Chapter 5

Investigating the two point correlation function.

In this chapter we investigate the lowest order non-linear correction to the two point correlation function for various initial conditions. We also investigate the non-linear correction to the pair velocity.

5.1 Effect of small scales on large scales.

In this section we study how the small scales affect the large scales. We consider a situation where initially $\xi^{(1)}(x) = 0$ for $x > x_0$. In the linear regime $\xi(x)$ will continue to have the value zero for $x > x_0$. This will not be true in the non-linear regime and there will be non-zero correlation for $x > x_0$ due to the streaming of particles and the non-local nature of gravity. In this section we want to find out, to the lowest order of non-linearity, the nature of the induced correlations.

We first consider an extreme case of the above situation where the initial two point correlation is a Dirac delta function and $\xi^{(1)}(x) = 0$ for $x > 0$. We initially have

$$\xi^{(1)}(x, \lambda_0) = \frac{1}{2}\lambda_0 \nabla^4 \phi(x) = \frac{1}{2}\lambda_0 A \delta^3(x). \tag{5.1}$$

Solving for the potential we get

$$\phi(x) = -\frac{A}{8\pi} x \tag{5.2}$$

and using this in equation *(4.42)* we obtain for $x > 0$.

$$\xi^{(2)}(x, \lambda) = \frac{5A^a}{98\pi^2} \frac{\lambda_0^{10}}{\lambda^8} \frac{1}{x^6}. \tag{5.3}$$

This shows the influence of the small scales on the large scales.

We next write equation *(5.3)* in a form that can be compared with a general case where the initial two point correlation function has compact support. This is done by introducing a quantity $M(\lambda)$ which is related to the integral of $\xi^{(1)}(x,\lambda)$ over the whole region $x \leq x_0$ where it is non-zero i.e.

$$M(\lambda) = \int \xi^{(1)}(x,\lambda) x^2 dx \,. \tag{5.4}$$

Expressing A in terms of M we can then write equation *(5.3)* for $x \gg x_0$ as

$$\xi^{(2)}(x,\lambda) = 3.264 \frac{M^2(\lambda)}{x^6} \,. \tag{5.5}$$

The general case where the initial two point correlation has compact support cannot be treated analytically because the integrals in equation *(4.42)* cannot be done analytically.

To check whether the conclusions drawn from the case where the initial condition is a delta function can be generalised we have considered a situation where

$$\nabla^4 \phi(x) = 16 - 1 \quad + x^3 \quad \text{for} \quad x \leq 2 \tag{5.6}$$

and

$$\nabla^4 \phi(x) = 0 \quad \text{for} \quad x > 2 \,. \tag{5.7}$$

This corresponds to the self-convolutionof a spheres of unit radius. The corresponding power spectrum has the form

$$P(k) \propto \left[\frac{\sin(k) - k \cos(k)}{k^3} \right]^2 \tag{5.8}$$

Doing the integrals in equation (*4.42)* numerically we find that for large x *i.e.*, in the interval $30 \leq x \leq 40$, the induced two point correlation can be fitted by the form

$$\xi^{(2)}(x,\lambda) = 3.277 \frac{M^2(\lambda)}{x^6} \,. \tag{5.9}$$

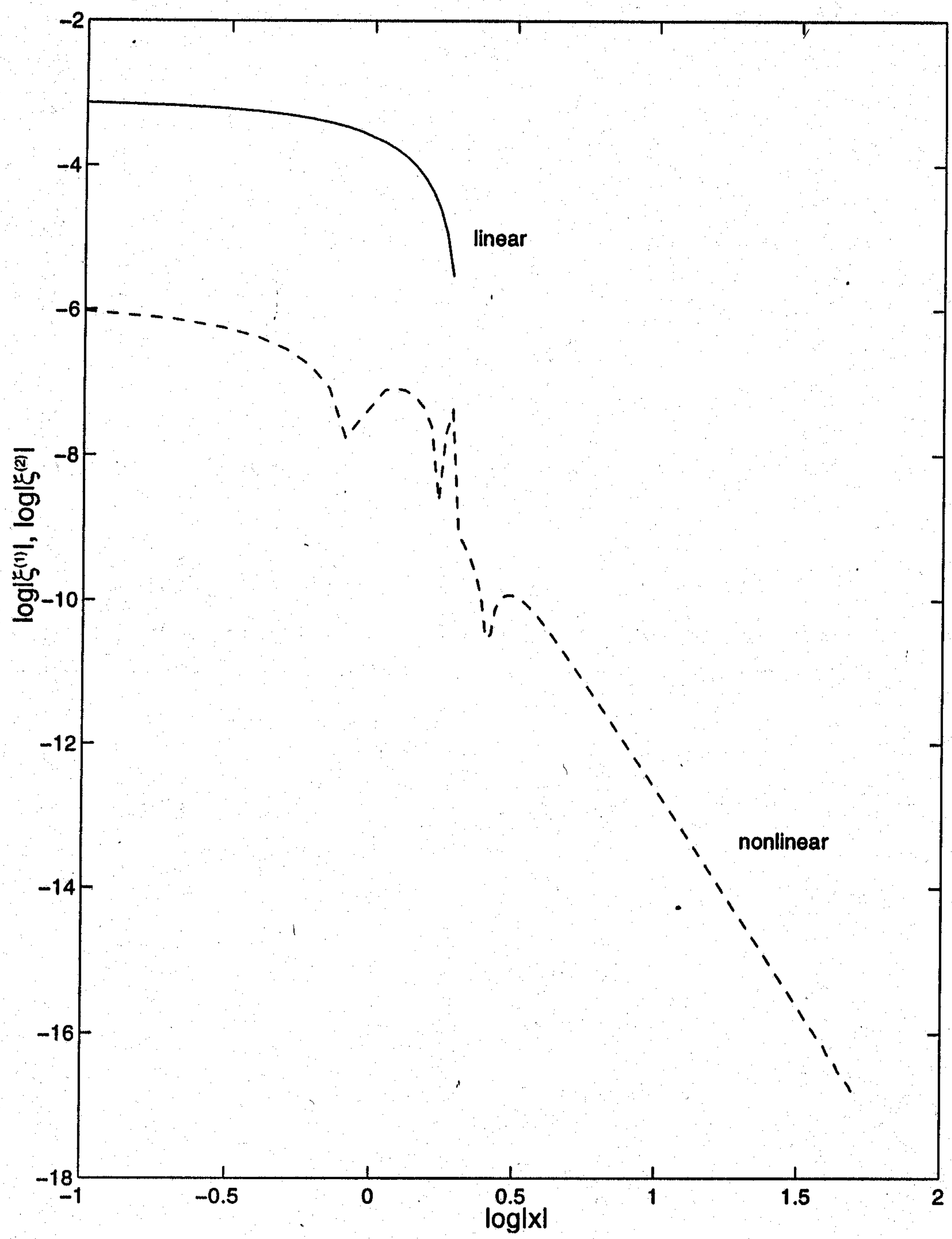
which is in satisfactory agreement with equation *(5.5)*. For larger values of x the integrals could not be reliably evaluated numerically as the numbers involved are extremely small. The initial and induced two point correlation are shown in figure *5.1*. We propose that the form *(5.5)* holds for the two point correlation induced are large distances in all cases where the initial two point correlation has compact support and a non-zero total integral.

We next consider the pair current. The initial pair current for the delta function initial condition is

$$j_\mu(x,\lambda) = \frac{4}{\lambda} \frac{x_\mu}{x^3} M(\lambda) \tag{5.10}$$

This current is divergence free everywhere except at the origin. Since it does not change the correlation at any non-zero separation it may be thought of as matter flowing into the overdense regions from infinity and flowing out of the underdense regions to infinity (Zel'dovich and Novikov *1983*, section *10.4)*.

Figure 5.1: The linear two point correlation and its lowest order non-linear correction for the case where the initial two point correlation corresponds to the self convolution of a sphere. The two curves have been given arbitrary displacements along the y axis for convenience of displaying.



If we try to calculate the non-linear correction to the pair current for the delta function initial condition we get a divergent answer. This is because the integral $\int_0^\infty \xi^{(2)}(y) y^2 dy$ diverges.

For the other initial condition considered above, i.e. equation (5.6) we find numerically that the induced pair current (figure 5.2) has a x^{-2} behaviour at large x. This is the leading part of the induced pair current and it too represents the flow of matter from infinity. In addition to this divergence-freepart, the pair current also has a part that has a x^{-5} behaviour at large x and it corresponds to a local redistribution of matter. It is the latter that gives rise to the x^{-6} correlation at large x but is swamped by the x^{-2} part in figure 5.2.

For large separations (x $>$ 2) the pair velocity has the same spatial dependence as the pair current.

5.2 The spatial dependence of the non-linear correction.

The expressions for the contribution to the two point correlation and the pair velocity at order ϵ^4 are rather complicated. To understand them better we proceed to evaluate them numerically for different initial conditions.

First we consider initial two point correlations such that the corresponding power spectrum has the form $P(k) = k^n e^{-k}$. We consider cases where n takes the values n $= .5, 1, 1.5, 2$ and 3. These initial conditions have just one length scale which is introduced by the exponential cut-off which becomes effective for k $>$ 1. In all these cases the correlation function has a power law behaviour $x^{-\gamma}$ for large separations and the corresponding values of the index γ are $\gamma = 3.5, 4, 4.5, 6$ and 6. We also consider the case $P(k) = ke^{-k^2}$ which has a Gaussian cut-off for the power spectrum at large k instead of the exponential cut-off used in all the other cases.

Figures 5.3 and 5.4 show the function $\xi^{(1)}(x)$ for the different initial conditions considered. This function is defined so that at any instant the two point correlation at order ϵ^2 is

$$\xi^{(1)}(x,t) = \left(\frac{S(t)}{S(t_0)}\right)^2 \xi^{(1)}(x). \tag{5.11}$$

Figures 5.3 and 5.4 also show the quantity $\overline{\xi^{(1)}}(x)$. This when multiplied by $\left(\frac{S(t)}{S(t_0)}\right)^2$ gives the $\sim \epsilon^2$ contribution to $\bar{\xi}(x,t)$ which is defined as

$$\bar{\xi}(x,t) = \frac{3}{x^3}\int_0^x \xi(y,t) y^2 dy. \tag{5.12}$$

This is the average of the two point correlation function over a sphere of radius x . All of the cases considered here satisfy the relation $\xi^{(1)}(x) \propto \overline{\xi^{(1)}}(x)$ for large x. This is cru-

Figure 5.2: The linear pair velocity and its lowest order non-linear correction for the case where the initial two point correlation corresponds to the self convolution of a sphere. The two curves have been given arbitrary displacements along the y axis for convenience of displaying.

