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The maximum mass of neutron stars

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The choice of a suitable topic for this talk posed a problem for me. Should I talk about some work I have done recently? Or, should I talk about something which is not yet resolved, but of great contemporary significance? The latter has the advantage that it might inspire some of the younger persons in the audience to think about it. And that is what I am going to do.

The topic I have chosen is the "Maximum Mass of Neutron Stars". This may appear at first sight to be an academic question, an esoteric one at that (!); but this is not so.

1. An overview of the problem

The present epoch is undoubtedly the golden age of Relativistic Astrophysics, the era of neutron stars and black holes. Recent observations have provided exciting evidence for the countless number of supermassive black holes that power the Quasars and the Active Galactic Nuclei. As for black holes of stellar mass, the sample has been growing steadily ever since the discovery of Cygnus X-1. It is in this latter context that the concept of the maximum mass of neutron stars plays a central role. As the number of black holes in binary x-ray sources grows, and as their "mass spectrum" begins to take shape, the limiting mass of neutron stars acquires greater significance. Indeed, the maximum mass of neutron star is one of the most important predictions of the general relativistic theory of stellar structure. Unfortunately, although there have been strenuous efforts to pin down the value of this limiting mass, the picture is unclear more than sixty years after the pioneering attempt by Oppenheimer and Volkoff (1939). In this talk I will attempt to outline some of the attempts made in recent years and, in particular, highlight the reasons why this problem has remained a challenging one.

The first question to ask is this: why should there be a limit to the masses of neutron stars? In a sense a possible answer is implicit in the theory of white dwarfs due to Chandrasekhar. Let us recall the remarkable discovery made by him in 1930. Chandrasekhar showed that the sequence of white dwarfs in which the gravitational pressure in balanced by the degeneracy pressure of the *electrons* has a limiting mass given by

$$M_{lim} = 0.197 \left[\left(\frac{hc}{G} \right)^{3/2} \frac{1}{{m_p}^2} \right] \frac{1}{{\mu_e}^2} \approx \frac{5.76 M_\odot}{{\mu_e}^2} \tag{1}$$

where μ_e is the mean molecular weight per electron (Chandrasekhar, 1931). This limiting mass is reached when the electrons become ultrarelativistic. Interestingly, **in this extreme relativistic limit the pressure of a degenerate ideal gas of fermions is independent of the rest mass of the particle.** Thus, if one were to construct a sequence of neutron stars in which gravity is balanced by the degeneracy pressure of an ideal neutron gas then this sequence, too, will have a limiting mass of 5.76 $M_\odot$.

But, of course, Chandrasekhar could not have envisaged such stars since the neutron had not been discovered when the limiting mass of white dwarfs was discovered! Very soon after the discovery of the neutron, Baade and Zwicky (1934) did hypothesize neutron stars. By 1936 Gamow had conjectured that ***"neutron cores"*** would form in massive stars when the supply of thermonuclear fuel is exhausted. In 1939, Oppenheimer and Volkoff wondered whether this conjecture by Gamow would be valid for arbitrarily massive stars, or to put it in their own words ***"to investigate whether there is some upper limit to the possible size of such a neutron core"***.

To calculate the radius of a star for a given mass, one has to integrate the equation of hydrostatic equilibrium with an assumed equation of state (EOS), i.e. pressure as a function of density. Oppenheimer and Volkoff realized that for a neutron star the modification of Newtonian gravity due to general relativity would have to be taken into account. The equation of hydrostatic equilibrium in general relativity reads as follows:

$$-\frac{dP}{dr} = \frac{G \left[M(r) + 4\pi r^3 \frac{P(r)}{c^2} \right] \left[\rho(r) + P(r)/c^2 \right]}{r^2 \left[1 - \frac{2GM(r)}{rc^2} \right]} \tag{2}$$

This generalization of the familiar Newtonian equation takes into account the fact that in general theory of relativity **all** forms of energy contribute to gravity. ρ in the above equation is the ***mass-energy density.*** Consequently, M is not the rest mass of the star but its ***gravitational mass.*** This is less than the rest mass or the baryonic mass by the binding energy of the star. The above equation, known as the Tolman-Oppenheimer-Volkoff equation, reduces to the Newtonian one

$$-\frac{dP}{dr} = \frac{GM(r)\rho(r)}{r^2} \tag{3}$$

in the limit $c \to \infty$.

For the equation of state governing the neutron star, Oppenheimer and Volkoff assumed that the neutrons could be treated as an ***ideal fermi gas***. The equilibrium configurations obtained by them is shown in the figure,

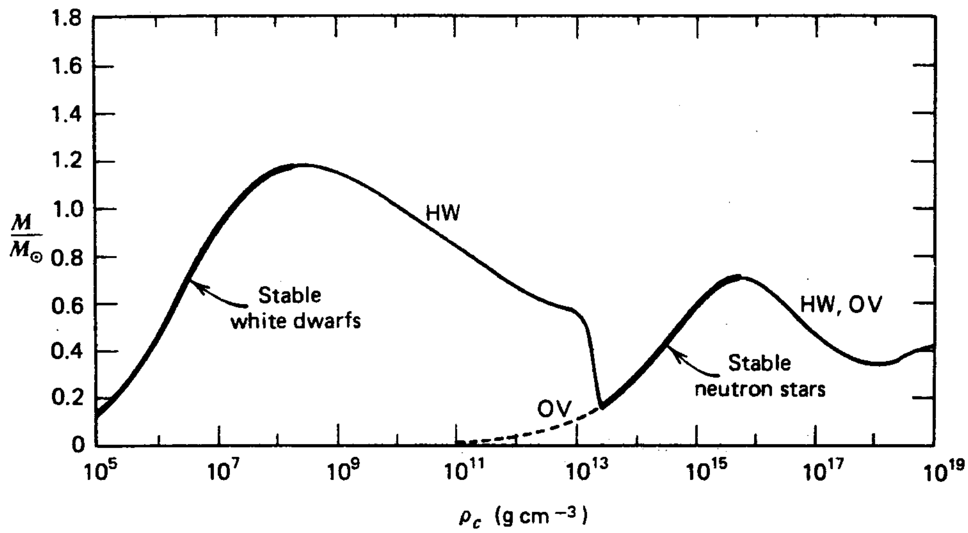

Figure 1. A plot of the gravitational mass *versus* central density (from Shapiro and Teukolsky, 1983). The curve labelled *OV* is the Oppenheimer-Volkoff result. The maximum mass for the neutron stars is $0.7M_{\odot}$, and there is no minimum mass. Also shown in the figure is the sequence of white dwarfs with a corresponding limiting mass. Anticipating our discussion of the equation of state of dense matter in section 3, one has attempted to smoothly join the white dwarf equation of state and the *OV* equation of state with the Harrison-Wheeler equation of state (HW).

which is a plot of mass versus the central density. It may be recalled that stable configurations correspond to the sections of the curve where $\frac{dM}{d\rho_c} > 0$. At the extrema - turning points - the frequency of the fundamental radial mode goes to zero and the star becomes unstable to compression. Consequently, the mass corresponding to the extremum represents the **maximum mass** of the star. From their calculations Oppenheimer and Volkoff concluded that the maximum mass for neutron stars is 0.7 $M_{\odot}$. For completeness the figure also shows the white dwarf branch, with a corresponding limiting mass equal to 1.4 $M_{\odot}$.

How good an estimate is this of the maximum mass? Not very good and the reason is rather easy to see. Although Oppenheimer and Volkoff had properly taken into account the gravitational effects, they had assumed that the neutrons could be regarded as an *ideal* fermi gas. Indeed, this is what Chandrasekhar had assumed for the electrons in a white dwarf. Curiously, the assumption of an ideal gas is quite rigorously valid for electrons, but is a very poor assumption for neutrons. Let us digress to appreciate this, since this continues to be at the heart of the difficulty in determining the maximum mass of neutron stars.

A gas may be regarded as *ideal* if the energy of interaction between the particles can be neglected in comparison with the kinetic energy. A degenerate electron gas has the peculiar property that as the density ***increases*** it becomes more and more ***ideal*** ! The kinetic energy of the electron increases inversely as the ***square*** of the inter-particle spacing (or $n^{2/3}$, where n is the number density of electrons), while the coulomb energy of interaction increases inversely as the mean particle spacing ($n^{1/3}$). Thus, in the limit of high densities kinetic energy dominates over the potential energy. But this is not so for a collection of nucleons because the nuclear energy of interaction increases very much more dramatically with increasing density than the kinetic energy. Thus the nucleons inside an atomic nucleus, or in a neutron star (whose density is roughly the same as that of atomic nuclei) represent a strongly interacting system. Indeed, this is also the case with a collection of atoms or molecules interacting via van der Waal's forces at terrestrial densities and temperatures. Interesting phenomena like ***phase changes*** are a manifestation of interaction effects.

