THE ASTROPHYSICAL JOURNAL, 428:419–432, 1994 June 20

PERTURBATIVE GROWTH OF COSMOLOGICAL CLUSTERING. I. FORMALISM

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Received 1993 August 26; accepted 1993 December 20

ABSTRACT

Here we rederive the hierarchy of equations for the evolution of distribution functions of various orders using a convenient parameterization. We use this to obtain equations for evolving the two- and three-point correlation functions in powers of a small parameter, viz., the initial density contrast. The correspondence of the lowest order solutions of these equations to the results from the linear theory of density perturbations is shown for an $\Omega = 1$ universe. These equations are then used to calculate, to the lowest order, the induced three-point correlation function that arises from Gaussian initial conditions in an $\Omega = 1$ universe. We obtain an expression which explicitly exhibits the spatial structure of the induced three-point correlation function. It is seen that the spatial structure of this quantity is independent of the value of Ω. We also calculate the triplet momentum. We find that the induced three-point correlation function does not have the "hierarchical" form often assumed. We discuss possibilities of using the induced three-point correlation to interpret observational data. The formalism developed here can also be used to test a validity of different schemes to close the BBGKY hierarchy.

Subject headings: galaxies: clustering — large-scale structure of universe — methods: analytical

1. INTRODUCTION

Most efforts at understanding the formation of structure in the universe use numerical simulations. Although this allows us to evolve given initial perturbations, there is still room for using analytic approaches to improve our understanding of the physics of gravitational clustering. Also, we cannot numerically evolve the observations backward in time to find the initial conditions, and there do not exist very strong observational reasons to choose one kind of initial conditions over another. On may then have to work through a very large number of initial conditions before getting one that matches observations. These arguments motivate analytic treatments, the one explored in this paper being perturbation theory.

In this paper we consider the purely gravitational growth of perturbations in an expanding universe filled with dust. The evolution of the statistical quantities characterizing these perturbations is described by the BBGKY hierarchy (§ 3). In the fluid limit this hierarchy has infinite equations, and they cannot be generally solved. We consider perturbations that are initially small (§ 4), and this provides a natural means for truncating the hierarchy. By taking moments in velocity space, we derive equations for perturbatively evolving the two- and three-point correlation functions (§ 5). To the lowest order the solutions to these equations correspond to the correlations from the linear theory of density perturbations. These equations can be solved to higher order to obtain the correlation functions in the weakly nonlinear regime.

The BBGKY hierarchy can also be used to study the correlations in the strongly nonlinear regime. In this case, one has to assume some scheme for closing the hierarchy. Such a scheme could be tested using the formalism developed in this paper. The study of the hierarchy in the weakly nonlinear regime may also yield clues that could be used to construct a scheme for closing the BBGKY hierarchy in the strongly nonlinear regime.

The general area of the application of the BBGKY hierarchy to cosmological correlations has been the subject of many studies (Peebles 1980 and references given there). In this paper we have given a self-contained treatment of the subject. This is in view of two technical differences from the earlier work, viz., a nonstandard parameter (§ 2) has been used in place of cosmic time and the entire treatment is in real space.

In § 6 we assume that the initial perturbations are Gaussian and use the equations developed in the previous section to calculate, to the lowest order, the induced three-point correlation of an $\Omega = 1$ universe. We also calculate the triplet momentum.

Fry (1984) has calculated the three-point correlation function for initial conditions that are Gaussian and have a power-law power spectrum by using second-order perturbation theory (Peebles 1980). The method used by Fry may be described as evolving one realization of the perturbations and then calculating the correlations, whereas in this work we dynamically evolve the statistical quantities themselves. In § 7 we present a comparison of the three-point correlation function as calculated by us and by Fry. Inagaki (1991) also has calculated the Fourier transform of the three-point correlation function perturbatively for Gaussian initial conditions using the BBGKY hierarchy.

In § 7 we also briefly discuss the possibility of applying the calculated three-point correlation function to observational data and the possibility of using the formalism developed here to test the validity of schemes for closing the BBGKY hierarchy. We make some remarks on a scheme used by Davis & Peebles (1977) to close the hierarchy.

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2. THE EVOLUTION PARAMETER

Consider a system of a large number (N) of collisionless particles interacting through Newtonian gravity. The Lagrangian for such a system is

$$L = \frac{m}{2} \sum_a \left(\frac{dr^a}{dt}\right)^2 + \frac{m^2 G}{2} \sum_{a \neq b} \frac{1}{|r^a - r^b|}, \tag{1}$$

where r_μ^a refers to the μ Cartesian component of the a particle. When there is no subscript, it refers to the vector r^a. We transform to a time-dependent coordinate system with new coordinate x, with

$$r_\mu^a(t) = S(t) x_\mu^a(t), \tag{2}$$

where $S(t)$ is a function of time. The Lagrangian becomes

$$L = \frac{mS^2}{2} \sum_a \left(\frac{dx^a}{dt}\right)^2 + \frac{m^2 G}{2S} \sum_{a \neq b} \frac{1}{|x^a - x^b|} - \frac{mS}{2} \frac{d^2 S}{dt^2} \sum_a (x^a)^2 . \tag{3}$$

The extra potential arises due to the change to an accelerating coordinate system. The function $S(t)$ (scale factor) is dimensionless and is chosen such that it satisfies

$$\frac{d^2 S}{dt^2} = -\frac{4\pi G \rho}{3S^2} \tag{4}$$

where $\rho\, d^3x$ is the mass that would be in the volume d^3x if all the particles were uniformly distributed.

The Lagrangian then becomes

$$L = \frac{mS^2}{2} \sum_a \left(\frac{dx^a}{dt}\right)^2 + \frac{m^2 G}{2S} \sum_{a \neq b} \frac{1}{|x^a - x^b|} + \frac{2\pi G \rho m}{3S} \sum_a (x^a)^2 . \tag{5}$$

If the particles are all uniformly distributed, the attractive force of gravity is exactly canceled by the repulsive harmonic oscillator force described by the last term in equation (5). In this case, if all the particles start with $dx_\mu^a/dt = 0$, then the solution is $x_\mu^a(t) = x_\mu^a(t_0)$, i.e., the coordinate system moves with the particles. Thus x is a comoving coordinate system, and ρ is the comoving density, which remains a constant. In this case the system corresponds to a part of a homogeneous and isotropic universe where all the dynamical information is in $S(t)$. In this paper we consider how a system with an initial configuration slightly different from the above-mentioned one evolves.

Next, time is replaced by a parameter λ, where

$$d\lambda = \frac{dt}{S(t)^2} . \tag{6}$$

The condition

$$A = \int L\, dt = \int L'\, d\lambda \tag{7}$$

defines the new Lagrangian,

$$L' = \frac{m}{2} \sum_a \left(\frac{dx^a}{d\lambda}\right)^2 + \frac{Sm^2 G}{2} \sum_{a \neq b} \frac{1}{|x^a - x^b|} + \frac{2\pi S G \rho m}{3} \sum_a (x^a)^2 . \tag{8}$$

For evolution in λ the Hamiltonian is

$$H = \frac{1}{2m} \sum_a (p^a)^2 - \frac{Sm^2 G}{2} \sum_{a \neq b} \frac{1}{|x^a - x^b|} - \frac{2\pi S G \rho m}{3} \sum_a (x^a)^2 , \tag{9}$$

where

$$p_\mu^a = m \frac{dx_\mu^a}{d\lambda} \tag{10}$$

is the momentum conjugate to x_μ^a. The main advantage of using λ instead of the cosmic time is that no S appears explicitly in equation (10). As a result, the equation of motion for a particle, which is

$$\frac{d^2 x_\mu^a}{d\lambda^2} = SGm \sum_b \frac{x_\mu^b - x_\mu^a}{|x^a - x^b|} + \frac{4}{3} \pi S G \rho x_\mu^a , \tag{11}$$

resembles the equation of motion of a particle in an inertial reference frame with a time-dependent force. If instead we use cosmic time as the evolution parameter, derivatives of the scale factor appear in the equation of motion. These terms have been avoided by using the parameter λ.

