leq \theta_1 < 2\pi, \quad 0 \leq \theta_i < \pi, \quad i = 2, 3, \ldots,$ n.

If we choose units $c =$ G $= 1$, then $[M] \sim L^{n-1}$ and P can be written in terms of dimensionless variables

$$\rho = r(M)^{-1/(n-1)}, \qquad (II.4a)$$

$$a = \bar{\alpha}\kappa(M)^{-2/(n-1)} \qquad (II.4)$$

The variable ρ is our new dimensionless radial coordinate, so that

$$P = 1 + \frac{\rho^2}{2a}\left[1 - \left[1 + \frac{8a}{\rho^{n+1}}\right]^{1/2}\right]. \qquad (II.5)$$

In the limit $a \to 0$, P can be written in its limiting form

$$P = 1 - \frac{2}{\rho^{n+1}}. \qquad (II.6)$$

This represents the higher-dimensional Schwarzschild solution, which we would have obtained in its 'exact' form, if we had started with the Einstein action of ordinary D-dimensional spacetime :

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,R. \qquad (II.7)$$

The horizon is given by $P = 0$ and, for a Boulware-Deser black hole, this is located at $\rho = \rho_h$ or $r = r_h$ and is given by

$$\begin{aligned} &\rho_h^{n-1} + a\rho_h^{n-3} - 2 = 0 \\ \text{or,}\quad &r_h^{n-1} + \bar{\alpha}\kappa r_h^{n-3} - 2GM = 0 \end{aligned} \qquad (II.8)$$

Correspondingly, for the higher-dimensional Schwarzschild solution (a=0), the horizon is located at

$$\rho_h = 2^{1/(n-1)} \quad \text{or,} \quad r_h = (2GM)^{1/(n-1)}. \qquad (II.9a)$$

For $a > 0$, there is only one horizon. For the five dimensional Boulware Deser solution the horizon is at

$$\rho_h = \sqrt{2-a} \qquad \text{or} \qquad r_h = \sqrt{2GM - \bar{\alpha}\kappa}, \qquad (II.9b)$$

so that we have a black hole solution only if $a < 2$.

II.(A) Geodesic motion

This section contains the results of our investigation on the geodesic motion in the Boulware- Deser Black Hole (BDBH) spacetime [Bhawal, 1990]. The study of geodesics is very useful for probing into the geometry of a spacetime. In the present context, this provides us information about the effect of the presence of extra dimensions and of the higher derivative terms on the motion of geodesics. We study only the cases corresponding to the positive values of a. Since this solution is asymptotic to the Schwarzschild spacetime of corresponding dimensions [Tangherlini,1963], we also compare the results of the two cases. This will help us in isolating the effects originating due to the presence of higher derivative terms from those present in ordinary higher dimensional gravity.

The symmetries of the Boulware-Deser spacetime give us ${}^{n+1}C_2$ 'rotational' Killing vectors and one 'time-translational' Killing vector given respectively as

$$\xi_{ij} = x_{[i}\partial_{j]}, \qquad i,j = 1,2,\ldots,(n+1) \tag{II.10}$$

$$\xi_t = \partial_t. \tag{II.11}$$

Note that different combinations of the assigned values of the indices i, j correspond to different Killing vectors and not their tensor components.

The solution of the geodesic equations is then considerably facilitated if we employ integrals of motion by using the theorem that if **u** is the **(n** + 2)-velocity of a geodesic, then for any Killing vector ξ, we have a constant of motion $\xi.\mathbf{u}$.

In terms of the polar coordinates, X^μ, we have u, $= g_{\mu\nu}(dX^\mu/d\tau)$ where r is the proper time along the geodesic. So. from the Killing vector $\xi_\mathbf{t}$, we obtain the constant E :

$$\xi_\mathbf{t}.\mathbf{u} = u_t = -P\frac{dt}{d\tau} = -E \tag{II.12}$$

Similarly, the rotational Killing vectors give us constants ℓ_{ij},

$$\xi_\mathbf{ij}.\mathbf{u} = \ell_{ij} \tag{II.13}$$

These are analogous to angular momenta. This leads us to define the 'total angular momentum' to be

$$L^2 = \sum_{i<j} \ell_{ij}^2 = \rho^2[u^2 - (\mathbf{u}.\hat{e}_\rho)^2 + (\mathbf{u}.\hat{e}_t)^2] \qquad (II.14)$$

$\hat{e}_\rho$, $\hat{e}_t$ being unit vectors in ρ and t directions respectively. So,

$$L^2 = \rho^2\left[u^2 - \frac{\dot{\rho}^2}{P} + \frac{E^2}{P}\right], \qquad (II.15)$$

where 'dot' denotes differentiation with respect to τ. The quantity E is defined to be the energy. Rearranging, we get

$$-\frac{E^2}{P} + \frac{\dot{\rho}^2}{P} + \frac{L^2}{\rho^2} = u^2 = \begin{cases} -1 & \text{for 'timelike' geodesics} \\ 0 & \text{for 'null' geodesics} \end{cases} \qquad (II.16)$$

Since this solution is spherically symmetric, we can always describe the geodesics in an invariant hyperplane described by all $\theta_k = \pi/2$, k $= 2, 3, \ldots$ n. Then the Killing vector corresponding to the symmetry in the direction of the azimuthal angle θ_1 gives us

$$\rho^2\dot{\theta}_1 = L \qquad (II.17)$$

The set of three equations 11.12, 16 & 17 describes the motion of geodesics in Boulware-Deser black hole spacetime.

(i)*Timelike* Geodesics $(u^2 = -1)$

Considering the above equations of motion, the behaviour of the trajectory of a particle both in proper time and coordinate time can be investigated. The situation in the case of an infalling particle has been illustrated in Figure 1 for a particular combination of values of 'E' and 'a' [E $=$ 2, a $=$ 1, ρ(initial) $=$ 10]. As usual, with respect to an observer stationed at infinity, the particle describing a timelike trajectory will take an infinite time to reach the horizon, even though it will cross the horizon and arrive at the singularity in a finite proper time. It is found that as 'a' decreases, the finite proper time needed to reach the singularity also reduces. This reduction is however very small as compared to the usual value of proper time needed.

