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A classical derivation of the Compton effect

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Abstract

With the aid of an atomic model in which the electrons are regarded as a gas distributed in a spherical enclosure surrounding the nucleus, it is shown that the classical wave-principles lead directly to a quantitative theory of the Compton effect and an explanation of the known experimental facts in connection with it.

1. Introduction

As the simplest possible model of a spherically symmetrical atom, consider a spherical enclosure of radius R within which are imprisoned a certain number Z of electrons. We shall assume that the electrons move about within the enclosure like the molecules of a gas. They are subject to the influence of a central field of force, but are prevented from coalescing with it by the energy of their movements and their mutual repulsions. Our atom model is placed in the path of a plane train of waves of definite frequency. What would be the nature of the secondary radiation emitted by the atom? We shall proceed to discuss this problem on the classical wave-principles.

It will be assumed that each of the electrons is periodically accelerated along the direction of the electric force in the incident waves and emits secondary waves whose amplitude at a distance r is given by the expression $-e^2\sin\theta/mc^2r$, where e, m and c have the usual meaning, and θ is the angle between r and the electric force in the incident waves. The problem of determining the secondary radiation at a distance from the atom resolves itself into that of finding the sum of Z vibrations of equal amplitude but whose phases are different. An important point is that as the electrons have been assumed to be mobile, the phase-relations between the Z vibrations are continually variable with time. Has the problem of finding the resultant of Z such vibrations any definite meaning or answer?

2. Theory of random interferences

The question raised in the foregoing paragraph is very similar to those which continually arise in such optical problems as the theory of coronas, the scattering of light by the molecules of a fluid and the like. We shall consider the mathematical formulation of our problem a little later, but the answer to it can be given forthwith by analogy with the known results in the optical cases referred to. The resultant of the Z vibrations can be divided into two parts. *The first part* is entirely determinate, its amplitude being a function of the angle between the primary and secondary rays which is *invariable* with time, its frequency is the same as that of the incident radiation, and its phase is definitely connected with it for each given direction. The *second part* is entirely *indeterminate*, so that neither the amplitude nor the phase can be specified at any given time or in any given direction, and consequently the frequency is also variable. Nevertheless, it is possible to specify the *statistical expectation of intensity* of this second and highly fluctuating type of secondary radiation, that is to say, the average value to which it tends in the mean of a large number of trials, and to determine the manner in which it tends to be distributed with reference to direction. This possibility of subdividing the resultant into two parts, one of which is stationary and determinate, and the other is fluctuating and indeterminate except in a statistical sense, follows in a strictly mathematical way from the theory of random interferences.

3. Analysis of secondary radiation

We may now proceed to demonstrate the foregoing statements.

To enable the secondary radiation to be evaluated in a simple manner, we shall assume that the electrons move independently of each other in the spherical enclosure within which they are confined, and that the probability of any specified electron being found within a volume element dV of the enclosure is $\chi(\rho)dV$, where $\chi(\rho)$ is a known function of the distance ρ of the volume element from the centre of the sphere. As the probability of the electron being found within the enclosure is unity, $\chi(\rho)$ must satisfy the condition

$$\int_0^R 4\pi\rho^2\chi(\rho)\mathrm{d}\rho = 1. \tag{1}$$

The sum of the Z vibrations at some distant point in some specified direction making an angle 2ϕ with the incident rays is

$$-\frac{e^2 \sin\theta}{mc^2 r}[\cos(2\pi\nu t+\chi_1)+\cos(2\pi\nu t+\chi_2)+\cdots+\cos(2\pi\nu t+\chi_2)] \tag{2}$$

where $\chi_1, \chi_2, \ldots\ldots \chi_z$ are the phases of the vibrations to be added. The resultant intensity is

$$I=\frac{e^4\sin^2\theta}{m^2c^4r^2}[Z+2\sum\cos(\chi_1-\chi_2)] \tag{3}$$

where the summation within the brackets extends over the $\frac{1}{2}Z(Z-1)$ terms of the form $\cos(\chi_1 - \chi_2)$. The phase differences $(\chi_1 - \chi_2)$ etc, may be expressed in terms of the positions of the two electrons concerned in each case. These positions are sufficiently specified by the coordinates $\mu_1, \mu_2, \ldots \rho_1, \rho_2, \ldots$ where ρ is the radial distance of the electron from the centre of the sphere and μ is the direction-cosine of ρ with respect to the bisector of the angle between the incident and scattered rays. The statistical expectation of the value of $\cos(\chi_1 - \chi_2)$ in a large number of trials is given by the real part of the quadruple integral

