Chapter 6

The evolution of correlation functions in the Zel'dovich approximation and its implications.

6.1 Introduction.

In chapter IV we have used the moments of the equations of the BBGKY hierarchy to perturbatively calculate the lowest order non-linear correction to the two point correlation function. The *equations* that we have used are valid even in the multi-streamed regime. The inviscid hydrodynamic equations without pressure and vorticity (referred to as the HD equations in the rest of this chapter) are often used for similar perturbative calculations (references in chapter I). These equations are valid only in the single streamed regime and they break down once multi-streaming occurs. The disturbances that have been considered are such that initially the flow is single streamed. Such a situation is correctly described by the HD equations. As the disturbances evolve the particle trajectories intersect and there are particles with different velocities at the same point i.e. the flow becomes multi-streamed. When this occurs the HD equations are no longer valid. This is because the HD equations neglect the local stress tensor associated with the moments of the velocity about the mean velocity at a point. In chapter IV we have found that at the lowest order of non-linearity the results obtained using the two methods discussed above match. Based on this we concluded that there were no effects of multi-streaming at the lowest order of non-linearity. In this chapter we investigate if by going to higher orders in the perturbative expansion we get any effects of multi-streaming or whether it is a limitation of the perturbative technique that it is not possible to use it to follow the transition from a single streamed flow to a multi-streamed flow.

Because of the difficulty in calculating the higher order terms in a perturbative treatment of gravitational dynamics (GD), we look at a simpler system where we use the Zel'dovich approximation (ZA, Zel'dovich 1970) to determine the motion of the particles. In this situation too the transition from a single streamed flow to a multi-streamed flow occurs and we can analyse it to see if in a perturbative calculation using distribution functions we can include any effects of multi-streaming which would be missed if the HD equations were used instead.

In section 2 we discuss the evolution equations. In section **3** we use distribution functions to calculate the evolution of the two point correlation function. The equations used in section 3 are valid even in the multi-streamed regime. In section 4 we do the same calculation using the HD equations and compare the result with that obtained in section **3.**

Bond and Couchman (1988) have studied the evolution of the two point correlation function using ZA and the calculation presented in section **3** is on similar lines. In a recent paper Schneider and Bartelrnann (1995) have also studied the two point correlation function and the power spectrum using ZA. For a comprehensive article on various aspects of ZA the reader is referred to a review by Shandarin and Zel'dovich (1989). In section 5 we consider the evolution of the pair velocity and its dispersion using ZA. In section 6 we numerically investigate the formulae for the evolution of the two point correlation and the pair velocity derived in sections 3 and 5. In this section we focus our attention on the behaviour at small scales.

In chapter V we investigated the lowest order non-linear correction (using GD) to the two point correlation for initial power spectra of the form $P(k) \propto k^n$ at small k and an exponential or Gaussian cut-off at large k. We found that for $0 < n \leq 3$, the numerical results for the non-linear correction to the two point correlation function at large x could be fitted by a simple formula . We also interpreted this formula in terms of a simple diffusion process. In section 7 of this chapter we investigate the evolution of the two point correlation function at large separations using ZA and compare it with the results from GD.

In section 8 we look at the evolution of the induced three point correlation function using ZA. We first calculate the three point correlation function at the lowest order of non-linearity and compare it to the results from GD (chapter III). We then go on to study the effect of the higher order non-linear terms at large separations.

The calculations using ZA are valid for any value of Ω_0, but whenever we make comparisons with GD it is for the specific value $\Omega = 1$.

A similar calculation has been done by Grinstein and Wise (1987) who have used ZA to study the evolution of skewness of the density field averaged over a Gaussian ball. Also, Munshi and Starobinsky (1994) have considered the evolution of the skewness of the density field for ZA and various other approximations, and Bernardeau et. al. (1993) have calculated the evolution (using ZA) of the skewness of the density field averaged over top hat filters. All of these calculations have been done at the lowest order of non-linearity.

In section 7 we present a discussion of the results obtained and the conclusions.

We would also like to point out that the notation used in this chapter is slightly different and largely independent of the notation introduced in the previous chapters. This chapter is more or less self contained as far as notation is concerned.

6.2 Evolution of the distribution function

The Zel'dovich approximation (ZA) defines a map from the initial position of a particle to its position at any later instant. If $x_\mu(t)$ is the comoving co-ordinate of a particle at any time t, the initial instant being t_0, and $b(t)$ the growing mode in the linear analysis of density perturbations, this map is

$$x_\mu(t) = x_\mu(t_0) + b(t)u_\mu \,. \tag{6.1}$$

For a particle the quantity u_μ is a constant and it is related to the peculiar velocity $v_\mu(t)$ at any instant by

$$v_\mu(t) = a(t)\frac{d}{dt}x_\mu(t) = a(t)\dot{b}(t)u_\mu \tag{6.2}$$

where $a(t)$ is the scale factor.

We consider a system of particles whose motion is governed by this mapping. This can be described by a distribution function $f(x, u, t)$, where $f(x, u, t)d^3x d^3u$ is the number of particles in the volume d^3x around the point x and having a value of u in an interval d^3u around u. We can think of equation (6.1) as a mapping in the phase space of the variables x and u. We can also see that Liouville theorem is true for this map. Using this we can obtain the equation for the time evolution of the distribution function f,

$$f(x, u, t) = f(x - b(t)u, u, t_0) \,. \tag{6.3}$$

We can also use equation (6.1) to obtain a differential equation for the evolution of the distribution function

$$\frac{\partial}{\partial b} f(x, u, b) + u_\mu \frac{\partial}{\partial x_\mu} f(x, u, b) = 0 \,, \tag{6.4}$$

where we use the growing mode b instead of time as the evolution parameter.

We are interested in the evolution of the statistical properties of an ensemble of such systems.

Every member of the ensemble initially has the particles uniformly distributed. Initially each particle can be labelled by its co-ordinate x_μ. The particles are given velocities $u_\mu(x)$. The velocity field is the gradient of a function $\psi(x)$ which for each system is a different realisation of a Gaussian random field. It is assumed that ψ is statistically homogeneous and isotropic. The statistical properties of the ensemble are initially fully specified by the two point correlation of ψ which is defined as $\phi(x) =< \psi(0)\psi(x) >$, where the angular brackets $<>$ denote ensemble averaging. It should be noted that this $\phi(x)$ is slightly different from the potential ϕ used in the previous chapters. The function used here is half the function used in the previous chapters,

The statistical quantity whose evolution we shall focus on in this chapter is the density two point correlation function $\xi(x,t)$.

6.3 The two point correlation using distribution functions

In this section we look at the evolution of the ensemble averaged two point distribution functions ρ_2. This is defined as

$$\rho_2(x^1, x^2, u^1, u^2, t) =< f(x^1, u^1, t) f(x^2, u^2, t) > . \tag{6.5}$$

From homogeneity and isotropy we can also say that

$$\rho_2(x^1, x^2, u^1, u^2, t) = \rho_2(x, u^1, u^2, t) \tag{6.6}$$

where

$$x_\mu = x^2_\mu - x^1_\mu. \tag{6.7}$$

The density two point correlation function is related to the zeroth moment of the two point distribution function with respect to u.

$$< \rho >^2 (1 + ((x,t)) = \int \rho_2(x, u', u^2, t) d^3u^1 d^3u^2. \tag{6.8}$$

In this chapter we normalise $< \rho >= 1$.

The initial two point distribution is a Gaussian in the velocities and hence specified by the covariance matrix

$$T^{ab}_{\mu\nu}(x) =< u^a_\mu u^b_\nu > (x) = \int u^a_\mu u^b_\nu \rho_2(x, u^1, u^2, t_0) d_3u_1 d^3u^2 \tag{6.9}$$

where a, b take values 1, 2. The initial two point distribution function then is the Gaussian distribution

$$\rho_2(x, u^1, u^2, t_0) = \frac{1}{(2\pi)^3} \exp\left[-\frac{1}{2} u^a_\mu u^b_\nu (T^{-1})^{ab}_{\mu\nu}(x)\right] , \tag{6.10}$$

where $\Delta T(x)$ is the determinant of the covariance matrix. In terms of the potential 4 we have

$$< u^1_\mu u^2_\nu >= -\partial_\mu \partial_\mu \phi(x) \tag{6.11}$$

and

$$< u^1_\mu u^1_\nu >= -\frac{1}{3}\nabla^2 \phi(0) \delta_{\mu\nu}. \tag{6.12}$$

We use equation *(6.3)* to obtain the time evolution of ρ_2

$$\rho_2(x, u^1, u^2, t) = \rho_2(x - (u^a - u^l) b(t), u^l, u^a, t_0). \tag{6.13}$$

This may also be written as

$$\rho(x, u^1, u^2, t) = \int \delta^3 \left[x' - (x - (u^2 - u^1) b(t))\right] \rho_2(x', u^1, u^2, t_0) d^3 x'. \tag{6.14}$$

Using the Fourier expansion of the Dirac delta function and using equation *(6.10)* we have

$$\begin{aligned} \rho(x, u^1, u^2, t) &= \int \left(\frac{1}{2\pi}\right)^3 \exp\left[ik_\mu \left(x'_\mu - x_\mu\right)\right] \exp\left[ik_\mu \left(u^2_\mu - u^1_\mu\right) b(t)\right] \\ &\times \frac{1}{(2\pi)^3 \sqrt{\Delta T(x')}} \exp\left[-\frac{1}{2} u^a_\mu u^b_\nu (T^{-1})^{ab}_{\mu\nu}(x')\right] d^3k d^3x' \end{aligned} \tag{6.15}$$

Using this in equation *(6.8)* and doing the u integrals we get-

$$1 + \xi(x, t) = \left(\frac{1}{2\pi}\right)^3 \int \exp\left[ik_\mu \left(x'_\mu - x_\mu\right)\right] \exp\left[-\frac{b^2(t)}{2} k_\mu k_\nu F_{\mu\nu}(\quad \right] d^3x' d^3k, \tag{6.16}$$

where

$$F_{\mu\nu}(x) = -\frac{2}{3}\nabla^2 \phi(0) \delta_{\mu\nu} + 2\partial_\mu \partial_\mu \phi(x) \tag{6.17}$$

Doing the k integral we obtain the two point correlation as

$$1 + \xi(x, t) = \frac{1}{(2\pi)^{\frac{3}{2}} b^3(t)} \int \frac{1}{\sqrt{\Delta F(x')}} \exp\left[-\frac{1}{2b^2(t)} \left(x - x_\mu\right) \left(x - x_\nu\right) F^{-1}_{\mu\nu}(x')\right] d^3x'. \tag{6.18}$$

Instead of integrating equation (6.16), if we do a Taylor expansion of

$$\exp\left[-\frac{b^2(t)}{2} k_\mu k_\nu F_{\mu\nu}(x')\right]$$

and then do the k and the x' integrals, we obtain

$$\begin{aligned} 1 + \xi(x, t) &= \sum_{n=0}^{\infty} \frac{b^{2n}}{n!} \partial_{\mu_1} \partial_{\nu_1} \ldots \partial_{\mu_n} \partial_{\mu_n} \left[\left(\partial_{\mu_1} \partial_{\nu_1} \phi(x) - \delta_{\mu_1 \nu_1} \frac{\nabla^2 \phi(0)}{}\right) \ldots\right. \\ &\left. \ldots \left(\partial_{\mu_n} \partial_{\nu_n} \phi(x) - \delta_{\mu_n \nu_n} \frac{\nabla^2 \phi(0)}{3}\right)\right]. \end{aligned} \tag{6.19}$$

Nowhere above has any assumption been made about the number of streams in the flow. Equation (6.18) obviously has the effects of multi-streaming built into it. Equation (6.19) is what one would obtain if one did a perturbative **expansion** of the distribution function and calculated the two point correlation function. Whether by doing the perturbative analysis this way (i.e. using distribution functions) we are able to include the effects of multi-streaming is what has to be checked.

