Chapter 3

Three point correlation for Gaussian initial conditions.

3.1 Calculating the three point correlation

Here we shall go one step beyond linear perturbations for Gaussian initial conditions. If the initial perturbations are Gaussian they are completely specified by the one and two point distribution functions 'f' and 'c' at some initial instant λ_0. All non-zero moments of the three point distribution function 'd', and the four point distribution function 'e' and all other higher distribution functions can be expressed in terms of moments of 'f' and 'c' at λ_0. The distribution function 'd' has no moments of order ϵ^3, but it has moments of order ϵ^4. These are

$$< p^a_\mu p^a_\nu >_3 = < p^a_\mu >_2 (a, a;, \lambda_0) < p^a_\nu >_2 (a, a;, \lambda_0) \tag{3.1}$$

and

$$< p^a_\mu p^b_\nu p^c_\sigma p^d_\gamma >_3 = \sum \delta^{ab} \left[< p^a_\mu p^c_\sigma >_2 < p^a_\nu p^d_\gamma >_2 + < p^a_\nu p^c_\sigma >_2 < p^a_\mu p^d_\gamma >_2 \right] \tag{3.2}$$

where the sum is over all possible pairs of particle indices in the delta function. There are no moments of 'e' of order ϵ^4 or lower. All this implies that

$$F_4 = F_5 = F_6 = F_7 = f_{11}(\lambda_0) = 0$$

and

$$f_5(\lambda_0) = f_7(\lambda_0) = f_9(\lambda_0) = 0 \tag{3.3}$$

to order ϵ^4. The functions f_{10} and f_{12} can be expressed in terms of moments of 'c' using equations *(3.1)* and *(3.2)*. Thus, the equation for the three point correlation function to order ϵ^4 is

$$\frac{\partial^4}{\partial\lambda^4}\zeta - 40\pi G\rho S\frac{\partial^2}{\partial\lambda^2}\zeta - 40\pi G\rho\frac{dS}{d\lambda}\frac{\partial}{\partial\lambda}\zeta - 12\pi G\rho\left[\frac{d^2S}{d\lambda^2} - 12\pi G\rho S^2\right]\zeta$$

$$= f_{12} + f_{10} - f_8 + \frac{\partial}{\partial X}f_6 + \left[\qquad - \frac{\partial^2}{\partial\lambda^2}\right] f_4 . \tag{3.4}$$

The terms on the right hand side are all products of two terms of order ϵ^2 and can be calculated using the equation of the previous chapter. For an $\Omega = 1$ universe, keeping only the growing mode, we can write the terms on the right hand side as

$$f_4(1,2,3,\lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{10} f_4(1,2,3,\lambda_0)\,, \tag{3.5}$$

$$f_6(1,2,3,\lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{11} f_6(1,2,3,\lambda_0)\,, \tag{3.6}$$

$$f_8(1,2,3,\lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{12} f_8(1,2,3,\lambda_0)\,, \tag{3.7}$$

$$f_{10}(1,2,3,\lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{12} f_{10}(1,2,3,\lambda_0) \tag{3.8}$$

and

$$f_{12}(1,3 \quad \lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{12} fir\ (1,2,3,\lambda_0)\,. \tag{3.9}$$

Using these, the equation for the three point correlation function is

$$\frac{\partial^4}{\partial\lambda^4}\zeta - \frac{60}{\lambda^2}\frac{\partial^2}{\partial\lambda^2}\zeta + \frac{120}{\lambda^3}\frac{\partial}{\partial\lambda}\zeta + \frac{216}{\lambda^4}\zeta$$

$$= \left(\frac{\lambda_0}{\lambda}\right)^{12}\left[f_{12} + f_{10} - f_8 - \frac{11}{\lambda_0}f_6 - \frac{92}{\lambda_0^2}f_4\right]_{\lambda_0}\,, \tag{3.10}$$

which has a solution,

$$\zeta(1,2,3,\lambda) = \frac{1}{2856}\left(\frac{\lambda_0}{\lambda}\right)^{8}\left[\lambda_0^4(f_{12} + f_{10} - f_8) - 11\lambda_0^3 f_6\right.$$

$$\left. - 92\lambda_0^2 f_4\right]_{\lambda_0} + \left(\frac{\lambda_0}{\lambda}\right)^6 E(1,2,3) \tag{3.11}$$

where 'E' is some function to be decided by the initial conditions.

Imposing the initial condition $\zeta(1,2,3,\lambda_0)=0$ on the solution, we get

$$\zeta(1,2,3,\lambda) = \frac{1}{2856}\left(\frac{\lambda_0}{\lambda}\right)^6\left[\left(\frac{\lambda_0}{\lambda}\right)^2-1\right]\left[\lambda_0^4(f_{12}+f_{10}-f_8)\right.$$

$$\left. - \quad 11\lambda_0^3 f_6 - 92\lambda_0^2 f_4\right]_{\lambda_0} . \qquad (3.12)$$

Actually, for a complete solution of the equations, four initial conditions have to be given. However the function E can be neglected if one is concerned only with the fastest growing part of the three point correlation function that is induced by the two point correlation function. Written explicitly this is

$$\zeta(1,2,3,\lambda) = \frac{1}{2856}\left(\frac{\lambda_0}{\lambda}\right)^8 \times$$

$$[12\left(\frac{\lambda_0}{m}\right)^4 \frac{\partial^4}{\partial x_\mu^a \partial x_\nu^a \partial x_\gamma^{a_1'} \partial x_\sigma^{a_2'}}\left(< p_\mu^a p_\sigma^{a_1'} >_2 (\lambda_0) < p_\nu^a p_\gamma^{a_2'} >_2 (\lambda_0)\right)$$

$$+ \quad 1 \qquad \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left(< p_\mu^a >_2 \left(a, a_1', \lambda_0\right) < p_\mu^a >_2 (a, a_2', \lambda_0)\right)$$

$$- \quad \frac{33\lambda_0}{n^4 nm}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a}\int p_\nu^b c\left(a, a_1', \lambda_0\right) c\left(a;,4,\lambda_0\right) X_\mu^{4a} d^3x^4 d^{12}p$$

$$- \quad \frac{9}{2n^4\pi}\left(\frac{\lambda_0}{m}\right)^2 \frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a}\int p_\nu^b p_\sigma^c c\left(a, a_1', \lambda_0\right) c\left(a;,4,\lambda_0\right) X_\mu^{4a} d^3x^4 d^{12}p$$

$$- \quad \frac{138}{\pi}\frac{\partial}{\partial x_\mu^a}\left(\xi\left(a, a_1', \lambda_0\right)\int \xi\left(a_2', 4, \lambda_0\right) X_\mu^{4a} d^3x^4\right)] \qquad (3.13)$$

This solution can be further simplified if we use the potentials introduced earlier. Using the potential and doing the integrals over space by parts we have

$$\frac{\partial}{\partial x_\mu^a}\left(\xi\left(a, a_1', \lambda_0\right)\int \xi\left(a;,4,\lambda_0\right) X_\mu^{4a} d^3x^4\right)$$

$$= \quad - \quad \pi\lambda_0^2 \frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi\left(a, a_1'\right)\frac{\partial}{\partial x_\mu^a}\nabla^2\phi\left(a, a_2'\right)\right], \qquad (3.14)$$

$$\frac{1}{nn^4 m}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a}\int p_\nu^b c\left(a, a_1', \lambda\right) c\left(a;,4,\lambda\right) X_\mu^{4a} d^3x^4 d^{12}p$$

$$= \quad - \quad 2\lambda_0 \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left[\frac{\partial}{\partial x_\nu^b}\left(\nabla^2\phi\left(a, a_1'\right)\right)\frac{\partial}{\partial x_\mu^a}\left(\nabla^2\phi\left(a, a_2'\right)\right)\right]$$

$$- \quad 4\lambda_0 \frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi\left(a, a_1'\right)\frac{\partial}{\partial x_\mu^a}\nabla^2\phi\left(a, a_2'\right)\right], \qquad (3.15)$$

and

$$\frac{1}{\pi n^4 m^2}\frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a}\int p_\nu^b p_\mu^a c\left(c, c_1', A\right) c\left(c_2', 4, A\right) X_\sigma^{4c} d^3x^4 d^{12}p$$

$$= - 16\frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left[\frac{\partial}{\partial x_\nu^a}\left(\nabla^2\phi\left(a, a_1'\right)\right)\frac{\partial}{\partial x_\mu^a}\left(\nabla^2\phi\left(a, a_2'\right)\right)\right]$$

$$- 8\frac{a}{\partial x_\mu^a}\left[\nabla^4\phi\left(a, a_1'\right)\frac{g}{\partial x_\mu^a}\nabla^2\phi\left(a, a_2'\right)\right] . \qquad (3.16)$$