The relation between this momentum and the peculiar velocity is

$$p_\mu^a = mSv_\mu^a \,. \tag{12}$$

In terms of λ equation (4) becomes

$$\frac{d}{d\lambda}\left(\frac{1}{S^2}\frac{dS}{d\lambda}\right) = -\frac{4\pi G\rho}{3} \,. \tag{13}$$

Any solution of this equation is given by a parabola,

$$\frac{1}{S} = \frac{2\pi G\rho}{3}\,[(\lambda + \lambda_1)^2 + K] \,. \tag{14}$$

Since the range of λ can be chosen arbitrarily, we set $\lambda_1 = 0$. The case with $K = 0$ corresponds to a $\Omega = 1$ universe where we have

$$S(\lambda) = \frac{3}{2\pi G\rho\lambda^2} \,, \tag{15}$$

with λ going from $-\infty$ to 0. In this paper for all calculations we assume $\Omega = 1$, and equation (15) is used for $S(\lambda)$.

3. THE BBGKY HIERARCHY AND EVOLUTION OF REDUCED DISTRIBUTION FUNCTIONS

It is assumed that (1) there is a large spatial scale on which the universe is homogeneous and isotropic, and (2) volumes of this size located at different parts of the universe are independent realizations of the same physical processes with different initial conditions. Such volumes can be "assembled" to form an ensemble.

The system defined in the previous section is a model for one member of such an ensemble. Such a system can be described by a distribution function on phase space. This function $f(x, p, \lambda)$ gives the number of particles in a unit volume of the phase space at the point (x, p) at the instant λ.

The ensemble described above can be used to define a M-particle distribution function $\rho_M(x^1, p^1, x^2, p^2, \ldots, x^M, p^M, \lambda)$ defined as

$$\rho_M(x^1, p^1, x^2, p^2, \ldots, x^M, p^M, \lambda) = \langle f(x^1, p^1, \lambda) \ldots f(x^M, p^M, \lambda) \rangle \,, \tag{16}$$

where the angular brackets indicate an average over all the systems in the ensemble.

This function gives the joint probability density of finding a particle in the volume $d^3x^1\, d^3p^1$ at the point (x^1, p^1) and in the volume $d^3x^2\, d^3p^2$ at the point (x^2, p^2) and in the volume $d^3x^3\, d^3p^3$ at the point (x^3, p^3), etc., at the instant λ. The evolution of the distribution functions is governed by the BBGKY hierarchy (Peebles 1980 and references given there). The first three equations of this hierarchy are given below in the fluid limit.

Because of homogeneity, the one-particle distribution function does not depend on position. This has been used in equation (18) below. We define a function f, where $f(p^1, \lambda) = \rho_1(x^1, p^1, \lambda)$.

In the equations below the numbers 1, 2, 3,... are used to refer to phase-space points $x^1, p^1, x^2, p^2, x^3, p^3, \ldots$. Later on the numbers 1, 2, 3,... are used to denote points $x^1, x^2, x^3, \ldots$ in space, as will be clear from the context. We also use the notation

$$X_\mu^{ab} = \frac{x_\mu^a - x_\mu^b}{|\boldsymbol{x}^a - \boldsymbol{x}^b|^3} \tag{17}$$

throughout the text.

$$\frac{\partial}{\partial\lambda}\rho_1(1, \lambda) + Sm^2G\frac{\partial}{\partial p_\mu^1}\int \rho_2(1, 2, \lambda)X_\mu^{21}\, d^3x^2\, d^3p^2 + \frac{4}{3}\pi SG\rho m x_\mu^1 \frac{\partial}{\partial p_\mu^1}\rho_1(1, \lambda) = 0 \,, \tag{18}$$

$$\frac{\partial}{\partial\lambda}\rho_2(1, 2, \lambda) + \frac{p_\mu^a}{m}\frac{\partial}{\partial x_\mu^a}\rho_2(1, 2, \lambda) + \frac{4}{3}\pi SG\rho m x_\mu^a\frac{\partial}{\partial p_\mu^a}\rho_2(1, 2, \lambda) + Sm^2G\frac{\partial}{\partial p_\mu^a}\int \rho_3(1, 2, 3, \lambda)X_\mu^{3a}\, d^3x^3\, d^3p^3 = 0 \,, \tag{19}$$

where a takes the values 1, 2, and

$$\frac{\partial}{\partial\lambda}\rho_3(1, 2, 3, \lambda) + \frac{p_\mu^a}{m}\frac{\partial}{\partial x_\mu^a}\rho_3(1, 2, 3, \lambda) + \frac{4}{3}\pi SG\rho m x_\mu^a\frac{\partial}{\partial p_\mu^a}\rho_3(1, 2, 3, \lambda) + Sm^2G\frac{\partial}{\partial p_\mu^a}\int \rho_4(1, 2, 3, 4, \lambda)X_\mu^{4a}\, d^3x^4\, d^3p^4 = 0 \,, \tag{20}$$

where a takes the values 1, 2, and 3.

The hierarchy continues up to the equation for the N-particle distribution function.

For special initial conditions it is possible to truncate the hierarchy at some level, the error from the terms dropped being of a higher order in some small parameter compared to the terms retained. In order to do this explicitly, it is convenient to work in terms of the reduced distribution functions defined below.

The probability density for finding a particle at x^1 with momentum p^1 and another with position x^2 and momentum p^2 has a contribution from the one-particle distribution function. This is $f(1)f(2)$. The reduced two-particle distribution function is defined by the equation

$$\rho_2(1, 2) = f(1)f(2) + c(1, 2) \,. \tag{21}$$

The reduced three- and four-particle distribution functions are similarly defined by

$$\rho_3(1, 2, 3) = f(1)f(2)f(3) + \sum_P f(1)c(2, 3) + d(1, 2, 3) , \tag{22}$$

$$\rho_4(1, 2, 3, 4) = f(1)f(2)f(3)f(4) + \sum_P f(1)f(2)c(3, 4) + \sum_P c(1, 2)c(3, 4) + \sum_P f(1)d(2, 3, 4) + e(1, 2, 3, 4) , \tag{23}$$

where $\sum_P$ means a sum over all cyclic permutations of the particle indices. Equations (18), (19), and (20) and the definition of the reduced distribution functions can be combined to obtain equations for the evolution of the reduced distribution functions. These equations are

$$\frac{\partial}{\partial \lambda} f(1, \lambda) + Sm^2G \frac{\partial}{\partial p^1_\mu} \int c(1, 2, \lambda) X^{21}_\mu \, d^3x^2 \, d^3p^2 = 0 , \tag{24}$$

$$\frac{\partial}{\partial \lambda} c(1, 2, \lambda) + \frac{p^a_\mu}{m} \frac{\partial}{\partial x^a_\mu} c(1, 2, \lambda) + Sm^2G \frac{\partial}{\partial p^a_\mu} f(a) \int c(a', 3, \lambda) X^{3a}_\mu \, d^3x^3 \, d^3p^3 + Sm^2G \frac{\partial}{\partial p^a_\mu} \int d(1, 2, 3, \lambda) X^{3a}_\mu \, d^3x^3 \, d^3p^3 = 0 , \tag{25}$$

where $a' = 2$ when a takes the value 1, and $a' = 1$ when $a = 2$, and

$$\frac{\partial}{\partial \lambda} d(1, 2, 3, \lambda) + \frac{p^a_\mu}{m} \frac{\partial}{\partial x^a_\mu} d(1, 2, 3, \lambda) + Sm^2G \frac{\partial}{\partial p^a_\mu} f(a) \int d(a'', 4, \lambda) X^{4a}_\mu \, d^3x^4 \, d^3p^4$$
$$+ Sm^2G \frac{\partial}{\partial p^a_\mu} c\,(a, a'_1) \int c(a'_2, 4, \lambda) X^{4a}_\mu \, d^3x^4 \, d^3p^4 + Sm^2G \frac{\partial}{\partial p^a_\mu} \int e(1, 2, 3, 4, \lambda) X^{4a}_\mu \, d^3x^4 \, d^3p^4 = 0 . \tag{26}$$

The symbol a'' represents two position indices, and the various values a and a'' are to be summed over whenever they appear together are shown below.

a:	1	2	3
a'':	2, 3	3, 1	1, 2

The various values over which the symbols a, a'_1, and a'_2 are to be summed over whenever they appear together are shown below.

a:	1	1	2	2	3	3
a'_1:	2	3	3	1	1	2
a'_2:	3	2	1	3	2	1

4. INITIAL CONDITIONS

We next specify the initial conditions which we are going to evolve using these equations. These initial conditions have to be such that we can have some meaningful evolution using only a few equations of the whole hierarchy.

We choose initial conditions where the deviation of the particles from the uniform distribution is small. The fractional density perturbation at any point is of order ϵ (a small number). The peculiar velocities are also of this order. In other words, it is a cold system (dust) where the peculiar velocities are only those that arise due to the gravitational acceleration.