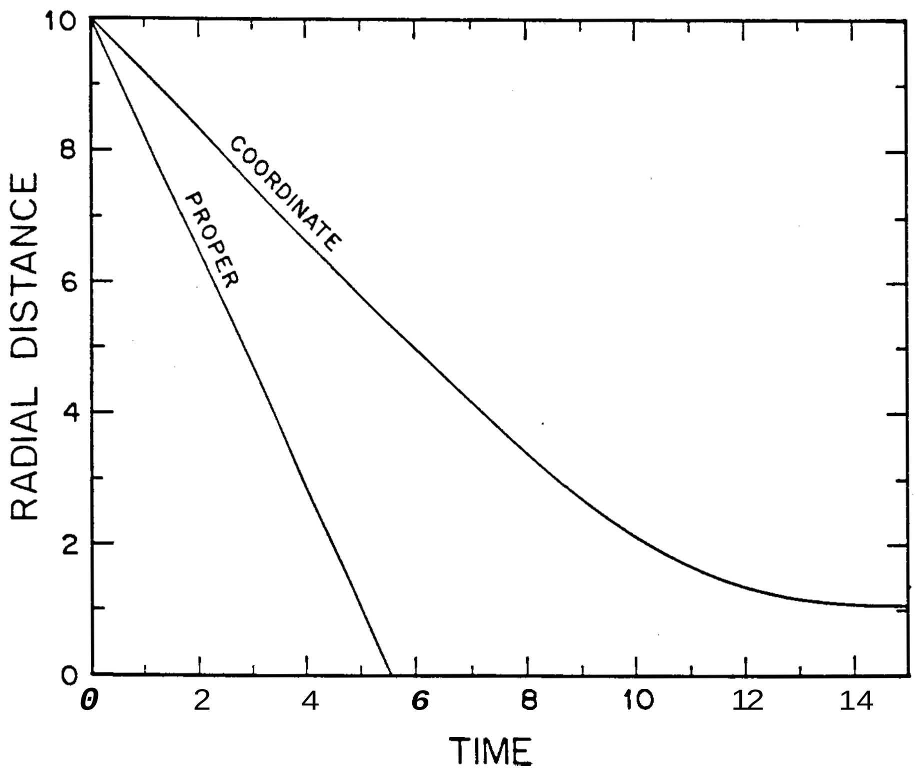

Fig.1 *Time in* 5–dimensional *BDBH.*

The field equation (11.16) may be written as

$$\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}(1 + \frac{L^2}{\rho^2})P = \frac{1}{2}E^2. \qquad (II.18)$$

This shows that the radial motion of a geodesic is the same as that of a unit mass particle of energy $E^2/2$ in ordinary one dimensional non-relativistic mechanics moving in the effective potential

$$V_{\text{eff}} = \frac{1}{2}\left[1 + \frac{L^2}{\rho^2}\right]P. \qquad (II.19)$$

In the case of the five-dimensional Schwarzschild black hole,

$$V_{\text{eff}} = \frac{1}{2} - \frac{1}{R^2} + \frac{L^2}{2R^2} - \frac{L^2}{R^4}, \qquad (II.20)$$

the extremum of which is at $\rho_e = 2L/(L^2 - 2)^{1/2}$.

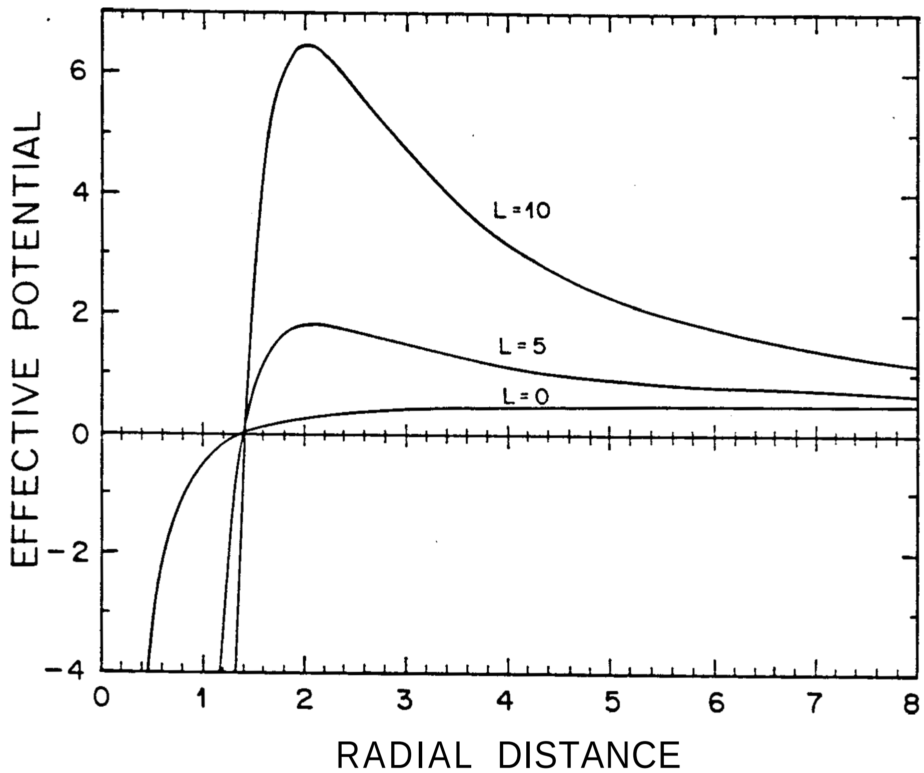

Fig. 2 V_{eff} for *timelike* geodesics in 5-dim. *Schwarzschild* black hole ($a = 0$)

If $L^2 < 2$, there is no extremum of V_{eff}. A particle heading toward the center of attraction with $L^2 < 2$ will fall directly to the horizon and will continue its fall into the spacetime singularity at $\rho = 0$. From now on, we shall refer to such a monotonous type of potential as type I.

