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The kinematics of bowed strings

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1. Introduction

It has long been known that if a string be bowed *exactly* at one of its points of aliquot division, the harmonics having a node at that point fail to be elicited, but that if the bow be moved to a point only slightly distant from the node and applied with suitable pressure, the same harmonics are elicited with great vigour, and in certain circumstances may even transcend the fundamental component of the vibration in their intensity. This fact attracted the attention of experimenters early in the history of the subject and naturally assumes prominence in any exposition of the theory of bowed strings. Among the more recent workers who have studied the phenomenon may be mentioned Krigar-Menzel and Raps,[1] and also Davis[2] who noticed a similar effect in the case of the longitudinal vibration of rubbed strings. Neither these experimenters nor later theorists such as Lippich[3] and Andrew Stephenson[4] have however succeeded in clearing up the nature of the transition between the cases in which the partial vibrations fail to appear, and those in which they reassert themselves with great intensity. The question is of considerable interest in relation to the mechanical theory of the action of the bow, and has (along with various other problems) been discussed by me in the first part of a monograph on bowed stringed instruments which has recently been published.[5] The treatment of the theory given in the monograph is largely of a graphical character, and though the formulae for the harmonic analysis of the motion were given, they were not fully discussed. It is thought that an exposition of the subject from a more strictly analytical point of view may be of advantage. This is given in the present paper.

2. Analysis of discontinuous wave-motion

It is useful to commence by deriving the formulæ which give the amplitude and phase of the harmonics of a vibration involving only discontinuous changes of

[1] Sitzungberichte of the Berlin Academy, 1891.
[2] *Proc. Am. Acad. Sci.*, 1906.
[3] *Wien Berichte*, 1914.
[4] *Philos. Mag.* January, 1911.
[5] *Bull. Indian Assoc. Cultiv. Sci.*, Calcutta, 1918, No. 15.

velocity. The distribution of transverse velocity over the whole string at any epoch may be represented by a diagram (which we shall refer to as the velocity diagram) in much the same way as the configuration of the string at any instant may be represented by a displacement diagram. Since, by assumption, the vibration at every point on the string is determined solely by discontinuous changes of velocity, the velocity diagram of the string must consist of a number of straight lines inclined to the x-axis at the same angle and separated by discontinuities which move to the right or the left according as they belong to the positive or the negative wave, and on reaching an end of the string suffer reflection and return. The transverse velocity of every point on the string remains unaltered except when a discontinuity passes over it in one direction or the other, when it suddenly alters by a quantity equal to the magnitude of the discontinuity.

Let d_1, d_2, d_3, etc. be the magnitudes of the discontinuities and let their positions on the string at time t be given by $x = c_1, c_2, c_3$ etc. If y be the transverse displacement at any point on the string, it is readily shown by applying the Fourier analysis to the velocity diagram that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_t = \sum_1^\infty A_n \sin\frac{n\pi x}{l},$$

where

$$A_n = -\frac{2}{n\pi}\left[\mathrm{d}_1 \cos\frac{n\pi c_1}{l} + \mathrm{d}_2 \cos\frac{n\pi c_2}{l} + \text{etc.}\right] \tag{1}$$

The quantities c_1, c_2, c_3 etc. are not constant but vary with time. Accordingly, we may write $c_r = (c_r + at)$, or $2l - (c_r + at)$ according as the discontinuity d_r belongs to the positive or the negative wave. On making the necessary substitutions, and separating the sine and cosine components, we get as the equation for the vibration

$$y = \sum_1^\infty \sin\frac{n\pi x}{l}\left[a_n \sin\frac{2n\pi t}{\tau} + b_n \cos\frac{2n\pi t}{\tau}\right]$$

where

$$a_n = -\frac{1}{n^2\pi\tau}\left[\mathrm{d}_1 \cos\frac{n\pi c_1}{l} + \mathrm{d}_2 \cos\frac{n\pi c_2}{l} + \text{etc.}\right]$$

and

$$b_n = -\frac{1}{n^2\pi^2\tau}\left[\mathrm{d}_1 \sin\frac{n\pi c_1}{l} + \mathrm{d}_2 \sin\frac{n\pi c_2}{l} + \text{etc.}\right]. \tag{2}$$

It will be seen that, generally speaking, both sine and cosine functions of the time are involved in the expression for the displacement, and that it is not possible to get rid of the cosine terms by merely changing the origin of time. In general, therefore, the string does not coincide with its position of equilibrium at any epoch of the vibration. The cosine terms will however vanish provided that the positive and negative waves are of identical form and coincide in position at the epoch chosen as the origin of time. This is readily verified from (2) above, as the coefficients b_n all vanish, and in the resulting vibration, the string everywhere passes through the position of equilibrium at two epochs in each period of vibration.