Using these assumptions we can estimate orders of magnitude for the initial values of various moments of the distribution functions as powers of ϵ. We give this for some of the moments we encounter later.

$$\int f(1) d^3p^1 = n \qquad (nm = p) , \tag{27}$$

$$\int p^1_\mu f(1) d^3p^1 = 0 , \tag{28}$$

where n is the number density of particles;

$$\int (p^1_\mu)^2 f(1) d^3p^1 = n \langle (p^2_\mu)^2 \rangle_1 \sim \epsilon^2 , \tag{29}$$

$$\int c(1, 2) d^3p^1 \, d^3p^3 = n^2 \xi(x^1, x^2) \sim \epsilon^2 , \tag{30}$$

$$\int p^i_\mu c(1, 2) d^3p^1 \, d^3p^2 = n^2 \langle p^i_\mu \rangle_2 (x^1, x^2) \sim \epsilon^2 , \tag{31}$$

$$\int p^i_\mu p^j_\nu c(1, 2) d^3p^1 \, d^3p^2 = n^2 \langle p^i_\mu p^j_\nu \rangle_2 (x^1, x^2) \sim \epsilon^2 , \tag{32}$$

where i and j take values 1 and 2, and $i \neq j$; ξ is the two-point correlation function. All other moments of c are of higher order in ϵ, and

$$\int d(1, 2, 3)d^3p^1\, d^3p^2\, d^3p^3 = n^3\zeta(x^1, x^2, x^3) \sim \epsilon^3 \tag{33}$$

$$\int p^i_\mu\, d(1, 2, 3)d^3p^1\, d^3p^2\, d^3p^3 = n^3\langle p^i_\mu\rangle_3(x^1, x^2, x^3) \sim \epsilon^3 , \tag{34}$$

$$\int p^i_\mu\, p^j_\nu\, d(1, 2, 3)d^3p^1\, d^3p^2\, d^3p^3 = n^3\langle p^i_\mu\, p^j_\nu\rangle_3(x^1, x^2, x^3) \sim \epsilon^3 , \tag{35}$$

$$\int p^i_\mu\, p^j_\nu\, p^k_\sigma\, d(1, 2, 3)d^3p^1\, d^3p^2\, d^3p^3 = n^3\langle p^i_\mu\, p^j_\nu\, p^k_\sigma\rangle_3(x^1, x^2, x^3) \sim \epsilon^3 , \tag{36}$$

where i, j, and k run over the values 1, 2, 3, and $i \neq j \neq k$. All other moments of d are of higher order in ϵ. All moments of e are also of higher order in ϵ; ζ is the three-point correlation function, and χ the four-point correlation function.

These initial conditions correspond to a situation where the linear theory of density perturbations (Peebles 1980) can be applied. The initial conditions are all specified at some instant λ_0.

5. PERTURBATIVE EVOLUTION AND LINEAR THEORY

We now want to see how the various moments of the reduced distribution function evolve from the given initial conditions. The equations are too complicated to solve outright. We have to treat the problem perturbatively by initially keeping terms only up to the lowest order in ϵ and solving the equations, and then putting in the contribution from the higher order terms as corrections.

The derivations are described in this section and worked out in detail in the Appendix. We first deal with the two-particle reduced distribution function c. We proceed by taking moments of the evolution equation for c. The zeroth moment of equation (25) is

$$\frac{\partial}{\partial\lambda}\, \xi(1, 2, \lambda) + \frac{1}{m}\frac{\partial}{\partial x^a_\mu}\, \langle p^a_\mu\rangle_2(1, 2, \lambda) = 0 . \tag{37}$$

This equation relates the evolution of the two-point correlation function to the divergence of the first moment of c. This equation is the continuity equation for pairs. It was two unknown functions both the order ϵ^2, so we cannot ignore any of them. We cannot solve this equation either. We then look at the evolution of the first moment, which is given by the first moment of equation (24). We can take the divergence of this equation and differentiate the continuity equation with respect to λ, and combine the two to get an equation involving the two-point correlation function and the second moment of c. This equation still has two unknown functions of order ϵ^2. We take the evolution equation for the second moment of c and go through a similar procedure to obtain an equation relating the two-point correlation function to the third moment of c and moments of d. This equation is

$$\frac{\partial^3}{\partial\lambda^3}\, \xi - 8\pi G\rho\left[S\, \frac{\partial}{\partial\lambda}\, \xi + \frac{\partial}{\partial\lambda}\, (S\xi)\right] = f_2 - f_3 - \frac{\partial}{\partial\lambda} f_1 , \tag{38}$$

where

$$f_1(1, 2, \lambda) = SG\rho\, \frac{\partial}{\partial x^a_\mu} \int \zeta(1, 2, 3, \lambda)X^{3a}_\mu\, d^3x^3 ,$$

$$f_2(1, 2, \lambda) = 2SGn\, \frac{\partial^2}{\partial x^b_\nu\, \partial x^a_\mu} \int \langle p^a_\mu\rangle_3(1, 2, 3, \lambda)X^{3b}_\nu\, d^3x^3 ,$$

$$f_3(1, 2, \lambda) = \frac{1}{m^3}\, \frac{\partial^3}{\partial x^c_\sigma\, \partial x^b_\nu\, \partial x^a_\mu}\, \langle p^a_\mu\, p^b_\nu\, p^c_\sigma\rangle_2(1, 2, 3, \lambda) .$$

In this equation the only unknown function of order ϵ^2 is the two-point correlation function ξ. The functions f_1, f_2, and f_3 are of higher order in ϵ. Initially we neglect terms of higher order in ϵ and deal with an equation correct to order ϵ^2 only.

As the system evolves the higher order terms become important, and they have to be considered. They can be thought of as giving rise to corrections to the lowest order solution.

Keeping terms of order ϵ^2 only, equation (38) becomes

$$\frac{\partial^3}{\partial\lambda^3}\, \xi - 8\pi G\rho\left[S\, \frac{\partial}{\partial\lambda}\, \xi + \frac{\partial}{\partial\lambda}\, (S\xi)\right] = 0 . \tag{39}$$

This is a third-order differential equation for the two-point correlation function.

For an $\Omega = 1$ universe this is

$$\frac{\partial^3}{\partial\lambda^3}\, \xi - \frac{24}{\lambda^2}\frac{\partial}{\partial\lambda}\, \xi + \frac{24}{\lambda^3}\, \xi = 0 , \tag{40}$$

which has solutions of the form

$$\xi(1, 2, \lambda) = \left(\frac{\lambda}{\lambda_0}\right)^{-4} F_1 + \left(\frac{\lambda}{\lambda_0}\right) F_2 + \left(\frac{\lambda}{\lambda_0}\right)^6 F_3 , \tag{41}$$

where F_1, F_2, and F_3 are functions of x^1 and x^2. The two-point correlation function at λ_0 is expressed in terms of these functions which have to be given as initial conditions. We have three initial conditions because we have a third-order differential equation. Instead of these three functions, one could have given the two-point correlation function and the first two moments of c at λ_0 as initial conditions.

One can derive the same result by evolving one realization of the ensemble using the linear perturbation theory and then calculating the correlation function. The growing mode for density perturbations, usually denoted by $D_1(t)$, grows proportionally to the scale factor, and the decaying mode, usually denoted by $D_2(t)$, is proportional to t^{-1}. In terms of λ this is λ^3. The three modes of growth for the two-point correlation function correspond to D_1^2, $D_1 D_2$, and D_2^2 (Peebles 1980), which is also what we get above.

If the two-point correlation function starts as a mixture of the three modes, after some time it will be noted by the growing mode $D_1^2(\lambda)$. For most purposes it suffices to just keep this mode. If we consider a situation where only the growing mode is present, we can introduce a potential $\phi(x^1, x^2)$. All the quantities of interest can, to order ϵ^2, be expressed in term of this potential:

$$\phi(x^1, x^2) = \phi(x^1 - x^2) , \tag{42}$$

$$\xi(x^1, x^2, \lambda) = \frac{1}{2}\frac{\lambda_0^5}{\lambda^4} \nabla^4 \phi(x^1 - x^2) , \tag{43}$$

$$\langle p_\mu^a \rangle_2(x^1, x^2, \lambda) = m \frac{\lambda_0^5}{\lambda^5} \frac{\partial}{\partial x_\mu^a} \nabla^2 \phi(x^1 - x^2) , \tag{44}$$

$$\langle p_\mu^a p_\nu^b \rangle_2(x^1, x^2, \lambda) = 2m^2 \frac{\lambda_0^5}{\lambda^6} \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^b} \phi(x^1 - x^2) . \tag{45}$$

In the above equations the ∇^2 is with respect to either x^1 or x^2, and $a, b = 1, 2$ with $a \neq b$. It can be verified that the above relations are consistent with all of the two-particle evolution equations.