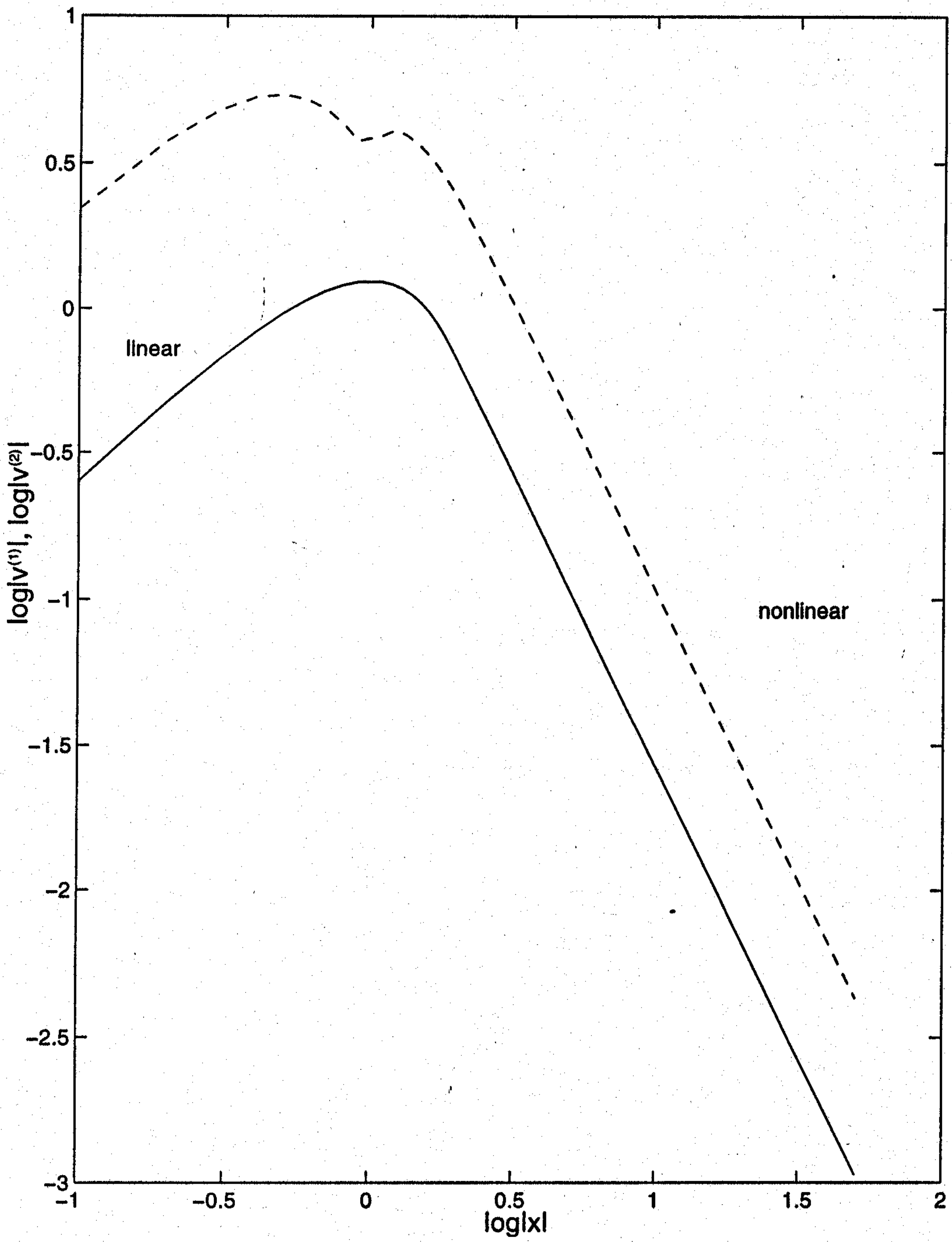


Figure 5.3: The initial two point correlation $\xi^{(1)}(x)$ shown by the solid line and it average $\overline{\xi^{(1)}}(x)$ shown by the dotted line for the different initial power spectra considered. This figure shows the small x behaviour.

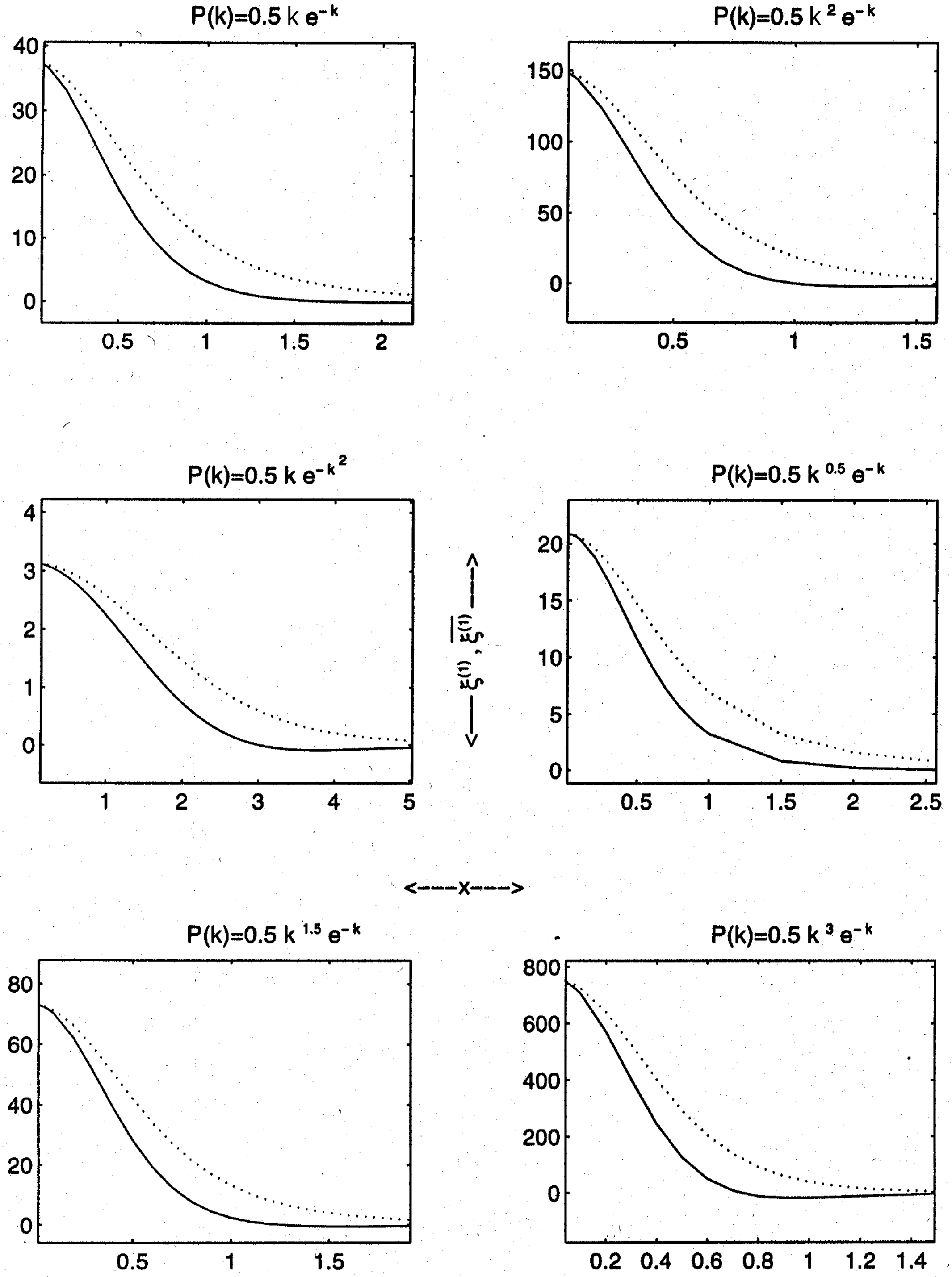
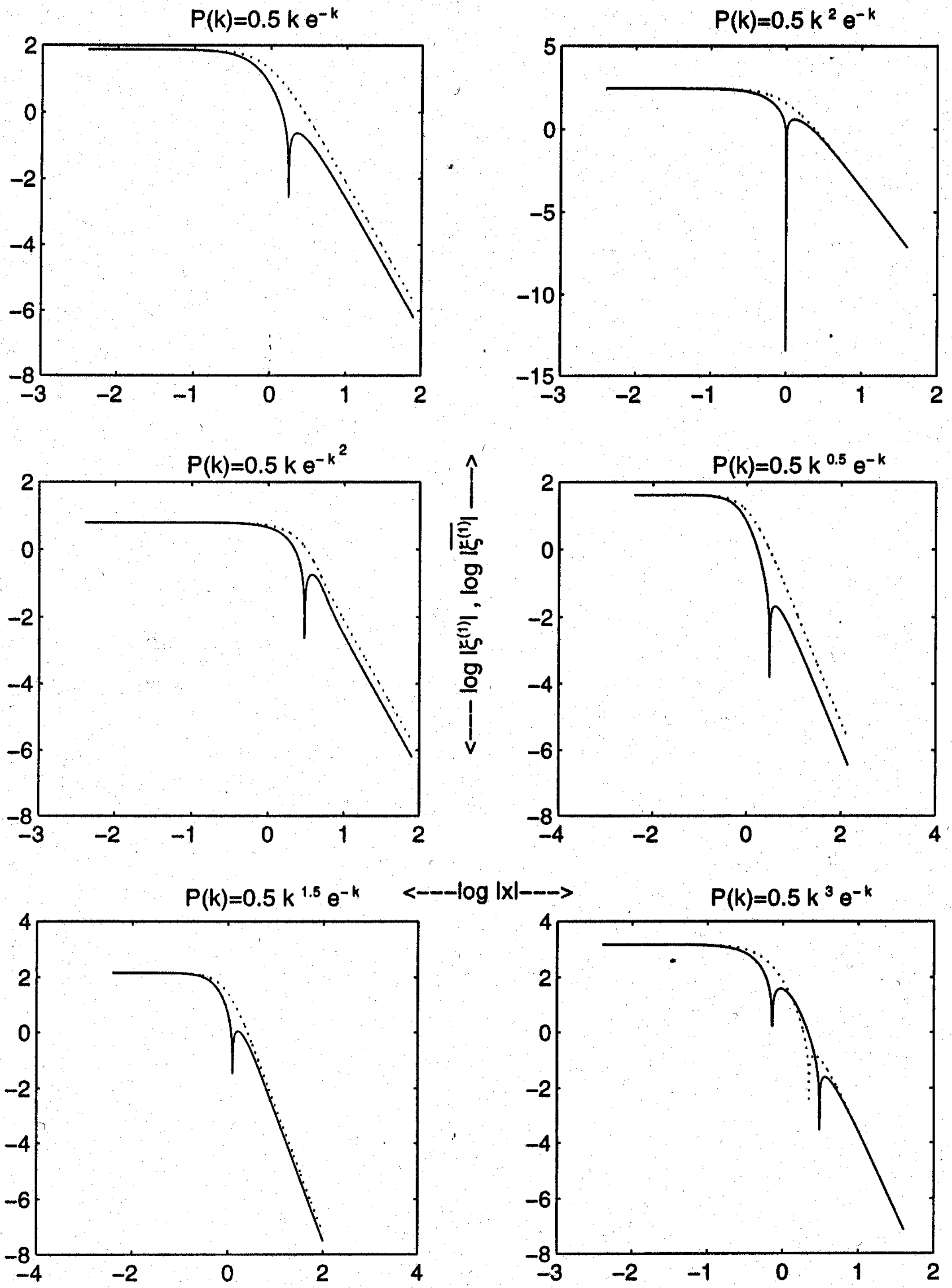


Figure 5.4: The initial two point correlation $\xi^{(1)}(x)$ shown by the solid line and it average $\overline{\xi^{(1)}}(x)$ shown by the dotted line for the different initial power spectra considered.



cial in deciding the behaviour of the induced three point correlation function $\zeta(1,2,3)$ at large separations and for this class of initial conditions we have $\zeta(1,2,3) \sim \xi(1,2)\xi(2,3)$ + permutations (chapter III).

Figures 5.5 and 5.6 show the function $v^{(1)}(x)$ which is related to the radial component (the only non-zero component) of the pair velocity at order ϵ^2 at any instant by the relation

$$v^{(1)}(x,t) = S(t)\frac{d}{dt}\left(\frac{S(t)}{S(t_0)}\right)^2 v^{(1)}(x) \,. \tag{5.13}$$

For all these cases the initial pair velocity too has a power law behaviour $x^{-\beta}$ at large separations and we have $\beta = \gamma - 1$ as expected.

It should be pointed out that the initial conditions when $n = 3$ differs from the other cases. For all the other cases $\xi^{(1)}(x)$ crosses zero only once, it is positive for small x and goes over to a negative value at large x where it has the power law form. The quantities $\overline{\xi^{(1)}}(x)$ and $v^{(1)}(x)$ do not change sign , the former is positive and the latter is negative. In the case when n = 3, $\xi^{(1)}(x)$ crosses zero twice, and $\overline{\xi^{(1)}}(x)$ and $v^{(1)}(x)$ cross zero once. Thus in this case (n = 3) at large x the signs of all the quantities are opposite to the signs in the other cases.

We have calculated the ϵ^4 contribution to all of the above mentioned quantities in the range $0 \leq x \leq 40$.

We first discuss the large separation behaviour of $\xi^{(2)}(x)$ which is defined such that the ϵ^4 contribution to the two point correlation is $\left(\frac{S(t)}{S(t_0)}\right)^4 \xi^{(2)}(x)$. This function is shown in figures 5.7 and 5.8. For all the cases we find that at large separations $\xi^{(2)}(x)$ has a power law form $\xi^{(2)}(x) \sim x^{-\eta}$ with $\eta = 7 - 2$ i.e. $\xi^{(2)}(x) \quad \nabla^2\xi^{(1)}(x)$. Motivated by this we investigated whether there is any simple relation between $\xi^{(2)}(x)$ and $\xi^{(1)}(x)$ which holds for all the cases. We looked at the ratio

$$Z = \frac{\xi^{(2)}(x)}{\nabla^2\phi(0)\nabla^6\phi(x)} \tag{5.14}$$

at large x for the different cases and we find that the value of Z is nearly the same $(-.048 < Z < -.049)$ for all the cases. This relation can also be expressed as

$$\xi^{(2)}(x) = R\left[\int_0^\infty \xi^{(1)}(y)y\,dy\right]\nabla^2\xi^{(1)}(x) \tag{5.15}$$

with R = – 42 ≈ .194. In terms of the power spectrum this may be written as

$$P_2(k) = -\frac{R}{2\pi^2}\left[\int_0^\infty P_1(k')dk'\right]k^2 P_1(k) \tag{5.16}$$

where $P_1(k)$ is the initial power spectrum and $P_2(k)$ is the correction to the power spectrum at ϵ^4. The fact that $\int_0^\infty P(k)dk > 0$ tells us that $P_2(k) < 0$. In real space we can say that

Figure 5.5: The initial pair velocity for the different initial power spectra considered. These figures shows the small x behaviour.