6.4 The two point correlation using the hydrodynamic equations

In this section we shall work in the single stream approximation. We consider any one member of the ensemble described previously. Its evolution is described by equation (6.4). We take the zeroth moment of this equation with respect to u. Using the definitions

$$\rho(x,b) = m\int f(x,u,b)d^3u \tag{6.20}$$

and

$$\rho(x,b)v_\mu(x,b) = m\int u_\mu f(x,u,b)d^3u\,, \tag{6.21}$$

we have the continuity equation

$$\frac{\partial}{\partial b}\rho(x,b) + \partial_\mu(\rho(x,b)v_\mu(x,b)) = 0\,. \tag{6.22}$$

Next, taking the first moment of equation (6.4) and using equation (6.22) we have

$$\begin{aligned} &\rho(x,b)[\frac{\partial}{\partial b}v_\mu(x,b) + v_\nu(x,b)\partial_\nu v_\mu(x,b)] + \\ &+ m\int(v_\nu(x,b) - u_\nu)(v_\mu(x,b) - u_\mu)f(x,u,b)d^3u = 0\,. \end{aligned} \tag{6.23}$$

In the single stream approximation the last term in the above equation is dropped, and we have

$$\frac{\partial}{\partial b}v_\mu(x,b) + v_\nu(x,b)\partial_\nu v_\mu(x,b) = 0\,. \tag{6.24}$$

We shall use equations (6.22) and (6.24) to perturbatively evolve the density and velocity fields of the system and use it to calculate the two point correlation function.

Using equation (6.22) we can obtain an equation for the first derivative of the two point correlation function

$$\frac{\partial}{\partial b}[<\rho>^2(1+\xi(x,b))] = -<\partial^1_\mu(\rho(x^1)u_\mu(x^1))\rho(x^2)> - <\rho(x^1)\partial^2_\mu(\rho(x^2)u_\mu(x^2))>\,. \tag{6.25}$$

Using the normalisation $< p >= 1$, the above equation may be written as

$$\frac{\partial}{\partial b}\xi(x,b) = -\partial^{a_1}_{\mu_1} < \rho(1) u^{a_1}_{\mu_1} \rho(2) > . \tag{6.26}$$

We can'use equation *(6.22)* and *(6.24)* to obtain equations for the higher derivatives of the two point correlation

$$\frac{\partial^n}{\partial b^n}\xi(x,b)) = (-1)^n \partial^{a_1}_{\mu_1}\partial^{a_2}_{\mu_2}...\partial^{a_n}_{\mu_n} < \rho(1) u^{a_1}_{\mu_1} u^{a_2}_{\mu_2} ... u^{a_n}_{\mu_n} \rho(2) > . \tag{6.27}$$

Next we write the two point correlation function as a Taylor series in powers of the growing mode b

$$\xi(x,b) = \sum_{n=1}^{\infty} \frac{b^n}{n!}\frac{\partial^n}{\partial b^n}\xi(x,b)_{b=0} \tag{6.28}$$

and using equation *(6.27)* we get

$$\xi(x,b) = \sum_{n=1}^{\infty} \frac{b^n(-1)^n}{n!} \partial^{a_1}_{\mu_1}\partial^{a_2}_{\mu_2}\cdots\partial^{a_n}_{\mu_n} < \rho(1) u^{a_1}_{\mu_1} u^{a_2}_{\mu_2} ... u^{a_n}_{\mu_n} \rho(2) >_{b=0} .. \tag{6.29}$$

Then using the fact that the initial density is uniform, we have

$$\xi(x,b) = \sum_{n=1}^{\infty} \frac{b^n}{n!}(-1)^n \partial^{a_1}_{\mu_1}\partial^{a_2}_{\mu_2}...\partial^{a_n}_{\mu_n} < u^{a_1}_{\mu_1} u^{a_2}_{\mu_2} ... u^{a_n}_{\mu_n} >_{b=0} .. \tag{6.30}$$

Also the initial velocity field is Gaussian and hence all the odd terms in equation *(6.30)* are zero. We can then write this equation as

$$\xi(x,b) = \sum_{n=1}^{\infty} \frac{b^{2n}}{(2n)!} \partial^{a_1}_{\mu_1}\partial^{b_1}_{\nu_1}...\partial^{a_n}_{\mu_n}\partial^{b_n}_{\nu_n} < u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} ... u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} >_{b=0} .. \tag{6.31}$$

For a Gaussian field we have

$$< u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} u^{a_2}_{\mu_2} u^{b_2}_{\nu_2} ... u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} >= \sum_P < u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} >< u^{a_2}_{\mu_2} u^{b_2}_{\nu_2} > ... < u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} >, \tag{6.32}$$

where the sum is over all possible ways of pairing the u's.

Using this and the fact that the derivatives are symmetric in all the indices involved, we have

$$\begin{aligned} &\partial^{a_1}_{\mu_1}\partial^{b_1}_{\nu_1}\cdots\partial^{a_n}_{\mu_n}\partial^{b_n}_{\nu_n} < u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} ... u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} > \\ = &\frac{(2n)!}{n!2^n} \partial^{a_1}_{\mu_1}\partial^{b_1}_{\nu_1}...\partial^{a_n}_{\mu_n}\partial^{b_n}_{\nu_n} [< u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} > ... < u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} >] . \end{aligned} \tag{6.33}$$

This, when used in equation (6.31), give us

$$\xi(x,b) == \sum_{n=1}^{\infty} \frac{b^{2n}}{n!2^n} \partial^{a_1}_{\mu_1}\partial^{b_1}_{\nu_1}\partial^{a_2}_{\mu_2}\partial^{b_2}_{\nu_2}\cdots\partial^{a_n}_{\mu_n}\partial^{b_n}_{\nu_n} \left[< u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} >< u^{a_2}_{\mu_2} u^{b_2}_{\nu_2} > ... < u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} >\right] . \tag{6.34}$$

Summing the superscripts $a_1, b_1, ...a_n, b_n$ over the values 1 and 2 and using

$$< u^a_\mu u^b_\nu > = \frac{-\nabla^2\phi(0)}{3}\delta_{\mu\nu} \text{ if } a = b \tag{6.35}$$

and

$$< u^a_\mu u^b_\nu > = \partial^a_\mu \partial^b_\nu \phi(x) \text{ if } a \neq b \tag{6.36}$$

we have

$$\begin{aligned} 1 + \xi(x,t) &= \sum_{n=0}^{\infty} \frac{b^{2n}}{n!} \partial_{\mu_1}\partial_{\nu_1}...\partial_{\mu_n}\partial_{\mu_n} \left[\left(\partial_{\mu_1}\partial_{\nu_1}\phi(x) - \delta_{\mu_1\nu_1}\frac{\nabla^2\phi(0)}{3} \right) ... \right. \\ & \left. ... \left(\partial_{\mu_n}\partial_{\nu_n}\phi(x) - \delta_{\mu_n\nu_n}\frac{\nabla^2\phi(0)}{3} \right) \right] . \end{aligned} \tag{6.37}$$

This is the same as equation *(6.19)* which was obtained using distribution functions. Thus we see that the perturbative calculation of the two point correlation function using distribution functions is equivalent to doing it using the pressure free HD equations. It therefore has no effects of multi-streaming and hence we reach the conclusion that it is not possible to perturbatively follow the transition from a single streamed flow to multi-streamed flow.

6.5 The pair velocity and its dispersion.

The Zel'dovich approximation can be used to calculate the pair velocity which is the mean relative peculiar velocity at some separation x,. The pair velocity is

$$v_\mu(x,t) = a\dot{b}\frac{\int (u^2_\mu - u^1_\mu)\rho_2(x,u^1,u^2,t)d^3u^1d^3u^2}{\int \rho(x,u^1,u^2,t)d^3u^1d^3u^2} , \tag{6.38}$$

which is related to the first moment of the two point distribution function. Using equation *(6.15)* we obtain for the first moment of the distribution function

$$\int (u^2_\alpha - u^1_\alpha)\rho_2(x,u^1,u^2,t)d^3u^1d^3u^2 = \frac{1}{b(t)}\int (x_\alpha - x'_\alpha)g(x,x',b)d^3x' \tag{6.39}$$

where

$$g(x,x',b) = \frac{1}{(2\pi)^{\frac{3}{2}}\, b^3(t)}\frac{1}{\sqrt{\Delta F(x')}} \exp\left[-\frac{1}{2b^2(t)} \left(x'_\mu - x_\mu\right)\left(x'_\nu - x_\nu\right) F^{-1}_{\mu\nu}(x') \right] , \tag{6.40}$$

and

$$\int g(x,x',t)d^3x' = 1 + \xi(x,t) . \tag{6.41}$$

For $\Omega = 1$ this gives for the radial component (the only non-zero component) of the pair velocity

$$v(x,t) = \dot{a}x(1 - A(x,t)) \tag{6.42}$$

where

$$A(x,t) = \frac{\int g(x,x',t)x'd^3x'}{x\int g(x,x',t)d^3x'} \quad . \tag{6.43}$$

We also obtain for the dimensionless pair velocity

$$h(x,t) = \frac{-v}{\dot{a}x} = A(x,t) - 1\,. \tag{6.44}$$

We next use the Zel'dovich approximation to study the ensemble average of the dispersion of the relative peculiar velocity between two points. This dispersion has a contribution from the spread of velocities across the different realisations. In addition to this , in the multi-streamed epoch it also has a contribution from the velocity spread at a single point in any individual realisation in the ensemble. The dispersion of the relative peculiar velocity is defined as

$$< v_\mu v_\nu > (x,t) = (a\dot{b})^2 \frac{\int(u^2_\mu - u^1_\mu)(u^2_\nu - u^1_\nu)\rho_2(x,u^1,u^2,t)d^3u^1d^3u^2}{\int \rho_2(x,u^1,u^2,t)d^3u^1d^3u^2} \tag{6.45}$$

This is a symmetric tensor and since all statistical quantities are homogenous and isotropic we can write this as

$$< v_\mu v_\nu > (x,t) = \delta_{\mu\nu}B'(x,t) + \frac{x_\mu x_\nu}{x^2}C'(x,t) \tag{6.46}$$

where the $B'(x,t)$ is the dispersion in the component of the relative peculiar velocity in any direction perpendicular to $\vec{x}$ and $B'(x,t) + C'(x,t)$ is the dispersion in the component parallel to $\vec{x}$. We also define a dimensionless peculiar velocity dispersion tensor

$$e_{\mu\nu}(x,t) = \frac{< v_\mu v_\nu > (x,t)}{(\dot{a}x)^2} = \delta_{\mu\nu}B(x,t) + \frac{x_\mu x_\nu}{x^2}C(x,t). \tag{6.47}$$

Using equation *(6.15)* for the distribution function we have for its second moment

$$\int(u^2_\alpha - u^1_\alpha)(u^2_\beta - u^1_\beta)\rho_2(x,u^1,u^2,t)d^3u^1d^3u^2 = \frac{1}{b^2}\int(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)g(x,x',t)d^3x' \,. \tag{6.48}$$

Using this for $\Omega = 1$ we can write for the two components of the dimensionless velocity dispersion tensor

$$B(x,t) = \left(\int \frac{1}{2}\left[x'^2 - \frac{(\vec{x}\cdot\vec{x}')^2}{x^2}\right] g(x,x',t)d^3x'\right) \Big/ \left(x^2 \int g(x,x',t)d^3\right) \tag{6.49}$$

and

$$C(x,t) = 1 - 2A(x,t) + D(x,t) - B(x,t) \tag{6.50}$$

where

$$D(x,t) = \left(\int(\vec{x}\cdot\vec{x}')^2 g(x,x',t)d^3x'\right) \Big/ \left(x^4 \int g(x,x',t)d^3\right) \tag{6.51}$$

6.6 The two point correlation at small separations.

Here we consider the evolution of the two point correlation function and the pair velocity using the formulas derived for the Zel'dovich approximation. We consider a power spectrum $P(k) = .5k^2e^{-k}$ which has a power law form k^2 at small k with an exponential cut-off at large k. This cut-off introduces a length scale in the initial conditions. We first consider the evolution of the statistical quantities at scales much smaller than the scale introduced by the cut-off. Figure 6.1 shows the initial two point correlation function $\xi^{(1)}(x)$ and the initial dimensionless pair velocity $h^{(1)}(x)$ at small x. These quantities are defined so that in the linear regime

$$\xi^{(1)}(x,t) = \left(\frac{a(t)}{a(t_0)}\right)^2 \xi^{(1)}(x) \tag{6.52}$$

and

$$h^{(1)}(x,t) = \left(\frac{a(t)}{a(t_0)}\right)^2 h^{(1)}(x)\,. \tag{6.53}$$

We first look at the evolution of the two point correlation function and compare the results from ZA with the linear evolution and the evolution taking into account the lowest order non-linear effects in gravitational dynamics (GD) discussed in the preceding two chapters (i.e. IV and V). Figure 6.2 shows the two point correlation functibn as a function of the scale factor for all these three cases for different small spatial separations.