The potential ϕ is proportional to the correlation of the gravitational potential at the two points x^1 and x^2 and has dimensions $L^4 T^{-1}$. If the other modes are present, one can introduce potentials for them too. This is not considered here.

A similar procedure of taking moments can be used to derive an equation for the three-point correlation function. This equation is

$$\frac{\partial^4}{\partial \lambda^4} \zeta - 40\pi G \rho S \frac{\partial^2}{\partial \lambda^2} \zeta - 40\pi G \rho \frac{dS}{d\lambda} \frac{\partial}{\partial \lambda} \zeta - 12\pi G \rho \left(\frac{d^2 S}{d\lambda^2} - 12\pi G \rho S^2\right) \zeta$$
$$= f_{12} + f_{11} + f_{10} - f_9 - f_8 + \frac{\partial}{\partial \lambda}(f_7 + f_6) + \left(12\pi G \rho S - \frac{\partial^2}{\partial \lambda^2}\right)(f_5 + f_4) , \tag{46}$$

where

$$f_4(1, 2, 3, \lambda) = SG\rho \frac{\partial}{\partial x_\mu^a} \left[\xi(a, a_1', \lambda) \int \xi(a_2', 4, \lambda) X_\mu^{4a} d^3x^4 \right] ,$$

$$f_5(1, 2, 3, \lambda) = SG\rho \frac{\partial}{\partial x_\mu^a} \int \chi(1, 2, 3, 4, \lambda) X_\mu^{4a} d^3x^4 ,$$

$$f_6(1, 2, 3, \lambda) = \frac{2SG}{n^3} \frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a} \int p_\nu^b c(a, a_1', \lambda) c(a_2', 4, \lambda) X_\mu^{4a} d^3x^4 d^{12}p ,$$

$$f_7(1, 2, 3, \lambda) = 2SGn \frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a} \int \langle p_\nu^b \rangle_4(1, 2, 3, 4, \lambda) X_\mu^{4a} d^3x^4 ,$$

$$f_8(1, 2, 3, \lambda) = \frac{3SG}{n^3 m} \frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} \int p_\nu^b p_\mu^a c(c, c_1') c(c_2', 4) X_\sigma^{4c} d^3x^4 d^{12}p ,$$

$$f_9(1, 2, 3, \lambda) = 3SG \frac{n}{m} \frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} \int \langle p_\mu^a p_\nu^b \rangle_4 X_\sigma^{4c} d^3x^4 ,$$

$$f_{10}(1, 2, 3, \lambda) = \frac{12\pi SG\rho}{m^2} \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a} \langle p_\mu^a p_\nu^a \rangle_3(1, 2, 3, \lambda) ,$$

$$f_{11}(1, 2, 3, \lambda) = \frac{12\pi SG\rho}{m^2} \langle p_\mu^a p_\nu^a \rangle_1 \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a} \zeta(1, 2, 3, \lambda) ,$$

and

$$f_{12}(1, 2, 3, \lambda) = \frac{1}{m^4} \frac{\partial^4}{\partial x_\gamma^d \partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} \langle p_\mu^a p_\nu^b p_\sigma^c x_\gamma^d \rangle_3 .$$

The functions f_4 to f_{12} are of order ϵ^4 or higher. To order ϵ^3 we have a fourth-order differential equation for the three-point correlation function:

$$\frac{\partial^4}{\partial \lambda^4} \zeta - 40\pi G\rho S \frac{\partial^2}{\partial \lambda^2} \zeta - 40\pi G\rho\left(\frac{dS}{d\lambda}\right) \frac{\partial}{\partial \lambda} \zeta - 12\pi G\rho\left(\frac{d^2 S}{d\lambda^2} - 12\pi G\rho S^2\right)\zeta = 0 , \tag{47}$$

which for an $\Omega = 1$ universe becomes

$$\frac{\partial^4}{\partial \lambda^4} \zeta - \frac{60}{\lambda^2} \frac{\partial^2}{\partial \lambda^2} \zeta + \frac{120}{\lambda^3} \frac{\partial}{\partial \lambda} \zeta + \frac{216}{\lambda^4} \zeta = 0 . \tag{48}$$

The solution of this equation can be written as

$$\zeta(1, 2, 3, \lambda) = \lambda^{-6} F_4 + \lambda^{-1} F_5 + \lambda^4 F_6 + \lambda^9 F_7 , \tag{49}$$

where F_4, F_5, F_6, F_7 are functions of x^1, x^2, and x^3.

Thus we have obtained four modes D_1^3, $D_1^2 D_2$, $D_1 D_2^2$, and D_2^3 for the evolution of the three-point correlation function. This corresponds to what we would have obtained if we had used the linear theory of density perturbations to evolve some initial density perturbations and then calculated the three-point correlation function and compared it with the initial three-point correlation function of the density field. One could use a similar treatment for the higher correlation functions.

The solution we obtained for the two-point correlation function will be valid as long as the ϵ^3 terms may be neglected. As the evolution proceeds, the contribution from the higher order terms will increase and they will modify the evolution of the two-point correlation function. The evolution of the higher order functions f_1, f_2, and f_3 is calculated by solving to lowest order the equation for these quantities. For example, to lowest order the function f_1 will be of order ϵ^3, and its evolution is governed by equation (46). These functions are then to be incorporated as known functions into the equations for the two-point correlation function. These equations then have to be solved to obtain the two-point correlation function to a higher order. This method can in principle be used to calculate higher order terms for the other correlation functions also.

The perturbative approach breaks down when $\epsilon D_1 \sim 1$.

6. THREE-POINT CORRELATION FROM GAUSSIAN INITIAL CONDITIONS

Here we shall go one step beyond linear perturbations for Gaussian initial conditions. If the initial perturbations are Gaussian, they are completely specified by the functions f and c at the instant λ_0. All nonzero moments of d, e, and all other higher distribution functions can be expressed in terms of moments of f and c at λ_0. The distribution function d has no moments of order ϵ^3, but it has moments of order ϵ^4. These are

$$\langle p_\mu^a p_\nu^a \rangle_3 = \langle p_\mu^a \rangle_2(a, a_1', \lambda_0) \langle p_\nu^a \rangle_2(a, a_2', \lambda_0) \tag{50}$$

and

$$\langle p_\mu^a p_\nu^b p_\sigma^c p_\gamma^d \rangle_3 = \sum \delta^{ab}(\langle p_\mu^a p_\sigma^c \rangle_2 \langle p_\nu^a p_\gamma^d \rangle_2 + \langle p_\nu^a p_\sigma^c \rangle_2 \langle p_\mu^a p_\gamma^d \rangle_2) , \tag{51}$$

where the sum is over all possible pairs of particle indices in the delta function. There are no moments of e of order ϵ^4 or lower. All this implies that

$$F_4 = F_5 = F_6 = F_7 = f_{11}(\lambda_0) = 0$$

and

$$f_5(\lambda_0) = f_7(\lambda_0) = f_9(\lambda_0) = 0 , \tag{52}$$

to order ϵ^4. The functions f_{10} and f_{12} can be expressed in terms of moments of c using equations (50) and (51). Thus, the equation for the three-point correlation function to order ϵ^4 is

$$\frac{\partial^4}{\partial \lambda^4} \zeta - 40\pi G\rho S \frac{\partial^2}{\partial \lambda^2} \zeta - 40\pi G\rho \frac{dS}{d\lambda} \frac{\partial}{\partial \lambda} \zeta - 12\pi G\rho\left(\frac{d^2 S}{d\lambda^2} - 12\pi G\rho S^2\right)\zeta = f_{12} + f_{10} - f_8 + \frac{\partial}{\partial \lambda} f_6 + \left(12\pi G\rho S - \frac{\partial^2}{\partial \lambda^2}\right) f_4 . \tag{53}$$

