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On motion in a periodic field of force

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(Plates I–VIII)

SUMMARY OF CONTENTS

Vibrations maintained by a periodic field of force

The experimental study of the motion of a dynamical system in a periodic field of force leads to results of quite exceptional interest. One aspect of the problem, i.e. the oscillatory motion of the system about a position of equilibrium in the field, has some affinities to the case of vibrations maintained by a variable spring which I have dealt with in my previously published work, but the two classes of investigations lead to results which differ from one another, yet are related in a most remarkable way. By experimenting on stretched strings subjected to a variable tension, I showed that a normal variation of spring will enable the oscillations of the system to be maintained, when the frequency of these oscillations is sufficiently nearly equal to $\frac{1}{2}$ of, or $\frac{2}{2}$ times, or $\frac{3}{2}$ times, or $\frac{4}{2}$ times, etc. the frequency of variation of the spring, these ratios forming an ascending series. By experiments on the vibrations of a body about a position of equilibrium in a periodic field of force (to be described below), I have shown that the frequency of the oscillations maintained may be equal to, or half of, or one-third, or one-fourth etc. of the frequency of the field, in other words, it may be any one of a descending series of sub-multiples of the frequency of the field. It appears, in fact, that we have here an entirely new class of resonance-vibrations. It will be noticed that if the two series referred to above are both written in the same order of descending magnitudes of frequency, thus,

$$\frac{6}{2}, \frac{5}{2}, \frac{4}{2}, \frac{3}{2}, \frac{2}{2}, \frac{1}{2},$$
$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$$

the last two terms of the first series, and the first two of the second series coincide, and these two are to some extent typical of the rest. For, as I have shown in section IV of Bulletin no. 6, the 1st, the 3rd, the 5th and the odd types generally in the first series bear a family resemblance to each other, giving symmetrical vibration curves. The 2nd, the 4th and the other even types similarly resemble each other in giving markedly *asymmetrical* vibration curves. Since the first term in the ascending series is the 2nd in the descending series, we may expect that the 2nd, 4th, 6th etc. in the latter would give analogous types of motion, and that similarly the 1st, 3rd, 5th etc. would show resemblances amongst each other. These points will be dealt with more fully as we proceed.

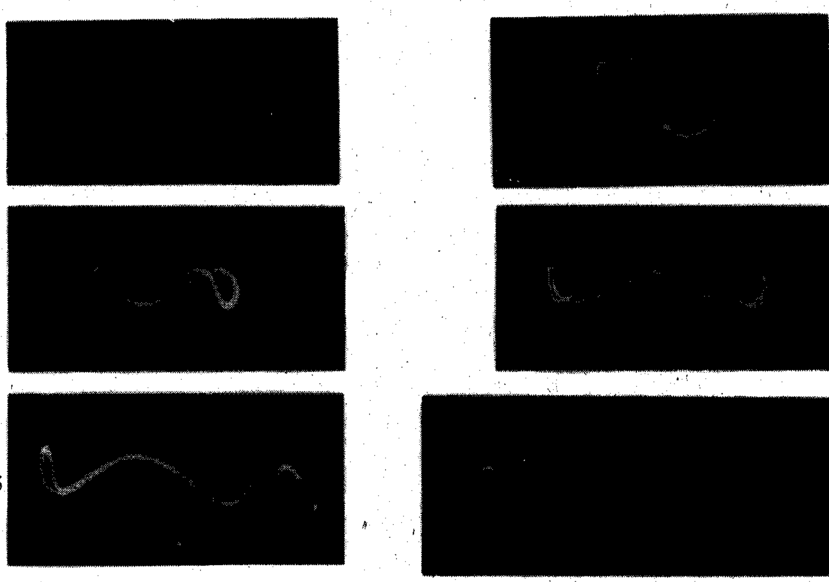
The vibrations studied which form the subject of this section were those of the armature-wheel of a synchronous motor of the attracted-iron type, about a position of equilibrium in the magnetic field produced by an intermittent current circulating in the coils of an electromagnet. The phonic wheel or synchronous motor devised by La Cour and Lord Rayleigh is, as is well known, of great service in acoustical investigations. In my own work on vibrations and their maintenance, it has been of considerable assistance, *vide* sections I, II and IV to VI of Bulletin no. 6. Apart however from the various uses of the instrument in different branches of Physics and in Applied Electricity, it possesses much intrinsic interest of its own as an excellent illustration of the dynamics of a system moving in a periodic field of force, and the present paper deals almost entirely with experiments carried out by its aid and with its applications to the study of vibrations.

The instrument used by me was supplied by Messrs. Pye & Co. of Cambridge and has given entire satisfaction. The motor consists of a wheel of soft iron mounted on an axis with ball-bearings between the two poles of an electromagnet placed diametrically with respect to it. The wheel has thirty teeth, and when a direct current is passed through the electromagnet, sets itself rigidly at rest with a pair of teeth at the ends of a diameter opposite the two poles of the electromagnet. The equilibrium under such conditions is of course thoroughly stable, and, in fact, the wheel possesses a fairly high frequency of free angular oscillation for displacements from this position of rest, and any motion set up by such displacement rapidly dies out, apparently on account of Foucalt currents induced in the iron by the motion. This, in general, is also true when an intermittent current supplied by a fork-interruptor is used to excite the electromagnet, except however in certain cases, when it is observed that the equilibrium becomes unstable of its own accord and the wheel settles down into a state of steady vigorous vibration about the line of equilibrium: or that an oscillation of sufficient amplitude once started maintains itself for an indefinitely long period.

An optical method can be conveniently used to study the frequency and the phase of the oscillations of the armature-wheel maintained in the manner described above. A narrow pencil of light is used, which first suffers reflection at the surface of a small mirror attached normally to one of the prongs of the fork-interruptor furnishing the intermittent current, and then falls upon a second similar mirror attached to the axle of the armature-wheel parallel to its axis of rotation. The apparatus is so arranged that the angular deflexions produced by the oscillations of the fork and the wheel are at right angles to each other, and the pencil of light which falls upon a distant screen, or which is focussed on the ground-glass of the photographic camera, is seen to describe a Lissajous figure from which the frequency, and the phase-relations between the oscillations of the fork-interruptor and of the armature-wheel, can be readily ascertained.