If $L^2 > 2$, we can easily see that the extremum ρ_e is the maximum point of V_{eff}. After that, the potential dies down to 0.500 as $\rho \to \infty$. From now on. we shall refer to such type of potential as type II.

Figure 2 shows different V_{eff} vs ρ plots obtained for different values of L. The lowest one ($L = 0$) is of type I. No stable bound orbit is possible. For a type II potential, only an unstable circular orbit can exist at ρ_e.

In a five dimensional Boulware-Deser black hole,

$$V_{\text{eff}} = \frac{1}{2} + \frac{L^2}{4a} + \frac{\rho^2}{4a} + \frac{L^2}{2\rho^2} - \frac{\rho^2 + L^2}{4a}\left[1 + \frac{8a}{\rho^4}\right]^{1/2} \qquad (II.21)$$

The extremum point $(\rho = \rho_e)$ of this potential is given by the equation

$$\rho_e^8(2 - L^2) + 4L^2\rho_e^6 + \rho_e^4(aL^4 - 8aL^2) + 8aL^4(a - 2) = 0 \qquad (II.22)$$

The solution of this quartic equation in ρ_e^2 is very complicated and can be written as

$$\rho_e = \sqrt{x}$$

$$x = \frac{-(p - l) \pm [(p - l)^2 - 4(k - b)]^{1/2}}{2} \quad \text{and} \quad \frac{-(p + l) \pm [(p + l)^2 - 4(k + b)]^{1/2}}{2},$$

where

$$l = (p^2 + 2k - q)^{1/2}, \qquad b = (k^2 - s)^{1/2}$$

$$k = \left[-\frac{\beta}{2} + \left[\frac{\beta^2}{4} + \frac{\sigma^3}{27}\right]^{1/2}\right]^{1/3} + \left[-\frac{\beta}{2} - \left[\frac{\beta^2}{4} + \frac{\sigma^3}{27}\right]^{1/2}\right]^{1/3} + \frac{q}{6},$$

$$\sigma = -\left[\frac{q^2}{12} + s\right], \qquad \beta = \frac{1}{3}sq - \frac{5}{432}q^3 - \frac{1}{2}p^2 s,$$

$$p = \frac{2L^2}{2 - L^2}, \qquad q = \frac{aL^4 - 8aL^2}{2 - L^2}, \qquad s = \frac{8aL^4(a - 2)}{2 - L^2}. \qquad (II.23)$$

From this solution, it is very difficult to make a general statement on the shape of the V_{eff} vs ρ characteristics in different cases. However, we have plotted these curves for different values of a and L, a typical one of which is shown in Figure **3.** In all cases, the effective potential curves are either of type I or II. Therefore, in no case, a stable bound orbit is possible. For a type II potential an unstable circular orbit can exist at a radius $\rho = \rho_e$. The five velocity of this circular orbit can be represented as

$$u^\alpha = N(1, 0, 0, 0, \omega), \qquad (II.24)$$

where

$$w^2 = \frac{1}{2a}\left[1 + \left[1 + \frac{8a}{\rho_e^4}\right]^{-1/2}\right] \quad \text{and} \quad N^{-2} = 1 - \frac{4}{\rho_e^2}\left[1 + \frac{8a}{\rho_e^4}\right]^{-1/2}.$$

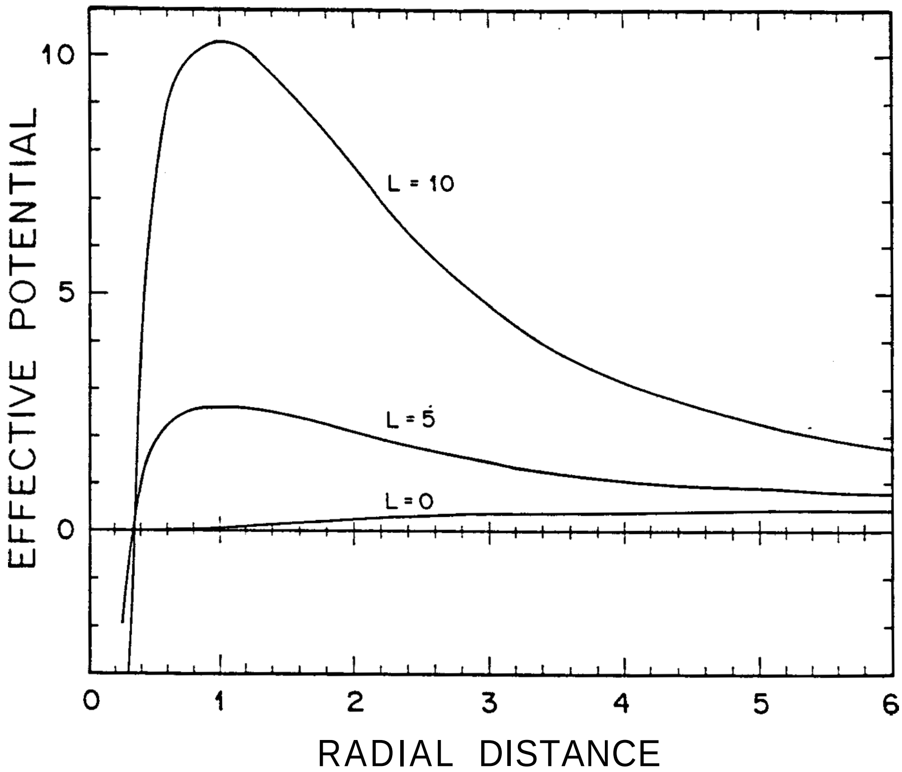

Fig.3 V_{eff} for *timelike* geodesics for a constant value of $a(= 1.9)$ in 5-dim. BDBH

For a fixed a as L increases, the height of the maximum increases. The effective potential becomes zero at the horizon $\rho_h = \sqrt{2-a}$ and, in all cases, asymptotically tends to *0.500* as ρ increases. For a fixed L, as a increases, the horizon consequently shifts to the left as does the location of the maximum (Figure 4).