The moral of this discussion is the following. If one wishes to model neutron stars better than Oppenheimer and Volkoff then one must come to grips with nucleon-nucleon interaction. Furthermore, it is naive to think that a neutron star consists only of free neutrons. During the past three decades there have been numerous attempts to construct realistic equations of state for a neutron star. Before reviewing these, let us recall the overall structure of a neutron star.

2. Internal structure

One may gain considerable insight into the internal structure of the star by regarding it as self-gravitating matter in its ground state. This first principle point of view may be justified as follows. In the formation of a neutron star nuclear processes take place sufficiently rapidly that matter in most of the star, except perhaps the outermost layers, can be considered to be in ***complete nuclear equilibrium***. In other words, strong, electromagnetic and weak interactions adjust the nuclear composition at each point to the thermodynamically most favourable one. Within a few years after its birth in a supernova explosion the neutron star would have cooled to temperatures $\sim 10^8 K$. This temperature is low compared to the characteristic excitation energies in the nuclei which are $\sim$ MeV ($10^{10} K$). Therefore, to a good approximation one may regard neutron star matter as being in its lowest energy state at each point, subject only to charge neutrality and conservation of baryons.

What is the absolute ground state of matter? For matter at zero pressure the ground state consists of (i) the neutrons and protons packaged into Fe^{56} nuclei, the most tightly bound nucleus, (ii) the nuclei arranged in a lattice to minimize the configurational energy, and (iii) the electrons being in a ferromagnetic state. So one may confidently expect the surface layers to be a ferromagnetic iron lattice.

As one goes deeper, and as the density increases due to gravitational pressure, the characteristics of the matter will change. By the time the density reaches $10^4 g\ cm^{-3}$, the atoms will be fully ionized. At a density $\sim 10^7\ g\ cm^{-3}$ the electron gas will be relativistically degenerate, with a Fermi energy $\sim 1MeV$. At this stage inverse beta decay will set in. Electrons at the top of the Fermi sea will combine with protons in the nuclei to form neutrons, with the neutrinos escaping and thus lowering the energy. Under these conditions, Ni^{62} is the most stable nucleus. At a depth where the density reaches $\sim 2.7 \times 10^8\ g\ cm^{-3}$, Ni^{64} will be the favoured nuclei. Thus, as one goes deeper one will encounter more and more neutron-rich nuclei. The sequence of nuclei up to a density of $4.3 \times 10^{11}\ g\ cm^{-3}$ will be as follows

Fe^{56}, Ni^{62}, Ni^{64}

Se^{84}, Ge^{82}, Zn^{80}, Ni^{78}, Fe^{76}

Mo^{124}, Zr^{122}, Sr^{120}, Kr^{118}

The nuclei listed above have closed shell configurations. The nuclei in the second line have 50 neutrons, those on the third line have 82 neutrons.

Above a density of $4.3 \times 10^{11}\ g\ cm^{-3}$, neutrons begin to "leak out" of the nuclei and form a degenerate **liquid**. This regime is known as the "***neutron drip regime***". This state of affairs, namely, a lattice of exotic neutron-rich nuclei immersed in a degenerate neutron fluid and a degenerate electron gas, will persist until one reaches a density $\sim 2.5 \times 10^{14}\ g\ cm^{-3}$ when the adjacent nuclei begin to "touch" one another. Beyond this density, the nuclei will merge into a fluid of neutrons and protons. The relative numbers of neutrons and protons will be determined by the condition of beta equilibrium

$$n \to p + e^- + \bar{\nu}_e, \quad e^- + p \to n + \nu_e. \tag{4}$$

This implies that the chemical potential of the neutrons must be equal to the sum of the chemical potentials of the protons and electrons. This condition yields a proton concentration $\sim 5\%$.

The very neutron-rich nuclei, as well as the free neutrons are stabilized against beta decay by the high degeneracy of the electrons. The beta decay of the neutron will be inhibited as long as the energy of the electrons produced is **less** than the Fermi energy of the electron gas. This has an important consequence for the ***minimum mass of neutron stars***.

There is a broad agreement up to this point in the description of the internal structure of a neutron star and this is summarized in the figure. The disagreement–sometimes passionate!– is about the nature of the matter at densities much in excess of terrestrial nuclear density ($2.5 \times 10^{14}\ g\ cm^{-3}$). We shall return to the origin of this disagreement a

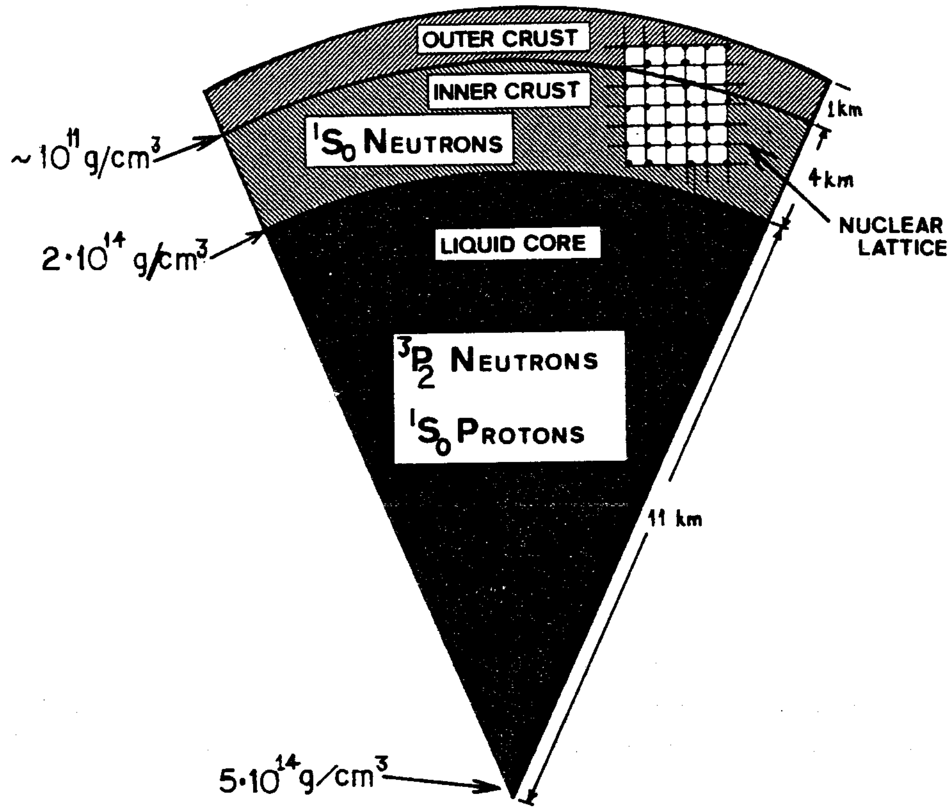

Figure 2. A schematic cross-section of a $1.4M_\odot$ neutron star based upon a *stiff* equation of state (from Sauls, 1989).

little later. For the moment let us accept the above picture of a neutron star, and discuss the equation of state, viz. the variation of pressure with density.

3. The equation of state

Now we shall briefly review some of the landmark equations of state. But first, some definitions are in order. Formally, the pressure is

$$P = -\left(\frac{\partial E}{\partial V}\right)_s = -\frac{\partial E}{\partial\left(\frac{1}{\rho_0}\right)} = \rho_0^2 \frac{\partial E}{\partial \rho_0} \tag{5}$$

where E is the specific energy per unit rest mass and $\rho_0 = nm_0$ is the rest mass density. Alternatively,

$$P = n^2 \frac{\partial\left(\varepsilon/n\right)}{\partial n} \tag{6}$$

where ε is the energy density and n the baryon number density. Broadly speaking one may delineate two regimes while calculating the equations of state: the solid crust and the fluid core, with 2.5×10^{14} $g\ cm^{-3}$ as the dividing density. Within the solid crust, one must treat the "neutron-drip regime" ($\rho > 4.3 \times 10^{11}$ $g\ cm^{-3}$) separately (and with care!).

The energy density in the solid crust may be formally written as

$$\varepsilon = n_N\, E_N\,(A, Z) + \varepsilon_e(n_e) + \varepsilon_n(n_n) + \varepsilon_{Lattice}. \tag{7}$$

In the above expression E_N is the energy of the nucleus (A,Z) including the rest mass of the nucleus and n_N is the number density of the nuclei; ε_e is the energy density of the electron gas of density n_e; ε_n is the energy density of the free neutron fluid (this is relevant only in the lower part of the solid crust); ε_L is the lattice energy. Of these, the energy of the electrons and the lattice energy are relatively easy to calculate. The nuclear energy is not known experimentally for the kind of very neutron-rich nuclei that one expects in the inner crust, and must therefore be deduced theoretically. Historically, this was done by adopting the ***liquid drop model*** of the nucleus and appealing to the semi-empirical mass formula so widely used in nuclear physics.

3.1 The Equation of State Above the Neutron Drip

Let us now consider the properties of the condensed matter above the neutron drip density of 4.3×10^{11} $g\ cm^{-3}$. This regime can be subdivided into two regions.(1) The inner crust with a free neutron fluid, extending roughly up to the nuclear density of 2.5×10^{14} $g\ cm^{-3}$, and (2) the fluid core.