The first thing to notice is that ZA, GD and the linear evolution start differing from one another at nearly the same epoch. The two point correlation calculated using ZA grows faster than the result obtained from the lowest order non-linear GD. This could be a result of the fact that we are considering only the lowest order of non-linearity in an epoch when all the terms in the perturbative series may be comparable and it is possible that if we take them into account GD may predict a faster growth. The linear two point correlation grows the slowest of all. We find that after initially increasing, the two point correlation function calculated using ZA reaches a maximum value and then decreases. This corresponds to the structures at that particular scale getting washed out after particle crossing. We find that the maximum value reached by the two point correlation function has a significant variation over the small range of separations that we consider here. We find that at smaller separations the two point correlation function reaches a higher value compared to larger separations. We nest consider the evolution of the dimensionless pair velocity. We see that the behaviour of this is quite similar to that of the two point correlation function. An important thing to notice is the fact that at the smallest scales that we have considered the dimensionless pair velocity reaches the value 1 before it starts decreasing. When $h(x,t)$ has the value

Figure 6.1: The linear two point correlation and the linear pair velocity at small separations for the initial power spectrum $P(k) = .5ke^{-k^2}$.

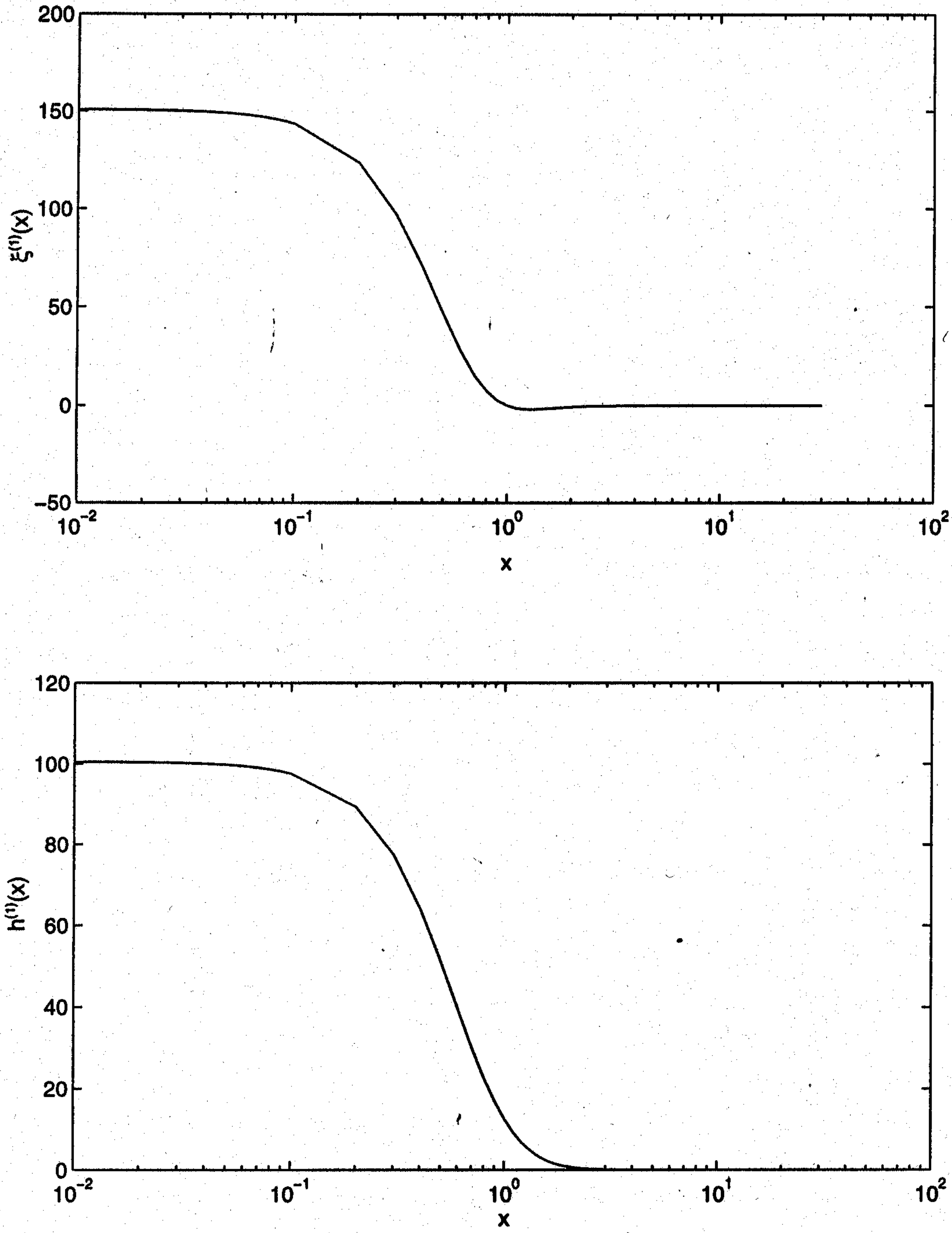
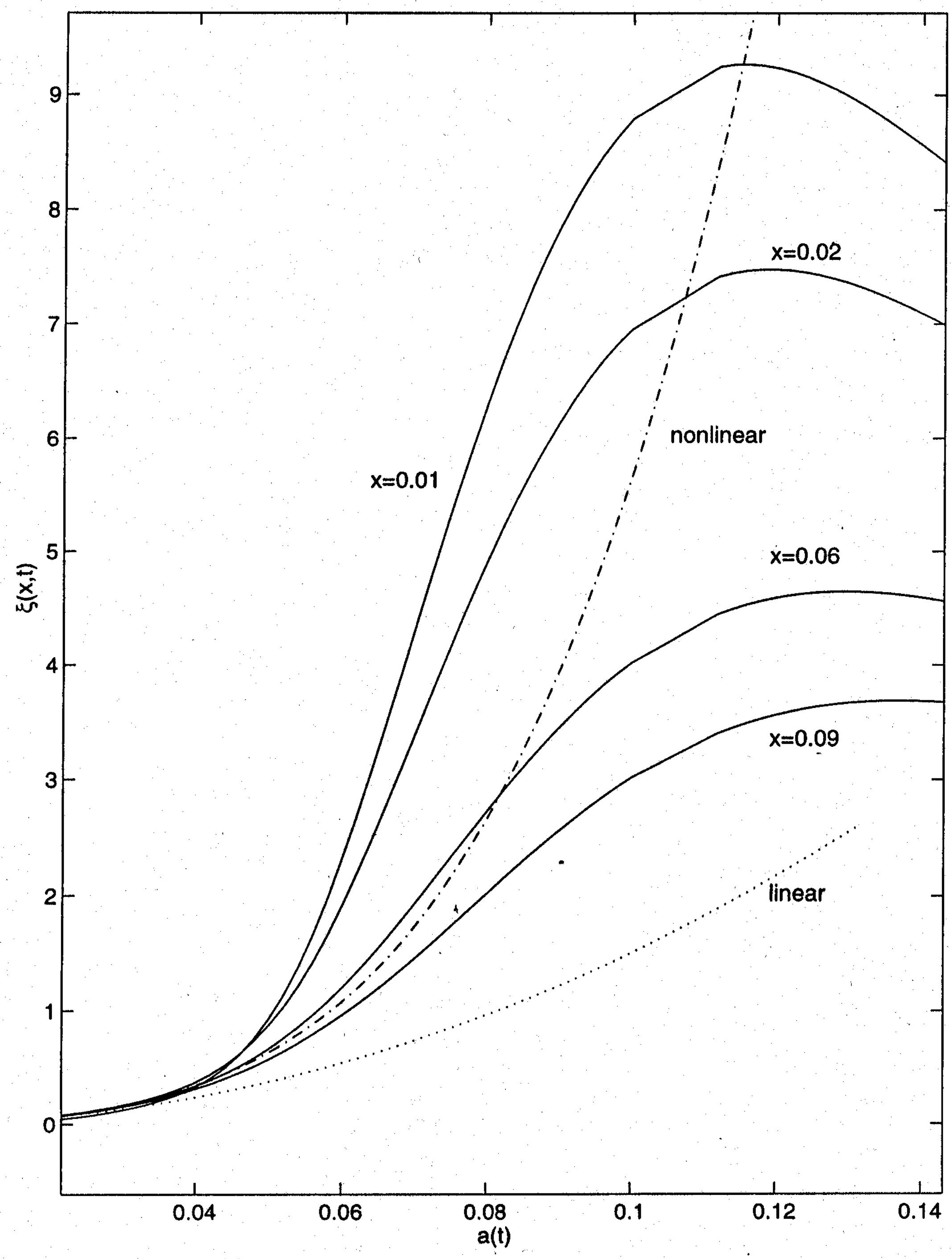


Figure 6.2: This shows the two point correlation function as a function of the scale factor for different small separations. The solid curves show the evolution in ZA. The dotted curve shows the prediction of linear theory for the separation $x = .01$ and the dashed curve shows the evolution at the same separation if we include the effects of lowest order non-linear GD.



one at a particular separation, in the proper (non expanding) co-ordinate system the mean relative peculiar velocity of the particles at this separation just balances the Hubble expansion between the two ends. In other words, in the proper co-ordinate system the particles which were initially moving away from each other are now at rest with respect to each other (on the average at a particular separation). This epoch is often referred to as the epoch when a certain scale turns around. We find that at small scales we can follow the pair velocity until turn around and beyond using ZA. At slightly larger scales we find that the pair velocity does not reach the value 1 but starts to fall off at a much smaller value. This is because at these separations not only are there particles which are moving towards one another but there are particles which have already undergone multi-streaming and are moving apart. It is the latter which causes the pair velocity to decrease. The results from ZA are quantitatively incorrect during this phase of the evolution. This is because in ZA the particles keep on moving away from one another after they have crossed, whereas in reality they form bound objects. This aspect of gravitational instability which is missed out by ZA is incorporated in the adhesion model (Gurbatov, Saichev and Shandarin, 1989) which makes the particles stick together by introducing an artificial viscosity. Another interesting epoch is the epoch when the pair velocity crosses zero. When the dimensionless pair velocity is positive it means that the particles at that separation are approaching each other (in the comoving co-ordinate system). Once the pair velocity becomes negative it means that at that particular separation the particles are on the average moving away from each other. This is when the structures at that scale start getting washed out and we see that the epoch when the pair velocity becomes negative corresponds to the epoch when the two point correlation at that scale starts to decrease. Thus is the epoch when the Zel'dovich approximation breaks down in that its predictions no longer tell us anything about the evolution under gravitational dynamics.

In figure 6.4 we show $h(x,t)$ as a function of $\bar{\xi}(x,t)$ for different separations. At the small scales considered here the quantities $\bar{\xi}(x,t)$ and $\xi(x,t)$ are nearly equal. The scaling relations discussed in the previous chapter assume that there is a unique relation between these two quantities. We find that in the ZA all the curves do not coincide but they do show a similar behaviour. We see that initially for small values of $\bar{\xi}(x,t)$ the various curves follow the prediction of linear theory $h = (2/3)\bar{\xi}(x,t)$. For larger values we find that the curves from ZA are steeper than the this. We also find that the curves are steeper at smaller scales.. At even large values of $\bar{\xi}(x,t)$ we find that the various curves flatten off and then start to fall off. Although the behaviour is similar for the different scales considered, we find that the curves do not coincide and they have a distinct spread. In contrast to this, the investigation based on gravitational N-body simulations indicate that $h(x,t)$ and $\bar{\xi}(x,t)$ are

Figure 6.3: This shows the dimensionless pair velocity as a function of the scale factor for different small separations. The solid curves show the evolution in ZA. The dotted curve shows the prediction of linear theory for the separation $x = .01$ and the dashed curve shows the evolution at the same separation if we include the effects of lowest order non-linear GD.