$$4\pi^2 \int_0^R \int_0^R \int_{-1}^{+1} \int_{-1}^{+1} \exp(i\psi\mu_1)\exp(-i\psi\mu_2)\chi(\rho_1)\chi(\rho_2)\rho_1^2\rho_2^2 d\rho_1 d\rho_2 d\mu_1 d\mu_2 \qquad (4)$$

where

$$\psi = 4\pi\rho\sin\phi/\lambda.$$

Since the variables ρ and μ are all independent, the integration with respect to μ_1 and μ_2 is readily effected. The quadruple integral (4) thus reduces to

$$\left[\int_0^R 4\pi\rho^2\chi(\rho)\frac{\sin\psi}{\psi}d\rho\right]. \qquad (5)$$

We shall write (5) in the form F^2, where F is evidently a function of R, λ and the angle ϕ. Multiplying F^2 by $\frac{1}{2}Z(Z-1)$ and substituting in (3), the latter reduces to the extremely simple form

$$I = \frac{e^4\sin^2\theta}{m^2c^4r^2}[Z^2F^2 + Z(1-F^2)]. \qquad (6)$$

Equation (6) may be written in the form

$$I = I_1 + I_2, \quad \text{where}$$

$$I_1 = \frac{e^4\sin^2\theta}{m^2c^4r^2}Z^2F^2 \qquad (7)$$

$$I_2 = \frac{e^4\sin^2\theta}{m^2c^4r^2}Z(1-F^2). \qquad (8)$$

The two parts I_1 and I_2 of the secondary radiation are fundamentally different in their physical nature as already remarked in the preceding section.

4. The two types of secondary radiation

We may point out that the first of the two types of secondary radiation into which we have resolved the emission from the atom is obviously the *diffracted radiation* from the atom. This is clear from its proportionality to Z^2, and from the form of the function appearing in (5). The clearest proof of its nature is however given when we attempt to find by the methods of the electromagnetic theory of light, the intensity of the diffraction-pattern due to a dielectric sphere. If the dielectric constant of the material of the sphere is a function of the radial distance from the centre and is given by the relation $K - 1 = \chi(\rho)$ where $\chi(\rho)$ is assumed small compared with unity, the intensity of the diffracted radiation is given by the expression

$$\frac{\pi^2 \sin^2 \theta}{r^2 \lambda^4} \left[\int_0^R 4\pi\rho^2 \chi(\rho) \frac{\sin \psi}{\psi} \, d\rho \right]^2.$$

The identity of form of this integral with that appearing in (5) and (7) makes the nature of the latter evident.

Thus, of the two terms appearing in (6), the first term namely (7) is a perfectly determinate and invariable part which is the diffraction pattern of the atom. Hence, it follows that the second term in (6), namely (8) represents the *statistical expectation of intensity* of a quantity whose value in individual trials is indeterminate. This is clear from the fact that this term arises from a summation of amplitudes with entirely indeterminate phase-relationships which consequently can give no definite result in any individual trial. The proportionality to Z instead of to Z^2 is significant in this connection. A summation of the intensities instead of the amplitudes of the effects of the electrons is permissible for the determination of a statistical average only when the effects under consideration are *completely uncorrelated* in phase. An important point to be noticed also from equation (1) and expression (5) is that when $\phi = 0$, that is, in the direction of the primary beam, $F^2 = 1$ and hence the expression (8) vanishes. This could have been expected *a priori*, for in the forward direction, the Z vibrations to be added have completely determinate phases, and hence the indeterminate part of the sum of the Z vibrations must vanish.

The first type of secondary radiation being the diffraction by the atom, what meaning should we attach to the second and highly fluctuating type of radiation which is indicated by our atom-model? In this connection, we may remark that the possibility, or rather, the necessity for separating the secondary radiation into two distinct parts arises only when we employ a dynamic atomic model. In static atomic models where the electrons are supposed to occupy fixed positions within the atom, the phase-relations between them have always fixed values for any fixed direction, and the entire effect produced by the electrons is single and indivisible; we would have merely a diffraction-pattern or "structure factor" for the entire atom, differing considerably no doubt from that due to a smoothed-out distribution, but belonging to the same physical class of phenomenon. With static atom-models, therefore, it is not permissible to speak of any "independent" scattering by the electrons.