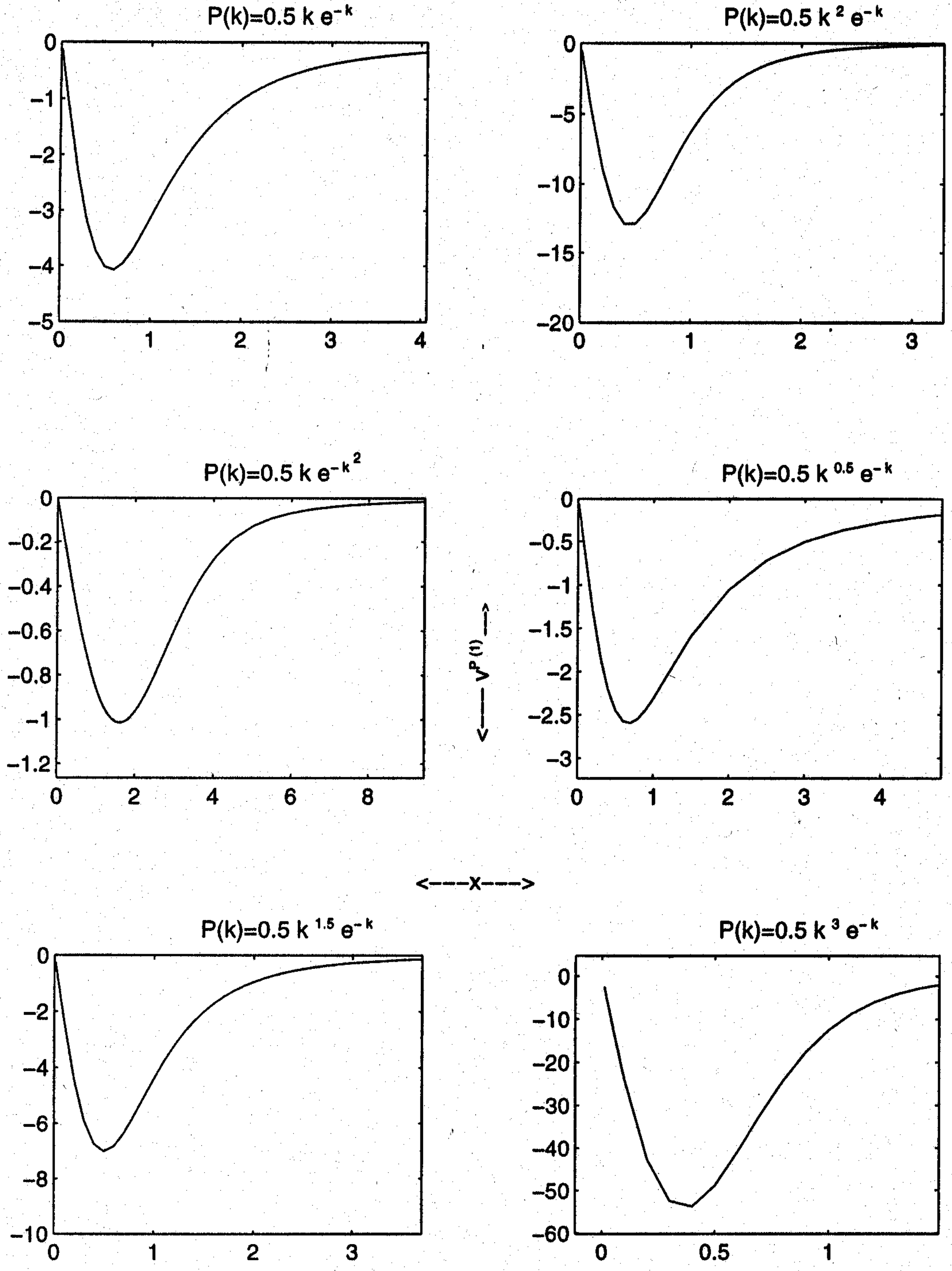


Figure 5.6: The initial pair velocity for the different initial power spectra considered.

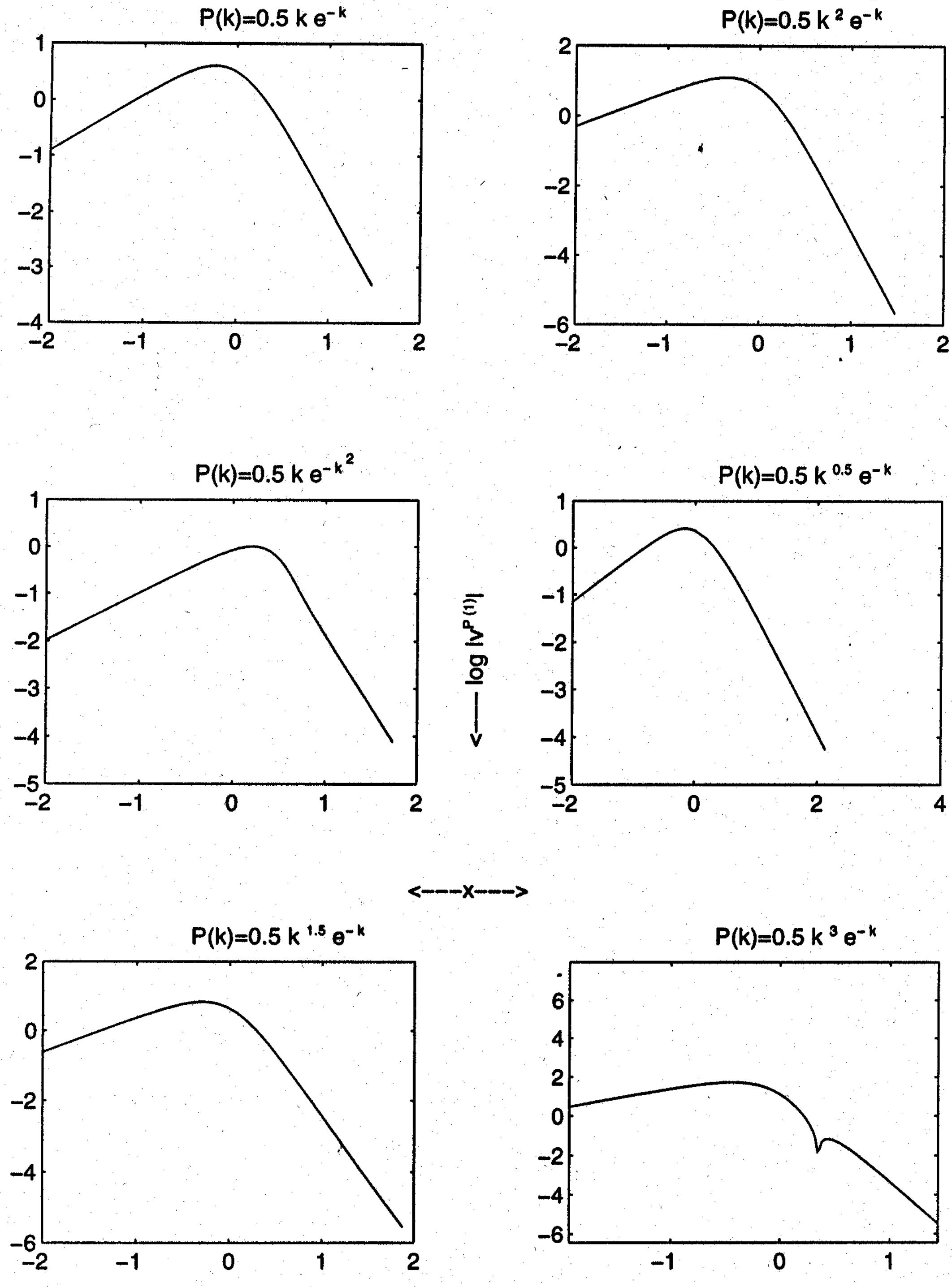


Figure 5.7: The non-linear correction to the two point correlation $\xi^{(2)}(x)$ shown by the solid line and it average $\overline{\xi^{(2)}}(x)$ shown by the dotted line for the different initial power spectra considered. This figure shows the small x behaviour.

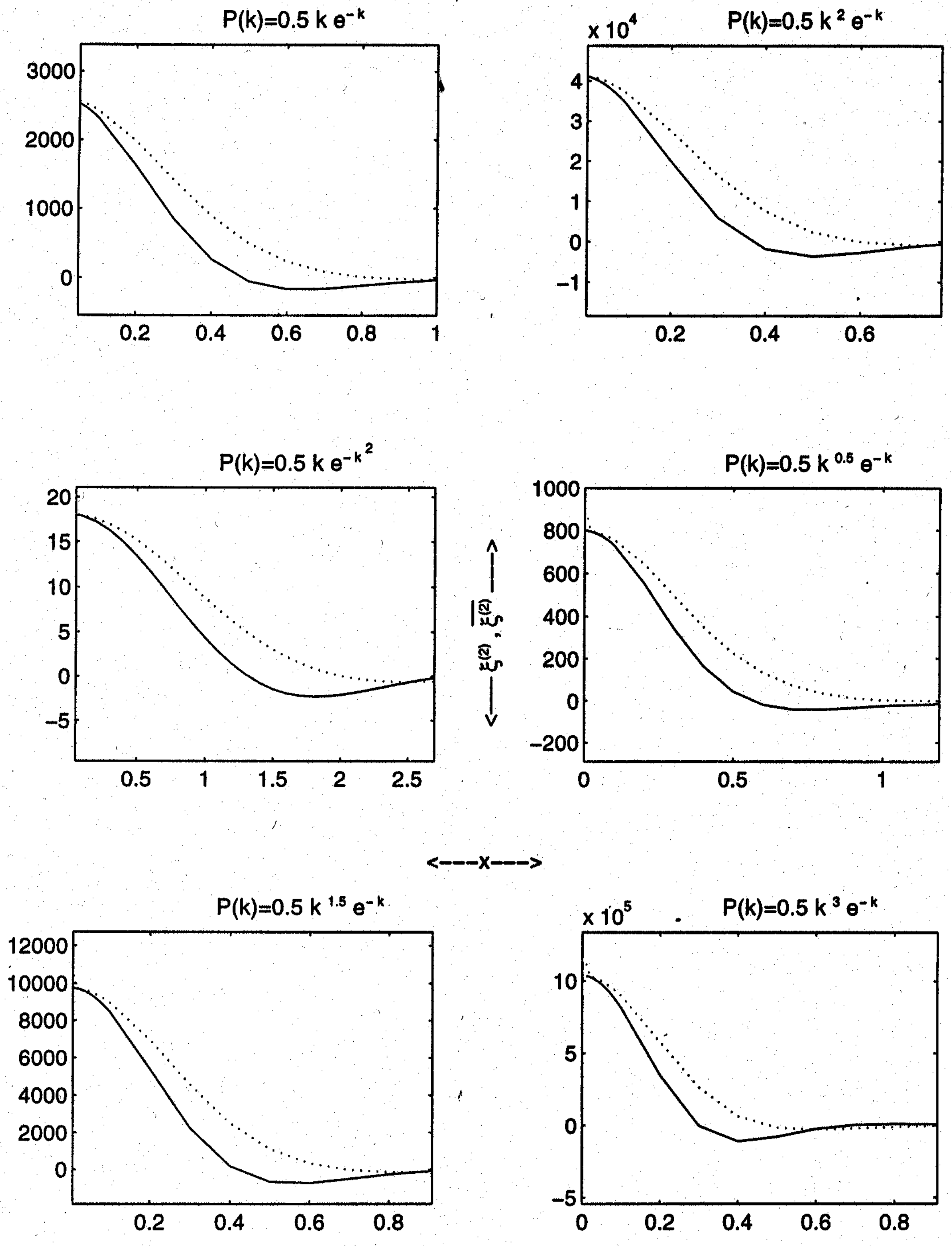
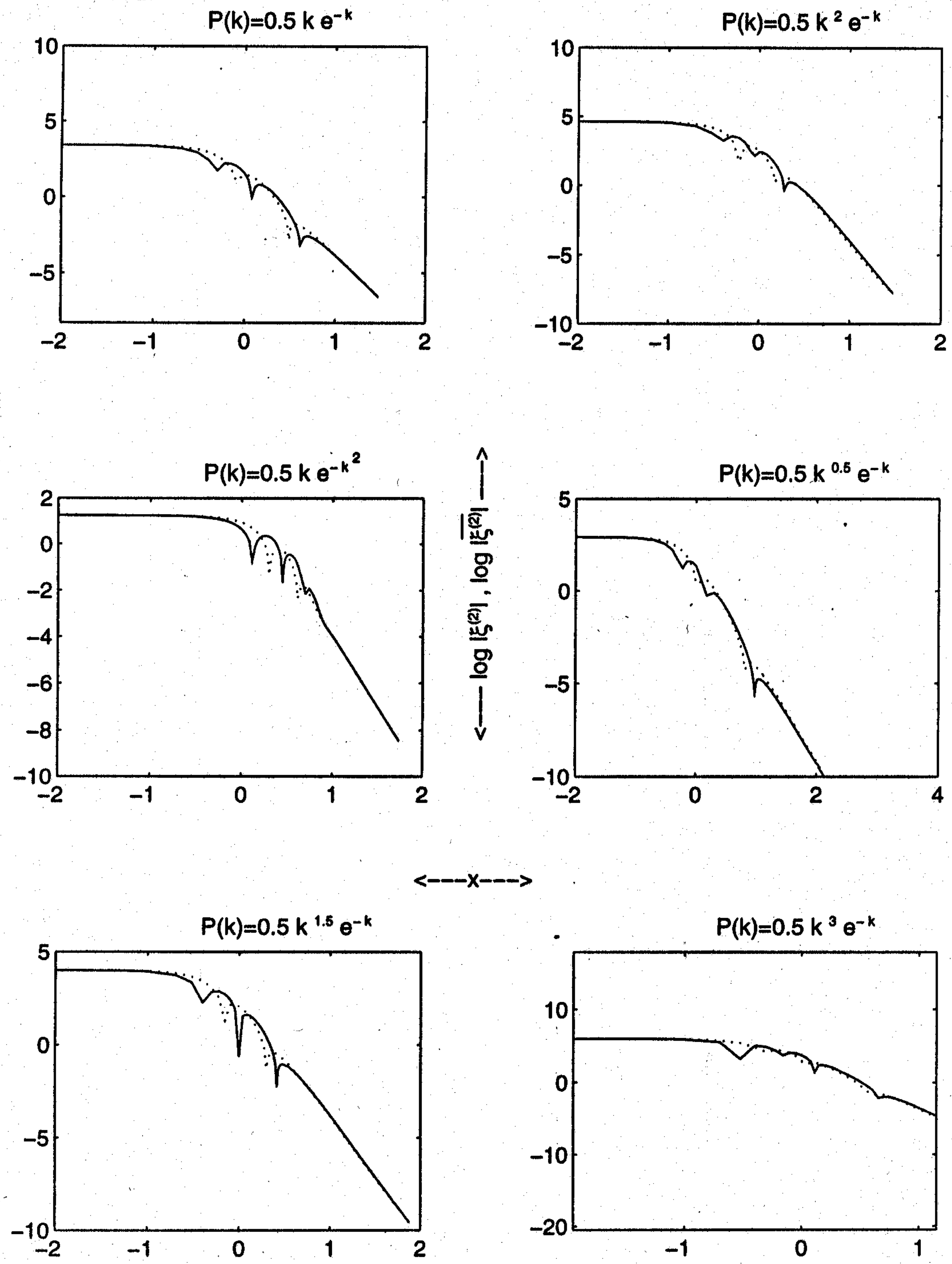