Figures 1 to 6, plate I, reproduce photographs of the curves secured in this manner. The vertical motion is that due to the fork and the horizontal motion that of the armature-wheel. Of these, figures 2 to 5 were obtained using a fork-interruptor of frequency about 24 per sec, and figure 1 with a fork of somewhat smaller frequency. It will be observed that the periods of the vibration of the armature-wheel as shown in figures 1 to 6 are respectively equal to, twice, thrice, four times, five times and six times the period of the fork: in other words the frequency is equal to or $\frac{1}{2}$ of or $\frac{1}{3}$ or $\frac{1}{4}$ or $\frac{1}{5}$ or $\frac{1}{6}$ that of the fork.

In taking the photographs, the motor-wheel was relieved of the large stroboscopic disk that is usually mounted upon it, and in working down the series, the adjustment of frequency is secured by suitably loading the wheel. The *fine* adjustment for resonance is effected by altering the current passing through



1 2 3 4 5 6

Plate I. To illustrate the maintenance of vibrations by a periodic field of force. The maintained vibration is horizontal and the vertical motion represents the periodicity of the field.

the interruptor with the aid of a rheostat and if necessary by regulating the contact-maker on the fork. Any oscillation of the wheel, when started, dies away except in the cases referred to above, in other words no frequencies intermediate between those of the series are maintained. In obtaining figure 1 in which the oscillations of the wheel and the fork are shown in unison, it is generally found necessary to increase the 'spring' of the wheel by passing a steady direct current through the electromagnet of the motor from a cell connected in parallel, in addition to the intermittent current flowing in the same direction from the interruptor circuit.

It will be seen that the Lissajous figures for the 1st, 3rd and 5th cases are distinctly asymmetrical in character; the 3rd (shown in figure 3) being markedly so. The 2nd, 4th and 6th types are quite symmetrical. This, it will be remembered, was what was anticipated above, and in fact the 1st, 3rd and 5th types differ rather markedly in their behaviour from the 2nd, 4th and 6th types. These latter are maintained with the greatest ease, while the former, particularly the 5th, are not altogether so readily maintained. In fact it is found advantageous, in order to maintain the 5th type steadily, to load the wheel somewhat unsymmetrically and to put it a little out of level, in order to allow the oscillations to take place about an axis slightly displaced from the line joining the poles of the electromagnet.

It will be noticed also that the lower frequencies of vibration have much larger amplitudes. This, I would attribute principally to the greatly reduced damping at the lower frequencies owing to the slower motion, the larger masses and the weaker magnetic fields employed.

We are now in a position to consider the mathematical theory of this class of maintained vibrations. To test the correctness of my theoretical work, I have prepared a series of photographs of the simultaneous vibration-curves of the fork and of the armature-wheel, which are reproduced as figures 1 to 3, plate II and figures 1 to 3, plate III. These curves were obtained by the usual method of recording the vibrations optically on a moving photographic plate, it being so arranged that the directions of movement of the two representative spots of light on the plate lie in the same straight line. The upper curve in each case shows the maintained vibration of the armature-wheel. The lower represents that of the fork-interruptor. The frequency of the former, it will be seen, is $\frac{1}{1}$ or $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$ or $\frac{1}{5}$ or $\frac{1}{6}\times$ that of the latter. The precise features of the vibration-curve noticed in each case will be referred to below, in connection with the mathematical discussion.

The equation of motion of a system with one degree of freedom moving in a periodic field of force, and subject also to the usually assumed type of viscous resistance, may be written in the following form,

$$\ddot{U} + k\ddot{U} + 2\alpha f(t)F(U) = 0 \tag{1}$$

where $F(U)$ gives the distribution of the field, $f(t)$ its variability with respect to time, and 2α is a constant. If we are dealing with oscillations about a position that would be one of stable equilibrium if the field were constant, $F(U)$ may as an

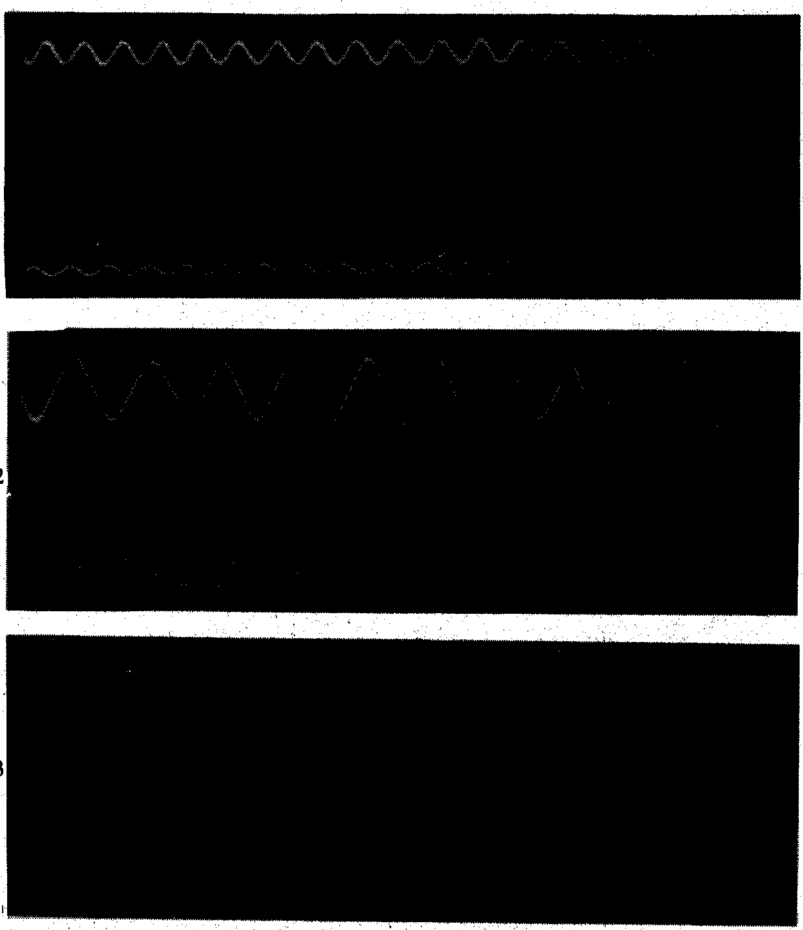


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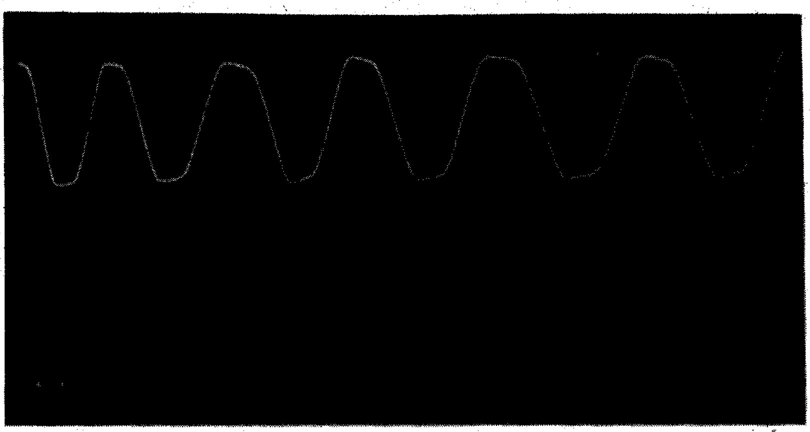
Plate II. Vibration-curves of oscillations maintained by a periodic impulsive field of force. The upper curve in each case represents the maintained motion.