The pioneering effort above the neutron drip density was by Baym, Bethe and Pethick (1971b). There are two subtle things to worry about:

1. As one goes towards the bottom of the crust there is less and less difference between the matter inside the nuclei and outside it! So one cannot have radically different approaches to calculating the energy of the nuclei and the energy of the free neutron fluid.

2. The importance of the ***surface energy*** in determining the energy of the nuclei will decrease dramatically. After all, as the matter inside the nucleus becomes similar to that outside, the surface energy should vanish.

The various equations of state mentioned above are shown in figure 3. Also plotted are the Chandrasekhar equation of state for an ideal degenerate electron gas with a nuclear composition of pure Fe^{56} (labelled Ch), as well as the equation of state of an ideal (n, p, e^-) gas. The *softening* of this EOS as neutronization sets in $\sim 10^7\ g\ cm^{-3}$ is clearly seen. Figure 4 shows the Baym-Bethe-Pethick EOS. Also shown for comparison is the Harrison-Wheeler EOS extrapolated to densities beyond the neutron drip density.

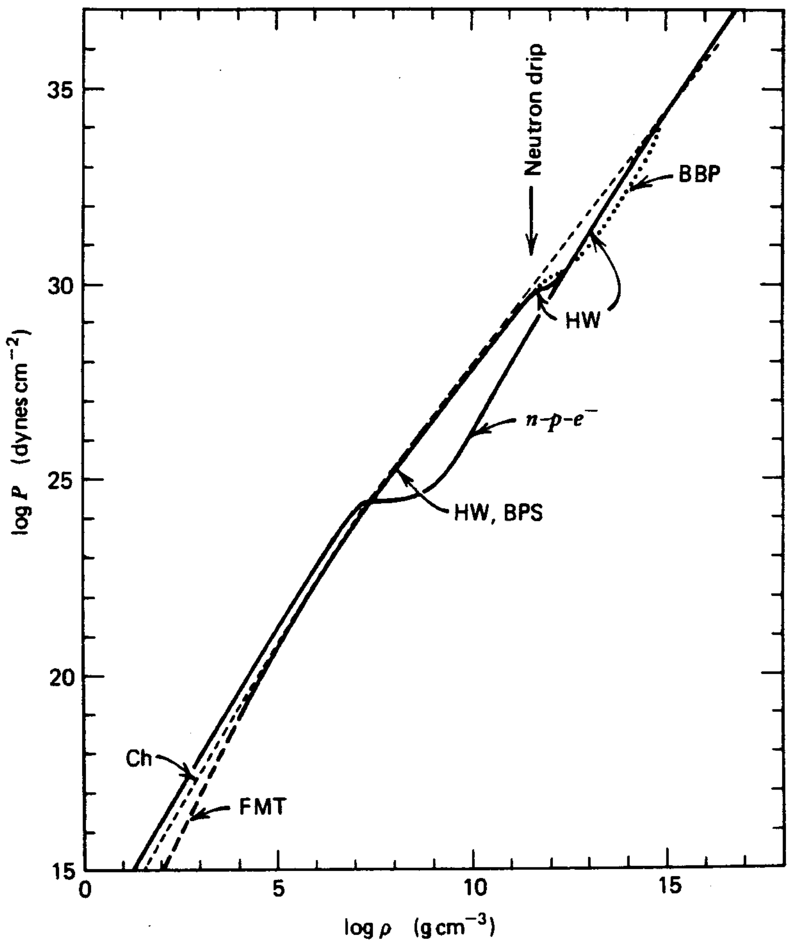

Figure 3. Several equations of state are shown in the figure: HW refers to the Harrison-Wheeler, and BPS to the Baym, Pethick, Sutherland equation of state. The ideal degenerate electron gas model (with $\mu_e = \frac{56}{26}$) is labelled Ch (for Chandrasekhar), and $n-p-e^-$ corresponds to the ideal $(n-p-e^-)$ gas. As one would expect, the latter deviates from the Chandrasekhar equation of state only for $\rho \geq 10^7\ gm\ cm^{-3}$ where inverse β-decay becomes important (from Shapiro and Teukolsky 1983).

3.2 The Liquid Interior

The problems of calculating the energy density in the fluid core are vastly more complex. Nearly 50 years ago Hans Bethe remarked that more man-hours of work had been devoted to understanding this problem than any other scientific question in history. And there have been Herculean efforts since that statement was made. The way one has gone about the problem is the following:

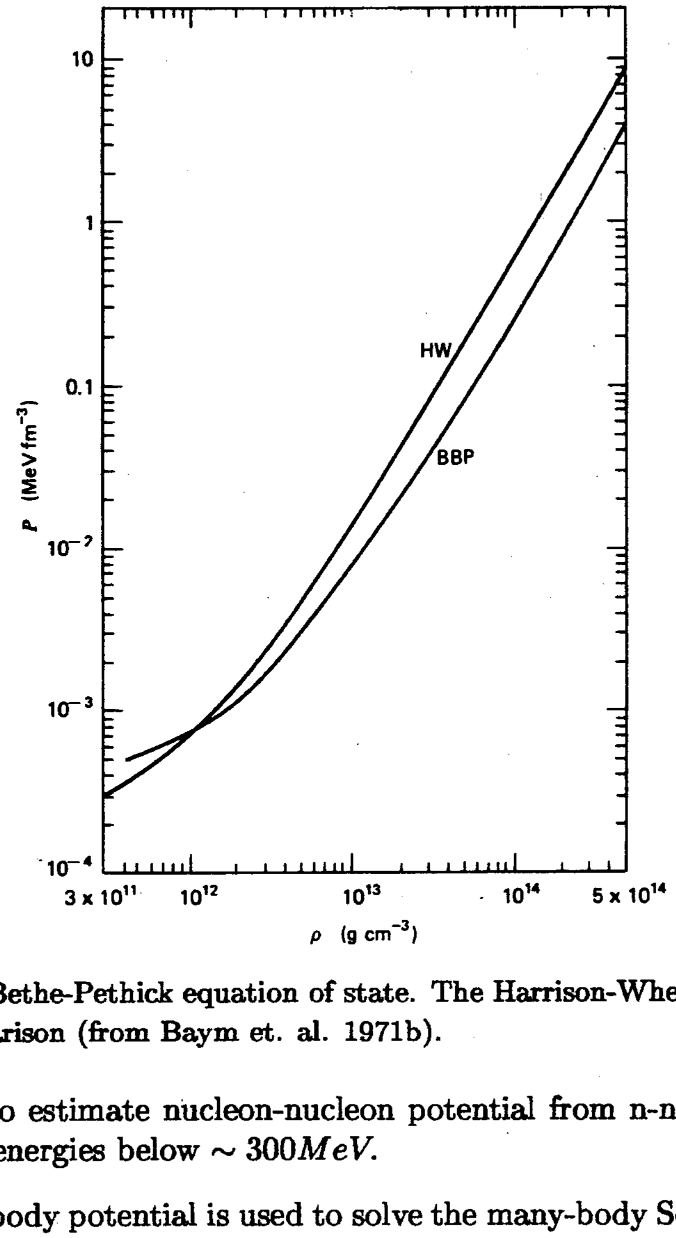

Figure 4. The Baym-Bethe-Pethick equation of state. The Harrison-Wheeler equation of state is also shown for comparison (from Baym et. al. 1971b).

1. One first tries to estimate nucleon-nucleon potential from n-n and n-p scattering experiments at energies below $\sim 300MeV$.
2. The deduced 2-body potential is used to solve the many-body Schrödinger equation and calculate the energy density as a functions of the baryon density. The most popular technique is some form of variational principle.

There are two checks: The 2-body potential is used to compute the saturation energy and the density of symmetric nuclear matter, as well as the binding energies of light nuclei with $A < 4$. To get these right, modern 2-body potentials include as many as 14 different contributions such as central potential, spin-orbit potential etc.!

A number of groups have worked hard over the past couple of decades to construct the equation of state of the liquid interior along the lines outlined above. One of the important conclusions that has emerged from these studies is that one must include 3-body forces as well (it may be recalled that 3-body potential cannot be described by a superposition of pair-wise potentials).

The next figure shows several calculations of the energy per nucleon as a function of the baryon number density.