Figure 5.8: The non-linear correction to the two point correlation $\xi^{(2)}(x)$ shown by the solid line and it average $\overline{\xi^{(2)}}(x)$ shown by the dotted line for the different initial power spectra considered.



$\xi^{(2)}(x)$ has the same sign as $\xi^{(1)}(x)$. Based on this numerical evidence we make a hypothesis that equation *(5.15)* holds for all initial conditions where the initial power spectrum index n satisfies $0 < n$. We have not considered any cases where n is negative. For such cases the integral in equation *(5.16)* does not converge and we do not expect this relation to hold..

One of the factors we believe to be responsible for the spread in the values of R (or Z) is that $\xi^{(2)}(x)$ goes to the power law form asymptotically and for all the cases we have not been able to calculate $\xi^{(2)}(x)$ to equally large separations. As of now we have no rigorous derivation for equation (5.15), but we give a heuristic interpretation in terms of a diffusion process in section *4.*

Makino et. al. *(1992)* have analytically calculated $P_2(k)$ for various power law initial conditions, of which the case where

$$P_1(k) = A\frac{k}{k_c} \quad \text{for} \quad k \leq k_c \tag{5.17}$$

and zero elsewhere is of interest to us. In the limit $k \to 0$ their analytic expression for $P_2(k)$ reduces to

$$P_2(k) = -\frac{61}{315\,(2\pi)^2}A^2k^3. \tag{5.18}$$

This can be written as

$$P_2(k) = -\frac{61}{315}\left(\frac{1}{2\pi^2}\right)\left[\int_0^\infty P_1(k')dk'\right]k^2P_1(k) \tag{5.19}$$

which on comparison with equation *(5.16)* gives us $R = \frac{61}{315}$ or $Z = -\frac{1}{4}\left(\frac{61}{315}\right) = -.0484$.

We see that this matches equation *(5.16)* and this serves as a test of our hypothesis.

At small x the behaviour of $\xi^{(2)}(x)$, as shown in figures *5.7* and *5.8* is rather complicated and is difficult to generalise. We find that it starts with a positive value at $a = 0$, which falls fast, becomes negative and oscillates around zero a few times before going over to the power law form.

We next consider the behaviour of $\bar{\xi}(x,t)$. At large separations this has a behaviour quite similar to $\xi^{(2)}(x)$ as shown in figures *5.7* and *5.8* and we find that

$$\overline{\xi^{(2)}}(x) = R\left[\int_0^\infty \xi^{(1)}(y)ydy\right]\frac{1}{x}\nabla^2(x\overline{\xi^{(1)}}(x))\,. \tag{5.20}$$

We see that at large separations $\overline{\xi^{(2)}}(x)$ has the same sign as $\overline{\xi^{(1)}}(x)$. The small x behaviour of $\overline{\xi^{(2)}}(x)$ is similar to $\xi^{(2)}(x)$.

The behaviour of $v^{(2)}(x)$, which is defined such that the ϵ^4 contribution to the pair velocity is

$$v^{(2)}(x,t) = S(t)\frac{d}{dt}\left(\frac{S(t)}{S(t_0)}\right)^4 v^{(1)}(x)\,. \tag{5.21}$$

is shown in figures *5.9* and *5.10.* For large values of x the behaviour can be described by

$$v^{(2)}(x) = R\left[\int_0^\infty \xi^{(1)}(y)y\,dy\right]\nabla^2 v^{(1)}(x)\,. \tag{5.22}$$

We see that $v^{(2)}(x)$ too has the same sign as $v^{(1)}(x)$ at large x. At small x the function $v^{(2)}(x)$ starts from zero and rises fast and then falls off and changes sign a few times before going over to the power law form. We see that at very small scales the ϵ^4 contribution acts to increase the correlation, whereas at intermediate scales it can act to increase or decrease the correlations.

We next consider a case where the initial power spectrum has the form $P_1(k) = e^{-k}$. The two point correlation function has the form $\xi^{(1)}(x) \sim x^{-4}$ for large x but $\overline{\xi^{(1)}}(x)$ does not have the same behaviour and we have $\overline{\xi^{(1)}}(x) \sim x^{-3}$ instead (figure *5.11).* Because of this, at large separations the three point correlation function exhibits a behaviour which is quite different from the one exhibited by the cases considered previously and in this case the three point correlation, as discussed in chapter III, does not have the 'hierarchical form'. We find that the large x behaviour of $\xi^{(2)}(x)$ too is somewhat different as compared to the previous cases. Although we find that $\xi^{(2)}(x) \propto \nabla^2\xi^{(1)}(x)$ the factor relating the two is different from that found for the previous cases and if we fit a formula like equation *(5.16)* we get $R \approx .496$ instead of .194.

We next consider the large x behaviour of $\overline{\xi^{(2)}}(x)$. We see that it has a behaviour of the form x^{-6} as compared with the initial function $\overline{\xi^{(1)}}(x)$ which has the form x^{-3} i.e. a difference of three in the power law index. This is different from the previous cases where there was a difference of two between the index for the linear function and the non-linear correction as seen in equation *(5.20).* This can be easily understood by noting that if we try to relate $\overline{\xi^{(2)}}(x)$ with $\overline{\xi^{(1)}}(x)$ using an expression like equation *(5.20)* we find that the fact that $\overline{\xi^{(1)}}(x) \sim x^{-3}$ implies that the right hand side is zero. We then deduce that, in addition to a part that behaves as x^{-3}, $\overline{\xi^{(1)}}(x)$ has part that behaves as x^{-4}. At large x the value of $\overline{\xi^{(1)}}(x)$ is determined solely by the former term as the latter falls off much faster, but the behaviour of the non-linear correction $\overline{\xi^{(2)}}(x)$ is determined by the latter as the first term does not contribute. We find that equation *(5.20)* gives a good fit for $R \approx .496$ which is consistent with the fit for $\xi^{(2)}(x)$. At large x the behaviour of $v^{(2)}(x)$ is similar to the behaviour of $\overline{\xi^{(2)}}(x)$.

At small x both $\xi^{(2)}(x)$ and $\overline{\xi^{(2)}}(x)$ start off with positive values which fall off fast. At intermediate values $\xi^{(2)}(x)$ changes sign twice and goes over to the power law form at large x whereas $\overline{\xi^{(2)}}(x)$ changes sign only once and hence the two quantities have opposite signs at large x. The behaviour of $v^{(2)}(x)$ is similar to the behaviour of $\overline{\xi^{(2)}}(x)$ except that it starts from the value zero.

Figure 5.9: The non-linear correction to the pair velocity for the different initial power spectra considered. This figure shows the small x behaviour.

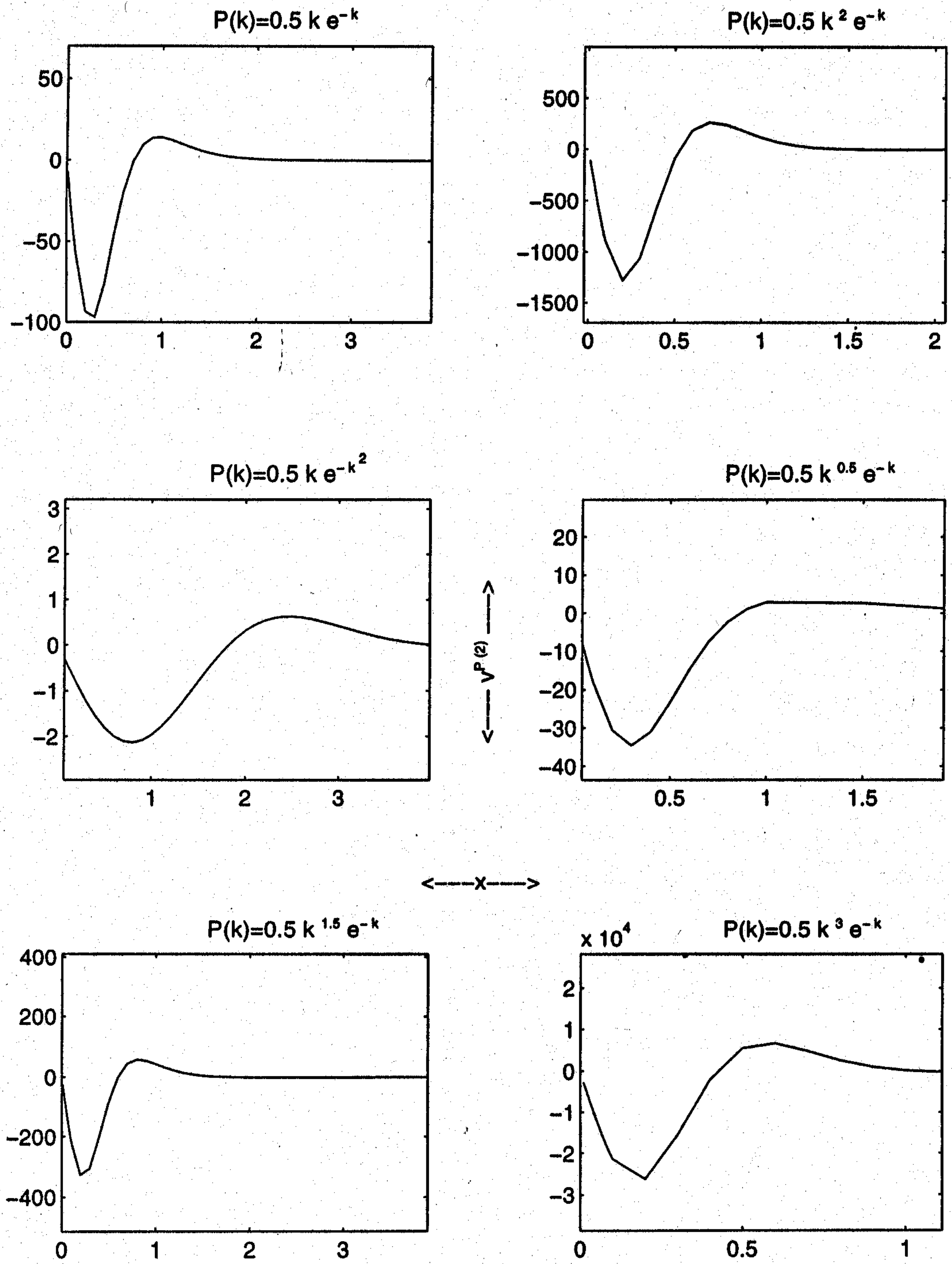


Figure 5.10: The non-linear correction to the pair velocity for the different initial power spectra considered.