Reference may be made here to three papers recently contributed to the *Philos. Mag.* in which it has been shown how certain simple discontinuous types of vibration may be experimentally realised.[6]

3. Vibrations with the complete series of partials

In order that the oscillation should include the complete series of partials, it is necessary to assume that the bowed "point" divides the string in an irrational ratio. If the motion at this point is known, the entire vibration of the string is kinematically determinate. For the purpose of the discussion, it may first be assumed that the bowed point of the string moves to and fro, once or oftener in each period of vibration, with constant and uniform velocities. Mechanical theory indicates that this is the type which the motion at the bowed point approximates to, but does not necessarily attain in any particular case. The velocity of the bowed point when it moves with the bow may be denoted by v_b and when it moves in the opposite direction, v_a. The velocity diagram of the string must be of such form that by the passage of the discontinuities over it, the velocity at the bowed point alternates between v_a and v_b once or oftener in each period of vibration. It is obvious that if the magnitude of all the discontinuities in the velocity diagram is the same and equal to $(v_a - v_b)$, and that two or more discontinuities do *not* pass over the bowed point in succession in the *same direction*, the required type of motion at the bowed point would be secured. For, the initial velocity at the bowed point being taken to be v_b the velocity changes to v_a when a discontinuity passes over it in one direction, and changes back to v_b when the same or another discontinuity passes over it in the opposite direction. By pursuing the argument on these lines, it may be shown that if the bowed point divides the string in an irrational ratio, and its motion is strictly of the type contemplated, the magnitudes of the discontinuities in the velocity diagram of the string are necessarily all equal to $(v_a - v_b)$. We shall here assume this result.

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[6]On discontinuous wave motion, *Philos. Mag.* January 1916, February 1918 and April 1918.

The magnitude of the discontinuities in the velocity-diagram being taken to be equal to $(v_a - v_b)$, the amplitudes and phases of the harmonics in the vibration may be readily calculated. The number of discontinuities in the velocity-diagram gives us a simple criterion for classification of the modes of vibration.

Case of one discontinuity

This is the simplest type of all. If the origin of time chosen be the epoch at which the discontinuity is at the end of the string, the cosine terms in the expression for the displacement vanish, and we get

$$y = \sum_{1}^{\infty} \frac{-(v_a - v_b)}{n\pi^2 T} \sin \frac{n\pi x}{l} \sin \frac{2n\pi t}{\tau}. \qquad (3)$$

This is the well-known principal type of vibration of a bowed string discovered by Helmholtz in which the vibration-curve at every point on the string is a simple two-step zig-zag. The ratio v_a/v_b is equal to the ratio in which the bowed point divides the string.

Case of two equal discontinuities

The kinematics of this case is readily worked out in detail. It is sufficient for our present purpose to give an analytical demonstration of two important features in regard to this type of vibration. If the origin of time chosen be the instant at which the two discontinuities coincide in position, the cosine terms vanish, and we get

$$y = \sum_{1}^{\infty} \frac{-2(v_a - v_b)}{n^2\pi^2 T} \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} \sin \frac{2n\pi t}{\tau}, \qquad (4)$$

where c is the distance from the end of the string of the point at which the two discontinuities cross. At the point $x = c$, we have

$$y = \sum_{1}^{\infty} \frac{-(v_a - v_b)}{n^2\pi^2 T} \sin \frac{2n\pi c}{l} \sin \frac{2n\pi t}{\tau} \qquad (5)$$

which evidently expresses a motion of the simple two-step zig-zag type, the ratio v_a/v_b being equal to the ratio in which the point $x = c$ divides the *half*-length of the string. The motion at other points may be found by graphical methods. If c be nearly equal to $l/_2$, the amplitude of the second, fourth harmonics etc. becomes large in comparison with the amplitude of the other partials, and may even transcend that of the fundamental component of the vibration. This is clear from (4) above.

Case of three equal discontinuities

In this case, the cosine components vanish only if at the instant at which two of the discontinuities pass each other, the third discontinuity is at the end of the string. This instant is taken as the origin of time. If the two discontinuities pass each other at the point $x = l/_3 \pm 2b$ (where $b \nless l/_6$), it may be readily shown as in the preceding case that the motion at the point $x = l/_3 \mp b$ is of the simple two-step zig-zag type, the ratio v_a/v_b being equal to the ratio in which that point divides the *third* of the length of the string.

If b be sufficiently small, the amplitudes of the third, sixth harmonics etc. become large in comparison with that of the fundamental and other components.