Using these expressions in equation (3.13) we get

$$\zeta(1,2,3,\lambda) = \frac{1}{28}\left(\frac{\lambda_0}{\lambda}\right)^8 \lambda_0^2\left[3\frac{\partial}{\partial x_\mu^a}\left(\nabla^4\phi\left(a, a_1'\right)\frac{\partial}{ax;}\nabla^2\phi\left(a, a_2'\right)\right)\right.$$

$$\left. + 2\frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left(\frac{\partial}{\partial x_\nu^a}\left(\nabla^2\phi\left(a, a_1'\right)\right)\frac{\partial}{\partial x_\mu^a}\left(\nabla^2\phi\left(a, a_2'\right)\right)\right)\right] ., \qquad (3.17)$$

which can also be written as

$$\zeta(1,2,3,\lambda) = \frac{1}{14m}\left[3\frac{\partial}{\partial x_\mu^a}\left(\xi\left(a, a_1', \lambda\right) < p_\mu^a >_2 \left(a, a_2', \lambda\right)\right)\right.$$

$$\left. + \frac{1}{m}\frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left(< p_\mu^a >_2 \left(a, a_1', \lambda\right) < p_\nu^a >_2 \left(a, a_1', \lambda\right)\right)\right] . \qquad (3.18)$$

We have here an expression for the three point correlation function that arises from perturbations that are initially Gaussian and have no three point correlation. This expression is of order ϵ^4 and is a local function involving only derivatives of the potential ϕ . This expression is valid as long as terms having higher powers of ϵ may be neglected. It has been assumed that the initial perturbation had only the growing mode. If other modes are present ϕ represents only the growing part of it. One can introduce two other potentials for the two other modes and the three point correlation function will have terms with all combinations. The expression calculated is the fastest growing component.

An interesting fact is that for all values of the density parameter Ω the three point correlation function has the same spatial dependence given by

$$\zeta(1,2,3,A) = F^A(\lambda)\frac{a}{\partial x_\mu^a}\left(\nabla^4\phi\left(a, a_1'\right)\frac{a}{\partial x_\mu^a}\nabla^2\phi\left(a, a_2'\right)\right)$$

$$+ F^B(\lambda)\frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left(\frac{\partial}{\partial x_\nu^a}\left(\nabla^2\phi\left(a, a_1'\right)\right)\frac{\partial}{\partial x_\mu^a}\left(\nabla^2\phi\left(a, a_2'\right)\right)\right) , \qquad (3.19)$$

where F^A and F^B are some function of λ. This is because equation (2.82) which governs the growth of the three point correlation function is a differential equation in' λ alone. The functions $F^A(A)$ and $F^B(A)$ have to be determined by solving equation (2.82) and will be different for different values of 52. In what follows we restrict ourselves to 52 = 1 and the fastest growing mode.

3.2 Discussion

To get a better understanding of the three point correlation function calculated in the previous section it is convenient to express it explicitly in terms of the two point correlation function ξ instead of the potential ϕ.

Using equation (2.60) which defines the potential and the fact that $\xi(x)$ is a spherically symmetric function we have

$$\frac{\partial}{\partial x_\mu}\nabla^2\phi(x) = \left(\frac{2\lambda^4}{\lambda_0^5}\right)\frac{x_\mu}{x^3}\int_0^x \xi(x',\lambda)x'^2dx' = \left(\frac{2\lambda^4}{3\lambda_0^5}\right)x_\mu\overline{\xi}(x), \tag{3.20}$$

where we have defined $\overline{\xi(x)}$, which is the average of $\xi(x)$ over a sphere of radius X, by the second equality above.

The above equation can be easily understood by an analogy to a spherical mass distribution where the gravitational force on a particle at any point can be found by replacing all the matter in the sphere between this particle and the center of the distribution by an equal point mass at the center, and ignoring all the matter outside this sphere. Using this in equation (3.17) we obtain

$$\zeta(1,2,3,t) = \frac{1}{7}\left(5+2cos^2\theta_{xy}\right)\xi(x)\xi(y) + \frac{1}{3}cos\theta_{xy}\frac{d}{dx}\xi(x)y\overline{\xi}(y)$$

$$+\ \frac{4}{21}\left(1-3cos^2\theta_{xy}\right)\xi(x)\overline{\xi}(y) + \frac{6}{63}\left(3cos^2\theta_{xy}-1\right)\overline{\xi}(y)\overline{\xi}(x), \tag{3.21}$$

where

$$x = |x^a - x^{a_1'}|$$

$$y = |x^a - x^{a_2'}|,$$

and

$$\cos\theta_{xy} = \frac{x_\mu y_\mu}{xy}.$$

We would like to remind the reader that a, a_1', and a_2' are to be summed over the values shown in the table in the previous chapter . Although the three point correlation function appears to be a local function when written in terms of the potential ϕ it is not local in terms of the two point correlation function ξ. The three point correlation function ζ does not depend only on the values of the two point correlation function ξ at the separations occurring in ζ. It depends on the two point correlation at all scales smaller than the scales where the three point correlation function is being evaluated. It should also-be noted that it involves a derivative of the two point correlation function ξ.

An interesting consequence of equation (3.21) arises when the two point correlation function has compact support i.e.

$$\xi(r) = 0 \; ; r > r_1 \, , \tag{3.22}$$

the three point correlation has the form

$$\zeta(1,2,3,t) = \frac{6}{7}\left(3cos^2\theta_{xy} - 1\right)\frac{M^2}{x^3y^3} \tag{3.23}$$

in the region where the separation between all the three points is more than r_1. M is defined as

$$M = \int_0^{x_1} \xi(x',\lambda)x'^2 dx' \tag{3.24}$$

and the three point correlation function here depends only on the integral of the two point correlation function over the volume where it is non-zero.

Fry (1984) has calculated the three point correlation function for the special case of power-law initial two point correlation function

((x) = Ax"'. (3.25)

The general result obtained by us (equation 3.21) agrees with Fry's result (equation 34 of Fry 1984) for the power law case when 'n' is less than three. For larger values of 'n' the integral of the two point correlation function diverges and deviations from the power law behaviour are required at small separations to obtain meaningful results.

If we assume deviations from the power law at small separations for the two point correlation function, keeping a power law behaviour at large x, our formula will give the same result as Fry's formula at large x if

$$\overline{\xi(x)} = \frac{3}{3-n}\xi(x) \tag{3.26}$$

for large x. Whether this happens or not depends crucially on the behaviour of the two point correlation function at small separations.

As an illustration of the above point we present two examples where the two point correlation function has a large x behaviour

$$\xi(x) \sim x^{-4} \tag{3.27}$$

but the three point correlation functions are quite different in the two cases.

First we consider

$$\xi(x) = A\frac{3\alpha^2 - x^2}{(x^2+\alpha^2)^3} \tag{3.28}$$

where A is some normalisation constant and α some length scale. This corresponds to a Harrison-Zel'dovich power spectrum ($\sim k^1$) with an exponential decay for large k. Using this we get

$$\overline{\xi(x)} = A\frac{3}{(x^2+\alpha^2)^2} \tag{3.29}$$

which satisfies equation (3.26) for large s. In this case we find that at large separations the three point correlation matches with the formula derived by Fry.

Next we consider

$$\xi(x) = A\frac{\alpha}{(x^2+\alpha^2)^2} \tag{3.30}$$

which corresponds to a power spectrum $\sim k^0$ with an exponential decay at large k and we get

$$\overline{\xi(x)} = \frac{3A}{2x^3}\left[\tan^{-1}\left(\frac{x}{\alpha}\right) - \frac{\left(\frac{x}{\alpha}\right)}{1+\left(\frac{x}{\alpha}\right)^2}\right]. \tag{3.31}$$

For large x we have

$$\overline{\xi(x)} = \frac{3\pi A}{4x^3} \tag{3.32}$$

which does not sgtisfy equation (3.26). In this case the three point correlation function that we calculate differs, even at large separations, from the expression that Fry has given. Because $\xi(x)$ behaves as x^{-4} and $\overline{\xi(x)}$ behaves as x^{-3} for large x, $\xi(x)$ falls off much faster than $\overline{\xi(x)}$ and the three point correlation is dominated by the term containing two $\bar{\xi}$s. The three point correlation function at large separations then is controlled by the contribution from the two point correlation at small separations.