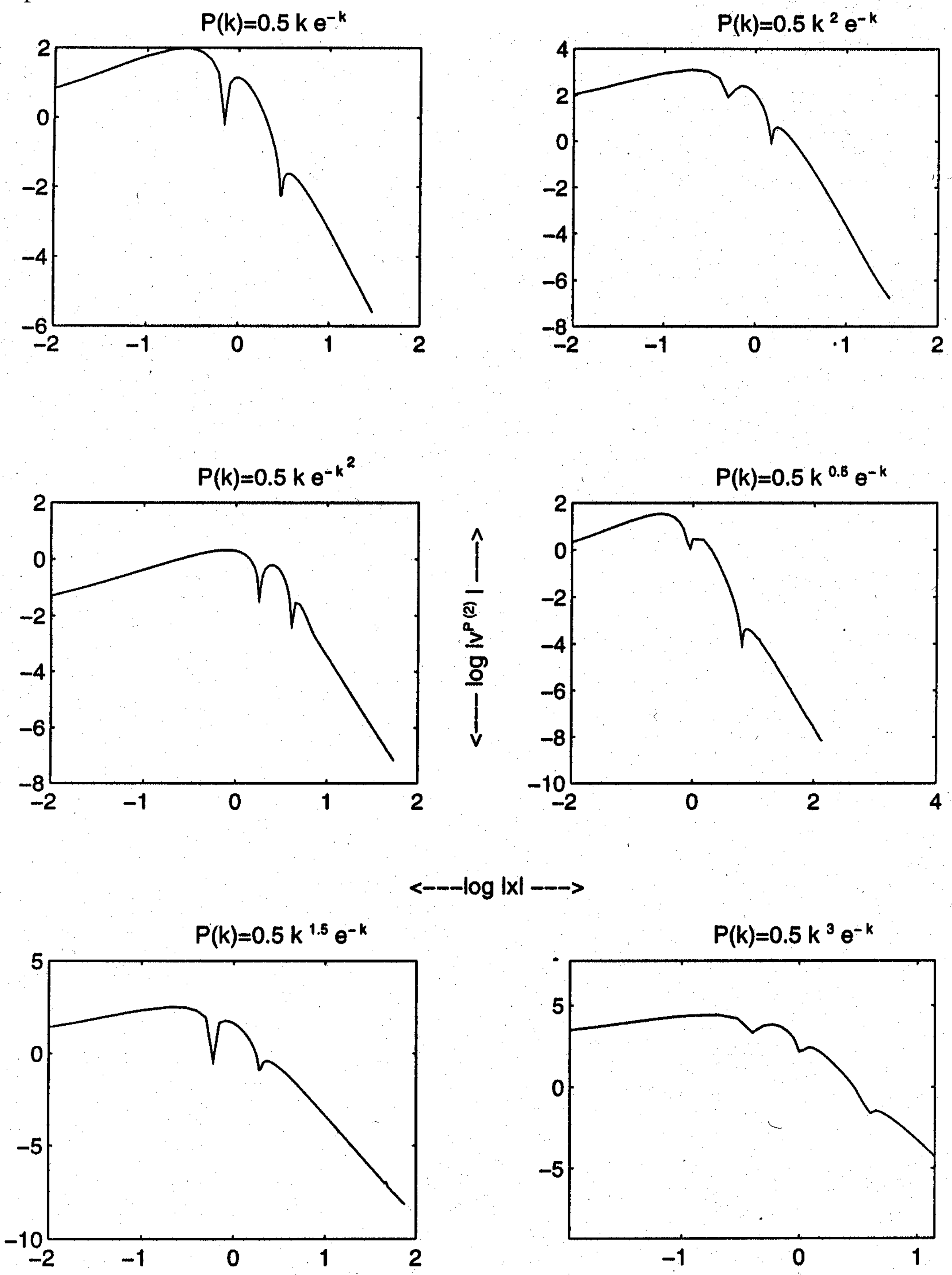
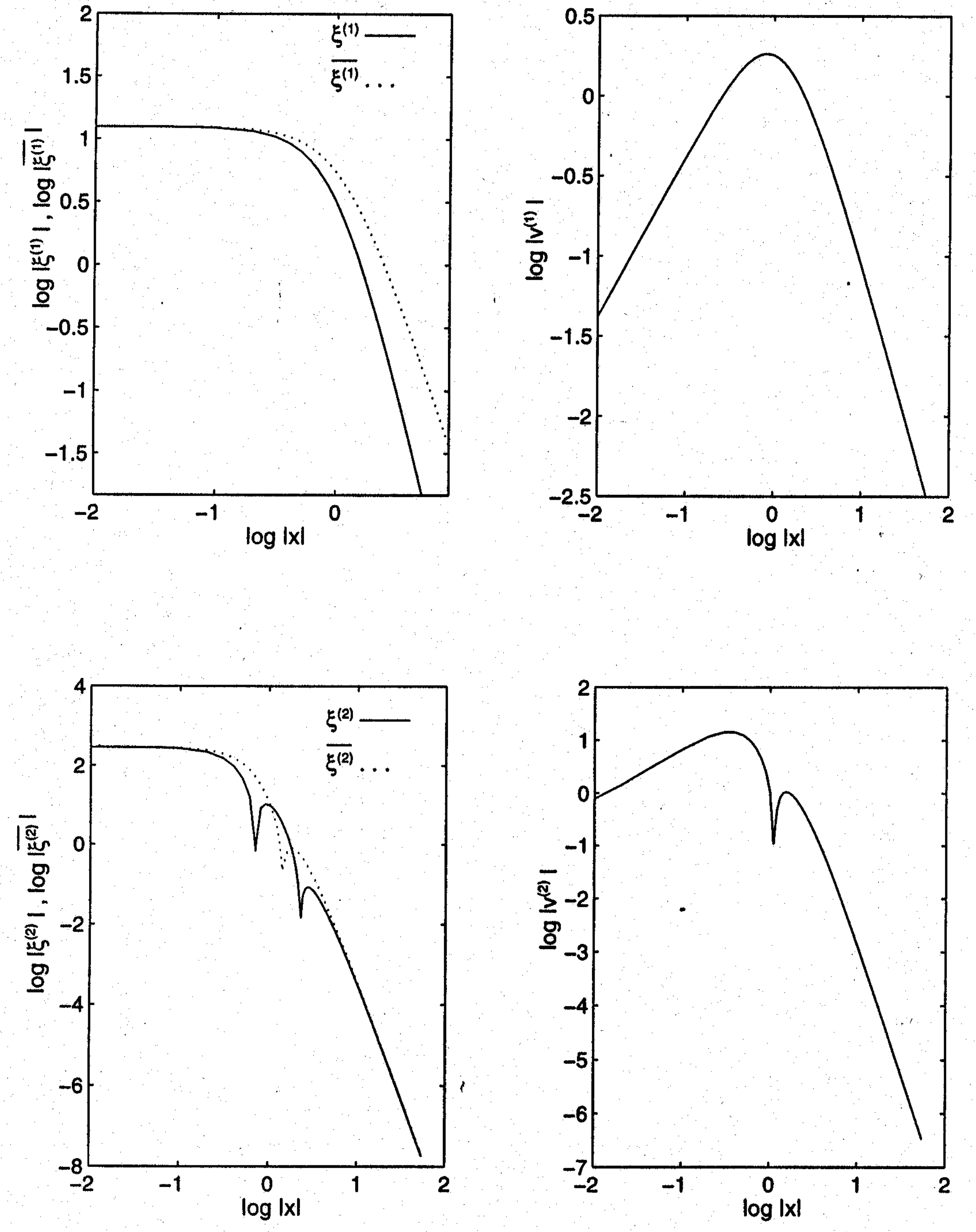


Figure 5.11: The linear two point correlation function $\xi^{(1)}(x)$, its average $\overline{\xi^{(1)}}(x)$ and the pair velocity $v^{(1)}(x)$, and the respective non-linear corrections $\xi^{(2)}(x)$,$\bar{\xi}(x,t)$ and $v^{(2)}(x)$ for the initial power spectrum $P(k) = e^{-k}$.



5.3 The temporal behaviour.

Here we would like to investigate the evolution of the correlation function and the pair velocity. It is generally believed that the linear results should hold at some length scale until the density contrast averaged over that length scale is of the order of unity. Having calculated the lowest order non-linear term we can see when this becomes of the order of the linear term. This would be a different criterion for determining when the linear results would no longer be applicable. Here we wish to compare these two criteria and investigate whether they are the same.

Before proceeding further we should remind the reader that we are working in the continuum (or fluid) limit and the initial conditions are such that the perturbative treatment is valid at all length scales. As a result of this there is a growth of clustering even at the smallest scales.

We only discuss the case with $P_1(k) \propto ke^{-k}$ here. We find that the other cases considered have a similar behaviour.

The smaller scales go non-linear first. We first consider the evolution of $\xi(0,t)$ which is the mean square density fluctuation (figure 5.12) . We find that the non-linear correction enhances the growth of the correlation. We also find that the linear term is equal to the correction when $\xi(0,t)$ is of the order of unity. This is as expected and here it happens at $S(t) = 0.121$. At $S(t) = 0.053$ the correction is one tenth of the linear term and we may expect that the other higher order corrections not considered here will not contribute before this epoch Next we consider the separation $x = 0.1$. Figure 5.13 shows the evolution of the various quantities of interest. There is no qualitative difference here with $x = 0$ except that at $x = 0$ the pair velocity is zero. When $x = 1$ (figure 5.14), there is a qualitative difference. Previously the effect of the non-linear terms was to increase the correlation and now it tends to decrease it, but here too $\overline{\xi^{(1)}}(x,t)$ and $\overline{\xi^{(2)}}(x,t)$ are equal when they both are of order unity. At very large separations e.g. $x = 20$ (figure 5.15) we find that $\overline{\xi^{(1)}}(x,t)$ and $\overline{\xi^{(2)}}(x,t)$ are equal when $\bar{\xi}(x,t) \sim 0.01$. The correlation function and the pair velocity show a similar behaviour too. Hence it appears that at this scale the linear theory is breaking down much before one would expect it to.

Actually it may not be so because this happens at a very large value of $S(t)$ (~ 10) and the perturbative approach has broken down much earlier on the small scales. The correction $\xi^{(2)}(x)$ is non-local and the contribution from the small scales keeps on growing in our calculation. In reality the clustering saturates at the small scales because of virialization. This saturation means that at late times ($\bar{\xi}(x,t) > 1$) the perturbative results overestimate the contribution from the small scales to the large scales. Thus we see that for small and

Figure 5.12: The solid line shows the two point correlation function $\xi(x,t) = \xi^{(1)}(x,t) + \xi^{(2)}(x,t)$ at $x = 0$ as a function of the scale factor $S(t)$. The dotted line shows just the linear contribution.