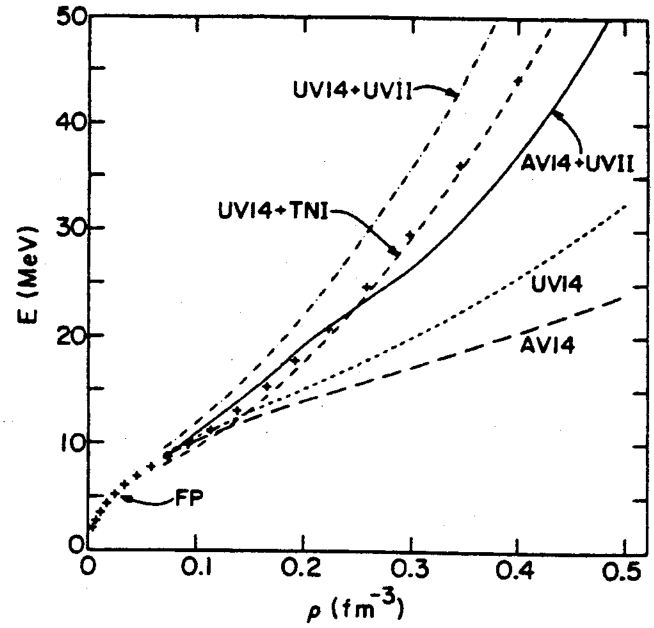

Figure 5. Energy per nucleon *versus* baryon density with and without 3-body forces in pure neutron matter. UV14 and AV14 use only 2-body forces, while the curves with the additional labels UVII and TNI include 3-body forces (from Wiringa, Fiks, Fabrocini 1988).

For reference it may be noted that the nuclear density of $2.5 \times 10^{14}\ g\ cm^{-3}$ corresponds to a number density of 0.16 baryons/fm^3. The curves marked $UV14$ and $AV14$ were obtained using the 2-body potential derived by the Paris-Urbana group and the Argonne group, respectively. The other three curves with additional labels $UVII$ or TNI include 3-body forces. It is clearly seen that the inclusion of 3-body interactions has the effect of **stiffening** the equation of state compared to those calculated using 2-body forces alone. The pressure P is calculated from the energy per nucleon, E, through the relation $P = n^2 \delta E/\delta n$. It should be noted that there is considerable uncertainty in the equation of state at densities above the nuclear density, with different calculations yielding different results. Despite this, there is growing consensus that (i) *the inclusion of 3-body forces is important, and (ii) that this inclusion will stiffen the equation of state.*

All this is fine, but is our basic premise correct? We may be doing a wonderful job, but the scenario may have nothing to do with the inner core of a real neutron star! Let me give you a couple of reasons why this may be so.

3.3 Soft Equations of State

In the entire discussion so far we have assumed that the constituent particles in the star are neutrons, protons and electrons. This is perfectly justifiable up to a density of the order of the nuclear density which, as we have said, is the boundary between the solid crust and the fluid core. Since in the inner parts of the core the densities will exceed nuclear density, there is some uncertainty about the composition of the neutron star matter. One may reasonably ask if, besides neutrons and protons, other particles, including exotic particles, are likely to exist. From a conservative point of view one needs to consider only the π meson and the K meson. The π meson can be neutral or charged, and has a rest mass $\sim 140MeV$. The reason why these mesons can spontaneously occur at high densities is the following. We already mentioned that the neutrons and protons in the core will be in beta equilibrium. This implies that $\mu_n = \mu_p + \mu_e$. Since μ_e increases with density, there will be a critical density at which it will exceed the rest mass energy of the π meson. It will then be energetically favourable for a neutron at the top of the Fermi sea to "decay" to a proton and a π^- meson. Let us recall our earlier remark that neutron matter in the core is stable because the β-decay of the neutron is inhibited by the extreme degeneracy of the electrons. What we have said above is that beyond a critical density a new decay channel becomes available for the neutron. At the interface between the crust and the fluid core the chemical potential of the electrons will already be $\sim 110MeV$. Consequently, one would expect π mesons (with a rest mass $\sim 140MeV$) to spontaneously occur at a density slightly in excess of nuclear density. But this expectation is somewhat naive. A spontaneously produced π meson will interact with the surrounding and this interaction energy will also have to be budgetted for. Simple estimates allowing for this suggest that π mesons may occur at densities $\sim$ twice the nuclear density.

In a similar fashion K-mesons, one of the "strange particles", can also spontaneously occur at even higher densities. Again, simple minded estimates suggest that the critical density for this will be ≥ 3 times the nuclear density.

Despite the uncertainties, let us accept these possibilities for a moment and ask in what way they are likely to affect the star. Unlike the electron, proton and the neutron which obey Fermi-Dirac statistics, ***the mesons obey Bose-Einstein statistics.*** A fundamental property of an ideal Bose gas is that at low enough temperatures it will undergo **"Bose-Einstein condensation"** or condense into the zero-momentum state.

The number of particles $N(0)$ in the zero momentum state is given by a simple expression

$$N(0) = N\left[1 - \left(\frac{T}{T_0}\right)^{3/2}\right], \quad T < T_0. \tag{8}$$

It may be seen from the above formula that *all* the particles will be in the zero momentum state at $T = 0K$. This phenomenon is known as Bose-Einstein condensation. It is, of course, a condensation in momentum state and not in coordinate space. This remarkable behaviour of a Bose gas is believed to be at the base of spectacular phenomena such as superfluidity. Since liquids such as Helium-4 (which exhibits superfluidity) are far from being "ideal", the Bose-Einstein condensation is not as pronounced as one would expect in an ideal gas. But recent experiments in which Rubidium atoms are cooled to incredibly low temperatures by laser light dramatically reveal this 'condensation' phenomenon, and the associated onset of superfluidity.

After this digression let us return to the question of the possible occurrence of π-mesons and K-mesons in a neutron star. If they do, then the majority of them will be in the zero momentum state. ***Consequently they will not contribute to the pressure.*** Since these particles appeared in place of the electrons (at supranuclear densities) the equation of state will be **'softened'**. This, as we shall see, can have a profound implication for the maximum mass of neutron stars.

Quark matter

Before taking up the discussion of models of neutron stars using modern equations of state, let us indulge in some more speculation. The key question is what will happen to an assembly of neutrons and protons as we go on increasing the density? This question acquires significance because according to the modern point of view neutrons and protons are not fundamental particles, but are made of three 'quarks', which are the fundamental building blocks. Therefore, it is not unreasonable to expect that at sufficiently high density the more appropriate description of 'nuclear matter' will be in terms of these 'quarks'. In the technical jargon one would say that at these densities the quarks have been 'deconfined'. The meaning of this is simply the following. Free quarks do not exist because the force between the quarks *increases* with distance. By the same token, at very short distances or very high temperatures the force between the quarks may be neglected and they may be regarded as essentially "free". Inside a baryon like a proton or neutron, the quarks and the gluons which are responsible for the force between them are said to be *'confined'*. But at high densities one may regard the nuclear matter as a Fermi sea of quarks. The quarks have been *deconfined* ! It is interesting to ask whether such a quark-gluon plasma will exist near the very centre of a neutron star. This is highly uncertain at the moment. Current estimates suggest that a phase transition from baryonic matter to a quark-gluon plasma might occur at a density somewhere between 5 to 10 times the nuclear density. If this situation obtains then it, too, can have a significant effect on the stability of the star since *free quark matter will be softer than baryonic matter.*

4. Models of neutron stars

After that lengthy digression into the equation of state of neutron star matter let us return to the task at hand, viz., to model neutron stars. The procedure is the same as the one used for constructing white dwarf models.

1. Pick a value of the central mass-energy density ρ_c. The assumed equation of state will give a corresponding value for the central pressure P_c.

2. The Tolman-Oppenheimer-Volkoff equation is integrated out from $r = 0$ using the initial condition at the centre. Each time a new value of the pressure is obtained the equation of state will give the corresponding density.

3. The value $r = R$ at which $P = 0$ is the radius of the star, and the mass contained within this is the radius of the star.

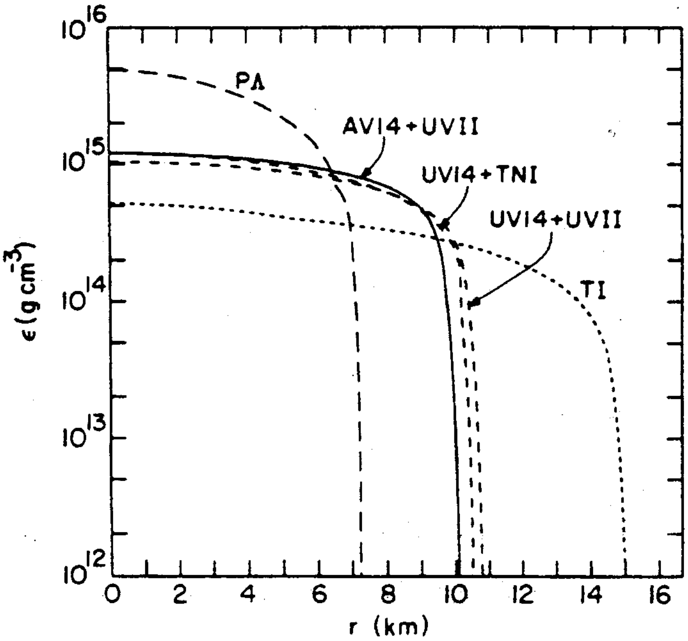

Figure 6. Density profiles of a $1.4M_\odot$ neutron star calculated with various equations of state. In addition to the three modern equations of state which include 3-body forces, the figure also shows the density profile corresponding to a very soft equation of state (PΛ) and a very stiff equation of state (TI). All the three modern models predict a radius for the star $\approx 10.5km$, and a central density $\sim 6\rho_0$. The crust is only ~1km thick in these models (from Wiringa, Fiks, Fabrocini 1988).