The terms on the right-hand side are all products of two terms of order ϵ^2 and can be calculated using the equation of the previous section. For an $\Omega = 1$ universe, keeping only the growing mode, we can write the terms on the right-hand side as

$$f_4(1, 2, 3, \lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{10} f_4(1, 2, 3, \lambda_0) , \tag{54}$$

$$f_6(1, 2, 3, \lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{11} f_6(1, 2, 3, \lambda_0) , \tag{55}$$

$$f_8(1, 2, 3, \lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{12} f_8(1, 2, 3, \lambda_0) , \tag{56}$$

$$f_{10}(1, 2, 3, \lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{12} f_{10}(1, 2, 3, \lambda_0) \quad (57)$$

and

$$f_{12}(1, 2, 3, \lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{12} f_{12}(1, 2, 3, \lambda_0) . \quad (58)$$

Using these, the equation for the three-point correlation function is

$$\frac{\partial^4}{\partial\lambda^4}\zeta - \frac{60}{\lambda^2}\frac{\partial^2}{\partial\lambda^2}\zeta + 120\frac{120}{\lambda^3}\frac{\partial}{\partial\lambda}\zeta + \frac{216}{\lambda^4}\zeta = \left(\frac{\lambda_0}{\lambda}\right)^{12}\left(f_{12} + f_{10} - f_8 - \frac{11}{\lambda_0}f_6 - \frac{92}{\lambda_0^2}f_4\right)_{\lambda_0} , \quad (59)$$

which has a solution,

$$\zeta(1, 2, 3, \lambda) = \frac{1}{2856}\left(\frac{\lambda_0}{\lambda}\right)^8 [\lambda_0^4(f_{12} + f_{10} - f_8) - 11\lambda_0^3 f_6 - 92\lambda_0^2 f_4]_{\lambda_0} + \left(\frac{\lambda_0}{\lambda}\right)^6 E(1, 2, 3) , \quad (60)$$

where E is some function to be decided by the initial conditions.

Imposing the initial condition $\zeta(1, 2, 3, \lambda_0) = 0$ on the solution, we get

$$\zeta(1, 2, 3, \lambda) = \frac{1}{2856}\left(\frac{\lambda_0}{\lambda}\right)^6\left[\left(\frac{\lambda_0}{\lambda}\right)^2 - 1\right][\lambda_0^4(f_{12} + f_{10} - f_8) - 11\lambda_0^3 f_6 - 92\lambda_0^2 f_4]_{\lambda_0} . \quad (61)$$

Actually, for a complete solution of the equations, four initial conditions have to be given. Also, the function E can be neglected if one is concerned only with the fastest-growing part of the three-point correlation function that is induced by the two-point correlation function.

Written explicitly, this is

$$\zeta(1, 2, 3, \lambda) = \frac{1}{2856\pi}\left(\frac{\lambda_0}{\lambda}\right)^8\left\{12\left(\frac{\lambda_0}{m}\right)^4 \frac{\partial^4}{\partial x_\mu^a \partial x_\nu^a \partial x_\gamma^{a_1'} \partial x_\sigma^{a_2'}} [\langle p_\mu^a p_\sigma^{a_1'}\rangle_2(\lambda_0)\langle p_\nu^a p_\gamma^{a_2'}\rangle_2(\lambda_0)]\right.$$
$$+ 18\left(\frac{\lambda_0}{m}\right)^2 \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}[\langle p_\mu^a\rangle_2(a, a_1', \lambda_0)\langle p_\mu^a\rangle_2(a, a_2', \lambda_0)] - \frac{33\lambda_0}{m}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a}\int p_\nu^b c(a, a_1', \lambda_0)c(a_2', 4, \lambda_0)X_\mu^{4a}\, d^3x^4\, d^{12}p$$
$$\left. - \frac{9}{2}\left(\frac{\lambda_0}{m}\right)^2 \frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a}\int p_\nu^b p_\sigma^c c(a, a_1', \lambda_0)c(a_2', 4, \lambda_0)X_\mu^{4a}\, d^3x^4\, d^{12}p - 138\frac{\partial}{\partial x_\mu^a}\left[\xi(a, a_1', \lambda_0)\int \xi(a_2', 4, \lambda_0)X_\mu^{4a}\, d^3x^4\right]\right\} . \quad (62)$$

This solution can be further simplified if we use the potentials introduced earlier. Using the potential and doing the integrals over space by parts, we have

$$\frac{\partial}{\partial x_\mu^a}\left[\xi(a, a_1', \lambda_0)\int \xi(a_2', 4, \lambda_0)X_\mu^{4a}\, d^3x^4\right] = -\pi\lambda_0^2 \frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi(a, a_1')\frac{\partial}{\partial x_\mu^a}\nabla^2\phi(a, a_2')\right] , \quad (63)$$

$$\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a}\int p_\nu^b c(a, a_1', \lambda_0)c(a_2', 4, \lambda_0)X_\mu^{4a}\, d^3x^4\, d^{12}p$$
$$= -2\pi m\lambda_0 \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left\{\frac{\partial}{\partial x_\nu^a}[\nabla^2\phi(a, a_1')]\frac{\partial}{\partial x_\mu^a}[\nabla^2\phi(a, a_2')]\right\} - 4\pi m\lambda_0 \frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi(a, a_1')\frac{\partial}{\partial x_\mu^a}\nabla^2\phi(a, a_2')\right] , \quad (64)$$

$$\frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a}\int p_\nu^b p_\mu^a c(c, c_1', \lambda_0)c(c_2', 4, \lambda_0)X_\sigma^{4c}\, d^3x^4\, d^{12}p$$
$$= -16\pi m^2 \frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left\{\frac{\partial}{\partial x_\nu^a}[\nabla^2\phi(a, a_1')]\frac{\partial}{\partial x_\mu^a}[\nabla^2\phi(a, a_2')]\right\} - 8\pi m^2 \frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi(a, a_1')\frac{\partial}{\partial x_\mu^a}\nabla^2\phi(a, a_2')\right] . \quad (65)$$

Using these expressions in equation (62), we get

$$\zeta(1, 2, 3, \lambda) = \frac{1}{28}\left(\frac{\lambda_0}{\lambda}\right)^8 \lambda_0^2\left[\!\!\left[3\frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi(a, a_1')\frac{\partial}{\partial x_\mu^a}\nabla^2\phi(a, a_2')\right] + 2\frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}\left\{\frac{\partial}{\partial x_\nu^a}[\nabla^2\phi(a, a_1')]\frac{\partial}{\partial x_\mu^a}[\nabla^2\phi(a, a_2')]\right\}\right]\!\!\right] , \quad (66)$$

which can also be written as

$$\zeta(1, 2, 3, \lambda) = \frac{1}{14m}\left\{3\frac{\partial}{\partial x_\mu^a}[\xi(a, a_1', \lambda)\langle p_\mu^a\rangle_2(a, a_2', \lambda)] + \frac{1}{m}\frac{\partial^2}{\partial x_\mu^a \partial x_\nu^a}[\langle p_\mu^a\rangle_2(a, a_1', \lambda)\langle p_\nu^a\rangle_2(a, a_1', \lambda)]\right\} . \quad (67)$$

We have an expression for the three-point correlation function that arises from perturbations that are initially Gaussian and have no three-point correlation. This expression is of order ϵ^4 and is a local function involving only derivatives of the potential ϕ. This expression is valid as long as terms having higher powers of ϵ may be neglected. It has been assumed that the initial perturbation had only the growing mode. If other modes are present, ϕ represents only the growing part of it. One can introduce two other potentials for the two other modes, and the three-point correlation function will have terms with all combinations. The expression calculated is the fastest-growing component.

It may also be pointed out that the three-point correlation is the divergence of some quantity. We can then use equation (92), the triplet continuity equation, to obtain an expression for the triplet momentum defined in equation (34).

$$\langle p_\mu^a \rangle_3(1, 2, 3, \lambda) = \frac{2m}{7}\left(\frac{\lambda_0}{\lambda}\right)^9 \lambda_0 \left[\!\left[3\left[\nabla^4\phi(a, a_1')\frac{\partial}{\partial x_\mu^a}\nabla^2\phi(a, a_2')\right] + 2\frac{\partial}{\partial x_\nu^a}\left\{\frac{\partial}{\partial x_\nu^a}[\nabla^2\phi(a, a_1')]\frac{\partial}{\partial x_\mu^a}[\nabla^2\phi(a, a_2')]\right\}\right]\!\right]. \tag{68}$$

This is the part of the triplet momentum $\langle p_\mu^a \rangle_3$ that has divergence and is coupled to gravity. This component grows the fastest and will dominate any other component of the triplet momentum.