Thus we see in the two cases above, though the two point correlation function has the same power law spatial dependence for large separations, the three point correlation functions are quite different.

This is further illustrated graphically in figures 3.1 and 3.2 which shows $Q(r)$ versus r for the two cases discussed above. Here $Q(r)$ is defined as

$$Q(r) = \frac{\zeta(1,2,3,\lambda)}{\xi(1,2)\xi(1,3)+\xi(2,3)\xi(2,1)+\xi(3,1)\xi(3,2)}, \tag{3.33}$$

where the three points 1, 2, and 3 are located at the three corners of an equilateral triangle of sides of size r.

Next we would like to make some cautionary remarks on the direct application of the three point correlation function calculated here to interpret observations. The calculation

Figure 3.1: This shows $Q(r)$ as a function of r for the first example considered. There is a singularity at the point where the value of the two point correlation function is zero.

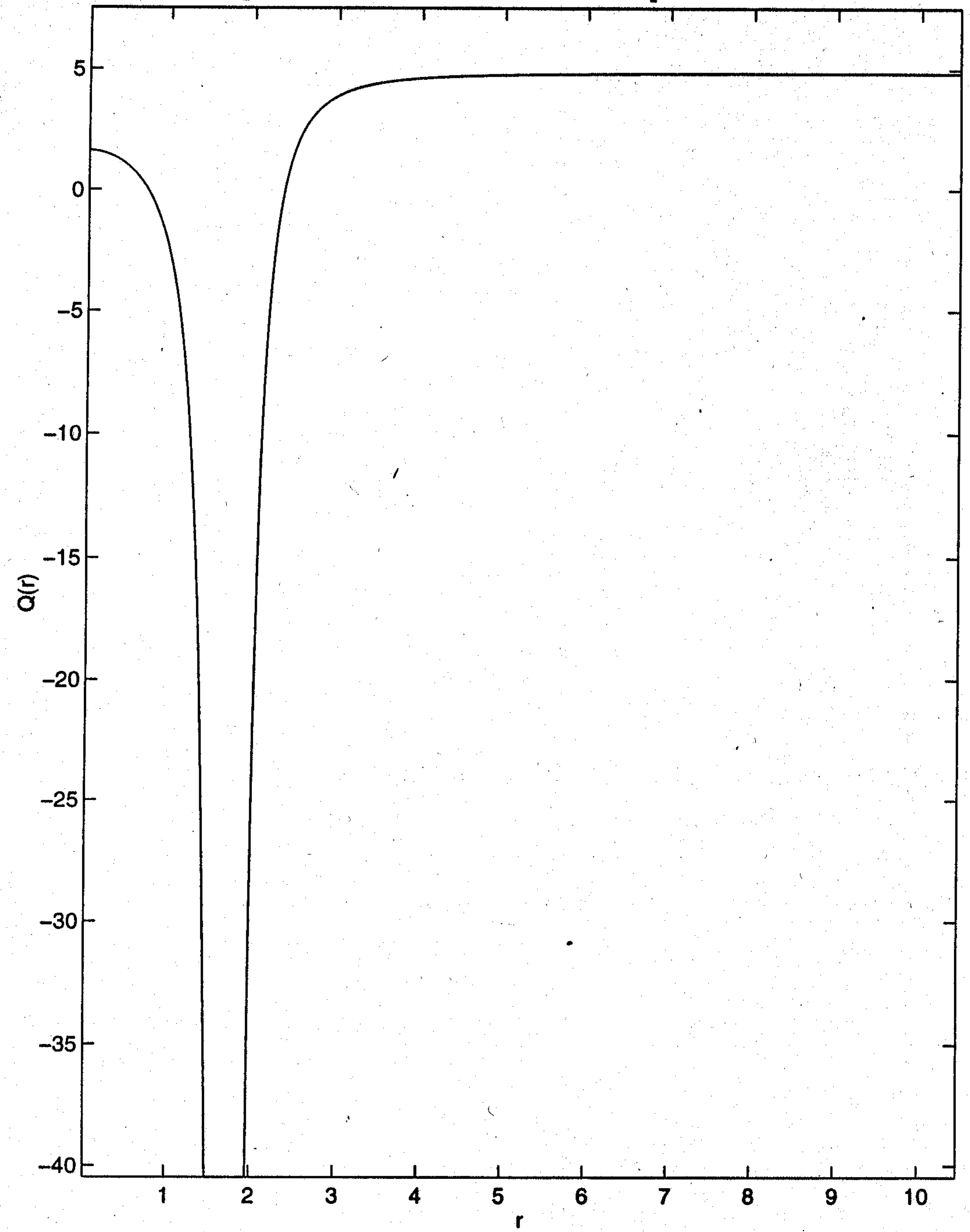
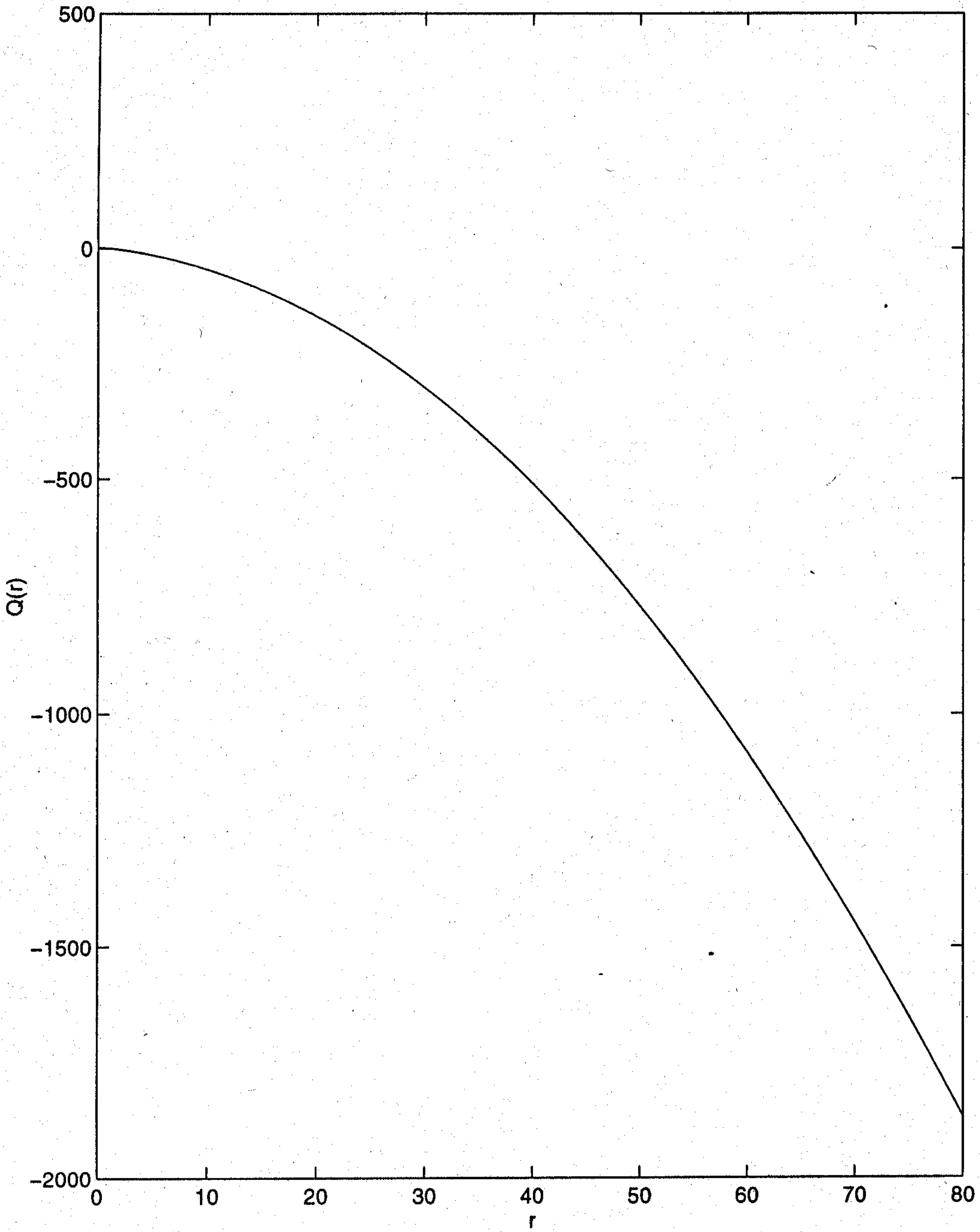


Figure 3.2: This shows $Q(r)$ for the second example considered. For large r we have $Q(r) \sim r^2$.



that has been done here is for the dominant matter component in the universe. If one wishes to use it to interpret galaxy correlations one should take the possibility that galaxies might be a biased tracer of matter into account. Secondly, although the galaxy correlations are small at large length scales, galaxies are strongly correlated at small length scales. Because of the non-local nature of the results, one has to check whether the perturbative results can be used at the large lengthscales when the small scales are strongly non-linear. In addition, even if the perturbative results are valid at large lengthscales, one cannot make a comparison of the three point correlation function at some lengthscales with just the two point correlation function at the same length scales. The three point correlation function is highly dependent on the shape of the initial two point correlation function at all scales smaller than the scales where the three point correlation is being evaluated.