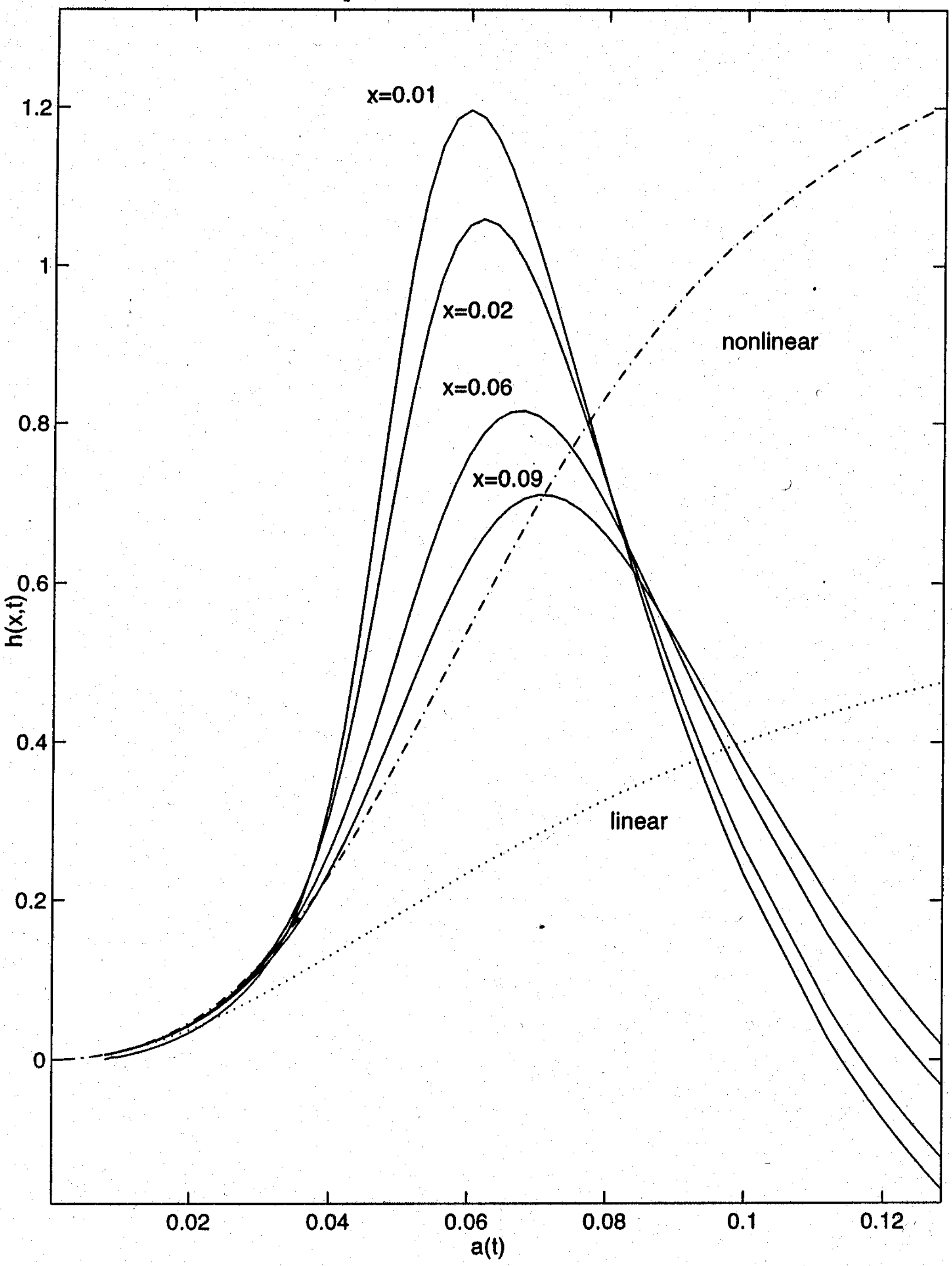
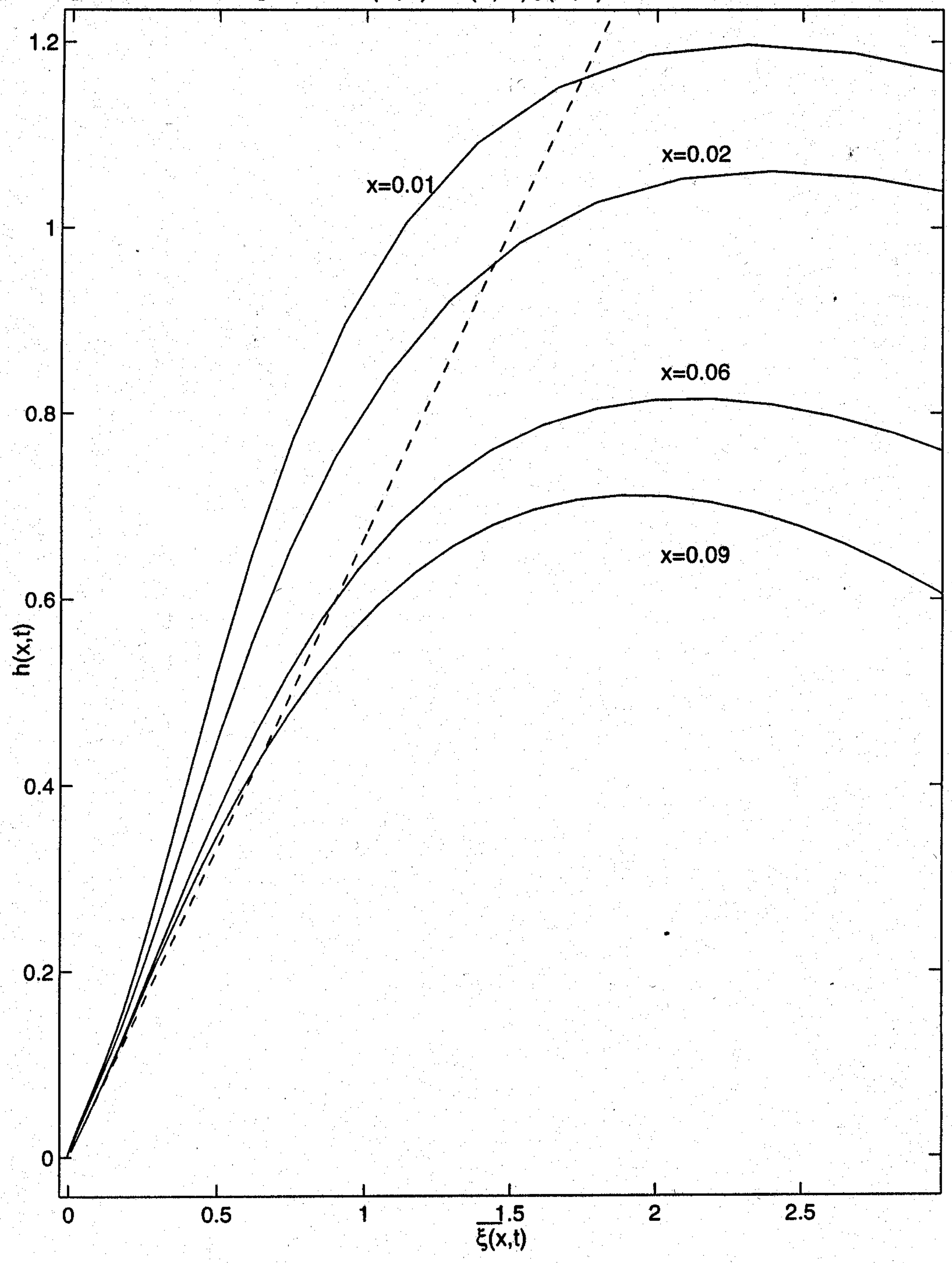


Figure 6.4: This shows the dimensionless pair velocity $h(x,t)$ as a function of $\bar{\xi}(x,t)$ for different small separations. The solid curves show the evolution in ZA.. The dashed curve corresponds to the equation $h(x,t) = (2/3)\bar{\xi}(x,t)$



related as $h(x,t) = (2/3)\bar{\xi}(x,t)$ until $h(x,t) \sim 1$ (Hamilton et.al 1991).

We next consider the evolution of the various statisttical quantities at intermediate separations which are comparable to the scale introduced by the cut-off. We first consider the evolution of the two point correlation function (figure 6.5). We see that at the separations $x = .4$ and $x = .7$ the behaviour is somewhat different from the behaviour at the small scales. We find that unlike the small separations where the non-linear correction increases the growth of the two point correlation function, here the effect of the lowest order non-linearity from GD is to slow down the growth of the two point correlation function relative to the linear evolution. We also see that at the separation $x = 0.4$ the evolution of $\xi(x,t)$ in ZA is faster than that predicted by linear theory where as at $x = 0.7$ it is slower. At the separation $x = 1$ the initial two point correlation function is zero and it remains so according to linear theory. The effect of the lowest order non-linearity from GD is to generate a positive correlation. ZA has the same effect but it grows slower compared to the lowest order non-linear GD. At the separation $x = 1.4$ the initial two point correlation function has a negative value which keeps on decreasing (i.e. increasing in magnitude). The effect of the non-linear correction in GD is to slow down this decrease (i.e. the growth of the magnitude slows down) and make the two point correlation function positive. The evolution of the two point correlation function in ZA is quite similar. In ZA the two point correlation function reaches a maximum value and then starts to decrease again. We next consider the evolution of $\bar{\xi}(x,t)$ (figure 6.6). We find that the evolution of this quantity in ZA is very similar to its evolution when we take into account the lowest order non-linear effects from GD. We find that when ZA predicts an evolution faster than linear theory, so does non-linear GD and when ZA predicts an evolution slower than linear theory non-linear GD predicts the same behaviour. The behaviour of $h(x,t)$ (figure 6.7) is very similar to the behaviour of $\bar{\xi}(x,t)$.

We have also considered the evolution of $h(x,t)$ as a function of $\bar{\xi}(x,t)$ at the different intermediate separations (figure 6.8). We find that in all the cases the curves obtained using ZA lie below the prediction $h(x,t) = (2/3)\bar{\xi}(x,t)$ (in contrast to the behaviour at small separations). We also find that although the curves at the various intermediate scales look very similar they are actually different. This is most obvious if we look at the value of $\bar{\xi}(x,t)$ at which $h(x,t)$ reaches its maximum value and starts to decrease. We find that this exhibits a distinct scale dependence and this value decreases as we go to larger separations.

6.7 The two point correlation at large separations.

In this section we investigate the evolution of the two point correlation function in the regime where it can be studied perturbatively and we look at the behaviour at large separations.