Another interesting fact is that for all values of the density parameter Ω the three-point correlation function has the same spatial dependence given by

$$\zeta(1, 2, 3, \lambda) = F^A(\lambda)\frac{\partial}{\partial x_\mu^a}\left[\nabla^4\phi(a, a_1')\frac{\partial}{\partial x_\mu^a}\nabla^2\phi(a, a_2')\right] + F^B(\lambda)\frac{\partial^2}{\partial x_\mu^a\,\partial x_\nu^a}\left\{\frac{\partial}{\partial x_\nu^a}[\nabla^2\phi(a, a_1')]\frac{\partial}{\partial x_\mu^a}[\nabla^2\phi(a, a_2')]\right\}, \tag{69}$$

where F^A and F^B are some functions of λ. This is because equation (46), which governs the growth of the three-point correlation function, is a differential equation in λ alone. The functions $F^A(\lambda)$ and $F^B(\lambda)$ have to be determined by solving equation (46) and will be different for different values of Ω.

7. DISCUSSION

To get a better understanding of the three-point correlation function calculated in the previous section, it is convenient to express it explicitly in terms of the two-point correlation function ξ instead of the potential ϕ.

Using equation (43), which defines the potential, and the fact that $\xi(x)$ is a spherically symmetric function, we have

$$\frac{\partial}{\partial x_\mu}\nabla^2\phi(x) = \left(\frac{2\lambda^4}{\lambda_0^5}\right)\frac{x_\mu}{x^3}\int_0^x \xi(x', \lambda)x'^2\,dx' = \left(\frac{2\lambda^4}{\lambda_0^5}\right)x_\mu\bar{\xi}(x), \tag{70}$$

where we have defined $\overline{\xi(x)}$, which is related to the average of $\xi(x)$ over a sphere of radius x, by the second equality above.

The above equation can be easily understood by an analogy to a spherical mass distribution where the gravitational force on a particle at any point can be found by replacing all the matter in the sphere between this particle and the center of the distribution by an equal point mass at the center, and ignoring all the matter outside this sphere. Using this in equation (66), we obtain

$$\zeta(1, 2, 3, t) = \frac{1}{7}(5 + 2\cos^2\theta_{xy})\xi(x)\xi(y) + \cos\theta_{xy}\frac{d}{dx}\xi(x)y\bar{\xi}(y) + \frac{4}{7}(1 - 3\cos^2\theta_{xy})\xi(x)\bar{\xi}(y) + \frac{6}{7}(3\cos^2\theta_{xy} - 1)\bar{\xi}(y)\bar{\xi}(x), \tag{71}$$

where

$$x = |\boldsymbol{x}^a - \boldsymbol{x}^{a_1'}|,$$

$$y = |\boldsymbol{x}^a - \boldsymbol{x}^{a_2'}|,$$

and

$$\cos\theta_{xy} = \frac{x_\mu y_\mu}{xy}.$$

We would like to remind the reader that a, a_1', and a_2' are to be summed over the values shown in the tabulation in § 3. Although the three-point correlation function appears to be a local function when written in terms of the potential ϕ, it is not local in terms of the two-point correlation function ξ. The three-point correlation function ζ does not depend only on the values of the two-point correlation function ξ at the separations occurring in ζ. It depends on the two-point correlation at all scales smaller than the scales where the three-point correlation function is being evaluated. It should also be noted that it involves a derivative of the two-point correlation function ξ.

An interesting consequence of equation (71) arises when the two-point correlation function has compact support, i.e.,

$$\xi(r) = 0\,; \qquad r > r_1\,; \tag{72}$$

the three-point correlation has the form

$$\zeta(1, 2, 3, t) = \frac{6}{7}(3\cos^2\theta_{xy} - 1)\frac{M^2}{x^3y^3} \tag{73}$$

in the region where the separation between all of the three points is more than r_1. M is defined as

$$M = \int_0^{x_1} \xi(x', \lambda)x'^2\, dx' , \tag{74}$$

and the three-point correlation function here depends only on the integral of the two-point correlation function over the volume where it is nonzero.

Fry (1984) has calculated the three-point correlation function for the special case of the power-law initial two-point correlation function

$$\xi(x) = Ax^{-n} . \tag{75}$$

The general result obtained by us agrees with Fry's result for the power-law case when n is less than 3. For larger values of n the integral of the two-point correlation function diverges, and deviations from the power-law behavior are required at small separations to obtain meaningful results.

If we assume deviations from the power law at small separations for the two-point correlation function, keeping a power-law behavior at large x, our formula will give the same result as Fry's formula at large x if

$$\overline{\xi(x)} = \frac{\xi(x)}{3 - n} \tag{76}$$

for large x. Whether this happens or not depends crucially on the behavior of the two-point correlation function at small separations.

As an illustration of the above point, we present two examples where the two-point correlation has a large-x behavior

$$\xi(x) \sim x^{-4} , \tag{77}$$

but the three-point correlation functions are quite different in the two cases.

First we consider

$$\xi(x) = A \frac{3\alpha^2 - x^2}{(x^2 + \alpha^2)^3} , \tag{78}$$

where A is some normalization constant and α some length scale. This corresponds to a Harrison-Zel'dovich power spectrum ($\sim k^1$) with an exponential decay for large k. Using this, we get

$$\overline{\xi(x)} = A \frac{1}{(x^2 + \alpha^2)^2} , \tag{79}$$

which satisfies equation (76) for large x. In this case we find that at large separations the three-point correlation matches with the formula derived by Fry (1984).

Next we consider

$$\xi(x) = A \frac{\alpha}{(x^2 + \alpha^2)^2} \tag{80}$$

which corresponds to a power spectrum $\sim k^0$ with an exponential decay at large k, and we get

$$\overline{\xi(x)} = \frac{A}{2x^3} \left[\tan^{-1}\left(\frac{x}{\alpha}\right) - \frac{x/\alpha}{1 + (x/\alpha)^2}\right] . \tag{81}$$

For large x we have

$$\overline{\xi(x)} = \frac{\pi A}{4x^3} , \tag{82}$$

which does not satisfy equation (76). In this case the three-point correlation function that we calculate differs, even at large separations, from the expression that Fry has given. Because $\xi(x)$ behaves as x^{-4} and $\overline{\xi(x)}$ behaves as x^{-3} for large x, $\xi(x)$ falls off much faster than $\overline{\xi(x)}$ and the three-point correlation is dominated by the term containing two $\bar{\xi}$'s. The three-point correlation function at large separations then is controlled by the contribution from the two-point correlation at small separations.

Thus we see in the two cases above that although the two-point correlation function has the same power-law spatial dependence for large separations, the three-point correlation functions are quite different.

This is further illustrated graphically in Figures 1 and 2, which show $Q(r)$ versus r for the two cases discussed above. $Q(r)$ is defined as

$$Q(r) = \frac{\zeta(1, 2, 3, \lambda)}{\xi(1, 2)\xi(1, 3) + \xi(2, 3)\xi(2, 1) + \xi(3, 1)\xi(3, 2)} , \tag{83}$$

where the three points 1, 2, and 3 are located at the three corners of an equilateral triangle with sides of size r.

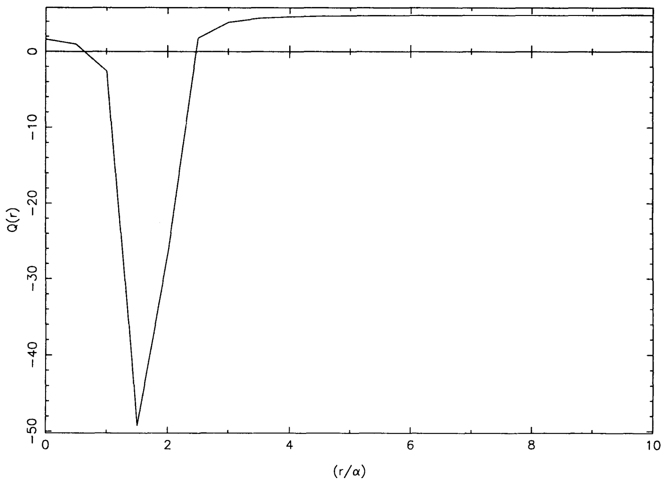

FIG. 1

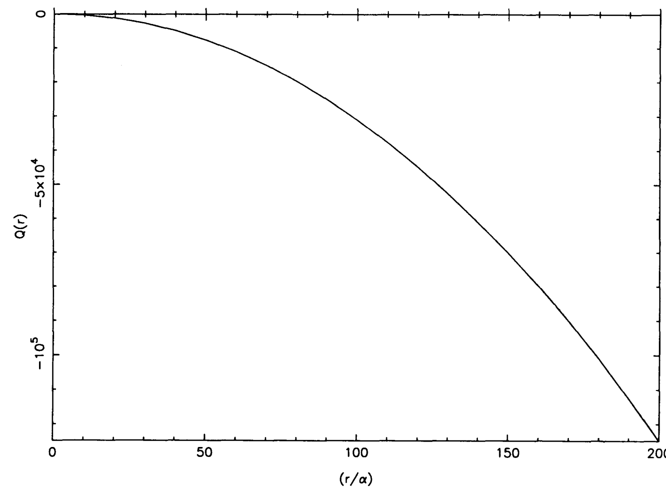

FIG. 2

FIG. 1.—$Q(r)$ for the first example considered in § 7. There is a singularity at $r = 3^{1/2}$, where $Q(r) = -\infty$.
FIG. 2.—$Q(r)$ for the second example considered in § 7. For large r we have $Q(r) \sim r^2$.