Finally, the formalism developed here can be applied to test the validity of any scheme to close the BBGKY hierarchy. Such a scheme involves assuming a relationship between some moments of the various distribution functions. The validity of these assumptions can be tested in the weakly non-linear regime using the formalism developed in this chapter. As an example consider the scheme proposed by Davis & Peebles (1977). They assume that the three point correlation function has the 'hierarchical' form, i.e.

$$\zeta(1,2,3) = Q\left(\xi(1,2)\,\xi(1,3) + \xi(2,1)\,\xi(2,3) + \xi(3,1)\,\xi(3,2)\right), \tag{3.34}$$

where Q is a constant and that the correlations arose from initially small Gaussian density perturbations. A comparison of the expression for the three point correlation in equation **(3.34)** with the three point correlation function calculated in this chapter shows that it is not possible to write the induced three point correlation function in the weakly non-linear regime' in the form assumed in equation **(3.34).** Thus, although using this formalism we cannot say anything about the assumptions made by Davis & Peebles in the strongly non-linear regime, we can say that it is invalid in the weakly non-linear regime.

Bibliography

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Chapter 4

Calculating the two point correlation function.

In chapter II we have considered the linear evolution of the two point correlation function. In this chapter we consider the lowest order non-linear effects in the evolution of the two point correlation function.

4.1 Notation and the Equations Governing the Two Point Correlation.

We present below the equation governing the perturbative evolution of the two point correlation function. This equation (2.41), which was derived in chapter II, is

$$\frac{\partial^3}{\partial\lambda^3}\xi - 8\pi G\rho\left[S\frac{\partial}{\partial\lambda}\xi + \frac{\partial}{\partial\lambda}(S\xi)\right] = f_2 - f_3 - \frac{\partial}{\partial\lambda}f_1\,, \qquad (4.1)$$

where,

$$f_1(1,2,\lambda) = SG\rho\frac{\partial}{\partial x^a_\mu}\int \zeta(1,2,3,\lambda)\,X^{3a}_\mu d^3x^3\,, \qquad (4.2)$$

$$f_2(1,2,\lambda) = 2SGn\frac{\partial^2}{\partial x^b_\nu \partial x^a_\mu}\int < p^a_\mu >_3 (1,2,3,\lambda)\,X^{3b}_\nu d^3x^3\,, \qquad (4.3)$$

$$f_3(1,2,\lambda) = \frac{1}{m^3}\frac{\partial^3}{\partial x^c_\sigma \partial x^b_\nu \partial x^a_\mu} < p^a_\mu p^b_\nu p^c_\sigma >_2 (1,2,\lambda)\,. \qquad (4.4)$$

Here the position indices take values 1 and 2 and are to be summed when they appear twice.

For an $\Omega = 1$ universe this becomes

$$\frac{\partial^3}{\partial\lambda^3}\xi - \frac{24}{\lambda^2}\frac{\partial}{\partial\lambda}\xi + \frac{24}{\lambda^3}\xi = f_2 - f_3 - \frac{\partial}{\partial\lambda}f_1\,. \qquad (4.5)$$

If we are interested in only the linear evolution, we can ignore the terms on the right hand side of this equation as they are initially of a higher order in powers of ϵ compared to the two point correlation function. The initial two point correlation function is of order ϵ^2 whereas the terms on the right hand side are of order ϵ^3 or higher. In this chapter we consider the terms on the right hand side of equation (4.5) and calculate their effect on the evolution of the two point correlation function.

We separately consider the various terms on the right hand side of equation (4.5). We first consider equation (4.2). This depends on three point correlation function ζ which has been considered in the previous chapter. For Gaussian initial conditions this has a non-zero value only at order ϵ^4 and higher. We reproduce the expression for the three point correlation function at order ϵ^4 from chapter III. This is

$$\begin{aligned}\zeta(1,2,3,\lambda) &= \frac{1}{28}\left(\frac{\lambda_0}{\lambda}\right)^8 \lambda_0^2 \sum_{a=1}^{3}\left[3\frac{\partial}{\partial x_\mu^a}\left(\nabla^4\phi\left(a,a_1'\right)\frac{\partial}{\partial x_\mu^a}\nabla^2\phi\left(a,a_2'\right)\right)\right. \\ &+ \left. 2\frac{d^2}{\partial x_\mu^a \partial x_\nu^a}\left(\frac{\partial}{\partial x_\nu^a}\left(\nabla^2\phi\left(a,a_1'\right)\right)\frac{\partial}{\partial x_\mu^a}\left(\nabla^2\phi\left(a,a_2'\right)\right)\right)\right]. \qquad (4.6)\end{aligned}$$

In the equation for the three point correlation function the following conventions are used
A. the position indices, e.g. a, take values 1,2, and 3 corresponding to the corners of the triangle for which the three point correlation function is being evaluated. Also, a position index which appears twice or more should be summed over the allowed values.
B. for a fixed value of the position index (e.g. $a = 1$), a_1' and a_2' are to be summed over the other two values (i.e. $a_1' = 2, a_2' = 3$ and $a_1' = 3, a_2' = 2$). This is to be done whenever such a combination of three position indices appear.

In some of the equations for the other moments of the three point distribution function, if indicated, the summation convention A may not hold, but the convention B always holds.

To calculate f_2 and f_3 we have to first calculate the following quantities: $< p_\mu^a >_3 (1,2,3,\lambda)$ and $\partial_\mu^a\partial_\nu^b\partial_\sigma^c < p_\mu^a p_\nu^a p_\sigma^a >_2 (1,2,\lambda)$.. These calculations are discussed in the following two sections. The overall strategy is the same as in chapter II. We take velocity moments of the BBGKY hierarchy and retain terms only up to order ϵ^4. As a result of this we obtain a set of ordinary differential equations in the parameter λ. These equations have complicated spatial dependences. The spatial calculation is simplified by taking spatial derivatives (curl and divergence) and we then obtain a set of equations that can be easily solved simultaneously. The lowest order at which the functions f_1,f_2 and f_3 have non-zero values for Gaussian initial condition; is ϵ^4. Thus the lowest order at which we have non-linear corrections to the two point correlation function is ϵ^4. This result is presented in the fourth section of this chapter.

4.2 The triplet momentum.

The triplet momentum $< p_\mu^a >_3 (1,2,3,A)$ is defined as the first moment of the three-point distribution function $d(1,2,3,A)$.

$$\int p_\mu^a d(1,2,3,\lambda)\, d^3p^1 d^3p^2 d^3p^3 = n^3 < p_\mu^i >_3 \left(x^1, x^2, x^3, \lambda\right) . \tag{4.7}$$

It is a function of three positions *1,2* and *3,* and the index *'a'* in $< p_\mu^a >_3$, which indicates at which vertex of the triangle we are considering the momentum, can refer to any one of them.

We want to calculate-this quantity to order ϵ^4. This is the lowest order for which it has non-zero value for Gaussian initial conditions.

The evolution of the triplet momentum is governed by the first moment of the third equation of the BBGKY hierarchy *(2.64)*

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} < p_\mu^a >_3 + \frac{1}{m}\frac{\partial}{\partial x_\nu^b} < p_\mu^a p_\nu^b >_3 (1,2,3,\lambda) \\
- \quad & SmG\rho \int \zeta\left(a'', 4\right) X_\mu^{4a} d^3x^4 - SmG\rho \int \chi(1,2,3,4)\, X_\mu^{4a} d^3x^4 \\
- \quad & SmG\rho\xi\,(a,a;) \int \xi\,(a;,4)\, X_\mu^{4a} d^3x^4 = 0 .
\end{aligned} \tag{4.8}$$

To evaluate $< p_\mu^a >_3$ we separately consider both its curl and divergence with respect to x^a and use these to construct it. All the equations given below for the curl and divergence are valid only to order c^4 and in all of them the summation convention **A** does not hold. In all these equations the indices *a,* b and *c* refer to the three different corners of the triangle that we are considering **i.e.** $a \neq b \neq c$.