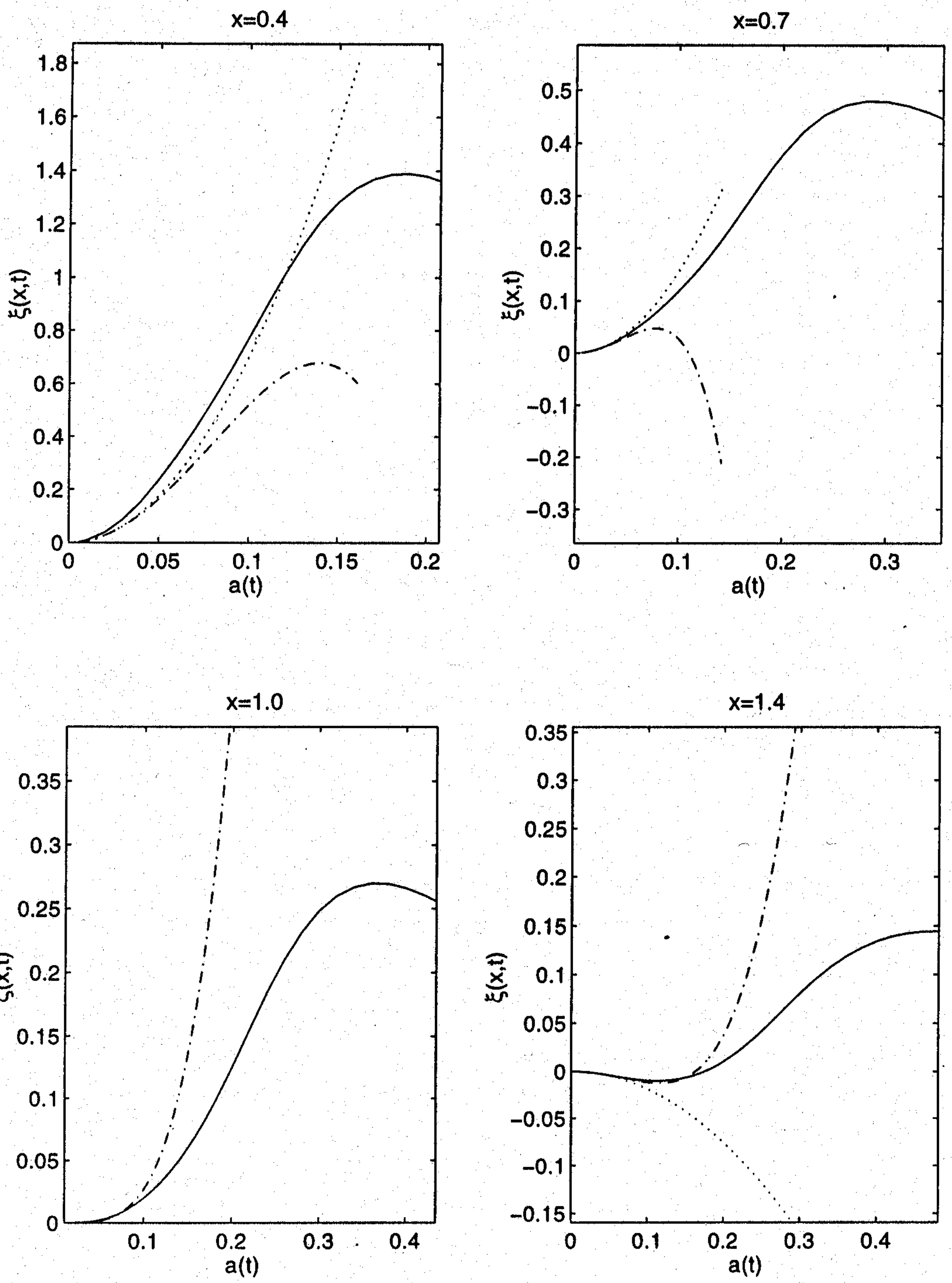


Figure 6.5: This shows the two point correlation function as a function of the scale factor for different intermediate separations. The solid curves show the evolution in ZA. The dotted curve shows the prediction of linear theory and the dashed curve shows the evolution if we include the effects of lowest order non-linear GD.

Figure 6.6: This shows the **average** of the two point correlation function $\bar{\xi}(x,t)$ as a function of the scale factor for different intermediate separations. The solid curves show the evolution in ZA. The dotted curve shows the prediction of linear theory and the dashed curve shows the evolution if we include the effects of lowest order non-linear GD.

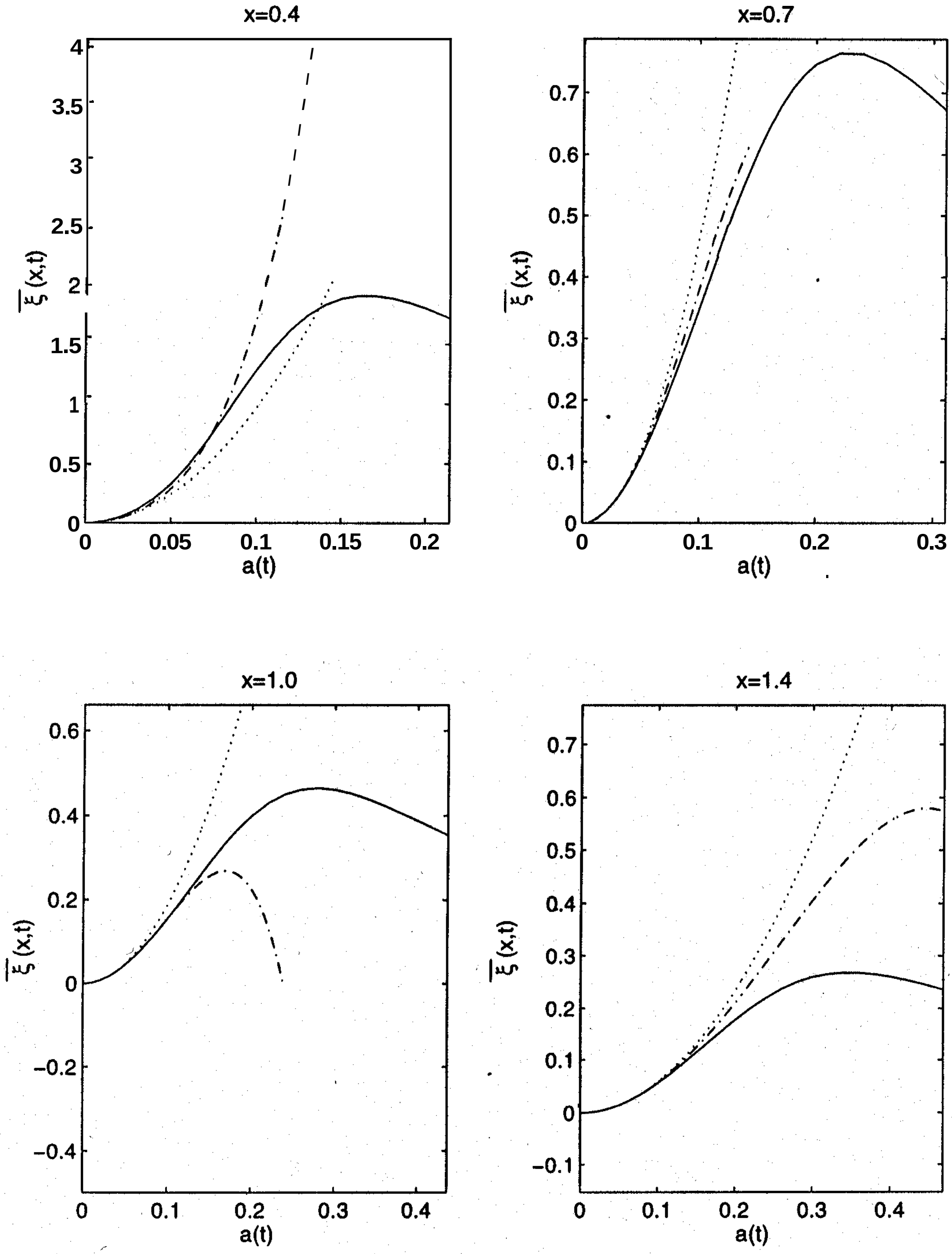


Figure 6.7: This shows the dimensionless pair velocity as a function of the scale factor for different small separations. The solid curves show the evolution in ZA. The dotted curve shows the prediction of linear theory and the dashed curve shows the evolution if we include the effects of lowest order non-linear GD.

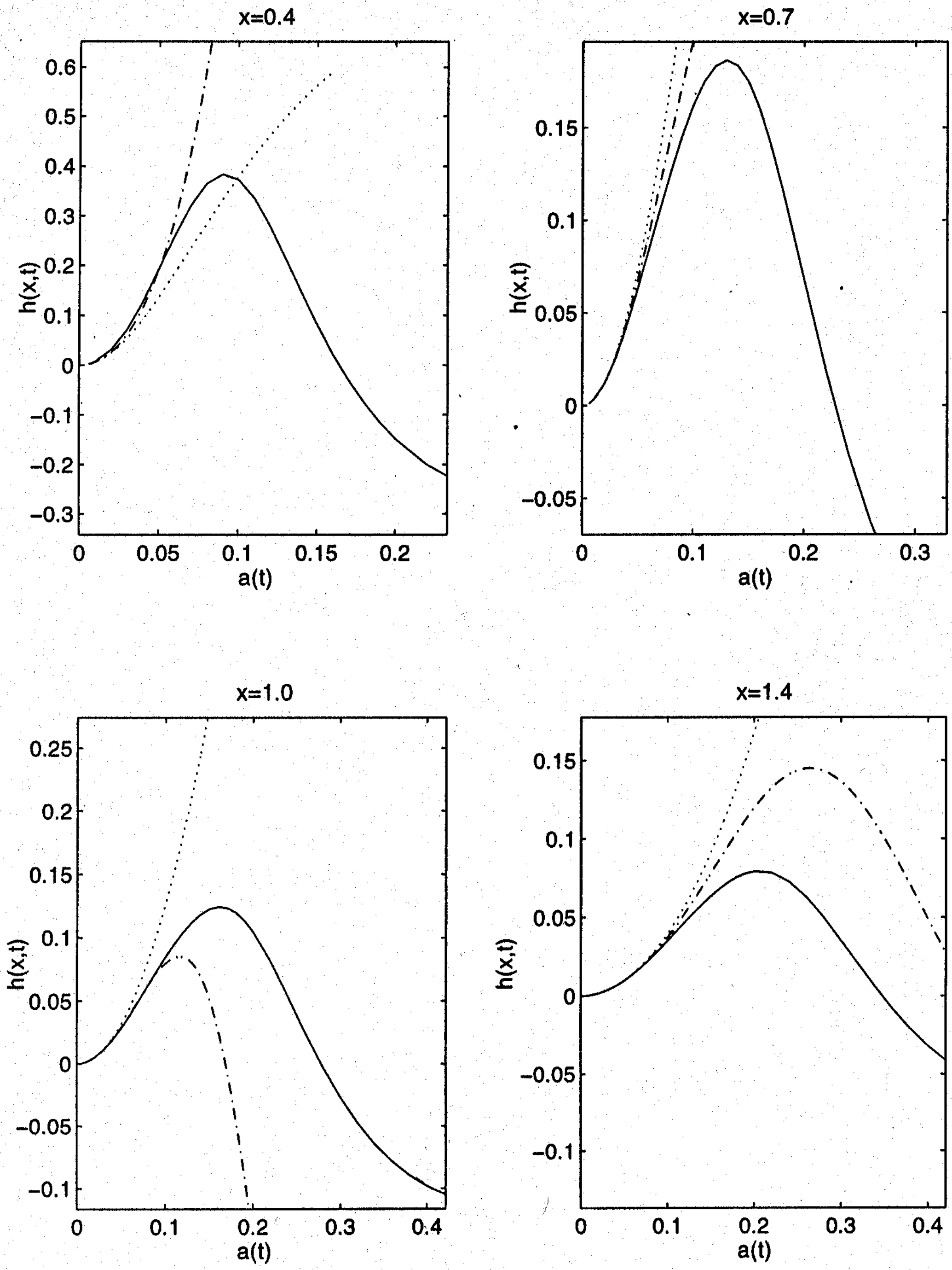
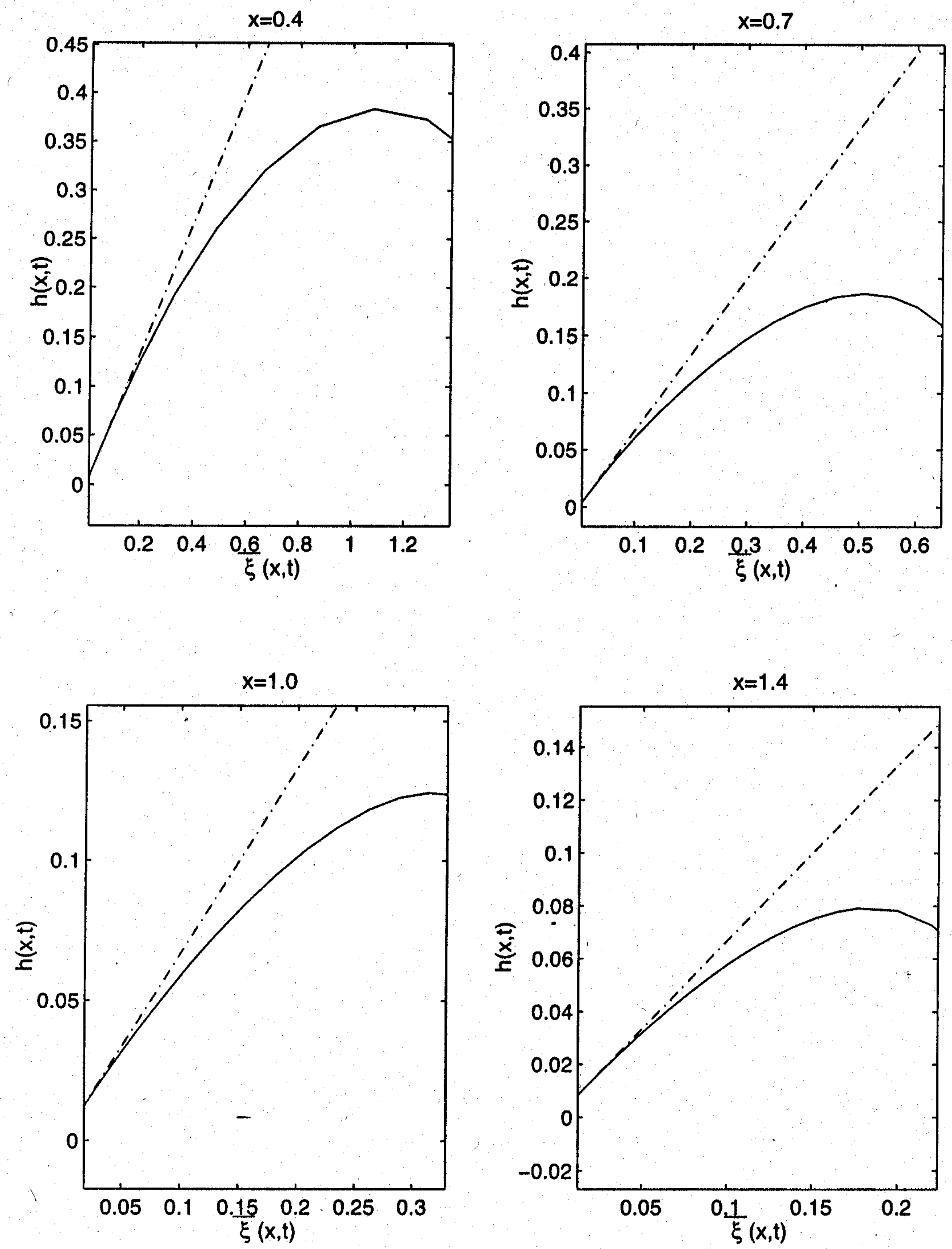