We have also investigated the field equations in other higher dimensions $(D > 5)$. But the results obtained are of similar nature as those in fire dimensions. Again. the effective potential is either of type I or II.

(ii)Null Geodesics (u=0)

For a five dimensional Schwarzschild black hole,

$$V_{\text{eff}} = \frac{L^2}{2\rho^2} - \frac{L^2}{\rho^4}. \qquad (II.25)$$

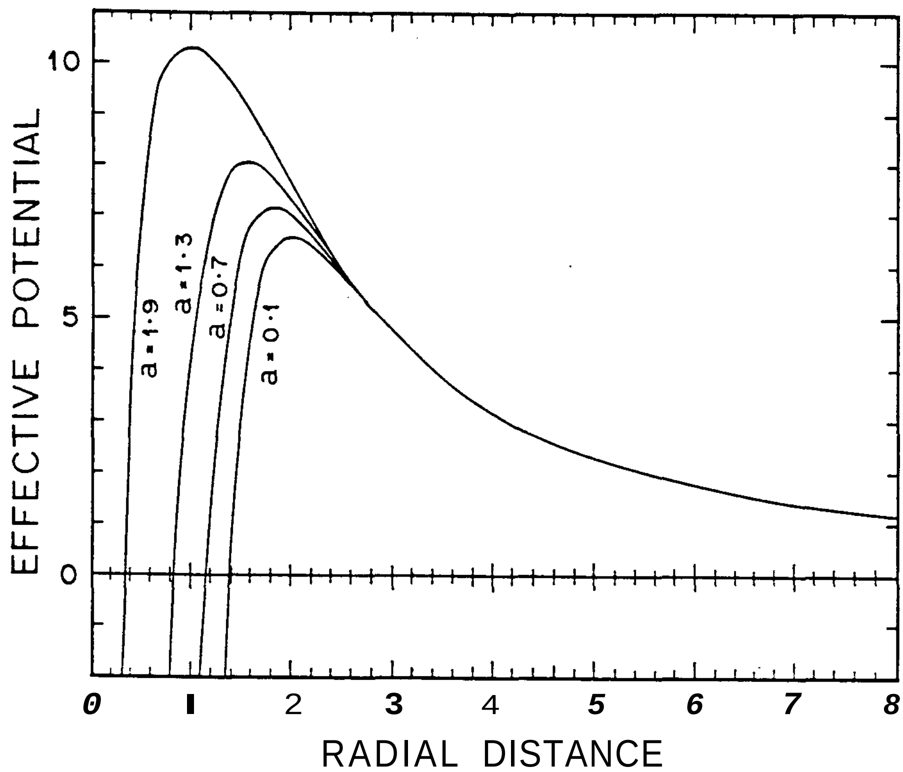

Fig.4 V_{eff} for *timelike* geodesics for a constant *value* of L(= 10) in 5-dim. BDBH

In this case, unlike the timelike geodesics, the potential is always of type II and the maximum point is at $\rho_e = 2$. The maximum value of the effective potential is dependent on L from which we deduce that the five dimensional Schwarzschild geometry will capture any photon sent toward it with an apparent impact parameter smaller than the critical value $b_c = 2\sqrt{2}$. The effective potential for null geodesics in five dimensional Boulware-Deser geometry is

$$V_{\text{eff}} = \frac{L^2}{2\rho^2} + \frac{L^2}{4a} - \frac{L^2}{4a}\left[1 + \frac{8a}{\rho^4}\right]^{1/2}. \qquad (II.26)$$

The position of the maximum is at $\rho_e = (16 - 8a)^{1/4}$. So, in this case also, for any value of L or a ($0 < a < 2$), the potential is of type II (Figures 5 & 6). The critical value of the 'apparent impact parameter' can be calculated to be

$$b_c = \left[\frac{2a\sqrt{2}}{\sqrt{2} - \sqrt{2-a}}\right]^{1/2} \qquad (II.27)$$

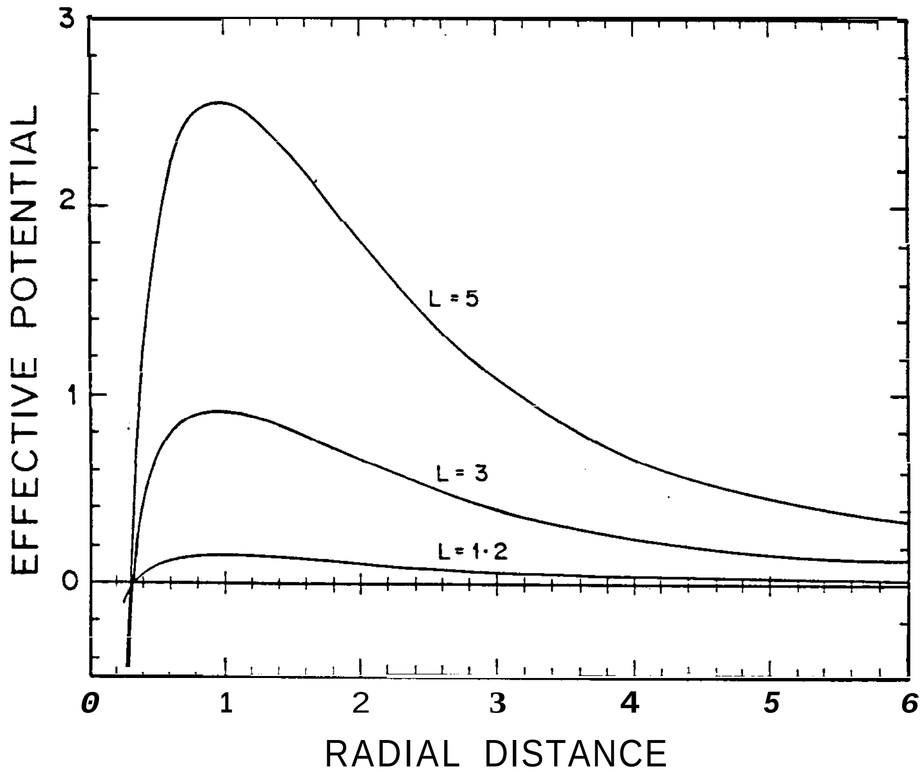

Fig.5 V_{eff} for null geodesics for a constant value of $a(= 1.9)$ in 5-dim. BDBH

(iii)Conclusions

We have observed that the nature of the effective potential curves for five dimensional Boulware-Deser spacetime is not different from those in five dimensional Schwarzschild solution. If we compare the two cases, we see that for a low or high value of L. the presence of a nonzero a (and, consequently, the presence of higher order terms in the action of the field) does not significantly affect the nature of geodesic orbits. except changing the position of the horizon and the maximum point of effective potential. However, for values of L around L $= \sqrt{2}$, the presence of a (or, sometimes, a high value of a) may change a type I potential to a type II one, being determined by Eq.(II.23).