Next we would like to make some cautionary remarks on the direct application of the three-point correlation function calculated here to interpret observations. The calculation that has been done here is for the dominant matter component in the universe. If one wishes to use it to interpret galaxy correlations, one should take into account the possibility that galaxies might be a biased tracer of matter. Second, although the galaxy correlations are small at large length scales, galaxies are strongly correlated at small length scales. Because of the nonlocal nature of the results, one has to check whether the perturbative results can be used at the large length scales when the small scales are strongly nonlinear. In addition, even if the perturbative results are valid at large length scales, one cannot make a comparison of the three-point correlation function at some length scales with just the two-point correlation function at the same length scales. The three-point correlation function is highly dependent on the shape of the initial two-point correlation function at all scales smaller than the scales where the three-point correlation is being evaluated.

Finally, the formalism developed in the paper can be applied to test the validity of any scheme to close the BBGKY hierarchy. Such a scheme involves assuming a relationship between some moments of the various distribution functions. The validity of these assumptions can be tested in the weakly nonlinear regime using the formalism developed in this paper. As an example, consider the scheme proposed by Davis & Peebles (1977). They assume that the three-point correlation function has the "hierarchical" form, i.e.,

$$\zeta(1, 2, 3) = Q[\xi(1, 2)\xi(1, 3) + \xi(2, 1)\xi(2, 3) + \xi(3, 1)\xi(3, 2)] , \tag{84}$$

where Q is a constant, and that the correlations arose from initially small Gaussian density perturbations. A comparison of the expression for the three-point correlation in equation (84) with the three-point correlation function calculated in the paper shows that it is not possible to write the induced three-point correlation function in the weakly nonlinear regime in the form assumed in equation (84). Thus, although using this formalism we cannot say anything about the assumptions made by Davis & Peebles in the strongly nonlinear regime, we can say that it fails in the weakly nonlinear regime.

We would like to thank Professor Rajaram Nityananda for his advice and encouragement.

APPENDIX

Here we give the derivation of equations (38) and (46), which govern the evolution of the two- and three-point correlation functions. The zeroth moment of equation (25) is equation (37):

$$\frac{\partial}{\partial \lambda} \xi(1, 2, \lambda) + \frac{1}{m} \frac{\partial}{\partial x^a_\mu} \langle p^a_\mu \rangle_2(1, 2, \lambda) = 0 .$$

The first moment of equation (25) is

$$\frac{\partial}{\partial \lambda} \langle p^a_\mu \rangle_2 + \frac{1}{m} \frac{\partial}{\partial x^b_\nu} \langle p^a_\mu p^b_\nu \rangle_2(1, 2, \lambda) - SmG\rho \int \xi(a', 3) X^{3a}_\mu d^3x^3 - SmG\rho \int \zeta(1, 2, 3, \lambda) X^{3a}_\mu d^3x^3 = 0 . \tag{85}$$

The last two terms have been obtained by integrating the p integral by parts and dropping the surface term. This will be done in the equations for all the other moments also. If we take the divergence of equation (85) and use it in equation (37), we have

$$\frac{\partial^2}{\partial \lambda^2} \xi - \frac{1}{m^2} \frac{\partial^2}{\partial x^b_\nu \partial x^a_\mu} \langle p^a_\mu p^b_\nu \rangle_2 - 8\pi SG\rho\xi = -f_1 , \tag{86}$$

where

$$f_1(1, 2, \lambda) = SG\rho \frac{\partial}{\partial x_\mu^a} \int \zeta(1, 2, 3, \lambda) X_\mu^{3a} d^3x^3 . \tag{87}$$

The second moment of equation (25) is

$$\frac{\partial}{\partial \lambda} \langle p_\mu^a p_\nu^b \rangle_2(1, 2, \lambda) + \frac{1}{m} \frac{\partial}{\partial x_\sigma^c} \langle p_\mu^a p_\nu^b p_\sigma^c \rangle_2(1, 2, \lambda) - \frac{m^2 GS}{n^2} \int (\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a) f(c) c(c', 3) X_\sigma^{3c} d^3x^3 d^9p - \frac{m^2 GS}{n^2} \int (\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a) d(1, 2, 3) X_\sigma^{3c} d^3x^3 d^9p = 0 . \tag{88}$$

Taking the divergence with respect to both the free indices, we have

$$\frac{\partial^3}{\partial \lambda\, \partial x_\nu^b\, \partial x_\mu^a} \langle p_\mu^a p_\nu^b \rangle_2 + \frac{1}{m} \frac{\partial^3}{\partial x_\sigma^c\, \partial x_\nu^b\, \partial x_\mu^a} \langle p_\mu^a p_\nu^b p_\sigma^c \rangle_2 + 8\pi SmG\rho \frac{\partial}{\partial x_\mu^a} \langle p_\mu^a \rangle_2 = m^2 f_2 , \tag{89}$$

where

$$f_2(1, 2, \lambda) = 2SGn \frac{\partial^2}{\partial x_\nu^b\, \partial x_\mu^a} \int \langle p_\mu^a \rangle_3(1, 2, 3, \lambda) X_\nu^{3b} d^3x^3 . \tag{90}$$

Differentiating equation (86) with respect to λ and using equations (89) and (37), we have equation (38), the equation for the two-point correlation function,

$$\frac{\partial^3}{\partial \lambda^3} \xi - 8\pi G\rho \left[S \frac{\partial}{\partial \lambda} \xi + \frac{\partial}{\partial \lambda} (S\xi) \right] = f_2 - f_3 - \frac{\partial}{\partial \lambda} f_1 ,$$

where

$$f_3(1, 2, \lambda) = \frac{1}{m^3} \frac{\partial^3}{\partial x_\sigma^c\, \partial x_\nu^b\, \partial x_\mu^a} \langle p_\mu^a p_\nu^b p_\sigma^c \rangle_2(1, 2, 3, \lambda) . \tag{91}$$

A similar treatment can also be done for equation (26), the third equation of the hierarchy. The zeroth moment of equation (26) is

$$\frac{\partial}{\partial \lambda} \zeta(1, 2, 3, \lambda) + \frac{1}{m} \frac{\partial}{\partial x_\mu^a} \langle p_\mu^a \rangle_3(1, 2, 3, \lambda) = 0 . \tag{92}$$

The first moment of equation (26) is

$$\frac{\partial}{\partial \lambda} \langle p_\mu^a \rangle_3 + \frac{1}{m} \frac{\partial}{\partial x_\nu^b} \langle p_\mu^a p_\nu^b \rangle_3(1, 2, 3, \lambda) - SmG\rho \int \zeta(a'', 4) X_\mu^{4a} d^3x^4 - SmG\rho \int \chi(1, 2, 3, 4) X_\mu^{4a} d^3x^4 - SmG\rho\xi(a, a_1') \int \xi(a_2', 4) X_\mu^{4a} d^3x^4 = 0 . \tag{93}$$

Taking the divergence of equation (93), we have

$$\frac{\partial^2}{\partial \lambda\, \partial x_\mu^a} \langle p_\mu^a \rangle_3 + \frac{1}{m} \frac{\partial^2}{\partial x_\nu^b\, \partial x_\mu^a} \langle p_\mu^a p_\nu^b \rangle_3 + 12\pi SmG\rho\zeta = m(f_4 + f_5) , \tag{94}$$

where

$$f_4(1, 2, 3, \lambda) = SG\rho \frac{\partial}{\partial x_\mu^a} \left[\xi(a, a_1', \lambda) \int \xi(a_2', 4, \lambda) X_\mu^{4a} d^3x^4 \right] \tag{95}$$

and

$$f_5(1, 2, 3, \lambda) = SG\rho \frac{\partial}{\partial x_\mu^a} \int \chi(1, 2, 3, 4, \lambda) X_\mu^{4a} d^3x^4 . \tag{96}$$