The curl of equation *(4.8)* is

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} F_\beta^a + \frac{1}{m}\sum_{b\neq a} G_\beta^{ab} \\
+ \quad & \frac{5m}{2}\left(\frac{\lambda_0}{\lambda}\right)^{10} \epsilon_{\beta\mu\nu}\left[\partial_\mu^a \nabla^4\phi\,(a,a_1')\, \partial_\nu^a \nabla^2\phi\,(a,a_2')\right] = 0
\end{aligned} \tag{4.9}$$

where,

$$F_\beta^a(1,2,3,\lambda) = \epsilon_{\beta\mu\nu}\partial_\mu^a < p_\nu^a >_3 \tag{4.10}$$

and

$$G_\beta^{ab}(1,2,3,A) = \epsilon_{\beta\mu\nu}\partial_\mu^a\partial_\sigma^b < p_\nu^a p_\sigma^b >_3 . \tag{4.11}$$

and we have used the fact that for Gaussian initial conditions to order ϵ^4

$$< p^a_\mu p^a_\nu >_3 = < p^a_\mu >_2 \left(a, a_1'\right) < p^a_\mu >_2 \left(a, a_2'\right) \qquad (4.12)$$

to evaluate G^{aa}_β.

The divergence of equation *(4.8)* is

$$\frac{\partial}{\partial\lambda} J^a + \frac{1}{m}\sum_{b\neq a} K^{ab} + m\left(\frac{\lambda_0}{\lambda}\right)^{10}\left[\sum_{q=1}^{3}\frac{3}{14}g(q) + \frac{1}{2}g(a)\right] = 0 \qquad (4.13)$$

where

$$J^a(1,2,3,\lambda) = \partial^a_\mu < p^a_\mu >_3 , \qquad (4.14)$$

$$K^{ab}(1,2,3,\lambda) = \partial^a_\mu \partial^b_\nu < p^a_\mu p^b_\nu >_3 \qquad (4.15)$$

and,

$$g(a) = 5\nabla^4\phi(a,a;)\nabla^4\phi\left(a,a_2'\right) + 7\partial^a_\mu\nabla^2\phi\left(a,a_1'\right)\partial^a_\mu\nabla^4\phi\left(a,a_2'\right)$$
$$+2\partial^a_\mu\partial^a_\nu\nabla^2\phi\left(a,a_1'\right)\partial^a_\mu\partial^a_\nu\nabla^2\phi\left(a,a_2'\right) . \qquad (4.16)$$

We have used equation *(4.13)* to evaluate K^{aa} and equation, *(4.6)* for the three point correlation function.

Next we consider the second moment of the three point distribution function

$$\frac{\partial}{\partial\lambda} < p^a_\mu p^b_\nu >_3 + \frac{1}{m}\frac{\partial}{\partial x^c_\sigma} < p^a_\mu p^b_\nu p^c_\sigma >_3 (1,2,3,\lambda)$$

$$- \frac{Sm^2G}{n^3}\int\left(\delta^{ca}_{\sigma\mu}p^b_\nu + \delta^{cb}_{\sigma\nu}p^a_\mu\right) f(c)\, d\left(c'',3\right) X x^{4c}_\sigma d^3x^4 d^{12}p$$

$$- \frac{Sm^2G}{n^3}\int\left(\delta^{ca}_{\sigma\mu}p^b_\nu + \delta^{cb}_{\sigma\nu}p^a_\mu\right) c\left(c,c_1'\right) c\left(c_2',4\right) X^{4c}_\sigma d^3x^4 d^{12}p$$

$$- \frac{Sm^2G}{n^3}\int\left(\delta^{ca}_{\sigma\mu}p^b_\nu + \delta^{cb}_{\sigma\nu}p^a_\mu\right) e(1,2,3,4)\, X^{4c}_\sigma d^3x^4 d^{12}p = 0 . \qquad (4.17)$$

We use this to get an equation for G^{ab}_β

$$\frac{\partial}{\partial\lambda}G^{ab}_\beta + \frac{1}{m}H^{abc}_\beta + \frac{6m}{\lambda^2}F^a_\beta$$
$$+ 5m^2\frac{\lambda_0^{10}}{\lambda^{11}}\epsilon_{\beta\mu\nu}\left[\partial^a\nu\nabla^2\phi(a,b)\partial^a_\mu\nabla^4\phi(a,c)\right.$$
$$\left. + \partial^a_\mu\nabla^4\phi(a,b)\,\partial^a_\nu\nabla^2\phi(a,c)\right] = 0 . \qquad (4.18)$$

where

$$H^{abc}_\beta = \epsilon_{\beta\mu\nu}\partial^a_\mu\partial^b_\alpha\partial^c_\sigma < p^a_\nu p^b_\alpha p^c_\sigma >_3 \qquad (4.19)$$

We have used the fact that for Gaussian. initial conditions to order ϵ^4

$$\begin{aligned} < p^a_\mu p^a_\nu p^b_\sigma >_3 &= < p^a_\mu p^b_\sigma >_2 (a,b) < p^a_\nu >_2 (a,c) \\ &+ < p^a_\nu p^b_\sigma >_2 (a,b) < p^a_\mu >_2 (a,c) \end{aligned} \tag{4.20}$$

to evaluate H^{abd}_β when $d = a$ or $d = b$.

By taking divergences of equation *(4.17)* we obtain

$$\frac{\partial}{\partial \lambda} K^{ab} + \frac{1}{m} L^{abc} + \frac{6\,m}{\lambda^2}\left(J^a + J^b\right) + m^2 \frac{\lambda_0^{n}}{\lambda^{11}} [g(a) + g(b)] = 0 \tag{4.21}$$

where

$$L^{abc} = \partial^a_\mu \partial^b_\nu \partial^c_\sigma < p^a_\mu p^b_\nu p^c_\sigma >_3 \,. \tag{4.22}$$

Finally we have the third moment of the equation for the three point distribution function

$$\begin{aligned} &\frac{\partial}{\partial \lambda} < p^a_\mu p^b_\nu p^c_\sigma >_3 + \frac{1}{m}\frac{\partial}{\partial x^d_\gamma} < p^a_\mu p^b_\nu p^c_\sigma p^d_\gamma >_3 (1,2,3,\lambda) \qquad (4.23) \\ &- \frac{Sm^2 G}{n^3} \int \left(\delta^{ea}_{\gamma\mu} p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu} p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma} p^a_\mu p^b_\nu\right) f(e)\, d\left(e'', 4\right) X^{4e}_\gamma d^3x^4 d^{12}p \\ &- \frac{Sm^a G}{n3} \int \left(\delta^{ea}_{\gamma\mu} p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu} p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma} p^a_\mu p^b_\nu\right) c\left(e, e'_1\right) c\left(e'_2, 4\right) X^{4e}_\gamma d^3x^4 d^{12}p \\ &- \frac{Sm^2 G}{n^3} \int \left(\delta^{ea}_{\gamma\mu} p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu} p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma} p^a_\mu p^b_\nu\right) e(1,2,3,4)\, X^{4e}_\gamma d^3x^4 d^{12}p = 0\,. \end{aligned}$$

This can be used to obtain the equation for H^{abc}_β

$$\begin{aligned} \frac{\partial}{\partial \lambda} H^{abc}_\beta &+ \frac{6m}{\lambda^2}\left(G^{ab}_\beta + G^{ac}_\beta\right) \\ &+ 10 m^3 \left(\frac{\lambda_0^{10}}{\lambda^{12}}\right) \epsilon_{\beta\mu\nu} \left[\partial^a_\nu \nabla^2 \phi(a,b)\, \partial^a_\mu \nabla^4 \phi(a,c)\right. \\ &+ \left.\partial^a_\mu \nabla^4 \phi(a,b)\, \partial^a_\nu \nabla^2 \phi(a,c)\right] = 0 \end{aligned} \tag{4.24}$$

To obtain this equation we have used the fact that for Gaussian initial conditions to order ϵ^4

$$< p^a_\mu p^a_\nu p^b_\sigma p^c_\gamma >_3 = < p^a_\mu p^b_\sigma >_2 < p^a_\nu p^c_\gamma >_3 + \text{permutations.} \tag{4.25}$$