1



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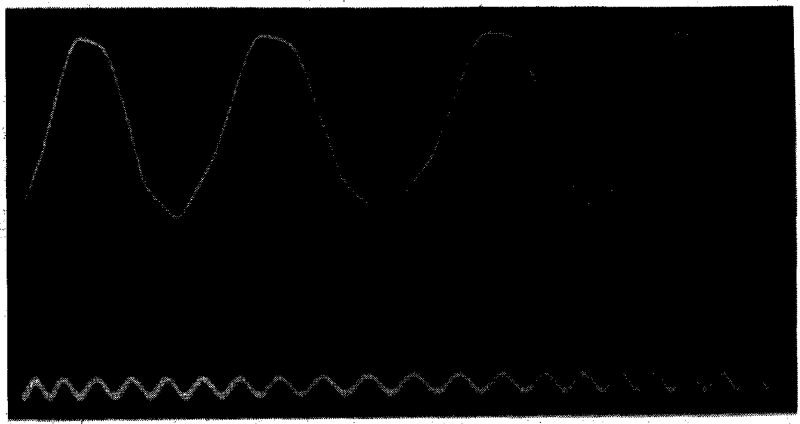

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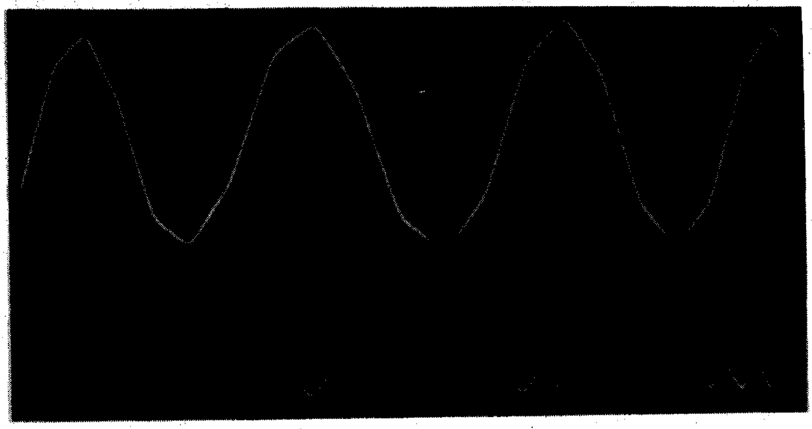

Plate III. Vibration-curves of oscillations maintained by a periodic impulsive field of force. The upper curve in each case represents the maintained motion.

approximation be put equal to U. We then have

$$\ddot{U} + k\ddot{U} + 2\alpha f(t)U = 0. \tag{2}$$

In the experiments described above, the periodicity of $f(t)$ is the same as that of the intermittence of the exciting current. If an alternating current had been used, the frequency of $f(t)$ would have been double that of the alternations. In any case we may write

$$\begin{aligned}\alpha f(t) = {} & a_1 \sin nt + a_2 \sin 2nt + a_3 \sin 3nt + \text{etc.} \\ & + b_0 + b_1 \cos nt + b_2 \cos 2nt + b_3 \cos 3nt + \text{etc.}\end{aligned} \tag{3}$$

Since U is shown to be periodic by experiment, we may write

$$\begin{aligned}U = {} & A_1 \sin pt + A_2 \sin 2pt + A_3 \sin 3pt + \text{etc.} \\ & + B_0 + B_1 \cos pt + B_2 \cos 2pt + B_3 \cos 3pt + \text{etc.}\end{aligned} \tag{4}$$

As a typical example of the even types of maintenance, we may take the cases in which $n = 4p$. We have

$$\begin{aligned}\alpha f(t) = {} & a_1 \sin 4pt + a_2 \sin 8pt + a_3 \sin 12pt + \text{etc.} \\ & + b_0 + b_1 \cos 4pt + b_2 \cos 8pt + b_3 \cos 12pt + \text{etc.}\end{aligned} \tag{5}$$

In this case, and also in the case of the second, sixth and in fact in all the *even* types of maintenance, we find that the quantities A_2, A_4, A_6, etc. and B_0, B_2, B_4, etc. do not enter into the equations containing A_1 and B_1. We therefore write them all equal to zero. The significance of this is that with the *even* types of vibration maintained by a periodic field of force, the *even harmonics are all absent from the maintained motion*. This result is fully verified by a reference to the vibration-curves of the 2nd, 4th and 6th types shown in figure 2, plate II, and figures 1 and 3, plate III. It will be seen that the vibratory motion of the armature-wheel has that type of symmetry so familiar in alternating current curves, in which all the even harmonics are absent. In other words, the image of one-half of the curve above the zero axis as seen by reflexion in a mirror placed parallel to this axis, is exactly similar to the other half below it.

Substituting now the odd terms alone left on the right-hand side of (4), for U in equation (2), we have the following series of equations:

$$\begin{aligned}
-(b_0 - p^2)A_1 + kpB_1 &= -b_1A_3 + a_1B_3 + b_1A_5 - a_1B_5 - \text{etc.} + \text{etc.} \\
-(b_0 - p^2)B_1 - kpA_1 &= a_1A_3 + b_1B_3 + a_1A_5 + b_1B_5 + \text{etc.} + \text{etc.} \\
-(b_0 - 9p^2)A_3 + 3kpB_3 &= -b_1A_1 + a_1B_1 - b_2A_5 + a_2B_5 + \text{etc.} - \text{etc.} \\
-(b_0 - 9p^2)B_3 - 3kpA_3 &= a_1A_1 + b_1B_1 + a_2A_5 + b_2B_5 + \text{etc.} + \text{etc.}
\end{aligned} \tag{6}$$

and so on.