It is a well-known result that in four dimensional Schwarzschild black hole geometry, stable bound orbits are possible for timelike geodesics. On the other hand, our numerical

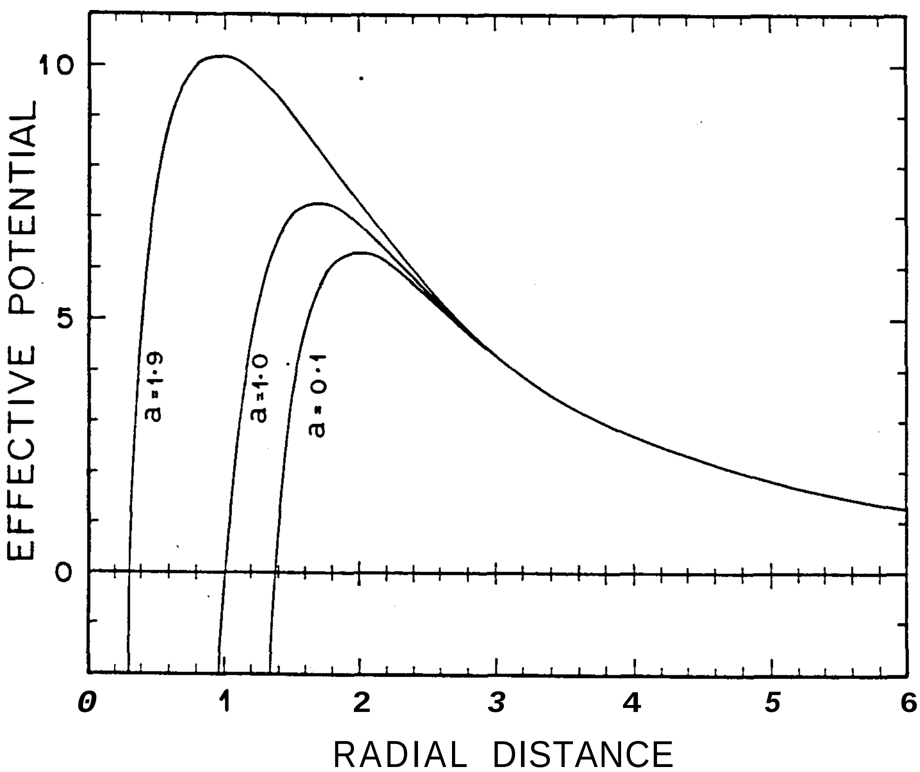

Fig.6 V_{eff} for *null* geodesics for a constant *value* of L(= 10) in *5-dimensional* BDBH

solutions have shown that the same is not true for higher dimensional Schwarzschild or Boulware-Deser spacetimes, where only unstable orbits are possible for suitable values of L. Considering the analytical solutions obtained, it is not very easy to arrive at the above conclusion because of the complexity of these solutions. However, drawing analogy with Newtonian mechanics, we can expect the above results.

It is well known that , also in Newtonian mechanics, a particle in a potential of the form $\phi \propto 1/r^{n-1}$ (n > 2) cannot describe a stable bound orbit*. The underlying connection between this fact and the present results can be readily understood. In the weal; field limit, $g_{00} = -(1 + 24)$. In the higher dimensional Schwarzschild solution, by Newtonian approximation, ϕ is taken to be $-GM/r^{(n-1)}$. The absence of stable bound orbits in higher

* A corollary of the Bertrand's theorem

Also, ℓ_1 and ℓ_i are integers. All $\ell_i \geq 0$ and

$$| \ell_1 | \leq \ell_2 \leq \ldots \leq \ell_n .$$

$C^{\mu_i}_{\lambda_i}(z_i)$ are Gegenbauer functions.

The radial function $\mathcal{R}(r)$ satisfies

$$\frac{d^2\mathcal{R}}{dr^{*2}} + (\omega^2 - V(r))\mathcal{R} = 0 \qquad (II.31)$$

where r^* is defined to be

$$\frac{dr^*}{dr} = \frac{1}{P} \qquad (II.32)$$

$$V(r) = e^{-2\lambda}\left[\frac{\bar{\ell}_n^2}{r^2} + \frac{n(n-2)}{4r^2}P + \frac{n}{2}\frac{P_{,r}}{r}\right]. \qquad (II.33)$$

In the asymptotic region ($r \to \infty$ or $r^* \to \infty$), Eq.(II.29) reduces to

$$\frac{N}{\omega^{1/2}r^{1/2}}\, e^{-i\omega u}\, \mathcal{A}(\theta_1, \theta_i) \qquad (11.34)$$

$$\text{and} \qquad \frac{N}{\omega^{1/2}r^{1/2}}\, e^{-i\omega v}\, \mathcal{A}(\theta_1, \theta_i) \qquad (11.35)$$

in terms of the null coordinates $u = t - r^*$ and $v = t + r^*$. N is the normalization constant.

At early times ($t \to -\infty$), the solutions $f_{\omega\ell}$ of the wave equation can be chosen so that on past null infinity $\mathcal{I}^-$ they form a complete family satisfying orthonormality conditions

$$(f_{\omega\ell}, f_{\omega'\ell'}) = \delta(\omega - \omega')\delta_{\ell_1,\ell_1'} \ldots \delta_{\ell_n \ell_n'} . \qquad (II.36)$$

In our compact notation the index $\ell \equiv (\ell_1, \ell_2, \ldots, \ell_n)$.