Figure 6.8: This shows the dimensionless pair velocity $h(x,t)$ as a function of $\bar{\xi}(x,t)$ for differebt separations. The solid curves show the evolution in ZA.. The dashed curve corresponds to the equation $h(x,t) = (2/3)\bar{\xi}(x,t)$



The initial conditions for the evolution of the cosmological correlations may be expressed in terms of the potential $\phi(x)$ or equivalently in terms of the matter two point correlation in the linear epoch, $\xi^{(1)}(x,t)$. The two are related by the equation

$$\xi^{(1)}(x,t) = b^2(t)V^4)(x). \tag{6.54}$$

Usually the initial conditions are given in terms of the matter two point correlation $\xi^{(1)}(x,t)$ or its Fourier transform $b^2(t)P_l(k)$ which is the power spectrum. One then has to invert equation *(6.54)* to obtain the potential $\phi(x)$ and its derivatives. In doing so one has the freedom in choosing boundary conditions and the effect of changing the boundary condition is

$$\nabla^2\phi(x) \rightarrow \nabla^2\phi(x) + C_1 \tag{6.55}$$

and

$$\phi(x) \rightarrow \phi(x) + \frac{C_1 x^2}{6} + C_2\,. \tag{6.56}$$

Equation *(6.19)* for the two point correlation function is invariant under these transformations and we are free to choose any boundary condition. For initial conditions where the integral $\int_o^\infty \xi^{(1)}(x)xdx$ (or $\int_o^\infty P_1(k)dk$) is finite we can choose the boundary condition $\text{limit}_{x\rightarrow\infty}\nabla^2\phi(x) = 0$. We then have

$$< u^2 >= -\nabla^2\phi(0) = \int_o^\infty \xi^{(1)}(x)xdx\,. \tag{6.57}$$

In addition, if at large x the function $\partial_\mu\partial_\nu\phi(x)$ is monotonically decreasing and $\partial_\mu\partial_\nu\phi(x) \ll (\delta_{\mu\nu}/3)\nabla^2\phi(0)$, we can then neglect all but one of the $\partial_\mu\partial_\nu\phi(x)$ terms that appear in equation *(6.19)*. 'For initial conditions where the power spectrum has the form $P(k) \propto k^n$ at small k and if it has a cut-off at large k, the conditions discussed above are satisfied for $n > -1$. For these cases we obtain for the two point correlation function at large x

$$\xi(x,t) = \sum_{n=0}^{\infty} \frac{b^{2(n+1)}}{n!}\left(\frac{-\nabla^2\phi(0)}{3}\right)^n \left(\nabla^2\right)^n \nabla^4\phi(x) \tag{6.58}$$

Using this we obtain the lowest order non-linear correction to the two point correlation function at large x,

$$\xi^{(2)}(x,t) = \frac{b^2}{3} < u^2 > \nabla^2\xi^{(1)}(x,t) \tag{6.59}$$

In chapter V we have considered the same quantity using GD and we found that for $0 < n \leq 3$ at large x the results can be fitted by the formula

$$\xi^{(2)}(x,t) = .194b^2 < u^2 > \nabla^2\xi^{(1)}(x,t) \tag{6.60}$$

We find that the two equations are very similar and they differ only in the numerical coefficient. In chapter V we also interpret equation *(6.60)* in terms of a simple heuristic model based on a diffusion process. We consider a particular member of the ensemble and look at the evolution of the density in volume elements located at a separation x from each other. We assume that the density in each volume element grows according to linear theory and the volume elements get rearranged randomly on small scales because of their peculiar velocities. Based on this model we obtained an equation identical to equation *(6.59)*. Thus we see that this model gives an exact description of what happens in ZA at large scales in the regime when the perturbative treatment is valid. In ZA, like in our heuristic model, the velocity of the particles is fixed whereas in GD the particle velocity changes as evolution proceeds. We believe that this is responsible for the smaller diffusion coefficient for GD compared to ZA.

Going back to equation *(6.68)* and writing it in Fourier space we obtain for the power spectrum

$$P(k,t) = \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-b^2k^2 < u^2 >}{3}\right)^n\right] b^2 P_1(k), \tag{6.61}$$

Summing up the terms in the square brackets we have

$$P(k,t) = \exp\left(\frac{-b^2k^2 < u^2 >}{3}\right) b^2 P_1(k). \tag{6.62}$$

which in real space gives us

$$\xi(x,t) = \frac{1}{(\sqrt{\pi}2L(t))^3}\int_o^{\infty} \exp\left[-\frac{(x-x')'}{4L(t))^2}\right] \xi^{(1)}(x',t.)d^3x', \tag{6.63}$$

where

$$L^2(t) = \frac{1}{3}b^2(t) < u^2 > . \tag{6.64}$$

The length scale $L(t)$ is the r.m.s. deviation of the particles from their Lagrangian (or initial) positions at any time t. We see that the non-linear evolution of the two point correlation function at large x corresponds to a convolution of the linear two point correlation with a Gaussian whose width is proportional to $L(t)$. This is consistent with our interpretation of the evolution in terms of a diffusion process..

For the case when the initial power spectrum has the form

$$P_1(k) = Ae^{-\alpha^2k^2}k^n, \tag{6.65}$$

using equation (6.58) at small k, we have for the non-linear power spectrum at small k

$$P(k,t) = Ae^{-(\alpha^2+L^2(t))k^2}b^2k^n, . \tag{6.66}$$

Using equation *(6.65)* and (6,66), and using the fact that

$$\int_{-\infty}^{\infty} e^{ikx}e^{-\beta^2\alpha^2k^2}k^n d^3k = \frac{1}{\beta^{3+n}}\int_{-\infty}^{\infty} e^{ik\frac{x}{\beta}}e^{-\alpha^2k^2}k^n d^3k \tag{6.67}$$

we obtain for the non-linear two point correlation function at large x

$$\xi_1(x,t) = \left[1 + \left(\frac{L(t)}{\alpha}\right)^2\right]^{-\frac{3+n}{2}} \xi_2^{(1)}\left(\frac{x}{\sqrt{1+\left(\frac{L(t)}{\alpha}\right)^2}}, t\right). \quad (6.68)$$

This formula relates the non-linear two point correlation at some separation x at a time t to the linear two point correlation at a smaller separation at the same time. Thus, at large x, for small values of the two point correlation, we have information being transferred out from the smaller scales to the larger scales.

We next numerically investigate the evolution of the two point correlation function at large separations for the initial power spectrum $P_1(k) = .5e^{-k^2}k$. Figure 6.9 shows the function $\xi^{(1)}(x)$ as a function of x for large separations.. This function multiplied by the square of the scale factor gives the linear two point correlation $\xi^{(1)}(x,t)$. At large x the function $\xi^{(1)}(x)$ has a negative sign and a power law behaviour x^{-4}. We investigate the evolution of the two point correlation function at the large separation $x = 10$. We do this using four different approximations which we list below:

(a). linear perturbation theory

(b). linear theory + the lowest order non-linear correction using GD (chapter V).

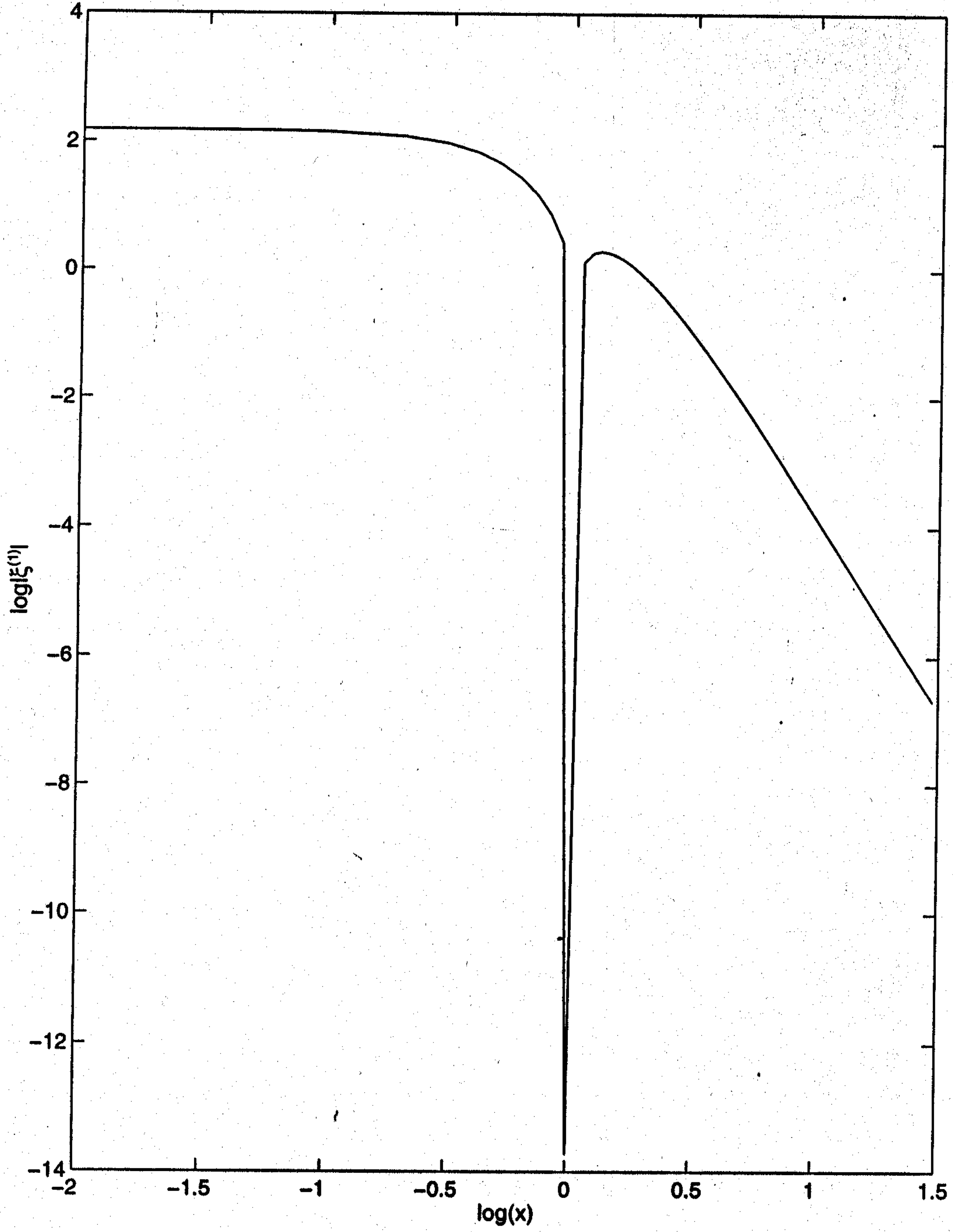
(c). the result obtained from summing the whole perturbation series for the ZA with the extra assumptions about the evolution at large separations made in this section i.e. equation (6.68)

(d). the non-perturbative two point correlation calculated using ZA (6.18). This exercise allows us to investigate two different issues. The first thing that we can check is how well ZA approximates GD. This can be done by comparing (b) with (c) and (d), In this section we have made some assumptions about the large x behaviour of the two point correlation function and arrived at the diffusion picture for the evolution. We can put these assumptions to test by comparing (c) with (d). The results are shown in figure 6.10. We find that all the four approximations match in the early stages of,the evolution. The two point correlation at this separation is initially negative and this value evolves according to linear theory where it gets multiplied by b^2. The different approximations start to differ as the evolution proceeds. The first thing to note is that they start to differ much before $\xi(x,t) \sim 1$ when one would naively expect the perturbation series to break down. This is a consequence of the non-local nature of the non-linear terms for the two point correlation. As discussed in chapter V, this can be understood using equation (6.57)

$$< u^2 > = \int_0^\infty \xi^{(1)}(x) x dx$$

which shows that the non-linear correction depends on the linear two point correlation condition at all the scales and the major contribution to this integral comes from the small scales.