Figure 6 shows the mass-energy density profile of a $1.4M_\odot$ neutron star calculated with modern equations of state which include 3-body forces. In addition to the three equations of state introduced earlier the figure also shows the mass-energy density profile for a very soft equation of state ($P\Lambda$) due to Friedman and Pandharipande (1981) and also for a very stiff tensor interaction equation of state (TI) due to Pandharipande and Smith (1975). A noteworthy feature of the models calculated with the modern potentials is that all of them predict roughly the same radius $\simeq 10.5 - 11$ km, and a central density $\sim 6\rho_{nuc}$. The crusts of these stars are only ~ 1 km thick.

The next Figure shows the gravitational mass as a function of the central density for sequences of neutron star models calculated with the same equations of state shown in the previous figure. The maximum mass of neutron stars predicted by the modern

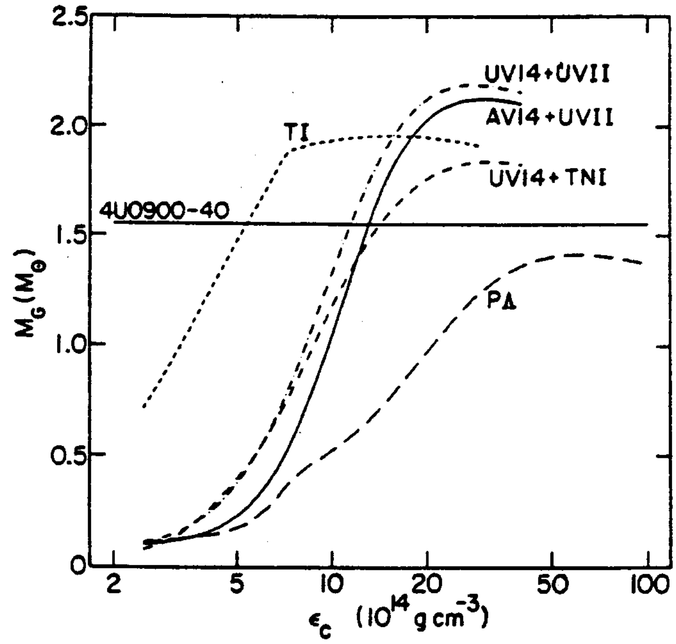

Figure 7. Gravitational mass *versus* central density. The modern equations of state predict a maximum mass $\sim 2\ M_\odot$, while the very soft equation of state (PΛ) predicts a maximum mass $\leq 1.5\ M_\odot$. The horizontal line is the estimated mass of the neutron star in the x-ray binary 4U0900-40 (from Wiringa, Fiks, Fabrocini 1988).

equations of state is $\sim 2M_\odot$. The 'soft' equation of state ($P\Lambda$) predicts a maximum mass which is just about $1.4M_\odot$! The reason why all these calculations yield a maximum mass significantly larger than $0.7M_\odot$, the value obtained by Oppenheimer and Volkoff, is clear. Oppenheimer and Volkoff treated the neutron star matter as an *ideal* neutron gas. At densities $\geq$ nuclear density the repulsion between the nucleons plays an important role in supporting the star against gravity. This is why one can make neutron stars with larger masses.

The mass-radius relation obtained for the families of models is shown in the next figure. A very interesting feature of this figure is that the modern equations of state predict roughly the same radii for neutron stars for all masses (except for the very low mass stars, $M \leq 0.5M_\odot$)!

Returning to the Mass *vs.* Central Density plot, the maximum mass predicted by the various stiff equations of state is comfortably larger than the measured masses of radio pulsars (which cluster around $1.4M_\odot$), as well as the (less accurately) deduced masses of the neutron stars in X-ray binaries (the horizontal line in the figure corresponds to the estimate of $1.55M_\odot$ for the neutron star in the binary 4U0900-40). But the one soft equation of state included in the figure predicts a dangerously low maximum mass. Recent calculations with other soft equations of state don't provide much comfort either. A recent equation of state with kaon condensate is that due to Thorsson, Prakash and Lattimer (1994). This yields a maximum mass $\sim 1.5M_\odot$. Given various uncertainties, this could plausibly be pushed up to $\sim 1.65M_\odot$ (see Brown, 1999).

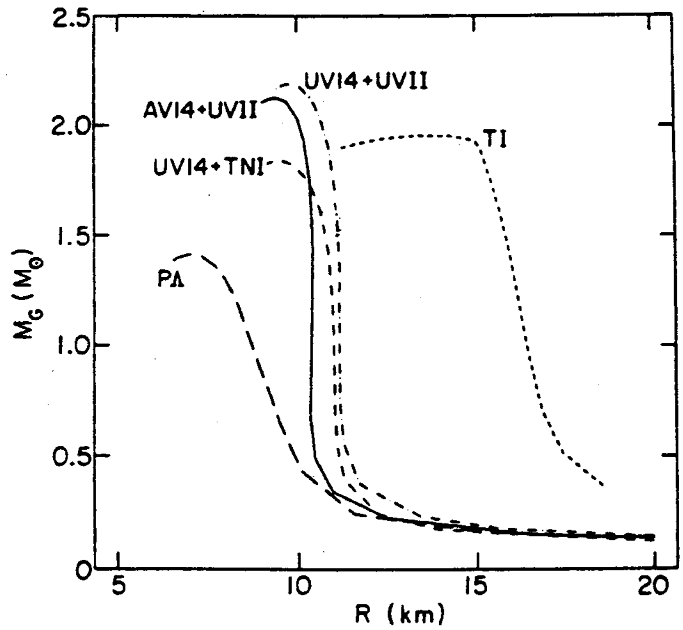

Figure 8. The mass-radius relation for neutron stars (from Wiringa, Fiks, Fabrocini 1988).

To summarize, all one can say more than sixty years after the pioneering calculation by Oppenheimer and Volkoff is that the maximum mass of neutron stars is most likely in the range $1.5M_\odot$ to $\geq 2M_\odot$.

5. Rapidly rotating neutron star models

So far we have considered non-rotating stars. Since in many interesting astrophysical situations one encounters rapidly rotating stars (such as the millisecond pulsars) one has to assess the extent to which the conclusions of the previous section will be altered by the inclusion of rotation. That rotation will have a stabilizing effect is intuitively clear. A star supported by both pressure and rotation will be less dense and will have smaller gravity than the corresponding non-rotating spherical star. Consequently the ***upper mass limits for rotating neutron stars will be larger than those of non-rotating stars.*** The question is, by how much?

The main conclusion arrived at by a number of very discerning physicists is that rotation increases the maximum mass by approximately 15 - 20%. The change in the limiting mass are constrained by the fact that neutron stars cannot maintain differential rotation. Although the neutron star has a fluid core, one doesn't expect much differential rotation between the crust and the core. This is due to large viscosity of the core. Interestingly, this conclusion remains even if the core is a ***superfluid*** (and the arguments for it being a superfluid are very compelling). The occurrence of superfluidity and strong viscous coupling to the crust may seem paradoxical, but there are beautiful and deep reasons for it!

To return to our discussion, we remarked that detailed calculations show that the maximum mass of a neutron star is increased only by about 15 - 20% by rotation, and that this is due to rigid body rotation of the star. White dwarfs, in contrast, can rotate differentially. And these can have masses as large as 2.5 times the Chandrasekhar limiting mass for non-rotating white dwarfs (Durisen, 1975).

6. Theoretical upper limit to the mass

Due to the considerable uncertainty in the EOS at supranuclear densities the maximum mass of neutron stars continues to be elusive. Given the great significance of this number one may ask, ***can one appeal to some very general principles to constrain the EOS at ultrahigh densities and derive a limiting mass.*** The answer is "yes". There are two condition one can confidently impose:

1. $\frac{dP}{d\rho} \geq 0$. This is the microscopic stability condition. If this is violated then the matter would spontaneously collapse.
2. $\frac{dP}{d\rho} \leq c^2$. This is the "causality condition" that the speed of sound should not exceed the speed of light.

Historically, and around the same time, two sets of investigators approached this problem from slightly different points of view. We shall first mention the treatment due to Rhoades and Ruffini (1974). Their starting point was the Tolman-Oppenheimer-Volkoff equation. As for the equation of state, they assumed that it was well determined up to a density ρ_0. They then performed a variational calculation to determine the equation of state at $\rho > \rho_0$, subject to the two general conditions mentioned above, which maximizes the mass. They found this way that

$$P = P_0 + (\rho - \rho_0)c^2, \quad \rho \geq \rho_0 \tag{9}$$

They chose $\rho_0 = 4.6 \times 10^{14}\ g\ cm^{-3}$ and adopted the Harrison-Wheeler equation of state for $\rho < \rho_0$. Given this choice for the combination of equations of state (above and below the matching density), by integrating the equation for equilibrium structure Rhoades and Ruffini found

$$M_{max} \simeq 3.2 M_{\odot}. \tag{10}$$

This is a much quoted result in the context of the formation of stellar mass black holes.