They contain only positive frequency modes and are chosen to reduce to the incoming spherical modes *(11.35)* in the remote past. The field Φ can be decomposed as

$$\Phi(-\infty) = \sum_{\ell} \int (a_{\omega\ell} f_{\omega\ell} + a^{\dagger}_{\omega\ell} f^{*}_{\omega\ell}) d\omega. \qquad (II.37)$$

The 'in vacuum' state corresponding to the absence of incoming radiation from $\mathcal{I}^-$ can be defined as

$$a_{\omega\ell} \mid 0 >_{\text{in}} = 0 \qquad \text{for all } \omega,\ \ell. \qquad (11.38)$$

Because of the presence of the potential $V(r)$ in Eq.(II.31), the standard incoming waves (11.35) will be partially scattered back by the background field to become a superposition of incoming and outgoing waves. These outgoing modes are totally different from those which arise as a result of the passage of incoming modes through the interior of the collapsing ball to the opposite side.

Since the interesting thermal effects arise only from the latter contribution, for the time being, we will remove $V(r)$ from Eq.(II.31), so that the field modes can be simply given by Eqs.II.34 and 35 everywhere.

At late times, the field is described by the superposition of two types of modes. First, there are the outgoing modes (11.34) which we call $p_{\omega\ell}$. At future null infinity $\mathcal{I}^+$, these are the positive frequency modes. Also, there will always be modes incoming at the event horizon which we call $q_{\omega\ell}$. So, at late times, the field is expanded as

$$\Phi(-\infty) = \sum_{\ell} \int d\omega \{ b_{\omega\ell} p_{\omega\ell} + \text{h.c} + c_{\omega\ell} q_{\omega\ell} + \text{h.c.} \} \qquad (II.39)$$

h.c→ hermitian conjugate.

Since massless fields are completely determined by their data on the past null infinity $\mathcal{I}^-$, one can express both $p_{\omega\ell}$ and $q_{\omega\ell}$ as linear combinations of $f_{\omega\ell}$ and $f^*_{\omega\ell}$. Let

$$p_{\omega\ell} = \sum_{\ell} \int d\omega' (\alpha_{\omega\omega'} f_{\omega'\ell} + \beta_{\omega\omega'} f^*_{\omega'\ell}). \qquad (II.40)$$

This is known as Bogolubov transformation. The coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are known as Bogolubov coefficients. These satisfy the normalization condition (or the Wronskian condition) :

$$1 = \int d\omega' (| \alpha_{\omega\omega'} |^2 - | \beta_{\omega\omega'} |^2). \qquad (II.41)$$

Therefore, the 'in vacuum' state will not appear to be a vacuum state to an observer at $\mathcal{I}^+$. Instead he will find that the expectation value of the Number operator in the 'in vacuum' to be

$$N_\omega = {}_{\text{in}}\!< 0 \mid b^\dagger_{\omega\ell} b_{\omega\ell} \mid 0 >_{\text{in}} = \int d\omega' \mid \beta_{\omega\omega'} \mid^2 \qquad (II.42)$$

$$\text{where} \qquad \beta_{\omega\omega'} = (p_{\omega\ell}, f^*_{\omega'\ell}). \tag{11.43}$$

Thus, in order to determine the number of particles created by the gravitational field and emitted to infinity, one has to calculate $\beta_{\omega\omega'}$.

To evaluate $\beta_{\omega\omega'}$, it was found to be convenient to take the surface of integration to lie in the in-region. This corresponds to the modes $p_{\omega\ell}$ being traced back along the null path. The modes $p_{\omega\ell}$ is of the form (11.34). **At** early times, the ray will be moving along constant v lines. But since the phase of the wave remains constant, it will still have the numerical value $e^{-i\omega u(v)}$ where the function $u(v)$ has to be determined. Following Hawking (1974), one may show that

$$u = -\ln[v_0 - v]/S + \text{constant} \tag{11.44}$$

where v_0 is the value of the ray surface that forms the event horizon. The quantity S is called 'Surface Gravity' and is given as

$$S = \frac{1}{2}\frac{dP}{dr}\bigg|_{r=r_h}. \tag{II.45}$$

r_h describes the horizon of the black hole where $P = 0$. Another derivation of Eq.(II.44) using moving mirror consideration can be found in Birrel and Davies (1982).

So, at early times, we have

$$p_{\omega\ell} = \begin{cases} N\omega^{-1/2}r^{-1}\mathcal{A}\exp\left[\frac{i\omega}{S}\ln(v_0 - v)/K\right], & \text{for } v < v_0 \\ 0 & \text{for } v > v_0. \end{cases} \tag{II.46}$$

where K is a constant.

Now, the ordinary in-vacuum is defined with respect to modes $f_{\omega\ell}$ given by Eq.(II.35). So, using Eq.(II.43), the Bogolubov coefficients can be determined as

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\int_{-\infty}^{v_0} dv\left(\frac{\omega'}{\omega}\right)^{1/2} e^{-i\omega' v}\exp\left[\frac{i\omega}{S}\ln(v_0 - v)\right]. \tag{II.47}$$

One may also calculate

$$\alpha_{\omega\omega'} = (p_{\omega\ell}, f_{\omega'\ell})\frac{1}{2\pi}\int_{-\infty}^{v_0} dv\left(\frac{\omega'}{\omega}\right)^{1/2} e^{+i\omega' v}\exp\left[\frac{iW}{S}\ln(v_0 - v)\right]. \tag{II.48}$$

These integrals can be evaluated in terms of Γ-functions. Using Eq.(II.41) one can have

$$| \alpha_{\omega\omega'} |^2 = \exp\left(\frac{2\pi\omega}{S}\right) | \beta_{\omega\omega'} |^2 \qquad (II.49)$$

Now, using Wronskian condition(II.41), we can obtain from Eq.(II.42)

$$N_\omega = \left[\exp\left(\frac{2\pi\omega}{S}\right) - 1\right]^{-1}. \qquad (II.50)$$

This corresponds to a Planck spectrum with temperature given by

$$T = \frac{S}{2\pi}. \qquad (II.50)$$

Therefore, we observe that the most important quantity in this treatment is the surface gravity S and the Hawking temperature of any black hole of topology R^2 x S^{D-2} of the form(II.1) can be given by Eq.(II.50).