Figure 6.9: The initial two point correlation as a function of the separation at large scales for the power spectrum $P(k) = .5e^{-k^2}k$.



The small scales become strongly non-linear very early in the evolution and it is because of this that the non-linear term starts contributing at large x even when $\xi(x,t) \ll 1$. In all the approximations (i.e. (b),(c) and (d)) the effect of the initial deviation from the linear theory is to make the growth rate faster than $b^2(t)$. In the initial stages approximations (b), (c) and (d) exhibit qualitatively similar behaviour but as the evolution proceeds we find that (d) starts showing a behaviour completely different from (b) and (c). We find that the rapidly decreasing function (d) slows down its decrease and then starts to increase, This is quite different from the behaviour of (b) and (c) which continue to decrease. This difference is because of the effects of multi-streaming. In ZA the correlations get washed but after multi-streaming occurs. Until the onset of multi-streaming the diffusion picture (c) matches quite well with the full ZA i.e. (d). A comparison of (b),(c) and (d) shows that ZA qualitatively predicts the same behaviour as GD and the quantitative difference may be attributed to the difference in the diffusion coefficients. In the case of the actual gravitational dynamics (non-perturbative) we expect that the results may be different because there the particles will get 'stuck' in bound objects once multi-streaming occurs (e.g. the adhesion model; Gurbatov, Saichev and Shandarin 1989) As a result of this the mean square displacement of the particles will be much less than in ZA or in perturbative GD. Although we expect this diffusion picture to hold for the actual evolution of the two point correlation function at large x, the perturbative treatment of GD and also calculations using ZA may overestimate what would be obtained in N-body simulations. Incidentally, the regime treated here would be difficult to study using such simulations since it involves the low amplitude tail of the two point correlation function which would be limited by the size of the box and it would require a large dynamical range.

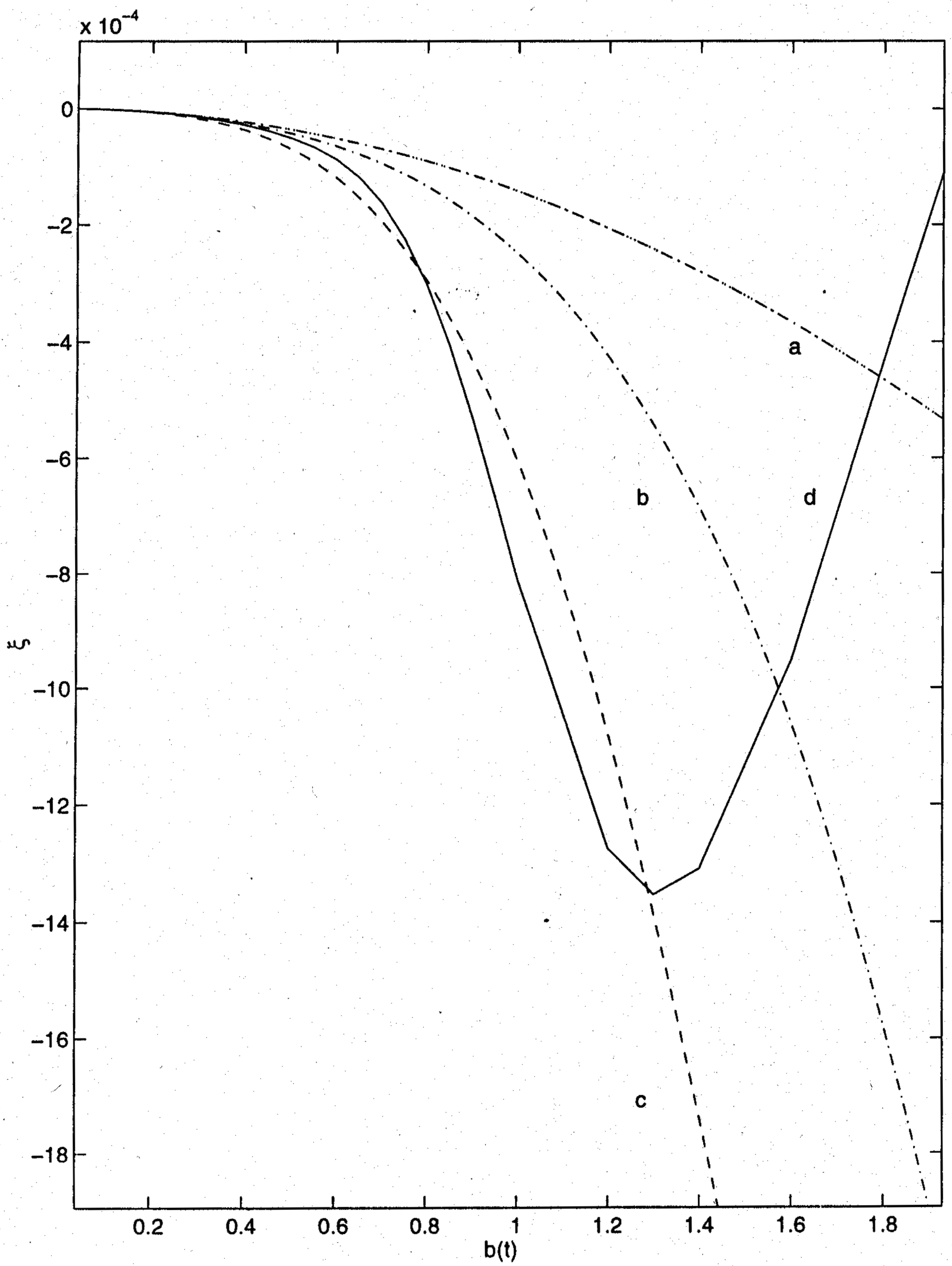
6.8 The 3 point correlation function.

We use ZA to follow the evolution of the N point correlation function. It is possible to do this non-perturbatively by following a line of reasoning very similar to that in section **3.** However since ZA is a good substitute for the gravitational dynamics only in the weakly non-linear regime we prefer to carry out the investigation perturbatively.

We first consider the evolution of the ensemble averaged N point distribution function $\rho_N(x^a, u^a, t)$. This is a generalisation of the ensemble averaged two point distribution function introduced in section **3** and the superscript a refers to the various points i.e., 1,2... N in phase space which are arguments of this function. Using equation (6.3) we obtain for the time evolution of this function

$$\rho_{\rm N}(x^a, u^a, t) = \rho_{\rm N}(x^a - b(t)u^a, u^a, t_0)\,. \tag{6.69}$$

Figure 6.10: The two point correlation at a fixed separation x = 10 as a function of the growing mode $b(t)$ for (a) linear theory, (b) linear theory + lowest order non-linear correction using GD (c) non-linear evolution using ZA and the assumptions made in section 5 about the large x behaviour, and (d) non-perturbative ZA



Expanding this in a perturbative series and using $a_1, a_2 \dots a_n$ for n indices that independently take values between 1 and N, and using $\mu_1, \mu_2 \dots \mu_n$ for n corresponding Cartesian components, we have

$$\rho_N(x^a, u^a, t) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} u^{a_1}_{\mu_1} u^{a_2}_{\mu_2} \dots u^{a_n}_{\mu_n} \partial^{a_1}_{\mu_1} \partial^{a_2}_{\mu_2} \dots \partial^{a_n}_{\mu_n} \rho_N(x^a, u^a, t_0) . \tag{6.70}$$

For both the kinds of indices the Einstein summation convention holds and all the a^is have to summed over the range 1 to N whenever they appear twice and the μ_is have to be summed over the three Cartesian components whenever the indices are repeated.

To calculate the N point correlation function we take velocity moments of the N point distribution function

$$\int \rho_N(x^a, u^a, t) d^{3N}u = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \partial^{a_1}_{\mu_1} \partial^{a_2}_{\mu_2} \dots \partial^{a_n}_{\mu_n} < u^{a_1}_{\mu_1} u^{a_2}_{\mu_2} \dots u^{a_n}_{\mu_n} > . \tag{6.71}$$

All the terms where n is odd are zero and only the terms with even n contribute. We also have

$$< u^{a_1}_{\mu_1} u^{a_2}_{\mu_2} \dots u^{a_n}_{\mu_n} u^{b_1}_{\nu_1} u^{b_2}_{\nu_2} \dots u^{b_n}_{\nu_n} > = < u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} > \dots < u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} > + \text{permutations} . \tag{6.72}$$

Using the fact that $\partial^{a_1}_{\mu_1} \partial^{a_2}_{\mu_2} \dots \partial^{a_{2n}}_{\mu_{2n}}$ is symmetric in all the indices, we can add up all the permutations to obtain for the terms with even n

$$\partial^{a_1}_{\mu_1} \partial^{b_1}_{\nu_1} \dots \partial^{a_n}_{\mu_n} \partial^{b_n}_{\nu_n} < u^{a_1}_{\mu_1} u^{b_1}_{\nu_1} \dots u^{a_n}_{\mu_n} u^{b_n}_{\nu_n} > = \frac{(2n)!}{2^n n!} \partial^{a_1}_{\mu_1} \partial^{b_1}_{\nu_1} \dots \partial^{a_n}_{\mu_n} \partial^{b_n}_{\nu_n} \left[T^{a_1 b_1}_{\mu_1 \nu_1} \dots T^{a_n b_n}_{\mu_n \nu_n} \right] . \tag{6.73}$$

where $T^{ab}_{\mu\nu} = < u^a_\mu u^b_\nu >$ is the covariance matrix introduced in section 3 evaluated for the N point distribution function.

Using this in equation (2.10), we have

$$\int \rho_N(x^a, u^a, t) d^{3N}u = \sum_{n=0}^{\infty} \frac{(b^2)^n}{2^n n!} \partial^{a_1}_{\mu_1} \partial^{b_1}_{\nu_1} \dots \partial^{a_n}_{\mu_n} \partial^{b_n}_{\nu_n} \left[T^{a_1 b_1}_{\mu_1 \nu_1} \dots T^{a_n b_n}_{\mu_n \nu_n} \right] \tag{6.74}$$

In the above equation, for a fixed value of n, there will be a term with n pairs $(a_1 b_1)$,$(a_1 b_2)$ ... $(a_n b_n)$ where each index is independently summed over values 1 to N. Thus, for a fixed value of n, the total contribution is a sum of N^{an} terms each corresponding to a different set of values for the position indices. In any one of these N^{an} terms there can be two kinds of pairs

A. if $a_i = b_i$, then $T^{a_i b_i}_{\mu_i \nu_i} = -\frac{1}{3} \delta_{\mu_i \nu_i} \nabla^2 \phi(0)$ is a constant

B. if $a_i \neq b_i$ then $T^{a_i b_i}_{\mu_i \nu_i} = \partial^{a_i}_{\mu_i} \partial^{a_i}_{\nu_i} \phi(a_i, b_i)$ is a function of the separation between these two points.