We turn next to a parallel effort by Nauenberg and Chapline (1973) who approached the problem from the point of view of the ***stability*** of the star. Earlier, while discussing the seminal work of Oppenheimer and Volkoff, we referred to the turning point criteria to isolate the extremum in mass. We also remarked that this point the star becomes unstable to compression. But this coincidence between the maximum mass and the instability point holds only for stars whose equilibrium and pulsations are governed by the same equation of state. This, however, need not be true in general.

In the 1960s Chandrasekhar undertook the first systematic study of the dynamical stability of stars in the exact framework of general relativity. This study yielded an exact stability criterion in general relativity. It is illuminating to look at the Newtonian limit of the stability criterion obtained by him since it gives an insight into the destabilizing effect of relativistic gravity. A Newtonian star will be unstable to radial oscillations if the adiabatic $\gamma = \frac{d\ln P}{d\ln\rho}$ becomes less than $\frac{4}{3}$; more precisely, when γ averaged over the star with respect to the pressure becomes less than $\frac{4}{3}$. In the post Newtonian limit, the stability conditions derived by Chandrasekhar reduces to

$$\gamma_c > \frac{4}{3} + K\left(\frac{2GM}{Rc^2}\right) \tag{11}$$

where K is a constant of order unity (Chandrasekhar, 1964). General relativity's stronger gravity leads to instability at larger values of γ. This, as Chandrasekhar pointed out, is what makes supermassive stars, in which radiation pressure dominates collapse (since $\gamma \approx \frac{4}{3}$). This is also true for massive white dwarfs in which the electrons are relativistic (again,$\gamma \approx \frac{4}{3}$). What is interesting is that this instability implied by general relativity sets in (in the two cases mentioned above) when the stars are nearly Newtonian with radius

$$R = K\frac{\left(\frac{2GM}{c^2}\right)}{\gamma - \frac{4}{3}} \tag{12}$$

As Chandrasekhar and Tooper (1964) showed, a white dwarf close to the limiting mass becomes dynamically unstable when its radius is $\sim$ 5000 Schwarzschild radii!

Let us now return to the effort by Nauenberg and Chapline. For an assumed equation of state they determined the ***maximum mass*** by locating the limit of stability against radial perturbations. The limiting case when the velocity of sound equals the velocity of light yielded a theoretical upper limit to the maximum mass of a neutron star

$$M_{max} \approx 3.6 M_{\odot}. \tag{13}$$

The agreement between the results obtained by the two groups discussed above is not surprising because the assumption of uniform density is remarkably good for masses near the maximum value. A couple of remarks are in order before leaving this topic.

1. The value of $3.2M_\odot$ for the maximum mass obtained by Rhoades and Ruffini is often given the status of a rigorous upper limit in astrophysical literature. It is *not!* The limit on the maximum mass obtained in this manner is sensitive to the "matching density" ρ_0. Hartle and Sabbadini (1977) have studied this carefully and give the following empirical formula:

$$M_{max} < 4.8M_\odot \left(\frac{2 \times 10^{14}\ g\ cm^{-3}}{\rho_0}\right)^{\frac{1}{2}} \tag{14}$$

   Where ρ_0 is the density at which the causality limited equation of state is "attached" to a known and "reliable" equation of state. The value of $3.2M_\odot$ corresponds to a particular choice of $\rho_0 = 4.6 \times 10^{14}\ g\ cm^{-3}$ that Rhoades and Ruffini chose to make.

2. So far we have considered non-rotating stars. Friedman and Ipser (1987) have derived the following empirical formula for uniformly rotating models

$$M_{max}^{rot} < 6.1M_\odot \left(\frac{2 \times 10^{14}\ g\ cm^{-3}}{\rho_0}\right)^{\frac{1}{2}} \tag{15}$$

7. Discussion

We began by remarking that the real stumbling block in determining the maximum mass of neutron stars is the equation of state of neutron star matter at densities above the nuclear density $\sim 2.5 \times 10^{14}$ g cm^{-3}. After four decades of strenuous effort by several groups there is still considerable uncertainty concerning the equation of state: *is the matter in the core of the star "stiff" or "soft"?* This depends on whether or not Bose-Einstein condensates, such as pion condensate or kaon condensate, occur at supranuclear densities, and whether asymptotically free quark matter occurs at even higher densities. Till this question is resolved all one can say is that the maximum mass of neutron stars is somewhere in the range $1.5 - 6M_\odot$. It seems to us that the best one can do at present is to appeal to observations.

7.1 The observed Masses of Neutron Stars

There are three types of observations that can effectively constrain the equation of state: the measured masses of neutron stars, their spin frequency and their cooling rate. Let us first summarize the situation concerning the measured masses.

Binary pulsars

In the case of the Hulse-Taylor system (two neutron stars in a tight, eccentric orbit) and a few other pulsars one has been able to precisely determine the masses of both the stars, thanks to general relativistic effects. The masses of neutron stars determined in the fashion are shown in the figure. It is extraordinary that the masses are remarkably close to $1.4M_{\odot}$.

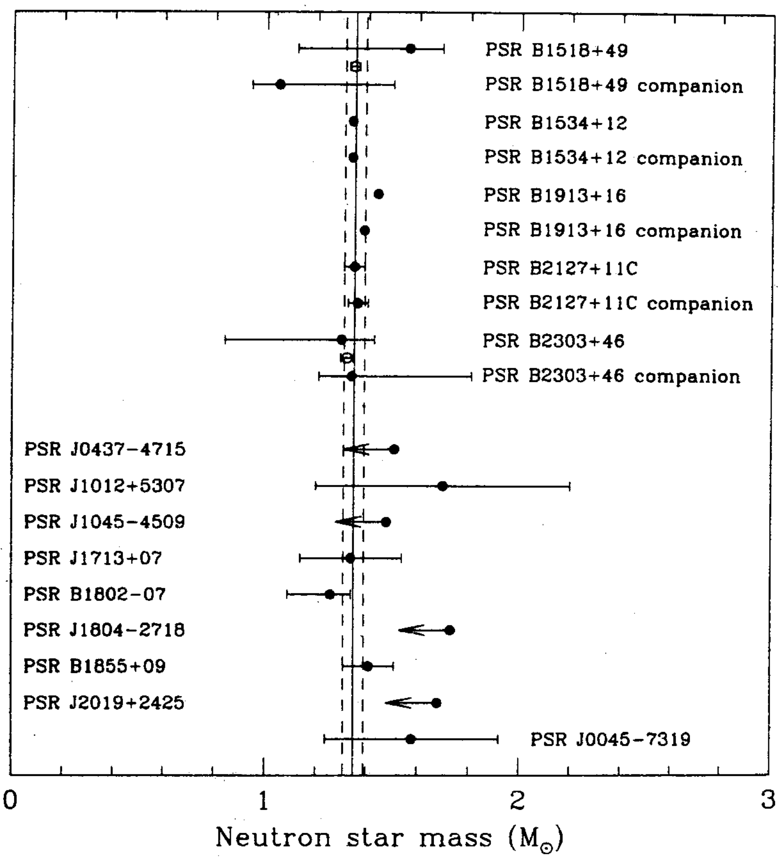

Figure 9. Neutron star masses from observations of radio pulsars in binary systems. All error bars indicate central 68% confidence limits. The masses of neutron stars in five *double neutron star systems* is shown at the top of the diagram. In two cases, PSR 1518+49 and PSR 2303+46, the *average* of the masses of the two neutron stars in the system is known with much better accuracy than the individual masses; these average masses are indicated with *open circles*. Eight *neutron star-white dwarf systems* are shown below these. The bottom most entry is the mass of a neutron star with a main sequence star as the companion. The vertical lines are drawn at $m = 1.35 \pm 0.04M_{\odot}$ (from Thorsett and Chakrabarty 1999).

The conclusion one can draw is that whatever be the nature of the equation of state, the maximum mass of neutron stars is greater than $1.4M_{\odot}$. Therefore, any equation of state which predicts a lower maximum mass can be ruled out!

7.2 Bosonic Condensates and Quark Matter

One of the central questions is whether π mesons and strange particles like the K meson spontaneously occur in neutron star matter. As mentioned before, these particles will be produced, in principle, if the chemical potential of the electrons exceeds the rest mass of these particles. For example, at sufficiently high densities electrons will change into K^- mesons

$$e^- \rightarrow K^- + \nu \tag{16}$$

with the neutrino leaving the star. The difficulty arises due to the fact that K^- meson so created is in a *medium* and therefore one has to know the interaction energy with the medium before one can estimate the density (and therefore the value of the chemical potential of the electrons) at which this process will occur. This is a complex many-body problem, with inherent uncertainties in estimating the interaction energy. This is where the ongoing relativistic heavy ion collision experiments will shed light. These experiments directly probe the stiffness of matter at densities well in excess of nuclear densities. The picture will become clear in a few years.