In five dimensional Boulware-Deser black hole given by Eq.(II.2),

$$S = \frac{\sqrt{2GM - \bar{\alpha}\kappa}}{(2GM + \bar{\alpha}\kappa)} \qquad (II.51)$$

In general, the temperature of a D-dimensional Boulware-Deser black hole is

$$T = \frac{(D-5)GM + r_h^{D-3}}{2\pi r_h(4GM - r_h^{D-3})} \qquad (II.52)$$

where r_h is a solution of Eq.(II.8).

Also, the temperature of an ordinary higher dimensional Schwarzschild black hole can be obtained by setting $\alpha = 0$

$$T = \frac{D-3}{4\pi(2GM)^{1/(D-3)}}. \qquad (II.53)$$

Similar expressions were also obtained by Myers and Simon (1988) and Wiltshire (1988) by making use of an alternative method which is described below in a sketchy way. [There are ,however, some minor differences in the equation for r_h and the expression for T because the definition of the constants and parameters of the black hole solution considered by them differ from those in the actual solution given by Boulware and Deser (1985).] By now, this method is also quite familiar in the literature [see Birrel and Davies,1982; Narlikar and Padmanabhan,1986].

The Hawking temperature of the spacetime metrics of the form (II.1) may be determined in each case by noting that if one analytically continues the metric to imaginary time, $t \rightarrow i\tau$, then the resulting manifold is regular if τ is identified with a period $\beta = 2\pi/S$.

After the analytical continuation, the metric (II.1) can be written with a Kruskal-like line element as

$$ds^2 = Pe^{-2Sr^*}[dX^2 + dY^2] + r^2 d\Omega_n^2 \qquad (II.54)$$

$$\text{where} \qquad X = S^{-1}e^{Sr^*}\cos S\tau$$

$$Y = S^{-1}e^{Sr^*}\sin S\tau \qquad (II.54)$$

This is a positive definite Riemannian space with topology $R^2 \times S^2$. Since the event horizon ($r = 2M$) is represented by the origin of the coordinate system, the singularity and the space inside the black hole are absent here. There is a rotational symmetry corresponding to the Killing vector ∂_τ. This endows τ with the properties of an angular coordinate with a periodicity $2\pi/S$. This periodicity in τ attributes the analytically continued Hartle-Hawking propagator [Hartle and Hawking,1976] all the properties of a thermal Green's function [Gibbons and Perry,1976,1978] from which the temperature can be easily identified to be $2\pi/S$.

Now, we may look back and correct the neglect of back scattering that depletes the outgoing flux by a factor $\mathcal{R}_\omega$, the reflection coefficient. This has the effect of replacing the left hand side of Eq.(II.41) by $(1 - \mathcal{R}_\omega)$. If we consider a sphere of radius r_0 centered on the collapsing ball, the density of states inside it will be $r_0 d\omega/(2\pi)$ [for all fixed ℓ_1, ℓ_i]. So, the number of particles per unit time in the frequency range ω to $\omega + d\omega$ passing out through the surface of the sphere can be calculated to be

$$\tilde{N}_\omega = \frac{d\omega}{2\pi}\frac{(1-\mathcal{R}_\omega)}{(\exp(2\pi\omega/S) - 1)}. \qquad (II.55)$$

Since $\mathcal{R}_\omega$ is a function of ω, the spectrum is not precisely Planckian in nature. The total luminosity of the black hole can be found by integrating (II.55) over all modes. The numerical study of the dependence of $\mathcal{R}_\omega$ on w for five and six dimensional Boulware-Deser black hole can be found in Iyer, Iyer and Vishveshwara(1988).

At the end of this section, we indicate here an interesting property of the five dimensional BDBH which one can guess from the expression of temperature in Eq.(II.51).

A problem faced in the process of Hawking radiation is that a black hole may vanish after radiating away its entire mass. In that case the incoming pure state fully converts to a mixed thermal state from the point of view of an observer located outside the event horizon, thus violating basic principle of quantum coherence— the time evolution should be described by a unitary operator, the Hamiltonian.

As is evident from Eq.(II.51), however, in this case, the black hole temperature may vanish at a finite mass and the evolution may end up with a zero temperature soliton. But the horizon also vanishes in that limit ($2GM \rightarrow \bar{\alpha}\kappa$) revealing the naked singularity. In this instance, then, the cosmic censorship hypothesis [Penrose. 19791 can be identified with the third law of black hole thermodynamics [Israel, 19861 which ensures that "No continuous process in which the energy tensor of accreted matter remains bounded and satisfies the weak energy condition in a neighbourhood of the apparent horizon can reduce the surface gravity of a black hole to zero within a finite advanced time". Such a black hole may, therefore, continue to radiate for ever without being fully evaporated away.

The Eq.(II.52) shows that a zero-temperature situation will never arise in BDBH solutions for D > 5. As shown by Myers and Simon (1988), this feature is always encountered in $2K+1$ dimensions for a Lovelock theory including 21i dimensional Euler density.

II.(C) Comments on Back Reaction

It is a natural question to ask whether the semiclassical effects or the leading order corrections can solve the problem of singularity faced in ordinary GR. Classically speaking, in higher dimensions, at short distances, Gravity is more attractive than in four dimensions due to the fact that the force varies inversely as distance to the power of $(D-2)$. In all dimensions, the centrifugal repulsive force varies inversely as distance cubed. So, in dimensions(> 5), the gravitational pull rises even more rapidly than the centrifugal repulsion. Also, we have observed in our study in sec.II.(A) that an object following timelike geodesic path takes less proper time to reach the singularity in the presence of higher order terms with a positive coupling constant. All these effects seem to make gravitational collapse and spacetime singularities even more likely in higher dimensional and higher order gravity.