From the fact that the statistical expectation of intensity of our second type of radiation is proportional to Z, the atomic number, the reader might be tempted to suppose that this is simply the J J Thomson–C G Barkla type of scattering by the Z electrons in the atom acting "independently", *Such a view would be erroneous.* For, in the first place, our formula is not simply the Thomson expression but appears with a very significant multiplying factor $(1 - F^2)$. But a more vital and fundamental difference is that our expression represents merely the statistical average of a quantity that *fluctuates with time.* This is essentially the result of using a dynamic atomic model, and the fluctuating effect involves also corresponding fluctuations with the time of the state of the atom itself, an idea not contemplated in the Thomson–Barkla theory. In fact, our investigation discloses that the so-called "Thomson scattering by the Z electrons acting independently" has no real existence either in a static or a dynamic atomic model.

As remarked above, the fluctuations with time of the secondary radiation from the atom involve corresponding fluctuations in the electrical state of the atom which we may attribute to the movements of the electrons. If we postulate that the atom does not or cannot fluctuate, the fluctuating type of secondary radiation cannot exist. On the other hand, if we believe that our atom-model is not a wholly erroneous picture of the real atom and are prepared to concede the inference from it that a fluctuating type of secondary radiation from the atom does exist, then we must be equally prepared to accept the corollary that its emission is accompanied by a simultaneous fluctuation in the electrical state of the atom, the two phenomena in fact being inseparably linked with each other.

We shall now proceed to identify the second or fluctuating type of secondary radiation with the Compton effect. It may seem surprising to be told that the classical wave-principles thus lead us directly to the existence of this effect and indeed also suffice to indicate its observed physical characters. The belief that the classical wave-principles are not easily reconcilable with the phenomena of the Compton effect must be ascribed, however, not to any defect of the wave-principles, but to the fact that they have not been interpreted correctly in the past in relation to the present problem. The existence of at least two different types of secondary X-radiation, one of which is of a highly "fluctuating" or incoherent character is the cardinal experimental fact which requires explanation, and we have seen already that the classical wave-principles taken together with a dynamic atom-model lead us to it very naturally. We shall presently see that they also explain the other facts known experimentally about the Compton effect.

Our identification of the classical fluctuating secondary radiation with the Compton effect is not merely qualitative. It may be developed mathematically and proves itself to be solidly based.

5. The characters of the Compton effect

From equations (7) and (8) written above, and from the nature of the function F^2 which involves R, λ and ϕ, it is readily seen that the ratio of the two types of secondary radiations depends on Z the atomic number, upon the ratio of the wavelength λ to the radius R of the atom, and upon the angle of diffraction 2ϕ. We shall consider these in order.

As already remarked, the fluctuating type of radiation vanishes when $\phi = 0$. With increasing angle of diffraction, the value of F^2 falls down from unity rather quickly, and ultimately reaches rather small values at large angles. The march of the function F^2 with ϕ depends of course on the structure-factor of the atom. The second type of radiation is, therefore, relatively to the first quite inconspicuous at small angles of diffraction. It reaches importance only when the angle of diffraction is such that F^2 is a small fraction of unity, and then becomes quite comparable with the regularly diffracted radiation.

The ratio of the second to the first type of radiation is largest for elements of low atomic number and becomes very small for elements of high atomic number. For elements of low atomic number, the two types of radiation are of comparable intensity even at very moderate angles of scattering; at larger angles, the fluctuating radiation becomes much the more conspicuous of the two.

For any given angle of diffraction, F^2 becomes smaller with decreasing wavelength of the incident radiation. Hence the ratio of the fluctuating to the stationary type of radiation increases with increasing hardness of the incident X-radiation. The angle of diffraction 2ϕ at which the two become of comparable intensity becomes smaller at the same time, $\sin\phi$ being in fact proportional to λ.

We see therefore that the experimentally known facts regarding the ratio of the "unmodified" and "modified" types of X-ray scattering and its variation with the atomic number of the scattering element, the wavelength of the incident radiation, and the angle of diffraction are correctly indicated by the very simple theory developed in the foregoing pages.

Our model atom is, of course, rather crude in its constitution. Its most defective feature is the assumption of an artificial barrier preventing the electrons from escaping outside. Such an artificial barrier is not really needed. In reality, any of the electrons is free to wander away from the atom into space if its kinetic energy should exceed the potential energy of its position in the central field of force. On the other hand, if its kinetic energy be less than this value, it cannot leave the atom and no barrier is needed to prevent its escape. If we imagine an electron to be so feebly bound that a small increment of energy would liberate it, such addition of energy would enable it to wander off from the atom. The escape of an electron from the atom may therefore be legitimately regarded as a possible mode of "fluctuation" in the electrical state of an atom, and one which is especially likely in the case of atoms in which the electrons are very loosely bound. We may therefore, without an undue stretch of language claim that considerations of a purely classical nature not only definitely predict the existence of a highly fluctuating type of secondary radiation from atoms, but also indicate that the fluctuation of the atom which must accompany the emission of such radiation consists of the ejection of an electron from it.