Evidently, the possibility of this being a consistent set of convergent equations depends upon the suitability of the values assigned to the constants k, p, b_0, a_1, b_1, etc.

It is not possible here to enter into a complete discussion of the solution of these equations. One point is however noteworthy. From the first two of the set of equations given above, it will be seen that such of the *harmonics* in the steady motion of the system as are present serve as the vehicles for the supply of the energy requisite for the maintenance of the fundamental part of the motion. Paradoxically enough, the frequency of none of these harmonics is the same as that of the field.

We now proceed to consider the odd types of vibration, i.e. the 1st, the 3rd, etc. Taking the 3rd as a typical case, we put $n = 3p$ and get

$$\alpha f(t) = a_1 \sin 3pt + a_2 \sin 6pt + a_3 \sin 9pt + \text{etc.}$$
$$+ b_0 + b_1 \cos 3pt + b_2 \cos 6pt + b_3 \cos 9pt + \text{etc.} \qquad (7)$$

Substituting (4) and (7) in equation (2) and equating the coefficients of sine and cosine terms of various periodicities to zero, we find that the quantities A_3, A_6, A_9, etc. and B_0, B_3, B_6, B_9, etc. do not enter into the equations containing A_1 and B_1. We therefore write them all equal to zero. The significance of this is that the maintained motion contains no harmonics, the frequency of which is the same as, or any multiple of the frequency of the periodic field of force. This remarkable result is verified by a reference to figure 3, plate I, from which it is seen, that the vibration curve is roughly similar to that of the motion of a trisection point of a string bowed near the end, the 3rd component, the 6th, the 9th, etc. being absent at the point of observation.

We then obtain the following set of relations by substitution:

$$-(b_0 - p^2)A_1 + kpB_1 = -b_1A_2 + a_1B_2 + b_1A_4 - a_1B_4 - \text{etc.}$$
$$-(b_0 - p^2)B_1 - kpA_1 = a_1A_2 + b_1B_2 + a_1A_4 + b_1B_4 + \text{etc.}$$
$$-(b_0 - 4p^2)A_2 + 2kpB_2 = -b_1A_1 + a_1B_1 + b_1A_5 - a_1B_5 + \text{etc.}$$
$$-(b_0 - 4p^2)B_2 - 2kpA_2 = a_1A_1 + b_1B_1 + a_1A_5 + b_1B_5 + \text{etc.} \qquad (8)$$

and so on.

It must be remembered that these relations are all only approximate, as $F(U)$ in general contains powers of U higher than the first which we have neglected, and which no doubt must be taken into account in framing a more complete theory. The general remarks made above with reference to equation (6) apply here also.

The exact character of the vibratory motion maintained by the periodic field of force in any case, depends upon the form of the functions $F(U)$ and $f(t)$ which determine respectively the disposition of the field and its variability with respect to time. One very simple and important form of $f(t)$ is that in which the field is of an impulsive character, in other words is of great strength for a very short interval of time comprised in its period of variation, and during the rest of the period is zero or nearly zero. Such a type of variation is not merely a mathematical possibility. In actual experiment, when a fork-interruptor is used to render the current passing through the electromagnet intermittent, the magnetization of the latter subsists only during the small fraction of the period during which the current flows and at other times is practically zero. When the current is flowing, the acceleration is considerable: at other times, the acceleration is nearly zero, and the velocity practically constant. These features are distinctly shown in all the vibration-curves (except those of the first type) reproduced in plates II and III, the sudden bends in the curves corresponding roughly to the extreme outward swings of the fork, i.e. to the instants when the magnetizing current was a maximum. It seems possible that a simpler mathematical treatment than that given above might be sufficient to discuss the phenomena of the maintenance of vibrations by a periodic field of force when the periodicity of the field is of the 'impulsive' type, in other words when the dynamical system is subject to periodic impulsive 'springs' one, two, three or more of which occur at regular intervals during each complete period of the vibration of the system.

These experiments on vibrations maintained by a periodic field of force are very well suited for lecture demonstration, as the Lissajous figures obtained by the method described above can be projected on the screen on a large scale, and form a most convincing demonstration of the fact that the frequency of the maintained motion is an exact sub-multiple of the frequency of the exciting current.

On synchronous rotation under simple excitation

It is well known that with an intermittent current passing through its electromagnet, the synchronous motor can maintain itself in 'uniform' rotation, when for every period of the current, one tooth in the armature-wheel passes each pole of the electromagnet. In other words, the number of teeth passing per second is the same as the frequency of the intermittent current. From a dynamical point of view it is of interest, therefore, to investigate whether the motor could run itself successfully at any speeds other than the 'synchronous' speed. Some preliminary trials with the motor unassisted by any independent driving proved very encouraging. The phonic wheel I have is mounted on ball-bearings, and runs very lightly when the large stroboscopic disc usually kept fixed upon it is taken off, and there is no current passing through the motor. When a continuous or intermittent current is flowing through the motor, the latter does not however run very lightly, being subject to very large electromagnetic damping apparently due to Foucalt currents in the iron. In the preliminary trials, however, I found that, using the intermittent unidirectional current from an interruptor-fork of frequency 60, the motor could run successfully of itself at *half* the synchronous speed, i.e. with 30 teeth passing per second. It of course ran very well at the usual synchronous speed, i.e. with 60 teeth passing per second. By increasing the speed, it was found that the motor could also run well of itself at *double* the synchronous speed, i.e. with 120 teeth passing per second. Using an interruptor-fork of low frequency

(23.5 per second) the motor, it was found, could also run of itself at *triple* the synchronous speed. No certain indication was however obtained of the intermediate speeds, i.e. $1\frac{1}{2}$ and $2\frac{1}{2}$ times respectively the synchronous speed.

To test these points, therefore, independent driving was provided. This was very satisfactorily obtained by fixing a small vertical water-wheel to the end of the axis of the motor and directing a jet of water against it. The water-wheel was boxed in to prevent any splashing of water on the observer. By regulating the tap leading up to the jet, the velocity of the latter could be adjusted. The speed of the phonic wheel was ascertained by an optical method, i.e. by observing the rim of the wheel as seen reflected in a mirror attached to the prong of the interruptor-fork. When the motor 'bites,' the pattern seen becomes stationary and remains so for long intervals of time or even indefinitely, and the speed of the wheel can be inferred at once from the nature of the pattern seen.

It was found in these trials that the motor could 'bite' and run at the following speeds. (Frequency of interruptor 60 per sec.).