Any of the terms can be represented by a directed graph with N vertices'and n edges. The N vertices correspond to the N points in the N point correlation function. Each $T^{ab}_{\mu\nu}$ in equation *(6.74)* corresponds to an edge. The pairs of the kind A correspond to an edge connecting a vertex to itself and a pair of the kind B corresponds to an edge connecting two different vertices. The integral $\int_o^\infty \rho_N(x^a, u^a, t) d^{3N}u$ then corresponds to a sum of graphs with N vertices and the number of edges going from 0 to infinity.

The quantity $\$ \rho_N(x^a, u^a, t) d^{3N}u \, d^3x^1 d^3x^2 .. d^3x^N$ is the mean number of particles we expect to find in the volume d^3x^1 at x^1 and d^3x^a at x^2 and ... d^3x^N around x^N. This has contribution from the lower (i.e., N-1,...1 point) correlation functions also, The residue when the contributions from the lower correlation functions have been removed, is called the reduced N point correlation function. Henceforth we shall refer to the reduced N point correlation function as the N point correlation function. The graphs that do not connect all the N points correspond to functions that do not refer to all the N points and these are the contributions from the lower correlations. The reduced N point correlation can be calculated by considering only the connected graphs with N vertices. The lowest order contribution to the N point correlation corresponds to the connected graphs with the least number of edges. These graphs are the tree graphs and they have N-1 edges. The other terms that contribute to the N point correlation can be generated by adding more edges to the tree graphs.

We use equation *(6.74)* to calculate the three point correlation function. The lowest order at which the three point correlation develops is $n = 2$ and this can be written as

$$\zeta^{(1)}(1,2,3,t) = \frac{b^4}{2} \partial^{a'_1}_{\mu_1} \partial^{a'_2}_{\mu_2} \partial^{a'_2}_{\mu_3} \partial^{a'_3}_{\mu_4} \left[T^{a'_1 a'_2}_{\mu_1 \mu_2} T^{a'_2 a'_3}_{\mu_3 \mu_4} \right] \tag{6.75}$$

where a'_1, a'_2 and a'_3 are to be summed over all possible permutations of *1,2* and 3. Equation *(6.75)* corresponds to the only possible tree graph with three vertices a'_1, a'_2 and a'_3, and two edges (a'_1, a'_2) and (a'_1, a'_3).

Using

$$\partial_\mu \nabla^2 \phi(x) = \frac{x_\mu}{x^3} \int_o^\infty \xi^{(1)}(y) y^2 dy = \frac{1}{3} x_\mu \overline{\xi^{(1)}}(x) \tag{6.76}$$

we have

$$\begin{aligned} \zeta^{(1)}(1,2,3,t) = {} & \frac{b^4}{2} [(1 + \cos^2 \theta_{xy}) \xi^{(1)}(x,t) \xi^{(1)}(y,t) \\ + {} & \cos \theta_{xy} \frac{2}{3} \frac{d}{dx} \xi^{(1)}(x,t) y \overline{\xi^{(1)}}(y,t) + \frac{2}{3}(1 - 3\cos^2 \theta_{xy}) \xi^{(1)}(x,t) \overline{\xi^{(1)}}(y,t) \\ - {} & \frac{1}{3}(1 - 3\cos^2 \theta_{xy}) \overline{\xi^{(1)}}(x,t) \overline{\xi^{(1)}}(y,t)] \end{aligned} \tag{6.77}$$

where

$$x = x^{a'_2} - x^{a'_3} \quad , \quad y = x^{a'_2} - x^{a'_3}$$

and

$$\theta_{xy} = \frac{x_\mu y_\mu}{xy} \tag{6.78}$$

This explicitly exhibits the dependence of the lowest order induced three point correlation function on the initial two point correlation function. We see that the three point correlation depends on both $\xi^{(1)}(x,t)$ and $\overline{\xi^{(1)}}(x,t)$. Thus we see that the small scales can influence the three point correlation at large scales through the quantity $\overline{\xi^{(1)}}(x,t)$. The lowest order induced three point correlation function calculated using ZA is very similar to that calculated by studying gravitational dynamics perturbatively at the lowest order beyond the linear theory (chapter III) and the difference is only in the numerical factors .

We next calculate the higher order terms that contribute to the three point correlation function. These are generated by adding more edges to the tree graphs. Consider a graph with n > 2 edges. In this graph the tree graph can be embedded in C_2^n ways. Using this in equation (6.74) we have

$$\zeta(1,2,3,t) = \sum_{n=0}^{\infty} \frac{b^{2(n+2)}}{2^{n+1}n!} \partial^{a_1}_{\mu_1} \partial^{b_1}_{\nu_1} \dots \partial^{a_n}_{\mu_n} \partial^{b_n}_{\nu_n} \partial^{a_1'}_{\alpha_1} \partial^{a_2'}_{\alpha_2} \partial^{a_2'}_{\alpha_3} \partial^{a_4'}_{\alpha_4} \left[T^{a_1 b_1}_{\mu_1 \nu_1} \dots \right.$$
$$\left. \dots T^{a_n b_n}_{\mu_n \nu_n} T^{a_1' a_2'}_{\alpha_1 \alpha_2} T^{a_2' a_3'}_{\alpha_3 \alpha_4} \right] \tag{6.79}$$

As discussed in the previous section, at large x the contributions from the terms with $a_i = b_i$ will dominate. Thus, at large x the three point correlation function may be written as

$$\zeta(1,2,3,t) = \sum_{n=0}^{\infty} \frac{b^{2n}}{2^n n!} (-\frac{1}{3}\nabla^2\phi(0))^n (\nabla^{a_1})^2 (\nabla^{a_1})^2 \dots (\nabla^{a_n})^2 \xi_3^{(1)}(1,2,3,t) \tag{6.80}$$

where the index a_i indicates at which point the Laplacian acts, and it is to be summed over the values 1,2 and 3. In Fourier apace we have

$$F_3(k^1,k^2,k^3,t) = \sum_{n=0}^{\infty} \frac{b^{2n}}{2^n n!} (\frac{1}{3}\nabla^2\phi(0))^n ((k^{a_1})^2 + (k^{a_2})^2 \dots (k^{a_n})^2) F_3^{(1)}(k^1,k^2,k^3,t) \tag{6.81}$$

where F_3 is the Fourier transform of the three point correlation and $F_3^{(1)}$ is the Fourier transform of the lowest order three point correlation. The terms can be summed up to obtain

$$F_3(k^1,k^2,k^3,t) = \exp\left[-\frac{1}{2}\frac{b^2 < u^2 >}{3}((k^1)^2 + (k^2)^2 + (k^3)^2)\right] F_3^{(1)}(k^1,k^2,k^3,t) \tag{6.82}$$

which gives us in real space

$$\zeta(x^1,x^2,x^3,t) = \frac{1}{(\sqrt{2\pi}L(t))^9} \int \exp\left[-\frac{(x^a - y^a)^2}{2L^2(t)}\right] \zeta^{(l)}(y^1,y^2,y^3,t) d^9y \,. \tag{6.83}$$

Thus, at large separation, the effect of including the higher order terms for the three point correlation function is to convolved the lowest order induced three point correlation with a Gaussian of width $L(t)$. As with the two point correlation function, this too can be interpreted in terms of a diffusion process.

6.9 Discussion and Conclusions.

We find that when we calculate the two point correlation function as a series in powers of the growing mode, we get the same answer if we do the calculation using distribution functions or if we do it in the single stream approximation. Since the first set of equations is valid even after multi-streaming occurs and the second method breaks down once multi-streaming occurs, we would expect to get different answers using the two different methods. But the two results match to all orders in the expansion parameter. We therefore conclude that even though these equations are valid in the multi-streamed epoch, if we start from single streamed initial conditions we cannot perturbatively calculate any effect due to multi-streaming e.g. vorticity, pressure. This limitation arises from the fact that the full two point correlation function for ZA, which includes the effects of multi-streaming, is an exponential in $\frac{1}{b^2}$. All the derivatives of the function $\frac{1}{b}e^{-\frac{A}{b^2}}$ vanish at $b = 0$. As a result, if we try to expand this function in a series in powers of b around $b = 0$, we find that coefficients of all the powers of b are zero. Shandarin and Zel'dovich (1989) present a formula for N, the mean number of streams at any point, in a situation where the particles are moving in one dimension under ZA. At small b this formula is of the form $N = 1 + e^{\frac{-A}{b^2}}$ where A is a constant characterising the initial conditions. If we expand this in powers of b, the coefficients for all the terms are zero and we find that the mean number of streams is one. This confirms that the effects of multi-streaming cannot be studied perturbatively. Although in this analysis we used ZA, we expect this to hold for the full gravitational dynamics too, as derived at the lowest order'of non-linearity in chapter IV.

We find that at small separations ZA predicts an increase in clustering that is faster than the linear evolution. It is also faster than the evolution if we take into account the effects of the lowest order non-linear corrections in GD. We also find that the deviation from linear theory predicted by ZA is of the same sign as that predicted by the non-linear correction from GD. We have also investigated the relation between $h(x,t)$ and $\bar{\xi}(x,t)$ and we find that this does not show an universal 'behaviour. We also find that at small separations in ZA $h(x,t)$ grows faster than predicted by the relation $h(x,t) = (2/3)\bar{\xi}(x,t)$ which is valid in the linear regime. At scales comparable to the length scale introduced by the cut-off in the initial power spectrum we find that evolution in ZA could be faster or slower than the linear evolution. We also find that the evolution in ZA is very similar to the evolution we get if we take into account the lowest order non-linear corrections from GD. In our investigation of the relation between $h(x,t)$ and $\bar{\xi}(x,t)$ we find that at intermediate scales $h(x,t)$ lies below the value predicted by the relation $h(x,t) = (2/3)\bar{\xi}(x,t)$.

In our comparison of the evolution of the two point correlation function at large separations we find that the results obtained using ZA are quite similar to the lowest order non-linear results obtained using GD and both of them can be interpreted in terms of a diffusion process where the rearrangement of matter on small scales affects the two point correlation at large scales. In ZA, for an initial power spectrum with $n > -1$, the mean square displacement of the particles from their original positions is $L^2(t) = b^2(t) < u^2 >$ and this makes its appearance in the formula for the non-linear corrections to the two point correlation function obtained using ZA. Interpreting the results from GD in a similar fashion, for an initial power spectrum with $n > 0$, we have $L^2(t) \sim .58b^2(t) < u^2 >$. In chapter V we also considered the case with $n = 0$ and for this case we found $L^2(t) \sim 1.49b^2(t) < u^2 >$. The differences can be understood in terms of the fact that in ZA the particles move along trajectories calculated using linear GD, whereas when we take into account non-linear corrections, the trajectories get modified by the tidal forces. In the equations for the evolution of the two point correlation function the tidal force acts through the three point correlation function. The tidal force of the third particle (in the three point correlation), will cause the other two particles to move towards or away from one another. This effect will be strongly dependent on the spatial behaviour of the three point correlation function. For the cases with $n > 0$ the induced three point correlation has the hierarchical form at large x wheres for the case with $n = 0$ the induced three point correlation does not have this form. We propose that it is because of this that the effect of the tidal forces is different in these two cases and in the former case the effect of the tidal forces is to reduce the mean square displacements relative to ZA whereas in the latter case it increases it. Thus indirectly, it is a diagnostic of the effect of the back-reaction of the three point correlation function on the pair velocity which in turn effects the two point correlation.

We find that for ZA, at large x, we'can sum up all terms in the perturbative series and the non-linear two point correlation function is related to the linear two point correlation by a convolution with a Gaussian of width $\propto L(t)$. We also find that for special initial conditions where the power spectrum has a Gaussian cut-off at large k, the evolution at large x can be described by a simple scaling relation according to which the information propagates outward.

We also find that this picture based on diffusion gives a good description of the evolution under ZA until the onset of multi-streaming. Based on this we suggest that the evolution of of the two point correlation function in GD can also be described by a diffusion process until the onset of multi-streaming.

We have calculated the lowest order induced three point correlation function using ZA and we find that it is very similar to **the** result obtained using GD and the two differ only in the numerical factors. We also investigate the effect of the higher order non-linear terms and we find that at large x we can sum the whole perturbation series. We find that the expression obtained after taking into account the non-linear corrections is related to the lowest order three point correlation function by a convolution with a Gaussian of width $\propto L(t)$. This is very similar to the evolution of the two point correlation function at large separations.

It can be shown that a similar relation holds for the higher correlation functions also but we do not pursue this matter here.