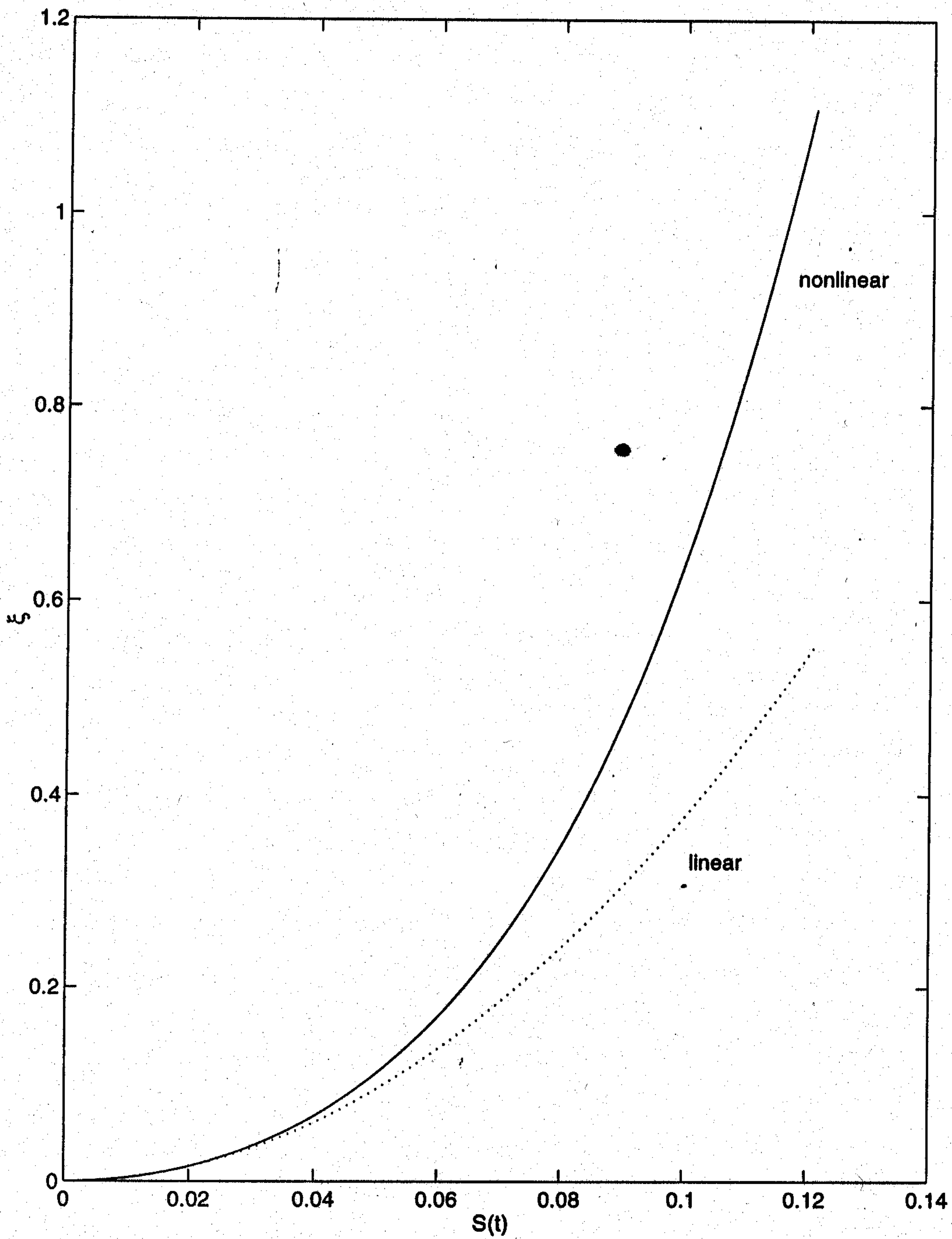


Figure 5.13: The solid curves show the two point correlation function $\xi(x,t) = \xi^{(1)}(x,t) + \xi^{(2)}(x,t)$, its average $\bar{\xi}(x,t)$ and the pair velocity $v(x,t)$ at $x = .1$ as a function of the scale factor $S(t)$. The dotted line shows just the linear contribution to these quantities.

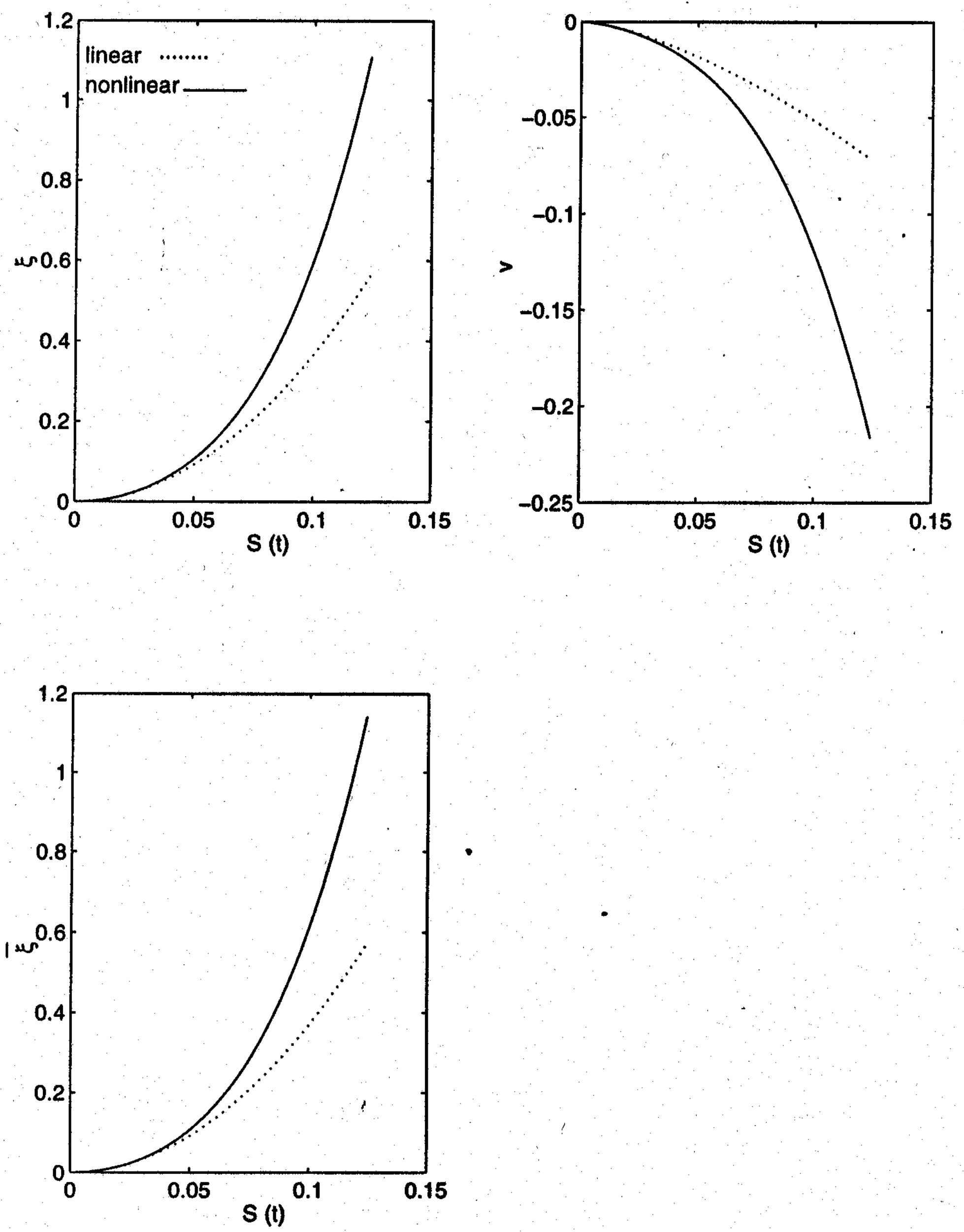


Figure 5.14: The solid curves show the two point correlation function $\xi(x,t) = \xi^{(1)}(x,t) + \xi^{(2)}(x,t)$, its average $\overline{\xi}(x,t)$ and the pair velocity $v(x,t)$ at $x = 1.0$ as a function of the scale factor $S(t)$. The dotted line shows just the linear contribution to these quantities.

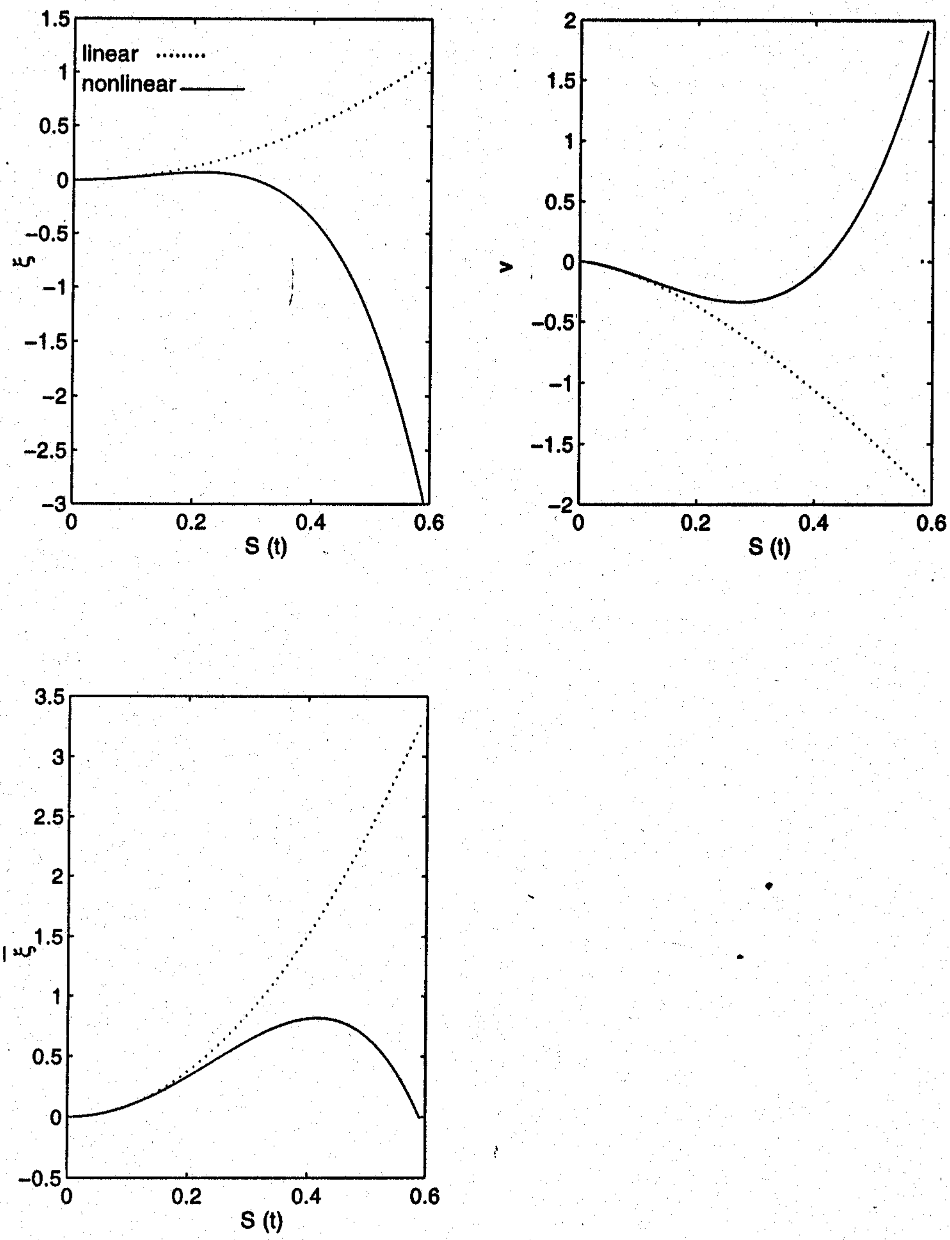
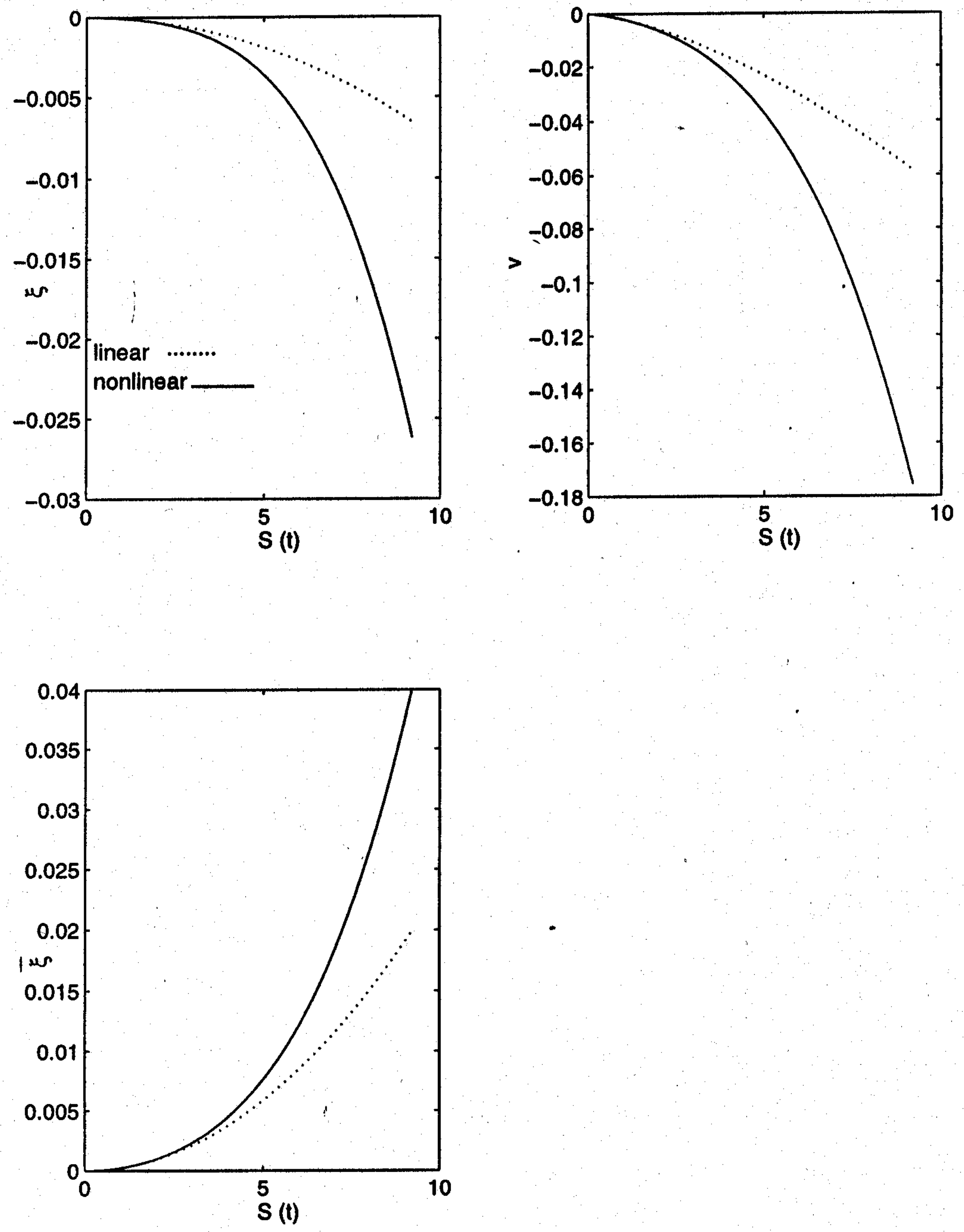