(*a*) $\frac{1}{2}$ the synchronous speed: stationary pattern of rim of moving wheel seen as a single sine-curve: wavelength $\frac{1}{2}$ the interval between teeth. Number of teeth passing electromagnet per second = 30.

(*b*) Synchronous speed: stationary pattern of rim of moving wheel seen as a sine-curve, wavelength = interval between teeth. Number of teeth passing electromagnet per second = 60.

(*c*) $1\frac{1}{2}$ times the synchronous speed: stationary pattern of rim of wheel seen as *three* interlacing waves. Number of teeth passing electromagnet per second = 90.

(*d*) 2 times the synchronous speed: stationary pattern seen as *two* interlacing curves. Number of teeth passing per second = 120.

(*e*) $2\frac{1}{2}$ times the synchronous speed: this was only obtained with difficulty. Number of teeth passing per second = 150.

(*f*) 3 times the synchronous speed: stationary pattern seen as *three* interlacing curves. Very satisfactory running. Number of teeth per second = 180.

(*g*) 4 times the synchronous speed: stationary pattern seen as 4 interlacing curves. Number of teeth per second = 240.

(*h*) 5 times the synchronous speed: stationary pattern seen as 5 interlacing curves. Number of teeth per second = 300.

The outstanding fact of observation is that while speeds which are equal to the 'synchronous' speed or any integral multiple of it are readily maintained, only the first two or three members of the other series (i.e. having ratios $\frac{1}{2}$, $1\frac{1}{2}$ etc. to the synchronous speed) can be obtained and the 'grip' of the wheel by the periodic magnetic forces, i.e. the stability of the motion, is hardly so great as in the integral series. This fact may be explained in the following general manner.

We may assume, to begin with, that the independent driving is less powerful than that required to overcome resistances, so that the wheel is a little *behind* the correct phases. In the case of the integral series, one or two or more teeth pass for every intermittence of the current, the wheel being in the same relative position, whatever this may be, to the electromagnet, at each phase of maximum magnetization of the latter. This is not, however, the case with the fractional speeds. It is only at every *alternate* phase of maximum magnetization that the wheel assumes the same position (whatever this may be) relative to the electromagnet. At the intermediate phases, it is displaced through a distance approximately equal to half the interval between the teeth. Whereas with the integral series, *every* phase of maximum magnetization *assists* the rotation, in the fractional series the wheel is alternately assisted and retarded by the successive phases of maximum magnetization, and it is the *net* effect of assistance that we perceive, this being of course comparatively small.

The explanation given above may be made clearer by reference to plates IV and V, in which are reproduced photographs of the armature-wheel in rotation taken under special arrangements for illumination. Figures 1 and 4, plate IV, show the motor in rotation at half the synchronous speed, photographed under intermittent illumination having the same frequency as the fork-interruptor. Figure 1 refers to the anti-clockwise rotation, and figure 4 to clockwise rotation. Figures 1 and 4, plate V, refer similarly to the cases of rotation at the normal synchronous speed in the anti-clockwise and clockwise directions respectively. In these four photographs the position of both the poles of the electromagnet is indicated, and the displacement of the position of the armature-wheel with respect to the poles on reversing the direction of rotation can readily be made out by comparison. From figures 1 and 4, plate IV, it will be seen that in the case of the half-speed rotation the attraction of the poles on the armature-wheel would tend alternately to encourage and retard the motion, the difference only being the surplus available to balance the loss of energy due to frictional forces.

In taking these photographs it was arranged that the axis of the motor should point towards the camera, and the normal illumination of the wheel was secured by an arc lamp and a silvered mirror placed at an angle of 45° between the motor and camera. The removal of a small circular patch of the silvering from the back of the glass opposite the lens of the camera, and the interposition between these two of the fork-interruptor with light overlapping plates fixed on its prongs periodically to open out a passage for the light to enter the camera, completed the arrangements. It was found that rotation at double the synchronous speed could be similarly photographed: the definition was however not so good on account of the greater velocity of motion. It was found that the arrangement described above was far more satisfactory than the employment of intermittent illumination from the spark of an induction coil worked with a fork-interruptor.

As the synchronous, half-synchronous and double-synchronous speeds can all be readily maintained without independent driving, they can be very effectively exhibited as lecture experiments by lantern projection in the following way. The synchronous motor (which is quite small and light when the stroboscopic disk is

1

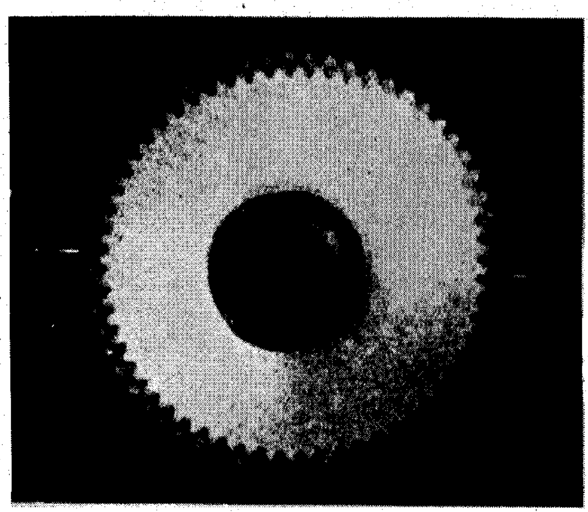

Figure 1. Synchronous motor, rotating at half-speed (anti-clockwise) photographed under periodic illumination.

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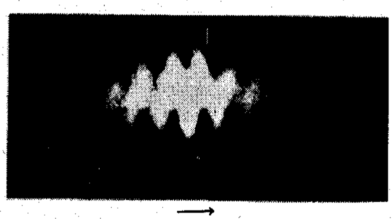

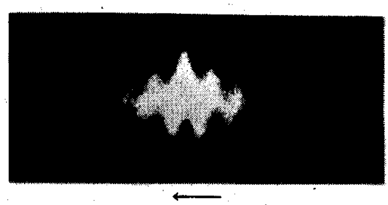

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Figures 2 and 3. Edge of wheel photographed through oscillating lens. Direction of rotation indicated by arrow.