Taking divergence of equation *(4.24)* we obtain

$$\frac{\partial}{\partial \lambda} L^{abc} + \frac{6m}{\lambda^2}\left(K^{ab} + K^{bc} + K^{ca}\right) + 2\sum_{p=1}^{3} m^3 \left(\frac{\lambda_0^{10}}{\lambda^{12}}\right) g(p) = 0\,. \tag{4.26}$$

The equations (4.9),(4.18) and (4.24) can be simultaneously solved to obtain

$$F_\beta^a = \frac{m}{2}\left(\frac{\lambda_0^{10}}{\lambda^9}\right)\epsilon_{\beta\mu\nu}\left[\partial_\mu^a\nabla^4\phi(a,a;)\,\partial_\nu^a\nabla^2\phi(a,a;)\right] \tag{4.27}$$

$$G_\beta^{ab} = m^2\left(\frac{\lambda_0^{10}}{\lambda^{10}}\right)\epsilon_{\beta\mu\nu}\left[\partial_\mu^a\nabla^4\phi(a,a_1')\,\partial_\nu^a\nabla^2\phi(a,a;)\right] \tag{4.28}$$

$$H_\beta^{abc} = 2m^3\left(\frac{\lambda_0^{10}}{\lambda^{11}}\right)\epsilon_{\beta\mu\nu}\left[\partial_\mu^a\nabla^4\phi(a,a_1')\,\partial_\nu^a\nabla^2\phi(a,a;)\right]. \tag{4.29}$$

Simultaneously solving equations (4.13),(4.21) and (4.26) we have

$$J^a = \frac{m}{14}\left(\frac{\lambda_0^{10}}{\lambda^9}\right)[2g(a)+g(b)+g(c)] \tag{4.30}$$

$$K^{ab} = \frac{m^2}{7}\left(\frac{\lambda_0^{10}}{\lambda^{10}}\right)[2g(a)+2g(b)+g(c)] \tag{4.31}$$

$$L^{abc} = \frac{4m^3}{7}\left(\frac{\lambda_0^{10}}{\lambda^{11}}\right)[g(a)+g(b)+g(c)]. \tag{4.32}$$

Using these we obtain the triplet momentum as

$$\begin{aligned}
<p_\mu^a>_3(1,2,3,A) = m\frac{\lambda_0^{10}}{\lambda^9}&\Big[\frac{1}{2}\partial_\mu^a\nabla^2\phi(a,a_1')\,\nabla^4\phi(a,a_2') \\
+&\ \frac{1}{7}\partial_\mu^a\left(\partial_\nu^a\nabla^2\phi(a,a_1')\,\partial_\nu^a\nabla^2\phi\ fa,a;)\right) \\
+&\ \frac{5}{7}\partial_\mu^a\nabla^2\phi(a,a;)\,\nabla^4\phi(a;,a_2') + \frac{1}{2}\partial_\mu^a\partial_\nu^{a_1'}\nabla^2\phi(a,a;)\,\partial_\nu^{a_1'}\nabla^2\phi(a;,a_2') \\
+&\ \frac{1}{2}\partial_\mu^a\partial_\nu^{a_1'}\phi(a,a_1')\,\partial_\nu^{a_1'}\nabla^4\phi(a_1',a_2') + \frac{2}{7}\partial_\mu^a\partial_\nu^{a_1'}\partial_\sigma^{a_1'}\phi(a,a_1')\,\partial_\nu^{a_1'}\partial_\sigma^{a_1'}\nabla^2\phi(a_1',a_2') \\
-&\ \frac{3}{56\pi}\int X_\mu^{4a}\partial_\nu^4\left(\partial_\nu^4\nabla^2\phi(4,a;)\,\nabla^4\phi(4,a;)\right)d^3x^4\Big]
\end{aligned} \tag{4.33}$$

and we also obtain for $a \# b$

$$\begin{aligned}
\partial_\nu^b<p_\mu^a p_\nu^b>_3(1,2,3,\lambda) = \frac{m^2}{7}\frac{\lambda_n^{90}}{\lambda^{10}}&\Big[7\partial_\mu^a\nabla^2\phi(a,a;)\,\nabla^4\phi(a,a;) \\
+&\ 2\partial_\mu^a\left(\partial_\nu^a\nabla^2\phi(a,a;)\,\partial_\nu^a\nabla^2\phi(a,a;)\right) + 20\partial_\mu^a\nabla^2\phi(a,b)\,\nabla^4\phi(b,c) \\
+&\ 10\partial_\mu^a\nabla^2\phi(a,c)\,\nabla^4\phi(b,c) + 14\partial_\mu^a\partial_\nu^b\nabla^2\phi(a,b)\partial_\nu^b\nabla^2\phi(b,c) \\
+&\ 14\partial_\mu^a\partial_\nu^b\phi(a,b)\partial_\nu^b\nabla^4\phi(b,c) + 7\partial_\mu^a\partial_\nu^c\nabla^2\phi(a,c)\,\partial_\nu^c\nabla^2\phi(b,c) \\
+&\ 7\partial_\mu^a\partial_\nu^c\phi(a,c)\partial_\nu^c\nabla^4\phi(b,c) + 8\partial_\mu^a\partial_\nu^b\partial_\sigma^b\phi(a,b)\,\partial_\nu^b\partial_\sigma^b\phi(b,c) \\
+&\ 4\partial_\mu^a\partial_\nu^c\partial_\sigma^c\phi(a,c)\,\partial_\nu^c\partial_\sigma^c\phi(b,c) \\
-&\ \frac{3}{4\pi}J\,X_\mu^{4a}\partial_\nu^4\left(\partial_\nu^4\nabla^2\phi(4,a;)\,\nabla^4\phi(4,a;)\right)d^3x^4\Big]
\end{aligned} \tag{4.34}$$

4.3 The third moment of the two point distribution function.

In this section the position indices takes' the values *1* and *2,* and they are to be summed whenever they appear twice or more.

The third moment of the two point distribution function is governed by the equation

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} < p^a_\mu p^b_\nu p^c_\sigma >_2 + \frac{1}{m}\frac{\partial}{\partial x^d_\gamma} < p^a_\mu p^b_\nu p^c_\sigma p^d_\gamma >_2 (1,2,\lambda) \\
&- \frac{Sm^2G}{n^2}\int \left(\delta^{ea}_{\gamma\mu} p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu} p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma} p^a_\mu p^b_\nu\right) f(e)\, c\left(e',3\right) X^{3e}_\gamma d^3x^3 d^9p \\
&- \frac{Sm^2G}{n^2}\int \left(\delta^{ea}_{\gamma\mu} p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu} p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma} p^a_\mu p^b_\nu\right) d(1,2,3)\, X^{3e}_\gamma d^3x^3 d^9p = 0\,.
\end{aligned}$$

If we take divergence with respect to all the three sets of free indices we have

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\partial^a_\mu\partial^b_\nu\partial^c_\sigma < p^a_\mu p^b_\nu p^c_\sigma >_2 + \frac{1}{m}\partial^a_\mu\partial^b_\nu\partial^c_\sigma\partial^d_\gamma < p^a_\mu p^b_\nu p^c_\sigma p^d_\gamma >_2 \\
&+ 12\pi SmG\rho\left[< p^a_\mu p^a_\nu >_1 \partial^a_\mu\partial^a_\nu\xi(1,2,A) + \partial^a_\mu\partial^a_\nu < p^a_\mu p^a_\nu >_2 (1,2,\lambda)\right] \\
&- 3SmG\rho\partial^a_\mu\partial^b_\nu\partial^c_\sigma \int X^{3c}_\sigma < p^a_\mu p^b_\nu >_3 (1,2,3,A)\, d^3x^3 = 0\,. \qquad (4.35)
\end{aligned}$$

For Gaussian initial conditions the terms in this equation have non-zero values only at order ϵ^4 and higher. To order ϵ^4 there are two unknown functions in the equation **i.e.** $\partial^a_\mu\partial^b_\nu\partial^c_\sigma < p^a_\mu p^b_\nu p^c_\sigma >_2$ and $< p^a_\mu p^a_\nu >_2$. We need one more equation to self-consistently solve this equations. This equation is obtained from the second moment of the two point distribution function