To be strictly logical, of course, we must be prepared to admit the possibility of other possible modes of fluctuation of the atom and the existence of corresponding special types of secondary radiation from it. Our simple atom-model with its virtually free electrons is not however capable of dealing with such cases quantitatively.

To avoid misapprehension, it should be made clear that the "fluctuations" of the atom we are considering are quite different in nature from the "fluctuations" contemplated in thermodynamics and kinetic theory. We are here concerned with the "fluctuations" of the atom from its normal condition under the influence of external radiation. Whether simultaneous thermal excitation would modify the results is a question into which we need not enter here.

6. The change of wavelength

It now remains to consider the question of the change of wavelength and its relation to the motion of the ejected electron. Any type of secondary radiation from particles in motion necessarily involves changes of wavelength. Since the fluctuating secondary radiation is associated with the motion of an electron from within to outside the atom, we may naturally expect its frequency to be altered by reason of the Doppler effect. If we assume the electron to move with a velocity v in the direction of propagation of the primary waves, the wavelength of the spherical secondary waves emitted by it and observed in a direction making an angle χ with the primary waves is

$$\lambda' = \lambda + \lambda\frac{v}{c}(1 - \cos\chi). \tag{9}$$

Since we have assumed the electron to be periodically accelerated in the direction of the electric force, the only other possible way in which the incident radiation can act on the electron is by way of radiation pressure or electromagnetic momentum. This must be in the *forward* direction, for the secondary radiations from the electron being spherically symmetrical, they can have no resultant momentum in any direction. Thus, classical wave-principles justify (9) at least in form. We shall be doing no violence to them, though we may be offending Newtonian mechanics, by taking the forward momentum of the scattering electron to be the same as that of finite train of plane waves having the total energy $h\nu$. We then obtain

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\chi) \tag{10}$$

which is the Compton equation derived on the classical wave-principles.

In obtaining (10), we have considered the *electron*, and altogether ignored the *atom* with which it is associated before expulsion. The radiation from the atom does not consist of simple spherical waves as contemplated above, but of the complex fluctuating disturbance given by equation (6) of which the second part represents only a "statistical expectation" of intensity. Hence, we should be very far from being justified in assuming without further examination that the argument adduced above is valid for the case of the atom as distinguished from that of the free electron. Nevertheless there is good reason for accepting (10) as representing *statistically* the observed relation between wavelength and direction of observation, at least as an approximation. For, when the angle of diffraction is not too small, and especially for elements of low atomic number, equation (6) gives the statistical expectation of intensity as practically Z times that of a single free electron, and hence (10) may be regarded as *statistically* valid to a close approximation.

It is a different matter if we ask what are the possibilities in any *individual* trial. To find an answer to this, we have to consider more closely the character of the fluctuating disturbance radiated from the atom. We must remember that not only is its intensity variable with time, but also its angular distribution, so that apparently we have no method, *a priori*, of finding what it would be in any individual trial. A way out of this difficulty is however indicated by the well-known general solution of the equation of wave-propagation discovered by Whittaker (See "*Modern Analysis*," Camb. University Press); Whittaker's solution shows that even a most arbitrary type of wave-disturbance can be represented as the superposition of plane trains of waves travelling in all directions through space. We may thus analyse our fluctuating radiation into sets of plane waves travelling in different directions. Since the incident waves are themselves plane and periodic, we may accept the hint given by equation (10), and assume that it gives, as an approximation, the wavelength of the plane train travelling in the direction specified by the angle χ as the result of such analysis. In any *particular trial*, therefore, the fluctuating part of the radiation emitted by the atom may be conceived as consisting, approximately, of a single such plane wave train or a group of closely adjacent plane wave-trains. In finding the corresponding velocity and direction of motion of the ejected electron, we have to take into account the momentum of the incident wave as well as that of the unsymmetrically scattered secondary wave. We understand in this way, why in any individual trial, the electron may be ejected in directions other than that of the primary ray. Its actual velocity and direction of motion depend on the incident and scattered waves jointly. Incidentally, it becomes evident why the argument of the "triangle of momenta" by which Compton obtained his formulae gives the same result for the change of wavelength as the simple classical theory of the emission of spherical secondary waves by the electron.