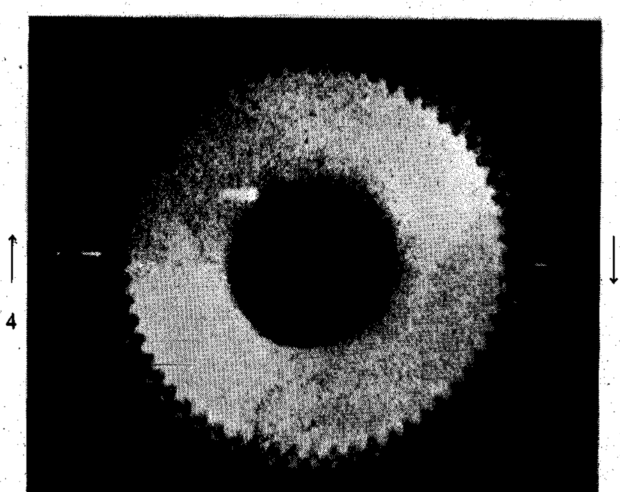

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Figure 4. Same as figure 1 with direction of rotation reversed.

Plate IV

1

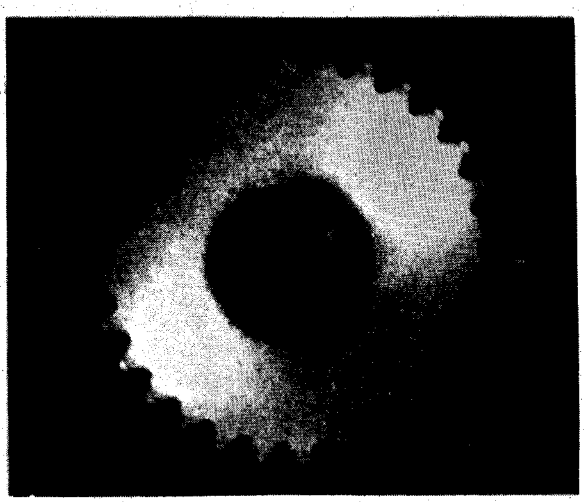
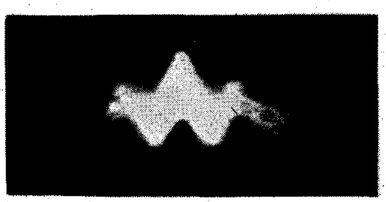


Figure 1. Synchronous motor, rotating at normal speed (anti-clockwise) photographed under intermittent illumination.

2



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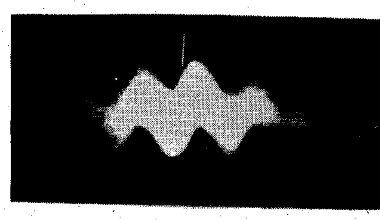

Figures 2 and 3. Edge of wheel photographed through oscillating lens. Direction of rotation indicated by arrow.

4

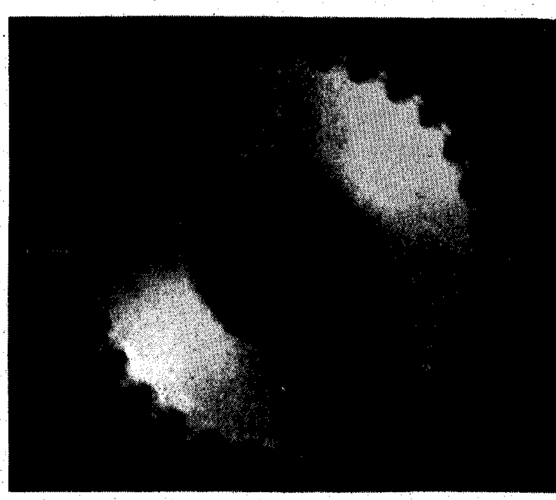

Figure 4. Same as figure 1 with direction of rotation reversed.

Plate V

removed) is placed on the horizontal stage of the lantern and the rim of the wheel is focussed on the screen. In front of the projection prism, where the image of the source of light is formed, is placed the fork-interruptor with the necessary device for intermittent illumination fitted to its prongs. When these are set into vibration and the synchronous motor is set in rotation, the "pattern" corresponding to the maintained speed becomes visible on the screen, and the effect of reversing the direction of rotation can also be demonstrated.

Figures 2 and 3 in plates IV and V are photographs of the edge of the phonic wheel revolving at the half-synchronous and the synchronous speeds, taken through a lens which was fixed to one of the prongs of the interruptor-fork, and which therefore caused the image of the wheel to vibrate on the plate in a direction at right angles to the plane of rotation. In these photographs also, the displacement of the position caused by reversing the direction of rotation is clearly shown.

We now proceed to discuss the mathematical theory of the maintenance of uniform rotation in each of these cases. The first step is obviously to show that with the assumed velocity of rotation, the attractive forces acting on the disc communicate sufficient energy to it to balance the loss due to frictional forces. Taking the line joining the poles as the axis of x, the position of the wheel at any instant may be defined by the angle θ which a diameter of the wheel passing through a given pair of teeth makes with the axis of reference. If n is the number of teeth in the wheel, the couple acting on the latter for any given field strength at the poles is obviously a periodic function of $n\theta$ which vanishes when $\theta = 2\pi r/n$, and also when $\theta = 2\pi(r+\frac{1}{2})/n$, where r is any integer.

We therefore write

$$\begin{aligned}\text{Couple} &= \text{Field strength} \times [a_1 \sin n\theta + a_2 \sin 2n\theta + a_3 \sin 3n\theta + \text{etc.}]\\ &= \text{Field stength} \times f(n\theta) \text{ say,}\end{aligned}$$

where the terms a_1, a_2, a_3, etc. rapidly diminish in amplitude. It will be seen that the cosine terms are absent. Since the field strength is periodic, we may write the expression for the couple acting on the wheel thus

$$\begin{aligned}\text{Couple} &= Lf(n\theta)[b_1 \sin(pt+\varepsilon_1) + b_2 \sin(2pt+\varepsilon_2) + \text{etc.}]\\ &= Lf(n\theta)F(t), \text{ say.}\end{aligned}$$

The work done by the couple in any number of revolutions

$$= \int Lf(n\theta)F(t)\mathrm{d}t.$$

It is obvious that this integral after any number of complete revolutions is zero, except in any of the following cases, when it has a finite value proportional to and increasing with t; i.e. when

$$n\theta = pt \text{ or } 2pt \text{ or } 3pt \text{ or } 4pt \text{ and so on}$$

or when

$$2n\theta = pt \text{ or } 2pt \text{ or } 3pt \text{ or } 4pt \text{ and so on}$$

or when

$$3n\theta = pt \text{ or } 2pt \text{ and so on.}$$

It is therefore a necessary but not, of course, always a sufficient condition for uniform rotation to be possible that one or more of the above relations should be satisfied. The first series corresponds to the synchronous speed and multiples of the synchronous speed. These have been observed experimentally by me up to the fifth at least. The second series includes the above and also the half-synchronous speed and odd multiples of the same. These latter have also been observed by me up to the fifth odd multiple. Since a_2 is much smaller than a_1, the relative feebleness of the maintenance of the half-speeds observed in experiment will readily be understood.