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} < p^a_\mu p^b_\nu >_2 (1,2,\lambda) + \frac{}{m}\frac{\partial}{\partial x^c_\sigma} < p^a_\mu p^b_\nu p^c_\sigma >_2 (1,2,\lambda) \\
&- \frac{m^2GS}{n^2}\int \left(\delta^{ca}_{\sigma\mu} p^b_\nu + \delta^{cb}_{\sigma\nu} p^a_\mu\right) f(c)\, c\left(c',3\right) X^{3c}_\sigma d^3x^3 d^9p \\
&- \frac{m^2GS}{n^2}\int \left(\delta^{ca}_{\sigma\mu} p^b_\nu + \delta^{cb}_{\sigma\nu} p^a_\mu\right) d(1,2,3)\, X^{3c}_\sigma d^3x^3 d^9p = 0\,. \qquad (4.36)
\end{aligned}$$

From equation *(4.36)* we get

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\partial^a_\mu\partial^a_\nu < p^a_\mu p^a_\nu >_2 + \frac{1}{3m}\partial^a_\mu\partial^b_\nu\partial^c_\sigma < p^a_\mu p^b_\nu p^c_\sigma >_2 \\
&+ \frac{2}{3m}\partial^a_\mu\partial^a_\nu\partial^a_\sigma < p^a_\mu p^a_\nu p^a_\sigma >_2 - 2SmG\rho\partial^a_\mu\partial^a_\nu \int < p^a_\mu >_3 X^{a3}_\nu d^3x^3 = 0\,. \qquad (4.37)
\end{aligned}$$

Simultaneously solving equations *(4.35)* and *(4.37)* we obtain

$$\begin{aligned}
\partial^a_\mu \partial^b_\nu \partial^c_\sigma < p^a_\mu p^b_\nu p^c_\sigma >_2 = & \frac{3}{26} \partial^a_\mu \partial^a_\nu \partial^a_\sigma < p^a_\mu p^a_\nu p^a_\sigma >_2 + \frac{5\lambda}{52m} \partial^a_\mu \partial^b_\nu \partial^c_\sigma \partial^d_\gamma < p^a_\mu p^b_\nu p^c_\sigma p^d_\gamma >_2 \\
+ & \frac{45m}{26\lambda} < p^a_\mu p^a_\nu >_1 \partial^a_\mu \partial^a_\nu \xi(1,2,\lambda) - \frac{27m^2}{52\lambda^2\pi} \partial^a_\mu \partial^a_\nu \int < p^a_\mu >_3 X^{3a}_\nu d^3x^3 \\
- & \frac{45m}{104\lambda\pi} \partial^a_\mu \partial^b_\nu \partial^c_\sigma \int < p^a_\mu p^b_\nu >_3 X^{3c}_\sigma d^3x^3
\end{aligned} \tag{4.38}$$

All the terms on the left hand side of equation *(4.38)* are known to order ϵ^4. Writing it in terms of ϕ we have

$$\begin{aligned}
\partial^a_\mu \partial^b_\nu \partial^c_\sigma < p^a_\mu p^b_\nu p^c_\sigma >_2 = & \left(\frac{\lambda_0^{10}}{\lambda^{11}}\right) \Big[-\frac{82}{7} \nabla^6 \phi(1,2) \nabla^2 \phi(0) \\
+ & \frac{240}{7} \nabla^4 \phi(1,2) \nabla^4 \phi(1,2) + \frac{816}{7} \partial^1_\mu \nabla^4 \phi(1,2) \partial^1_\mu \nabla^2 \phi(1,2) \\
+ & 96 \partial^1_\mu \partial^1_\nu \nabla^2 \phi(1,2) \partial^1_\mu \partial^1_\nu \nabla^2 \phi(1,2) + \frac{528}{7} \partial^1_\mu \partial^1_\nu \partial^1_\alpha \nabla^2 \phi(1,2) \partial^1_\mu \partial^1_\nu \partial^1_\alpha \phi(1,2) \\
+ & 48 \partial^1_\mu \partial^1_\nu \nabla^4 \phi(1,2) \partial^1_\mu \partial^1_\nu \phi(1,2) + \frac{96}{7} \partial^1_\mu \partial^1_\nu \partial^1_\alpha \partial^1_\beta \phi(1,2) \partial^1_\mu \partial^1_\nu \partial^1_\alpha \partial^1_\beta \phi(1,2) \\
- & \frac{1}{7\pi} \partial^1_\mu \partial^1_\nu \int X^{31}_\mu \Big(36 \partial^1_\nu \nabla^2 \phi(1,3) \nabla^4 \phi(2,3) + \frac{45}{2} \partial^1_\nu \partial^3_\alpha \nabla^2 \phi(1,3) \partial^3_\alpha \nabla^2 \phi(2,3) \\
+ & \frac{45}{2} \partial^1_\nu \partial^3_\alpha \phi(1,3) \partial^3_\alpha \nabla^4 \phi(2,3) + 9 \partial^1_\nu \partial^3_\alpha \partial^3_\beta \phi(1,3) \; \partial^3_\alpha \partial^3_\beta \nabla^2 \phi(2,3) \Big) d^3x^3 \Big]
\end{aligned} \tag{4.39}$$

4.4 The two point correlation.

Using the results derived in the two previous sections we can calculate f_2 and f_3. We first consider only the λ dependence of ξ. Equation *(4.5)* may be written as

$$\partial^3 \quad - \frac{24}{\lambda^2} \frac{\partial}{\partial \lambda} \xi + \frac{24}{\lambda^3} \xi = \left(\frac{\lambda_0}{\lambda}\right)^{11} \left[f_2(\lambda_0) - f_3(\lambda_0) + \frac{10}{\lambda_0} f_1(\lambda_0) \right], \tag{4.40}$$

This has a solution

$$\xi(1,2,\lambda) = -\frac{1}{504} \frac{\lambda_0^{11}}{\lambda^8} \left[f_2(\lambda_0) - f_3(\lambda_0) + \frac{10}{\lambda_0} f_1(\lambda_0) \right]. \tag{4.41}$$

Using equations (4.2),(4.3) and *(4.4)* we calculate the spatial dependence of the right hand side. This gives us

$$
\begin{aligned}
\xi^{(2)}(1,2,\lambda) &= \frac{1}{504}\left(\frac{\lambda_0^{10}}{\lambda^8}\right)\Big[-32\nabla^6\phi(1,2)\,\nabla^2\phi(0) - 14\nabla^4\phi(1,2)\,\nabla^4\phi(0) \\
&+ \frac{900}{7}\nabla^4\phi(1,2)\,\nabla^4\phi(1,2) + 360\partial^1_\mu\nabla^4\phi(1,2)\,\partial^1_\mu\nabla^2\phi(1,2) \\
&+ \frac{1602}{7}\partial^1_\mu\partial^1_\nu\nabla^2\phi(1,2)\,\partial^1_\mu\partial^1_\nu\nabla^2\phi(1,2) + 144\partial^1_\mu\partial^1_\nu\partial^1_\alpha\nabla^2\phi(1,2)\,\partial^1_\mu\partial^1_\nu\partial^1_\alpha\phi(1,2) \\
&+ 126\partial^1_\mu\partial^1_\nu\nabla^4\phi(1,2)\,\partial^1_\mu\partial^1_\nu\phi(1,2) + \frac{144}{7}\partial^1_\mu\partial^1_\nu\partial^1_\alpha\partial^1_\beta\phi(1,2)\,\partial^1_\mu\partial^1_\nu\partial^1_\alpha\partial^1_\beta\phi(1,2) \\
&- \frac{1}{\pi}\,\mathrm{J}\, X^{31}_\mu\Big(\frac{3}{2}\partial^1_\nu\nabla^2\phi(1,3)\,\partial^2_\mu\partial^2_\nu\nabla^4\phi(2,3) + \frac{\supset}{2}\partial^1_\alpha\partial^1_\nu\nabla^2\phi(1,3)\,\partial^2_\mu\partial^2_\alpha\partial^2_\nu\nabla^{\mathrm{a}}\phi(2,3) \\
&+ \frac{15}{2}\partial^1_\alpha\partial^1_\nu\phi(1,3)\,\partial^2_\mu\partial^2_\alpha\partial^2_\nu\nabla^4\phi(2,3) + 3\partial^1_\alpha\partial^1_\beta\partial^1_\nu\phi(1,3)\,\partial^2_\mu\partial^2_\alpha\partial^2_\beta\partial^2_\nu\nabla^2\phi(2,3) \\
&- 15\nabla^4\phi(1,3)\,\partial^2_\mu\nabla^4\phi(2,3) - \frac{21}{2}\partial^1_\nu\nabla^4\phi(1,3)\,\partial^2_\nu\partial^2_\mu\nabla^2\phi(2,3)\Big)\, d^3x^3\Big]
\end{aligned}
\qquad (4.42)
$$

This is the two point correlation function at order ϵ^4.