It turns out that not only laboratory experiments of heavy ion collision at relativistic energies will help to resolve this issue, but also some astronomical observations. The occurrence of pion codensate or Kaon condensate or quark matter will have a dramatic effect on the temperature evolution of neutron stars. This is because such states will have very high neutrino luminosities.

And this has a profound significance for the cooling rate of neutron stars. Till the core temperature of a neutron star drops to $\sim 10^8 K$ the predominant cooling mechanism is neutrino emission. Cooling due to photon emission is important only at lower temperatures. Thus the cooling rate of *young* ($\leq 10^5$years)neutron stars will be dramatically different if they have bosonic condensates or quark matter. This is illustrated in the figure.

X-ray observations of neutron stars with reliable age estimates (through, for example, their association with historical supernovae) will help to resolve this question. The already existing observations (with satellites such ROSAT) do not appear to need ultra rapid cooling of neutron stars. But more sensitive observations with CHANDRA Observatory should shed light on this important question.

To summarize, all one can say at present is that there are no astronomical observations that compel us to invoke Bose-Einstein condensates or quark matter in neutron stars.

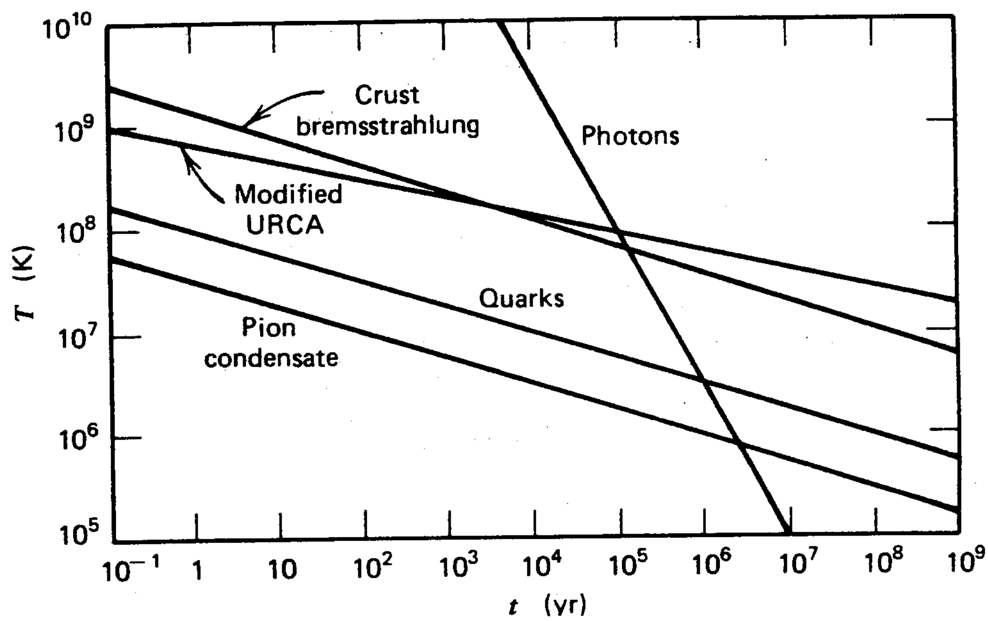

Figure 10. Cooling curves for neutron star interior. The various curves correspond to various processes acting alone. As may be seen, the presence of Bose condensates and/or Quark matter will result in an ultra rapid cooling of the neutron star (from Baym and Pethick 1979).

7.3 Constraints from Rotation Rates

As already mentioned, for a neutron star of a given mass the maximum angular velocity with which it can rotate is constrained by the equation of state. It is now generally accepted that the first-born neutron star in a binary system will be spun up during the phase when it accretes from the companion. Very rapidly spinning neutron stars, such as the millisecond pulsars, are believed to be such spun up pulsars. The question is, what limits the period to which they are spun up? According to the traditional point of view, this limitation arises from the *accretion rate* (with the Eddington rate providing an upper limit) and the magnetic field (smaller the field, smaller is the final period). But it could also be limited by instabilities arising due to rotation. Mass shedding at the Kepler frequency is an example. Let us briefly discuss some other important possibilities.

Nonaxisymmetric instability

It is well known that Newtonian stars that rotate sufficiently rapidly are unstable to a bar-mode instability associated with a perturbation having angular dependence cos $m\phi$, for $m = 2$. This is the point at which the sequence of Maclaurin spheroid bifurcates into the sequence of Jacobi ellipsoids. In 1970 Chandrasekhar made the important discovery that the Jacobi sequence is unstable after the bifurcation point, and that this instability is driven by *gravitational radiation* (Chandrasekhar, 1970). It should be stressed that this instability has no Newtonian analogue. In 1978 John Friedman made the remarkable discovery that a *non-axisymmetric instability driven by gravitational radiation is a*

generic feature of rotating perfect-fluid stars in general relativity (Friedman, 1978). Every rotating, self gravitating fluid is unstable in general relativity! This instability sets in not through $m = 2$ mode, but through modes of large m. This instability may be understood in the following way.

These are non-radial modes, and there will be a ***pair*** of them: one that moves ***forward*** with respect to the star and one that moves ***backward*** with respect to the star. In a slowly rotating star these modes will be damped out by gravitation waves in the following way. Gravitational waves will remove positive angular momentum from the forward moving mode and negative angular momentum from the backward moving wave (with respect to the star), thus damping out the oscillations. ***If the angular velocity of the star is sufficiently large, the mode travelling backward with respect to the star will be dragged forward with respect to an inertial observer.*** Gravitational waves will now remove ***positive angular momentum*** from this mode also. But since this mode is moving backwards with respect to the star, it has negative angular momentum. Thus the angular momentum (of the oscillation) will become more negative due to the emission of gravitational waves. In other words, gravitational radiation will drive this mode! Detailed calculations show that the limiting value of rotation rate when the above non-axisymmetric instability sets in will be reached before the Kepler frequency is reached (see Friedman and Ipser, 1992). This is true for most of the equations of state. The precise value of this limiting angular frequency will, of course, be determined by the stiffness of the equation of state. This is clearly illustrated in the figure from Friedman and Ipser (1992).

The angular velocity is plotted against the parameter T/W, the ratio of the rotational energy to the modulus of the potential energy. The various labelled curves correspond to sequences calculated with various equations of state. G is the ***softest*** and L and M the ***stiffest*** equation of state. The termination point of each of the sequence corresponds to the Kepler frequency Ω_K. The diagonal line crossing these curves is the estimate of the smallest rotation rate at which the gravitational wave instability will set in. It is interesting to note that since the curves are fairly "flat" this instability limit is within 15% of the Keplerian limit.

Presently, the population of millisecond pulsars is close to a couple of dozen. It is a curious and remarkable fact that the two fastest rotators among them PSR 1937+214 and PSR 1957+20 have very nearly the same angular frequencies, $4033s^{-1}$ and $3910s^{-1}$, respectively. These are within 3% of each other. Since the magnetic field of these pulsars differ by a factor ~ 2, the near coincidence of their angular frequencies would have to be accidental. A more appealing argument is that $\Omega \sim 4 \times 10^{-3}s^{-1}$is the limiting frequency for gravitational wave instability. The discovery of a couple of more pulsars near this frequency would clinch this argument. But let us take this conclusion seriously for a moment. A glance at the figure shows that of the 10 equations of state only the stiff ones (L, M, N, and O) are consistent with a mass of 1.4 $M_\odot$ ***and*** the period of these millisecond pulsars. By the same token, future discovery of even faster pulsars with submillisecond periods will require soft equations of state. While no definite statement can be made at