The third series has not so far been noticed in experiment. It is obvious that the maintaining forces in it should be excessively feeble compared with the first or the second. Perhaps, experiments with interruptor-forks of higher frequencies and independent driving of the motor may succeed in showing the existence of controlled rotation-speeds at these ratios.

Figure 1, plate VI, shows a record of the rotation of the motor maintained at double the synchronous speed, side by side with the vibration-curve of the fork-interruptor which supplied the intermittent current necessary. The record of the rotation of the motor was secured by the use of a moving photographic plate, and of an illuminated slit, the light from the latter being cut off periodically by the teeth of the armature-wheel, as it revolved. An illuminated pin-hole in a small piece of metal foil attached to a prong of the fork was simultaneously photographed on the plate. It will be seen that the number of teeth of the armature-wheel that passed in any given time is double the number of vibrations of the fork recorded in the same time.

Combinational rotation-speeds under double excitation

When the electromagnet of the synchronous motor is excited simultaneously by the intermittent currents from two separate interruptor-forks having different frequencies, maintenance of uniform rotation is possible not only at the various speeds related to the synchronous speeds due to either of the intermittent currents acting by itself, but also at speeds related jointly to the frequencies of the two currents.

The preliminary experiments on this point were made without the assistance of any independent driving of the motor and it found at once that differential rotation of the motor was easily maintained, the number of teeth passing per second being equal to the difference of the frequencies of the two interruptor-forks. This result is shown in figure 2, plate VI, the motion of the wheel and of the two forks being recorded in the same way as that described in a preceding paragraph. In this case it is seen that the teeth of the rotating armature-wheel have been clearly reproduced on the moving photographic plate.

When the 'differentially' revolving wheel was examined by reflexion in mirrors attached to the prongs of the two interruptor-forks, it was found that the patterns seen in neither of them was stationary. They were found to be moving steadily forward or backward with a definite speed, with occasional slight to and fro oscillations superposed thereon. This continuous rotation of the pattern seen was obviously due to the fact that the frequencies of the forks and their difference did not bear any simple arithmetical ratios to each other, and it enabled a rotation-speed maintained by joint action to be distinguished by mere inspection from one maintained by either of the two currents separately.

Using this optical method, and assisting the rotation of the motor with independent driving by a water-motor, various other combinational speeds were found to be maintained. Of these, the most powerfully and steadily maintained was the simple summational rotation. The summationals and differentials of the second series, i.e. those in which the half-frequencies of the fork enter, were also noticed. The rotation-speeds were determined by actual counting and a stop-watch.

The mathematical theory of these combinational speeds is very similar to that given for the case of excitation by one periodic current. For, the field strength in this case is also a periodic function of the time, and the function $F(t)$ which expresses its value at any instant may be expanded in the following form:

$$F(t) = a\Sigma\Sigma b \sin[(rp_1 \pm sp_2)t + E],$$

where $p_1/2\pi$ and $p_2/2\pi$ are the frequencies of the two interruptors, and r, s are any two positive integers. Using the same notation as before, we find that in any complete number of revolutions, a finite amount of energy proportional to the time is communicated to the wheel, only in any one out of the following sets of cases:

$$n\theta = (rp_1 \pm sp_2)t$$

or

$$2n\theta = (rp_1 \pm sp_2)t$$

or

$$3n\theta = (rp_1 \pm sp_2)t$$

and so on.

The cases actually observed in which rotation is maintained fall within the first two of the sets given above.

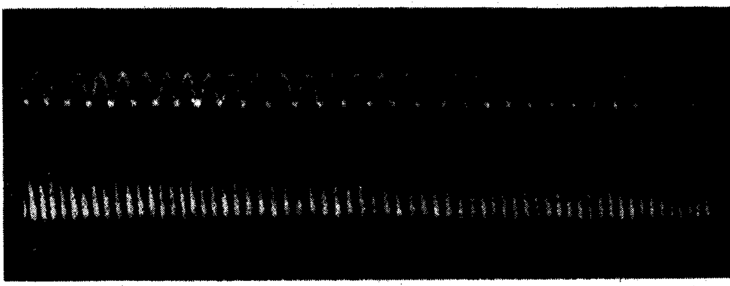

Figure 1. Double-synchronous rotation of phonic wheel.

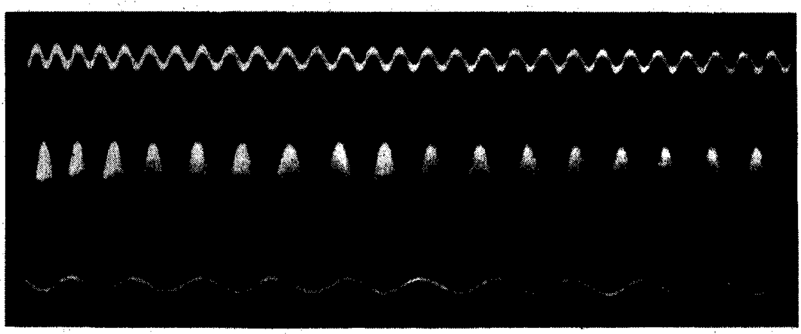

Figure 2. Differential rotation of phonic wheel.

Figure 3

Figure 4. Stroboscopic pictures of vibrating string. (See also plates XIII & XIV.)

Plate VI

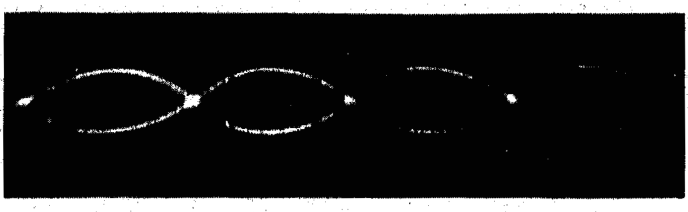

Figure 1. First type: 30-slot disk.

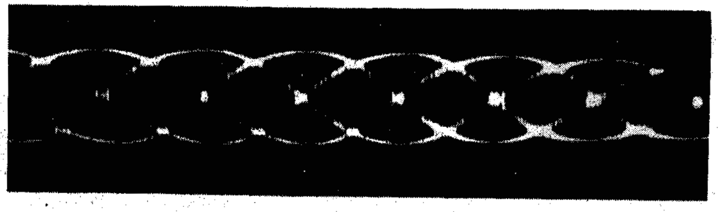

Figure 2. First type: 60-slot disk.