The singularity is guaranteed to exist inside the event horizon by virtue of the singularity theorems of Hawking and Penrose [For references and details, see Hawking and Ellis, 1973]. A basic condition for the validity of these theorems is that the timelike convergence condition be satisfied every here, namely, $R_{\mu\nu}\ell^{\mu}\ell^{\nu} \geq 0$, where ℓ^{μ} is any timelike or null vector. In GR these theorems remain to be valid provided the gravitational field is coupled only to sources which obey the strong energy condition. and the cosmological constant is either zero or negative. However, if one identifies $S_{\mu\nu}$ in Eq.(I.27) with some sort of stress energy tensor, one can see that this does not satisfy the strong energy condition (but we should point out that such a consideration is not a very comfortable idea since this quantity depends on the geometry itself). We do not yet know the details of the validity of the singularity theorems and black hole uniqueness theorems in such theories.

The effects of the higher order terms on the singularity formation can be properly understood only if one can solve the back reaction problem in such theories. The general procedure for studying back reaction problem in black hole is as follows :

(1) One has to choose a spherically symmetric metric which may be valid inside the black

hole

$$ds^2 = -e^{2p(r)}\,dr^2 + e^{2q(r)}\,dt^2 + r^2\,d\Omega^2. \qquad (II.56)$$

(2) Then one has to consider a field (say, massless scalar field) in such a background and calculate the renormalized vacuum expectation value of the corresponding stress energy tensor denoted by $< T_{\mu\nu} >$. The vacuum state under consideration is the Hartle-Hawking vacuum [Hartle and Hawking, 1976] which describes the blackbody radiation of the scalar massless particles in equilibrium with the thermal bath surrounding the black hole. The temperature at infinity of this bath is $T = S/(2\pi)$.

(3) Then $< T_{\mu\nu} >$ is considered to be the source of the semiclassical Einstein-Gauss-Bonnet equations

$$G_{\mu\nu} - \alpha\kappa S_{\mu\nu} = \kappa < T_{\mu\nu} >, \qquad (11.57)$$

to solve for $p(r)$ and $q(r)$.

Another important quantity for a real scalar field is the renormalized value of the mean square field $< \Phi^2 >$, which may determine the extent of symmetry restoration near a black hole in theories with spontaneous symmetry breaking [Hawking. 1981].

However, a detailed back reaction problem is, in general, notoriously difficult to handle. The procedure described above could never be completed because of the great difficulty in calculating $< T_{\mu\nu} >$. Some approximate calculations leading to one loop quantum correction to the metric in case of the four dimensional Schwarzschild solution ($\alpha = 0$) were, however, done [Howard, 1984; Balbinot and Barlith, 1989]. The problem of vacuum polarization in the gravitational field of a multidimensional black hole was considered by Frolov, Mazzitelli and Paz (1989) for a massless scalar field. One can show that the point splitting method employed by them can be generalised to BDBH as well. In practice, however, the calculation of both $< \Phi^2 >$ and $< T_{\mu\nu} >$ is hampered by the complicated form of the g_{00} component of the metric. Since we could not complete the calculation, we are not going to describe the procedure here, but would like to indicate a few difficulties. The paper by

Frolov et al (1989) is quite self explanatory and one may also refer to Birrel and Davies (1982).

The complicated form of 'P' in the BDBH metric poses a great problem for obtaining a solution for the radial part of the Euclidean Green's function $G_E(x, x')$. The solutions for the other parts can be easily obtained. To solve this, one may employ similar techniques used by Iyer, Iyer and Vishveshwara(1989) for the solution of scalar waves in the BDBH background. But the equations become very complicated and getting an analytical solution seems to be an impossible task. The calculation of the Schwinger-DeWitt expansion (SDWE) for the propagator which is very much important for getting the renormalized value also faces various difficulties. For example, we failed to solve an important integral representing geodesic interval $S(\rho, \rho_h) = \int_{\rho_h}^{\rho} P^{-1/2}\, dr$, which is essential for completing this calculation. Another difficulty arises, if one wants to solve the problem in dimensions greater than nine. In that case, one has to know the DeWitt coefficients a_k upto $k = D/2$ for even dimensions and $k = (D-1)/2$ for odd dimensions. However, DeWitt coefficients only upto $k = 4$ are known. Nobody seems to have calculated these coefficients for $k > 4$, which will be very difficult to do.

At the end of this section, we would like to point out an interesting study in the Einstein-Gauss-Bonnet theory, of the collapsing process by Poisson (1991). With suitable boundary conditions, he numerically studied the problem in a black hole spacetime with extra compactified dimensions to see whether the effect of the higher order terms can reduce the strength of curvature near the singularity.

The boundary conditions considered by him is as follows. Near the singularity, the observed four dimensional slice of the spacetime is assumed to be of the form of Eq.(11.56). The extra dimensions are compactified forming the internal space of the form $w(r)^2 g_{ab} dy^a dy^b$. On the other hand, very far away from the singularity, the spacetime is described by the usual four dimensional Schwarzschild metric, $g_{\mu\nu}$ with a constant radius $(D-4)$-torus added to the original line element :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + w_0^2 [d\chi_1^2 + \cdots + d\chi_{D-4}^2]. \qquad (II.58)$$

The constant radius w_0 is related to the radius of the internal space, $w(r)$, as $w(r) = w_0 \exp[z(r)]$. Then he numerically integrated the field equations inward to solve for $p(r)$, $q(r)$ and $d(r)$. The result shows that for a positive coupling constant α, the singularity occurs sooner than the classical description. For negative coupling the model breaks down because the radius of the internal space becomes zero.

One may attempt to conclude from various results that the extra higher order terms will not come to the rescue of the spacetime near the singularity. But one should remember that the first order quantum corrections should not be considered as the last word in the description of such an important issue like singularity formation. The equations governing the evolution of the spacetime near the singularity may be totally different from what we expect from our present understanding of the subject. Before the final quantum theory is arrived at, the questions related to the singularity theorems in the context of semiclassical gravity, uniqueness theorem and quantum coherence problems are to be rigorously studied.