Differentiating equation (92) with respect to λ and using equation (94), we have

$$\frac{\partial^2}{\partial \lambda^2} \zeta - \frac{1}{m^2} \frac{\partial^2}{\partial x_\nu^b\, \partial x_\mu^a} \langle p_\mu^a p_\nu^b \rangle_3 - 12S\pi G\rho\zeta = -(f_4 + f_5) . \tag{97}$$

The second moment of equation (26) is

$$\frac{\partial}{\partial\lambda}\langle p^a_\mu p^b_\nu\rangle_3 + \frac{1}{m}\frac{\partial}{\partial x^c_\sigma}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3(1, 2, 3, \lambda) - \frac{Sm^2G}{n^3}\int(\delta^{ca}_{\sigma\mu}p^b_\nu + \delta^{cb}_{\sigma\nu}p^a_\mu)f(c)d(c'', 3)Xx^{4c}_\sigma d^3x^4 d^{12}p$$
$$- \frac{Sm^2G}{n^3}\int(\delta^{ca}_{\sigma\mu}p^b_\nu + \delta^{cb}_{\sigma\nu}p^a_\mu)c(c, c'_1)c(c'_2, 4)X^{4c}_\sigma d^3x^4 d^{12}p - \frac{Sm^2G}{n^3}\int(\delta^{ca}_{\sigma\mu}p^b_\nu + \delta^{cb}_{\sigma\nu}p^a_\mu)e(1, 2, 3, 4)X^{4c}_\sigma d^3x^4 d^{12}p = 0\,. \quad (98)$$

Taking the divergence with respect to both the free indices, we have

$$\frac{\partial^3}{\partial\lambda\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu\rangle_3 + \frac{1}{m}\frac{\partial^3}{\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3 + 16\pi SmG\rho\frac{\partial}{\partial x^a_\mu}\langle p^a_\mu\rangle_3 = m^2(f_6 + f_7)\,, \quad (99)$$

where

$$f_6(1, 2, 3, \lambda) = \frac{2SG}{n^3}\frac{\partial^2}{\partial x^b_\nu\,\partial x^a_\mu}\int p^b_\nu c(a, a'_1)c(a'_2, 4)X^{4a}_\mu d^3x^4 d^{12}p \quad (100)$$

and

$$f_7(1, 2, 3, \lambda) = 2SGn\frac{\partial^2}{\partial x^b_\nu\,\partial x^a_\mu}\int\langle p^b_\nu\rangle_4(1, 2, 3, 4, \lambda)X^{4a}_\mu d^3x^4\,. \quad (101)$$

Using equation (92), this becomes

$$\frac{\partial^3}{\partial\lambda\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu\rangle_3 + \frac{1}{m}\frac{\partial^3}{\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3 - 16\pi Sm^2G\rho\frac{\partial}{\partial\lambda}\zeta = m^2(f_6 + f_7)\,, \quad (102)$$

which, when combined with equation (97), gives

$$\frac{\partial^3}{\partial\lambda^3}\zeta + \frac{1}{m^3}\frac{\partial^3}{\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3 - \pi G\rho\left[16S\frac{\partial}{\partial\lambda}\zeta + 12\frac{\partial}{\partial\lambda}(S\zeta)\right] = (f_6 + f_7) - \frac{\partial}{\partial\lambda}(f_4 + f_5)\,. \quad (103)$$

The third moment of equation (26) is

$$\frac{\partial}{\partial\lambda}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3 + \frac{1}{m}\frac{\partial}{\partial x^d_\gamma}\langle p^a_\mu p^b_\nu p^c_\sigma x^d_\gamma\rangle_3(1, 2, 3, \lambda) - \frac{Sm^2G}{n^3}\int(\delta^{ea}_{\gamma\mu}p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu}p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma}p^a_\mu p^b_\nu)f(e)d(e'', 4)Xx^{4e}_\gamma d^3x^4 d^{12}p$$
$$- \frac{Sm^2G}{n^3}\int(\delta^{ea}_{\gamma\mu}p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu}p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma}p^a_\mu p^b_\nu)c(e, e'_1)c(e'_2, 4)X^{4e}_\gamma d^3x^4 d^{12}p$$
$$- \frac{Sm^2G}{n^3}\int(\delta^{ea}_{\gamma\mu}p^b_\nu p^c_\sigma + \delta^{eb}_{\gamma\nu}p^a_\mu p^c_\sigma + \delta^{ec}_{\gamma\sigma}p^a_\mu p^b_\nu)e(1, 2, 3, 4)X^{4e}_\gamma d^3x^4 d^{12}p = 0\,. \quad (104)$$

Taking the divergence with all the three particle coordinates, we have

$$\frac{\partial^4}{\partial\lambda\,\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3 + \frac{1}{m}\frac{\partial^4}{\partial x^d_\gamma\,\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma x^d_\gamma\rangle_3 + 12\pi SmG\rho\frac{\partial^2}{\partial x^a_\mu\,\partial x^b_\nu}\langle p^a_\mu p^b_\nu\rangle_3 = m^3(f_8 + f_9 - f_{10} - f_{11})\,, \quad (105)$$

where

$$f_8(1, 2, 3, \lambda) = \frac{3SG}{n^3m}\frac{\partial^3}{\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\int p^b_\nu p^a_\mu c(c, c'_1)c(c'_2, 4)X^{4c}_\sigma d^3x^4 d^{12}p\,, \quad (106)$$

$$f_9(1, 2, 3, \lambda) = 3SG\frac{n}{m}\frac{\partial^3}{\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\int\langle p^a_\mu p^b_\nu\rangle_4 X^{4c}_\sigma d^3x^4\,, \quad (107)$$

$$f_{10}(1, 2, 3, \lambda) = \frac{12\pi SG\rho}{m^2}\frac{\partial^2}{\partial x^a_\mu\,\partial x^a_\nu}\langle p^a_\mu p^a_\nu\rangle_3(1, 2, 3, \lambda)\,, \quad (108)$$

and

$$f_{11}(1, 2, 3, \lambda) = \frac{12\pi SG\rho}{m^2}\langle p^a_\mu p^a_\nu\rangle_1\frac{\partial^2}{\partial x^a_\mu\,\partial x^a_\nu}\zeta(1, 2, 3, \lambda)\,. \quad (109)$$

Using equation (97), this becomes

$$\frac{\partial^4}{\partial\lambda\,\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma\rangle_3 + \frac{1}{m}\frac{\partial}{\partial x^d_\gamma\,\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma x^d_\gamma\rangle_3 + 12Sm^3G\rho\left[\frac{\partial^2}{\partial\lambda^2}\zeta - 12S\pi G\rho\zeta + (f_4 + f_5)\right] = m^3(f_8 + f_9 - f_{10} - f_{11}) . \quad (110)$$

Differentiating equation (103) with respect to λ and using equation (110), we get equation (46), the equation for the three-point correlation function,

$$\frac{\partial^4}{\partial\lambda^4}\zeta - 40\pi G\rho S\frac{\partial^2}{\partial\lambda^2}\zeta - 40\pi G\rho\frac{dS}{d\lambda}\frac{\partial}{\partial\lambda}\zeta - 12\pi G\rho\left(\frac{d^2S}{d\lambda^2} - 12\pi G\rho S^2\right)\zeta$$
$$= f_{12} + f_{11} + f_{10} - f_9 - f_8 + \frac{\partial}{\partial\lambda}(f_7 + f_8) + \left(12\pi G\rho S - \frac{\partial^2}{\partial\lambda^2}\right)(f_5 + f_4) ,$$

where

$$f_{12}(1, 2, 3, \lambda) = \frac{1}{m}\frac{\partial}{\partial x^d_\gamma\,\partial x^c_\sigma\,\partial x^b_\nu\,\partial x^a_\mu}\langle p^a_\mu p^b_\nu p^c_\sigma x^d_\gamma\rangle_3 . \quad (111)$$

REFERENCES

Davis, M., & Peebles, P. J. E. 1977, ApJS, 34, 45
Fry, J. N. 1984, ApJ, 279, 499
Inagaki, S. 1991, PASJ, 43, 661
Peebles, P. J. E. 1980, The Large-Scale Structure of the Universe (Princeton: Princeton Univ. Press)