Cases of four, five or more equal discontinuities

As regards these cases, it must suffice to mention the following kinematical result which emerges from a detailed discussion. If r, the number of discontinuities be a *prime* integer, e.g. 2, 3, 5, 7, 11 etc. it is always possible by suitably choosing the initial position of the discontinuities to secure that the motion at any specified point on the string (not lying near one end) is of the simple two-step zig-zag type. If the point specified lies near one of the points of section of the string into r equal parts, the rth, $2r$th, $3r$th harmonics etc. have relatively large amplitudes in the type of vibration thus set up.

But the case is entirely different when r, the number of discontinuities is not a prime integer, e.g. 4, 6, 8, 9 or 10, and is therefore a multiple of some smaller number. It is then found that no disposition of the discontinuities can secure a simple two-step zig-zag type of motion at any point lying elsewhere than within certain limited sections of the string. A simple example will make this clear. With 6 discontinuities on the velocity-diagram, it is kinematically possible to secure a two-step zig-zag type of motion at the bowed point if it lies in the vicinity of the node $l/_6$ on either side and the 6th harmonic is then powerful, in the motion elicited. If it lies elsewhere, that is, in the vicinity of the nodes $l/3$ or $l/2$, the motion is necessarily of a more complicated type, a four-step zig-zag and a six-step zig-zag being respectively the types of vibration of minimum complexity admissible at the bowed point in the two cases.

4. Vibrations with missing partials

We may now easily pass to the cases in which owing to the coincidence of the bowed point with a point of rational division of the string, certain partials fail to be maintained. It is obvious that the falling out of these partials leaves the motion at the bowed point unaffected. The mode of vibration of the string in these cases, can be very simply derived from the corresponding 'irrational' types by simple subtraction of the partials having a node at the bowed point. It will be shown here how this may be done in the most general case of discontinuous vibration. Assume that the bowed point coincides with a node of the sth harmonic which of course is also a node of the $2s$th, $3s$th harmonics etc. Taking the analysis of the velocity diagram given in (1) we have

$$A_1 = -\frac{2}{\pi}\left[d_1 \cos\frac{\pi C_1}{l} + d_2 \cos\frac{\pi C_2}{l} + \text{etc.}\right]$$

$$A_2 = -\frac{2}{s\pi}\left[d_1 \cos\frac{s\pi C_1}{l} + d_2 \cos\frac{s\pi C_2}{l} + \text{etc.}\right]$$

$$= -\frac{2}{\pi}\left[\frac{d_1}{s}\cos\frac{\pi C_1}{l/s} + \frac{d_2}{s}\cos\frac{\pi C_2}{l} + \text{etc.}\right] \tag{6}$$

Similarly we have

$$A_n = -\frac{2}{n\pi}\left[d_1 \cos\frac{n\pi C_1}{l} + d_3 \cos\frac{n\pi C_2}{l} + \text{etc.}\right]$$

$$A_{ns} = -\frac{2}{n\pi}\left[\frac{d_1}{s}\cos\frac{n\pi C_1}{l/s} + \frac{+d_2}{s}\cos\frac{n\pi C_2}{l/s} + \text{etc.}\right]. \tag{7}$$

Now the summation of the series $\sum_{n=1}^{n=\infty} A_n \sin(n\pi x/l)$ of which $A_1 \sin(\pi x/l)$ is the leading term gives us the original velocity diagram. From (6) and (7), it is evident that the subordinate series $\sum_{n=1}^{n=\infty} A_{ns} \sin(n\pi x/(l/s))$ of which $A_s \sin(s\pi x/l)$ is the leading term is of analogous type and gives us a velocity diagram with discontinuities d_1/s, d_2/s, etc. the ordinates of which have to be subtracted from the original diagram. The lines in the principal and subordinate diagrams are obviously inclined to the x-axis at the same angle, and the resulting figure in which the missing partials are excluded is therefore made up of straight lines parallel to the x-axis and separated by discontinuities. From this diagram, the nature of the vibration at any point of the string may be readily found by graphical methods.

It is evident that the effect of the falling out of the partials having a node at the bowed point is to introduce into the velocity diagram of the string, a number of discontinuities which are smaller than the original discontinuities in the ratio $1/s$ and of opposite sign.

5. Vibrations with the missing partials partially restored

A very complete graphical treatment of these cases is given in the monograph where they are referred to as 'transitional modes of vibration.' They are intermediate in form between the various types of vibrations discussed in the third and fourth sections of this paper, and have the distinguishing characteristic that the speed of the bowed point in the forward motion is generally constant, but that in the backward motion is not uniform. Generally speaking, they are of an unsymmetrical type, that involve both sine and cosine functions of the time. From a musical point of view, they are of great importance, and a fuller discussion of their features is being published elsewhere. Here, it must suffice to remark that the general analytical formula for discontinuous vibration given in section 2 is sufficient to cover these cases as well.