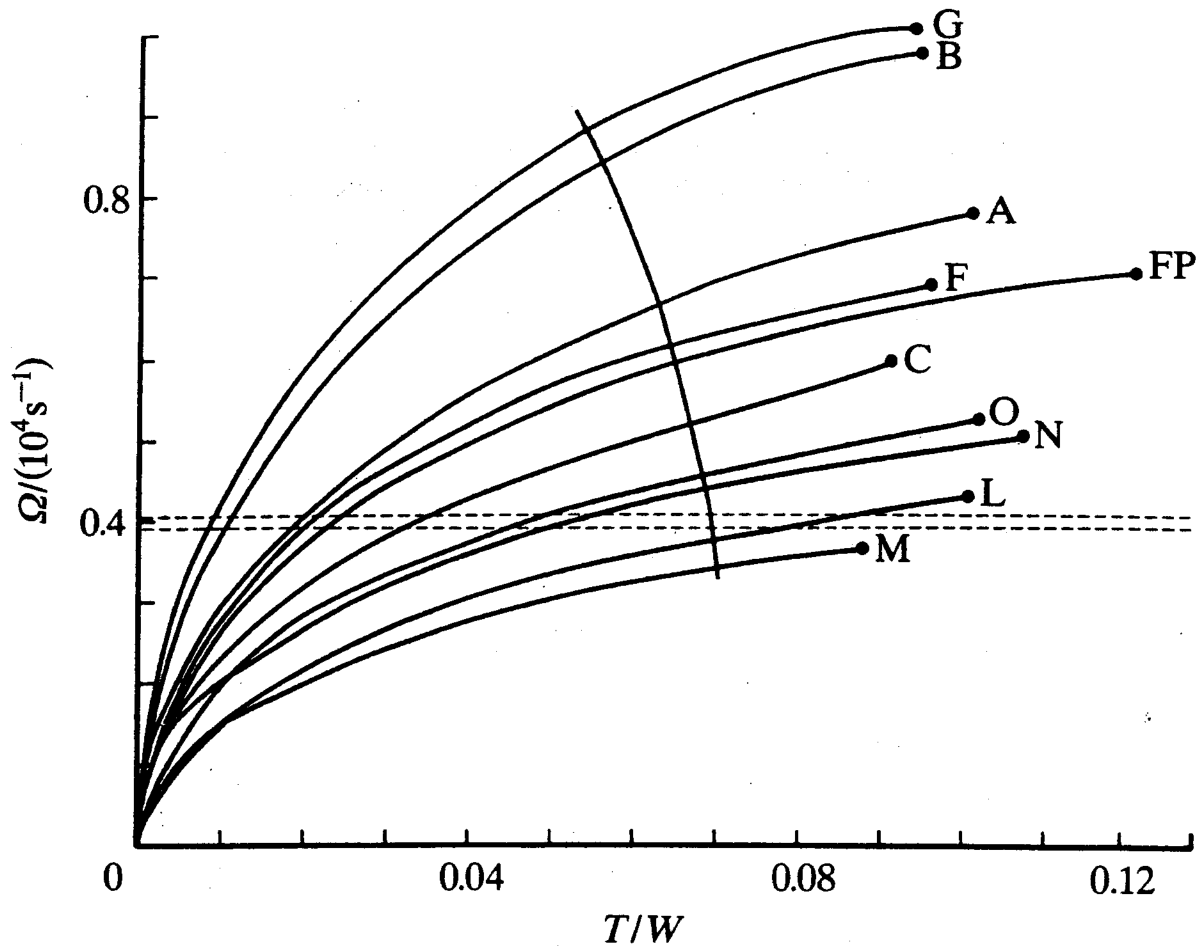

Figure 11. Sequences of uniformly rotating neutron stars of baryon masses $M_B = 1.4M_\odot$ for various equations of state. The angular velocity Ω is plotted against $T/|W|$, the ratio of rotational energy to the potential energy of rotating stars. The termination point of each sequence corresponds to $\Omega = \Omega_K$. The diagonal line crossing the sequences is a conservative estimate of the smallest rotation rate for which the model could be unstable to non-axisymmetric perturbations. The horizontal dashed lines are the angular velocities of the two fastest pulsars: PSR 1937+214 (upper line) and PSR 1957+20 (lower line). The main conclusion is that only the stiff equations of state (L, M, N and O) are consistent with a mass of $1.4M_\odot$ *and* the observed periods of these pulsars (from Friedman and Ipser 1992).

the moment, it is heartening that general relativity predicts an instability which can, in principle, constrain the equation of state.

But there is a qualification! For a perfect fluid the growth time of an unstable mode driven by gravitational radiation is the ***radiation reaction time***. If the viscous damping time is less than this then viscosity will stabilize the unstable modes (Detweiler and Lindblom, 1977; Lindblom and Hiscock, 1983). Consequently, ordinary gaseous stars are unlikely to develop these nonaxisymmetric instabilities. As for neutron stars, the picture is unclear. This is because of the relativity poor understanding of the viscosity of neutron star matter (we refer the reader to Lindblom and Mendell, 1995 and Lai and Shapiro, 1995 for a recent discussion of this topic). The current opinion may be summarized as follows. Although this may seem paradoxical, the viscosity of the fluid core may be **more** if it is in a *superfluid state*! This is because, as mentioned earlier, if superfluidity sets in it will be in a *vortex state*. It is this array of vortices, with their "normal cores", that enhances the viscosity (It may be recalled that electrons scatter off the vortex cores very efficiently). Having said this, it should be stressed that superfluid hydrodynamics is sufficiently complicated to introduce considerable uncertainty in the estimate of the viscous timescale. But if one accepts the above conclusion for a moment, viz., the viscosity of the fluid core may be *less* if the fluid is *normal*, than if it is a superfluid, then one may advance the following scenario.

Accretion induced collapse of a white dwarf: According to one point of view, rapidly spinning neutron stars with weak magnetic fields (such as the millisecond pulsars) may be formed when a white dwarf is pushed over the Chandrasekhar limit due to accretion from a companion. If this happens then the new-born neutron star will be hot, with core temperature in excess of $10^{10}K$. Interestingly, the superfluid transition temperature is estimated to be $\leq 10^{10}K$ (see Sauls, 1989). It is therefore likely that as the white dwarf collapses, and spins up due to the conservation of angular momentum, the stellar matter may be in the normal state with low viscosity. So one may expect the spin period of the new-born neutron star to be limited by nonaxisymmetric instabilities.

Recycled pulsars: The other and more favoured route for the formation of millisecond pulsars is that they are the first-born neutron stars in binary systems which are spun up during the mass transfer x-ray phase (Srinivasan, 1989). In this scenario, too, the minimum period to which the neutron star is spun up may be limited by gravitational wave instabilities if enough heat is generated during accretion, and superfluidity in the core is destroyed. Again, one is not in a position to make definitive statements at this state.

To conclude, although the maximum mass of neutron stars has acquired great significance in contemporary astronomy, it continues to be as elusive as ever!

References

Baade, W. Zwicky, F., 1934, *Phys. Rev.*, **45**, 138.
Baym, G., Pethick, C.J. Sutherland, P., 1971a, *Astrophys. J.*, **170**, 299.
Baym, G., Bethe, H.A. Pethick, C.J., 1971b, *Nuclear Physics*, **A175**, 225.
Baym, G. Pethick, C.J., 1979, *Ann. Rev. Astron. Astrophys.*, **17**, 415.
Brown, G.E., 1999, in *Pulsar Timing, General Relativity and the Internal Structure of Neutron Stars*,eds. Z Arzoumanian, F. van der Hooft and E.P.J. van den Heuvel, Royal Netherlands Academy of Arts and Sciences, Amsterdam.
Chandrasekhar, S., 1931, *Astrophys. J.*, **74**, 81.
Chandrasekhar, S., 1964, *Astrophys. J.*, **140**, 417.
Chandrasekhar, S., 1970, *Astrophys. J.*, **161**, 561.

Chandrasekhar, S. Tooper, R.F., 1964, *Astrophys. J.*, **139**, 1396.
Detweiler, S.L. Lindblom, L., 1977, *Astrophys. J.*, **213**, 193.
Durisen, R.H., 1975, *Astrophys. J.*, **183**, 215.
Harrison, B.K. Wheeler, J.A., 1958 in *La Structure et l'evolution de l'univers*, eds. B.K. Harrison, K.S. Thorne, M. Wakano and J.A. Wheeler, Onzième Conseil de Physique Solvay, Brussels.
Friedman, B., Pandharipande, V.R., 1981, *Nuclear Physics*, **A361**, 502.
Friedman, J.L., 1978, *Comm. Math. Phys.*, **62**, 247.
Friedman, J.L., Ipser, J.R., 1987, *Astrophys. J*, **324**, 594.
Friedman, J.L., Ipser, J.R., 1992, *Phil. Trans. R. Soc. Lond.*, **A340**, 391.
Hartle, J.B., Sabbadini, A.G., 1977, *Astrophys. J.*, **213**, 831.
Lindblom, L., Hiscock, W.A., 1983, *Astrophys. J.*, **267**, 384.
Lindblom, L., Mendell, G., 1995, *Astrophys. J*, **444**, 804.
Lai, D. Shapiro, S.L.,1995, *Astrophys. J.*, **442**, 259.
Nauenberg, M., Chapline, G.Jr., 1973, *Astrophys. J.*, **179**, 277.
Oppenheimer, J.R., Volkoff, G.M., 1939, *Phys. Rev.*, **55**, 507.
Pandharipande, V.R., Smith, K.A.,1975, *Nuclear Physics*, **A237**, 507.
Rhoades, C.E., Ruffini, R., 1974, *Phys. Rev. Lett.*, **32**, 324.
Sauls, J.,1989, in *Timing Neutron Stars*, eds. H. Ogelman and E.P.J. can den Heuvel, Kluwer: Dordrecht.
Shapiro, S.L. Teukolsky, S.A., 1983, *Black Holes, White Dwarfs and Neutron Stars*, John Wiley & Sons.
Srinivasan, G., 1989, *Astron. Astrophys. Rev.*, **1**, 209.
Thorsson, V., Prakash, M., Lattimer, J.N., 1994, *Nuclear Physics*, **A572**, 693
Throsett, S.E., Chakrabarty, D., 1999, *Astrophys. J.*, **512**, 288.
Wiringa, R.B., Fiks, V., Fabrocini, A., 1988, *Phys. Rev.*, **C 38**, 1010.