We do a Fourier transform of (4.42) and compare it with the result obtained in the single stream approximation (Makino et. al 1992) and find that the two match.

The algebra involved in deriving equation (4.42) was checked using the mathematical package MATHEMATICA and the Fourier transform was done using this package.

4.5 Discussion

The calculation presented here, which is based on the equations of the BBGKY hierarchy, has the effects of multistreaming, if any, at the lowest order of non-linearity i.e. ϵ^4. This matches with the results obtained in the single stream approximation which does not take into account any effect of multistrearning. Hence we conclude that there are no effects of multistreaming at this order of non-linearity. The equivalence between the two calculations at this order become clear only at the final stage i.e. after we have done the calculation. and one does not obtain the HD equations as an intermediate step.

In chapter VI we discuss if we can study the effects of multistreaming by going to higher orders of the perturbative expansion or whether it is a limitation of the perturbative treatment that it does not allow us to study the transition from a single streamed to a multi-streamed flow.

Given the initial two point correlation function $\xi^{(1)}(x, \text{A})$, to evaluate the non-linear correction at order ϵ^4 we solve the equation

$$
\nabla^2\left(\nabla^2\phi(x)\right) = \frac{2}{\lambda_0}\xi^1(x,\lambda_0) \qquad (4.43)
$$

to obtain $\nabla^2\phi(x)$ and then solve for $\phi(x)$ and use these in equation (4.42). These calculations are simplified a lot if we use the fact that $\xi(x, \lambda_0)$ is spherically symmetric. Equation (4.43) does not uniquely determine the functions $\nabla^2\phi(x)$ and $\phi(x)$. If $\nabla^2\phi(x)$ is a solution of equation (4.43) then so is $\nabla^2\phi(x) + C$ where C is some constant. Under this transformation we also have $\phi(x) \rightarrow \phi(x) + \frac{Cx^2}{6}$. If we consider all the terms in equation (4.42) that are affected by this

$$
\begin{aligned}
&- \quad 32\nabla^6\phi(1,2)\,\nabla^2\phi(0) + 126\partial^1_\mu\partial^1_\nu\nabla^4\phi(1,2)\,\partial^1_\mu\partial^1_\nu\phi(1,2) \\
&- \quad \frac{15}{2\pi}\int X^{31}_\mu\,\partial^1_\nu\partial^1_\alpha\phi(1,3)\,\partial^2_\mu\partial^2_\nu\partial^2_\alpha\nabla^4\phi(2,3)\,d^3x^3
\end{aligned}
\tag{4.44}
$$

we find that the correction to the two point correlation function is unchanged by such transformations and independent of C. Similarly, we can add a constant to $\phi(x)$ but this obviously does not have any affect as only derivatives of $\phi(x)$ appear in equation (4.42). Hereafter we shall use the boundary condition that $\nabla^2\phi(x)$ should vanish as x goes to infinity to fix the constant C.

At this stage we should point out that it such a choice of C is not always convenient. For example if the initial two point correlation is such that the power spectrum has the form $P(k) \propto k^n$ with $n < -1$ at small k, then the boundary condition stated above implies that

$$
\nabla^2\phi(0) = -4\pi\int_0^\infty P(k)dk \tag{4.45}
$$

is infinite. Although $\nabla^2\phi(0)$ is infinite and this quantity appears in the non-linear correctian to the two point correlation function $\xi^{(2)}(x, t)$, we may still get a finite $\xi^{(2)}(x, t)$ for certain initial conditions. This is because now $\xi^{(2)}$ is the difference of two infinite large quantities which cancel out to give a finite result. The same problem is encountered if one does the analysis in Fourier space where for $-1 > n > -3$ the correction to the power spectrum is a finite quantity which is the difference of two divergent integrals (Vishniac 1983). In real space this situation is easily handled by changing the boundary condition used to calculate $\nabla^2\phi(x)$. If we use the boundary condition $\nabla^2\phi(0) = 0$ to fix the constant C than the situation discussed above does not occur and it is possible to calculate $\xi^{(2)}(x, t)$ solely in terms of finite quantities. Here we shall only deal with situations where the former boundary condition ($\lim_{x\rightarrow\infty} \nabla^2\phi(x) = 0$) can be applied.

4.6 The pair velocity

We next calculate the first moment of the two point distribution function, $< p^a >_2 (1, 2)$, to order ϵ^4 . This is a function of two positions and the index a indicates at which of the two positions the momentum is being considered. We use the pair continuity equation

$$\frac{\partial}{\partial\lambda}\xi(1,2,\lambda)+\frac{1}{m}\frac{\partial}{\partial x_\mu^a}<p_\mu^a>_2(1,2,\lambda)=0. \tag{4.46}$$

to obtain

$$<p_\mu^a>_2(1,2,\lambda)=\frac{m}{8\pi}\int\partial_\mu^a\left(\frac{1}{|y-y'|}\right)\frac{\partial}{\partial\lambda}\xi(y')d^3y', \tag{4.47}$$

where

$$y_\mu^a=x_\mu^a-x_\mu^{a'}$$

end a' refers to the complement of a (e.g. if a=1, a' =2). Using the two point correlation at order ϵ^4 calculated in the previous section we get at order e^4

$$\begin{aligned}
<p_\mu^a>_2 &= \frac{m}{126}\frac{\lambda_0^{10}}{\lambda^9}\left[\frac{y_\mu^a}{y^3}\int_0^y\left[-32\nabla^6\phi(x)\nabla^2\phi(0)-14\nabla^4\phi(x)\nabla^4\phi(0)\right.\right.\\
&+ \frac{900}{7}\nabla^4\phi(x)\nabla^4\phi(x)+360\partial_\mu\nabla^4\phi(x)\partial_\mu\nabla^2\phi(x)\\
&+ \frac{1602}{7}\partial_\mu\partial_\nu\nabla^2\phi(x)\partial_\mu\partial_\nu\nabla^2\phi(x)+144\partial_\mu\partial_\nu\partial_\alpha\nabla^2\phi(x)\partial_\mu\partial_\nu\partial_\alpha\phi(x)\\
&+ \left.126\partial_\mu\partial_\nu\nabla^4\phi(x)\partial_\mu\partial_\nu\phi(x)+\frac{144}{7}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\phi(x)\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\phi(x)\right]x^2dx\\
&- \frac{3}{\pi}\int X_\alpha^{3a'}\left[\frac{1}{2}\partial_\beta^{a'}\nabla^2\phi\left(a',3\right)\partial_\alpha^a\partial_\beta^a\partial_\mu^a\nabla^2\phi(a,3)\right.\\
&+ \frac{1}{2}\partial_\beta^{a'}\partial_\gamma^{a'}\nabla^2\phi(a',3)\,\partial_\alpha^a\partial_\beta^a\partial_\gamma^a\partial_\mu^a\phi(a,3)\\
&+ \frac{5}{2}\partial_\beta^{a'}\partial_\gamma^{a'}\phi\left(a',3\right)\partial_\alpha^a\partial_\beta^a\partial_\gamma^a\partial_\mu^a\nabla^2\phi(a,3)\\
&- \partial_\alpha^{a'}\partial_\beta^{a'}\partial_\gamma^{a'}\partial_\delta^{a'}\phi(a',3)\,\partial_\beta^a\partial_\gamma^a\partial_\delta^a\partial_\mu^a\phi(a,3)\\
&- 5\nabla^4\phi(a',3)\,\partial_\alpha^a\partial_\mu^a\nabla^2\phi(a,3)\\
&- \left.\left.\frac{7}{2}\partial_\beta^{a'}\nabla^4\phi(a',3)\,\partial_\alpha^a\partial_\beta^a\partial_\mu^a\phi(a,3)\right]d^3x^3\right].
\end{aligned} \tag{4.48}$$

A quantity related to the first moment of the two point distribution function is the pair current

$$j_\mu(x,\lambda)=\frac{<p_\mu^2>_2-<p_\mu^1>_2}{m} \tag{4.49}$$

whose divergence gives the rate at which the correlation at any separation is growing. We use this to calculate the pair velocity which is the ensemble average of the relative peculiar velocity between any two particles at a comoving separation x at time t (or A).

In terms of the pair current this is

$$v_\mu(x,t) = S\frac{d\lambda}{dt}\left[\frac{j_\mu(x,\lambda)}{1+\xi(x,\lambda)}\right] \tag{4.50}$$

Some consequences of the expressions for the lowest order non-linear corrections to the two point correlation function and the pair velocity are investigated in the next chapter.

Bibliography

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