Figure 3. Second type: 30-slot disk.

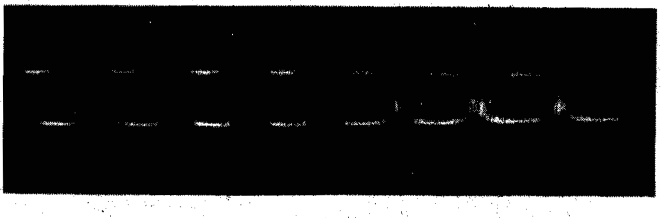

Figure 4. Second type: 60-slot disk.

Plate VII. Vibrations of a stretched string maintained by a variable tension, observed through a stroboscopic disk mounted on a synchronous motor.

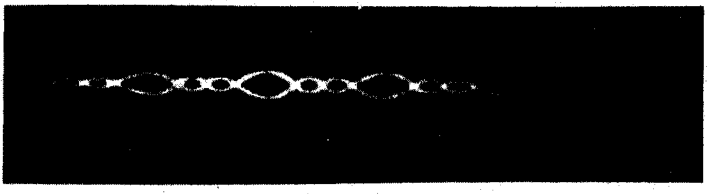

Figure 1. Third type: 30-slot disk.

Figure 2. Third type: 60-slot disk.

Figure 3. Fourth type: 30-slot disk.

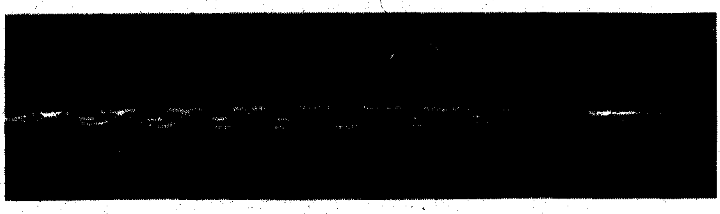

Figure 4. Fourth type: 60-slot disk.

Plate VIII. Vibrations of a stretched string maintained by a variable tension observed through a stroboscopic disk mounted on a synchronous motor.

Some applications of the synchronous motor

In acoustical observations, particularly where regularly intermittent illumination is desired, the synchronous motor is of great use. This is specially the case, when it is desired that the vibrations under observation should be seen under control, simultaneously or successively in various phases of motion. An excellent example where this requirement has to be met is in the study of the small motion at the nodes of a vibrating string. A series of 13 photographs of this small motion secured with a stroboscopic disc mounted on a synchronous motor was published as plate II of Bulletin no. 6. Another very interesting application of a stroboscopic disc with radical slits mounted on a synchronous motor is for the study of the oscillations of a stretched string under a variable tension. In this case, as I have shown, the vibrations of the string can be maintained when the frequency of its free oscillations is half of, or equal to, or one and half times, etc., etc. the frequency of the fork which varies its tension. This fork is electrically self-maintained, and the intermittent current from it also feeds the synchronous motor. The number of radial slits in the stroboscopic disc is either 30 or 60, i.e. equal to, or double the number of teeth on the armature-wheel. In making the observations, the stroboscopic disc is held vertically and the string is set horizontal and parallel to the disc is viewed through the top row of slits, i.e. those which are vertical and move in a direction parallel to the string as the disc revolves. It is advantageous to have the whole length of the string brilliantly illuminated and to let as little stray light as possible fall upon the reverse of the disc at some distance from which the observer takes his stand. A brilliant view is then obtained. Under these circumstances we see the string in successive cycles of phase along its length, and the peculiar character of the maintained motion in these cases is brought out in a very remarkable way. *The string is seen in the form of a stationary vibration curve*, which would be identical with those published in Bulletin no. 6, but for the fact that the amplitude of motion is not the same at all points of the string, being a maximum at the ventral segments and zero at the nodes.

Another point calls for remark. Using a fork with a frequency of 60 per second, the *free* oscillations of the string have a frequency of 30 in the case of the 1st type, 60 in the case of the 2nd, 90 with the 3rd, 120 with the 4th, 150 with the 5th, and so on. With the disc having 30 slits on it we get 60 views per second of any one point on the string, and with the even types of motion, i.e. the 2nd, 4th, etc. the 'vibration-curve' seen through the stroboscopic disc appears single. With the odd types, i.e. the 1st, 3rd, 5th, etc., *two* vibration-curves are seen, one of which is as nearly as can be seen the mirror-image of the other, intersecting it at points which lie or should lie upon the equilibrium position of the string. The reason why with the odd types we see the vibration-curve double is fairly clear and furnishes an excellent illustration of the principles of stroboscopic observation. The double pattern brings home to the eye in a vivid and convincing manner the fact that under the action of the variable spring the 'amplitude' and 'period' of the motion periodically increase and decrease after the manner of 'beats.'

An interesting variation on the experiment is made by using the disc with 60 slits. We then get 120 views per second and with the even types we get the vibration-curves double, but one of the curves is not the mirror-image of the other, the motion not being symmetrical. On the other hand, with the odd types we see the vibration-curves in quadruple pattern.

Plates VII and VIII, and figures 3 and 4, plate VI, reproduce photographs secured with the apparatus described above.

Figures 1 and 2, plate VII, represent the 1st type of maintenance observed through the 30-slot and the 60-slot discs respectively.

Figure 3, plate VII, represents the 2nd type seen through the 30-slot disc and figure 3, plate VI, and figure 4, plate VII, represent this type as seen through the 60-slot disc.

Figures 1 and 2, plate VIII, represent the 3rd type as seen through the two discs. Figure 3 and 4, plate VIII, similarly show the 4th type. Figure 4, plate VI, shows a compound type of vibration maintained by a simple harmonic variation of tension as seen through the 30-slot disc. It will be seen that in all these photographs, there is a slight degree of distortion towards the end of the string, due to the fact that the discs are relatively of small diameter and that the ends of the string are seen through the slits in a position of the latter in which they are inclined and move in a direction inclined to the string.

The arrangement described above may also be applied to the study of the motion of a bowed string. For this purpose, the string should be carefully tuned to be in unison with the fork-interruptor running the synchronous motor. With this arrangement, it is possible at one glance to recognize the modes of vibration at all points of the bowed string simultaneously. The definition of the curve seen is fairly good in the case of the simpler types, but it is not easy or possible to secure satisfactory photographs of the configuration of a bowed string by this method, on account of the appreciable width of the slots on the disc.