Figure 5.15: The solid curves show the two point correlation function $\xi(x,t) = \xi^{(1)}(x,t) + \xi^{(2)}(x,t)$,its average $\bar{\xi}(x,t)$ and the pair velocity $v(x,t)$ at $x = 20.$ as a function of the scale factor $S(t)$. The dotted line shows just the linear contribution to these quantities.



intermediate lengthscales the perturbative results are valid until $\bar{\xi}(x,t) \sim 1$ whereas at large separations the perturbative treatment seems to break down much before $\bar{\xi}(x,t) \sim 1$. This is because of a relatively large contribution to $\overline{\xi^{(2)}}(x,t)$ from the small scales through the $\int_0^\infty \xi^{(1)}(y,t) y dy$ which appears in $\overline{\xi^{(2)}}(x,t)$.

Peebles (1980) has shown that once virialized objects have formed on some small scales, those small scales have no influence on the evolution of the large scales. In our calculation clustering happens at all scales and we find a strong influence of small scales on large scales. The latter situation might be a better description of the early stages of structure formation which goes over to the former situation once bound objects have formed. The question as to how much the small scales influence the correlation on the large scales before the small scales get virialized has to be still be answered. Since virialization typically occurs when $\bar{\xi} \sim 10$ (Hamilton et. al. 1991) this effect may be quite significant.

5.4 Scaling relations.

Hamilton et. al. (1991) have suggested that in an $\Omega = 1$ universe the evolution of the two point correlation function can be described by a simple universal relation whose exact form they have obtained by fitting N-body simulations. More recently Nityananda and Padmanabhan (1994) have examined the possible origin of this universal scaling relation. The scaling relation can be based on the conjecture that the dimensionless pair velocity $h(a,x) = -\frac{v(x,t)}{\dot{S}(t)x}$ depends on $S(t)$ and x through $\bar{\xi}(x,t)$ alone. This conjecture is valid in the linear regime, Here we perturbatively test this conjecture at the lowest order of non-linearity.

We look at the behaviour of $h(x,t)$ and $\bar{\xi}(x,t)$ in some range where we are reasonably sure that the perturbative approach gives a good description of the clustering. Since the smallest scales go non-linear first, the criterion is based on the properties at $x = 0$ and we restrict our analysis to the epoch when $\xi^{(2)}(0,t) \leq 0.1\xi^{(2)}(0,t)$. Because of this conservative criterion we find that $h(x,t)$ differs very little from the linear value of $\frac{2}{3}\bar{\xi}(x,t)$. To look at the nature of this small deviation we consider the ratio $\frac{h(x,t)}{\bar{\xi}(x,t)}$. Figure 5.16 shows this ratio as a function of $\bar{\xi}(x,t)$ for different $S(t)$ for the initial condition $P_1(k) \propto ke^{-k}$. We find that for a fixed value of $\bar{\xi}(x,t)$ the ratio has a spread of values. Although this spread is not large (5 percent), it is comparable to the largest deviation of mean square density from the linear prediction (ten percent). We also find that this ratio has a systematic behaviour which can be understood by looking at the same quantity as a function of x for different $S(t)$ (figure 5.17). We see that there are points where the corrections to the pair velocity $v^{(2)}(x)$ and also to $\overline{\xi^{(2)}}(x)$ are both zero and $h(x,t)$ and $\bar{\xi}(x,t)$ there (figure 5.16) continues to follow the linear evolution. If we consider the first such point, then for smaller values of x (larger values of $\bar{\xi}(x, t)$) the deviation is positive with respect to the linear value. For larger values of x (smaller values of $\bar{\xi}(x, t)$) the deviation is negative with respect to the linear value. At large x (beyond the second zero crossing of the correction to the pair velocity) the deviation again becomes positive, but this is not seen in figure 5.16 because the whole range of x gets mapped to a very small range of $\bar{\xi}(x, t)$ between 0 and some very small value. Based on this we draw the conclusion that we cannot express $h(x, t)$ as a function of $\bar{\xi}(x, t)$ alone. We have carried out this exercise for all the different initial conditions discussed earlier and they too exhibit a similar behaviour.

As was discussed earlier, equations (5.20) and (5.22) give a good description of $\overline{\xi^{(2)}}(x)$ and $v^{(2)}(x)$ at large x for a large set of initial conditions of the form $\xi^{(1)}(x) \propto x^{-\gamma}$. Using these and the fact that $\frac{v^{(1)}(x)}{x} = -\frac{\overline{\xi^{(1)}}(x)}{3}$ we have

$$h(x,t) = 2S(t)^2\left[1 + \frac{2S^2(t)\gamma(\gamma-1)R}{x^2}\int_0^\infty \xi^{(1)}(y)y\,dy\right]\frac{x^{-\gamma}}{3} \tag{5.23}$$

and

$$\bar{\xi}(x,t) = S(t)^2\left[1 + \frac{S^2(t)\gamma(\gamma-1)R}{x^2}\int_0^\infty \xi^{(1)}(y)y\,dy\right]x^{-\gamma}\,. \tag{5.24}$$

This also show that $h(x, t)$ is not a function of $\bar{\xi}(x, t)$ alone.

5.5 An interpretation based on diffusion.

In the previous section we saw that for a certain class of initial conditions equation (5.15) gives a good fit for $\xi^{(2)}(x)$ in terms of $\xi^{(1)}(x)$. In this section we provide a possible interpretation for this equation. The cosmic energy equation (Irvine 1961; Dimitriev & Zel'dovich 1964; Peebles 1980), which is the second moment of the first equation of the BBGKY hierarchy, allows us to relate the integral that appears in equation (5.15) to the mean square momentum. In our notation the cosmic energy equation can be written as

$$\frac{d}{d\lambda} < (p^1)^2 >_1 (\lambda) = 4\pi S G\rho m^2 \frac{\partial}{\partial\lambda}\int_o^\infty \xi(x,\lambda)x\,dx \tag{5.25}$$

, where $x = \mid x^1 - x^2 \mid$.

Instead of looking at the evolution of $< (p^1)^2 >$, we consider the motion of the particles as a function of the growing mode which in this case is the scale factor $S(t)$. We define a velocity

$$u_\mu = \frac{dx_\mu}{dS} = \frac{1}{m}\frac{d\lambda}{dS}p_\mu \tag{5.26}$$

and $< u^2 > (S(t))$ is the mean square of this velocity. In terms of this the cosmic energy equation is,

$$\frac{d}{dS}\left[S^3 < u^2 >\right] = \frac{3S}{2}\frac{\partial}{\partial S}\int_o^\infty \xi(x,S)x\,dx \tag{5.27}$$

Figure 5.16: The ratio $\frac{h(x,t)}{\bar{\xi}(x,t)}$ for the initial power spectrum $P(k) = e^{-k}$ is shown as a function of $\bar{\xi}(x,t)$ for different values of the scale factor $S(t)$. S^{NL} corresponds to the value of the scale factor for which the non-linear correction to the two point correlation at $x = 0$ is one tenth of the linear quantity $(\xi^{(2)}(0,t) \sim 0.1\xi^{(1)}(0,t))$.

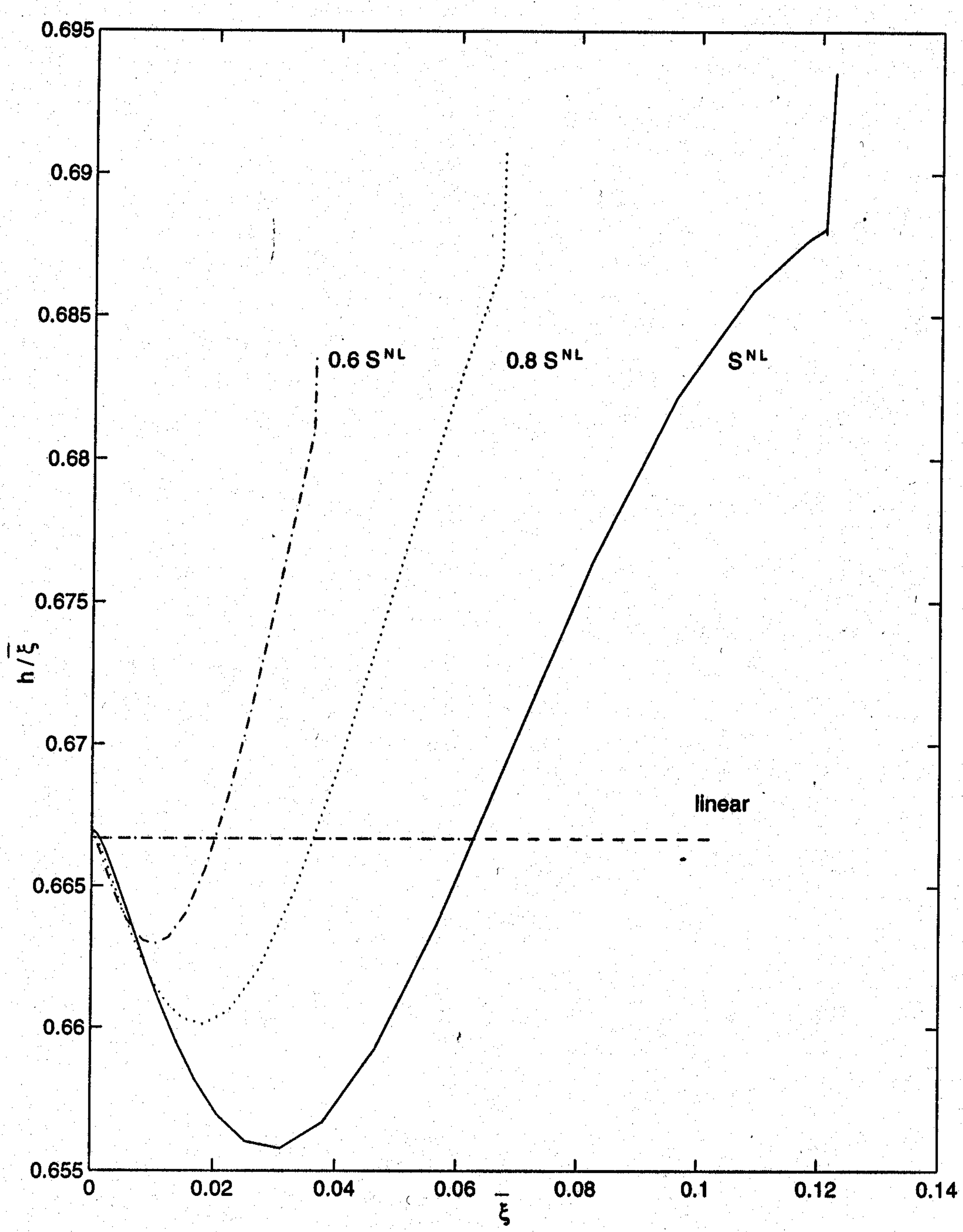
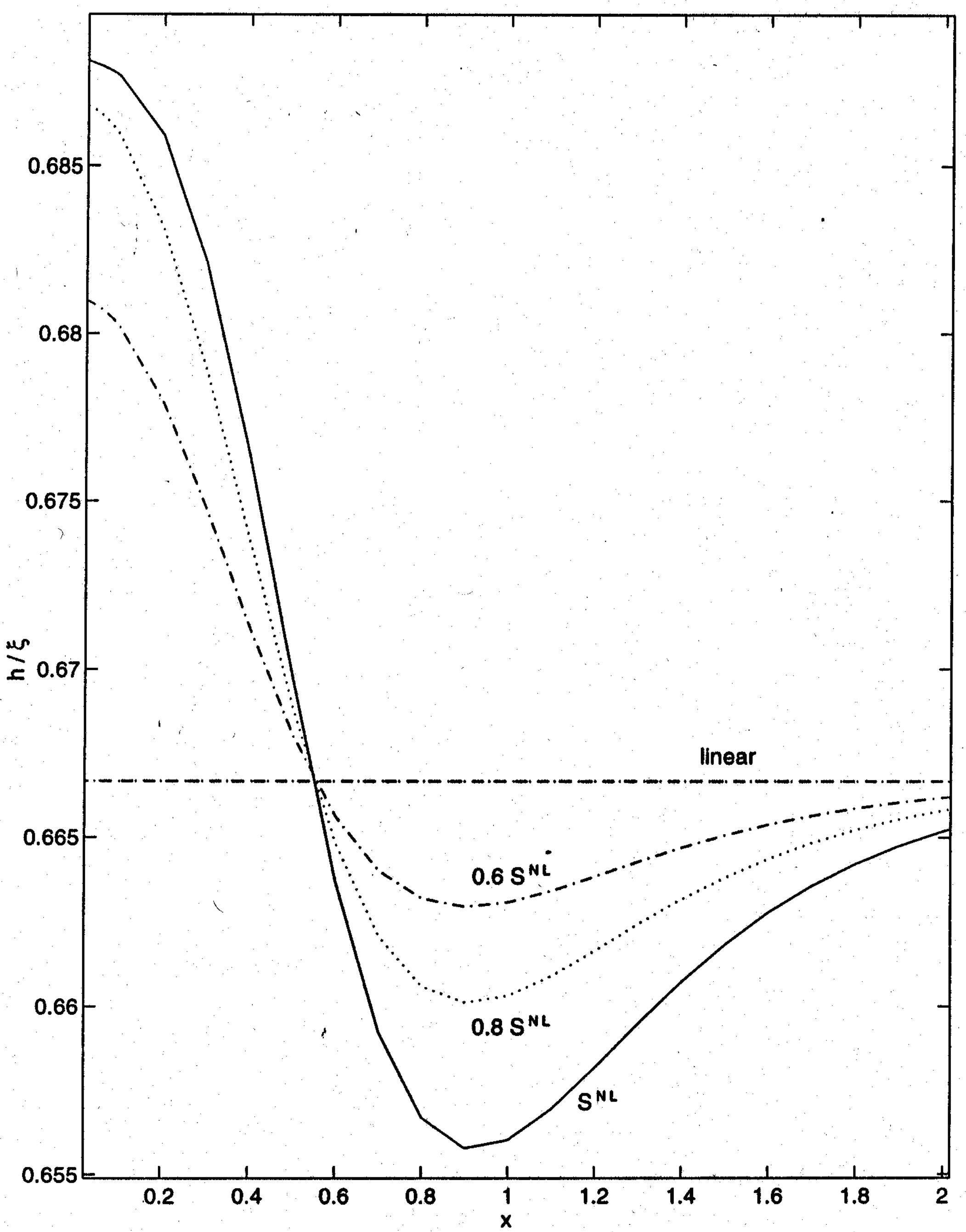


Figure 5.17: The ratio $\frac{h(x,t)}{\xi(x,t)}$ for the initial power spectrum $P(k) = e^{-k}$ is shown as a function of x for different values of the scale factor $S(t)$. S^{NL} corresponds to the value of the scale factor for which the non-linear correction to the two point correlation at $x = 0$ is one tenth of the linear quantity ($\xi^{(2)}(0,t) \sim 0.1\xi^{(1)}(0,t)$).



which in the linear regime gives us

$$< u^2 >= \frac{1}{} \int_o^\infty \xi^{(1)}(x) x dx, \quad (5.28)$$

i.e. $< u^2 >$ does not change, Note that in this equation we have not explicitly shown the superscript *(1)* over u to indicate that it the linear part of u. Henceforth we shall use u for the linear part of the same quantity. Using equation *(5.28)* we can write equation *(5.15)* as

$$\xi^{(2)}(x,t) = RS^2(t) < u^2 > \left(\frac{S(t)}{S(t_0)}\right)^2 \nabla^2 \xi^{(1)}(x) \quad (5.29)$$

where the coefficients in front of V^2 look like a diffusion coefficient. It is in light of this that we interpret equation *(5.15)*.

Consider a particular realisation of the Gaussian density fluctuation field. We consider two points x^1 and x^2 where the density fluctuations are $\Delta(x^1,t)$ and $A(x^2,t)$ respectively. The $\Delta(x,t)$ of the fluid element at any point x grows according to linear theory and the fluid element moves according to some random velocity u, which is assumed uncorrelated to the velocity at any other point. We then have

$$\begin{aligned} \Delta(x,t) &= \left(\frac{S(t)}{S(t_0)}\right)^2 \Delta(x+Su,t_0) = \left(\frac{S(t)}{S(t_0)}\right)^2 [\Delta(x,t_0) \\ &+ S(t) u_\mu \partial_\mu \Delta(x,t_0) + \frac{S^2(t)}{2} u_\mu u_\nu \partial_\mu \partial_\nu \Delta(x,t_0)\Big] . \end{aligned} \quad (5.30)$$

Using this and taking an ensemble average we have

$$\begin{aligned} \xi(x^1 - x^2, t &= < \Delta(x^1,t)\Delta(x^2,t) >= \left(\frac{S(t)}{S(t_0)}\right)^2 \xi(x^1 - x^2, t_0) \\ &+ \frac{1}{3} S^2(t) < u^2 > \left(\frac{S(t)}{S(t_0)}\right)^1 \nabla^2 \xi(x^1 - x^2, t_0) \end{aligned} \quad (5.31)$$

where

$$< u_\mu u_\nu >= \frac{\delta_{\mu\nu}}{3} < u^2 > . \quad (5.32)$$

The first term on the right hand side corresponds to the linear growth and the second term is the lowest non-linear correction. We see that by this process we have a non-linear correction which matches with equation *(5.29)* except for the numerical factor R. This difference can be attributed to the fact that we have kept the velocity u, constant and this is only valid in the linear regime. In the non-linear evolution this is not true. Based on equation *(5.31)* we suggest that the simple diffusion process described above gives a description of the evolution of the two point correlation at the lowest order of non-linearity for large separations. Thus, when we are looking at the correlation at a large separation x, we can consider the initial perturbation field to be growing linearly, but there is one more effect to be taken into account. This effect is the local rearrangement of matter on small scales and this may be thought of as being random. The local rearrangement is on a lengthscale L $\sim S(t)\sqrt{< u^2 >}$ and we expect this picture to be valid when L $\ll x$. When the mean displacement becomes comparable to the separation ($i.e.$L $\sim x$) we can longer treat the displacements as random and this picture is not valid.

5.6 Discussion and Conclusions.

To study the influence of small scales on the large scales we have considered two cases where the two point correlation is initially zero at large separations(i.e. $\xi^{(1)}(x) = 0$ for $x > x_0$). One of the cases has been treated analytically and the other numerically. We find that at large separations the induced two point correlation function has the form $\xi^{(2)}(x) \propto x^{-6}$. We also find that in both the cases the constant of proportionality is the same functional of the initial two point correlation function.

We have numerically investigated some cases where the initial two point correlation has a power law form $x^{-\gamma}$ at large x. The cases we have studied have a power spectrum of the form k^n with $0 \leq n \leq 3$ at small k and have an exponential or Gaussian cut off at large k. The cut off introduces a length scale and we find that at scales much smaller than this scale the non-linear term enhances the growth of clustering. At intermediate scales we find that the non-linear term changes sign more than once and it can act both ways i.e. to increase or decrease the clustering.

At large scales we find that the behaviour of the non-linear term depends on the condition whether $\overline{\xi^{(1)}}(x)$ a $\xi^{(1)}(x)$ or not. For the cases where this condition is satisfied we find that equation *(5.15)* gives a good fit to the lowest ordernon-linear correction $\xi^{(2)}(x)$ to the two point correlation. We find similar equations for the average of the two point correlation $\overline{\xi^{(2)}}(x)$ and the pair velocity $v^{(2)}(x)$ too. We have interpreted equation *(5.15)* in terms of a simple diffusion process. For all the quantities the non-linear term has the same sign as the corresponding linear term.

For the case where $\overline{\xi^{(1)}}(x)$ is not proportional to $\xi^{(1)}(x)$ we find that we obtain an equation similar to equation *(5.15)* with a different numerical coefficient. The quantities $\overline{\xi^{(2)}}(x)$ and $v^{(2)}(x)$ also have a similar behaviour, but for both of them it is not the leading linear part that contributes.

In all the cases equation *(5.15)* shows the effect of the various scales on the large scales. Equation (5.15) has an integral of the initial two point correlation over all scales, and for the initial conditions that we have considered the small scales contribute the most. Thus we see that at the lowest order of non-linearity the small scales effect the large scales.

Hansel et al (1985) have studied the effect of large scales on the small scales and they find that in the weakly non-linear regime the small scale perturbations get modulated by the large scale perturbations. This effect is like a diffusion in Fourier space because the effect of the disturbances at small k is to spread out the power spectrum at large k. It is very interesting that the effect of small scales on the large scales is a diffusion process in real space and the effect of the large scales on the small scales is like a diffusion process in Fourier space.

It is believed that perturbation theory is valid at a certain scale until $\bar{\xi}(x,t)$ at that scale is of the order of one. Another criterion for the applicability of perturbation theory is that the second order term should be smaller than the linear term. We have compared these two criteria and we find that the two break down at nearly the same epoch at small scales. At large scales we find that the second order term becomes of the same magnitude as the linear term at an epoch when $\bar{\xi}(x,t) \ll 1$. This happens because the second order term at large scales is influenced by the small scales. This happens at an epoch which is much later than the epoch when the perturbative treatment breaks down at small scales and because of the coupling of scales the second order term may be an overestimate of the actual non-linear effects at late times at large scales.

We have tested the hypothesis that the pair velocity $v(x,t)$ is an universal function of the average of the two point correlation $\bar{\xi}(x,t)$. We find that the pair velocity cannot be expressed as a function of the average of the two point correlation function and hence the scaling relations are not valid in the weakly non-linear regime. The scaling relations are based on the underlying idea that there exists a one to one map connecting any length scale in the non-linear regime to an unique length scale in the linear regime In the weakly non-linear regime the evolution can be thought of in terms of diffusion processes and hence there is no unique relation between the length scales. We propose that this is the reason why the scaling relations do not hold in the weakly non-linear regime.

Bibliography

[1] Dimitriev N. A. & Zel'dovich Ya. B., 1964, Sov. Phys. JETP, 18, 793

[2] Hamilton A. J. S., Kumar P., Lu E. & Mathews A. 1991, ApJ, 374, L1

[3] Hansel D., Bouchet F. R., Pellat R. & Ramani A. 1985, Phy Rev Let, 55, 437

[4] Irvine W. M., 1961, Ph.D. thesis, Harvard University (unpublished)

[5] Nityananda R. & Padmanabhan T. 1994, MNRAS, 271, 976

[6] Peebles, P. J. E. 1980, The Large Scale Structure of the Universe, (Princeton: Princeton University Press)

[7] Zel'dovich Ya. B. 1965, Adv. Astron. Ap., 3, 352

[8] Zel'dovich Ya. B. 1970, A & A, 1970, 5, 84

[9] Zel'dovich Ya. B. & Novikov I. D. 1983, The Structure and Evolution of the Universe, (Chicago: The University of Chicago Press)