by Ian Stewart

Juniper Green

About a year ago Ian Porteous, a mathematician at the University of Liverpool, told me about an elegant game. His son, Richard Porteous, invented it to teach children about multiplication and division. The game is called Juniper Green, after the school at which Richard taught. It is fun to play, and the search for a winning strategy is quite challenging.

To play Juniper Green, you should make 100 cards, numbered 1 through 100. Lay them face up on the table in numerical order, say, 10 rows of 10 cards each, so that it will be easy for players to locate the desired card. Here are the rules:

1. Two players take turns removing one card from the table. Cards removed are not replaced and cannot be used again.
2. Apart from the opening move, each number chosen must either be an exact divisor of the previous player's choice or an exact multiple.
3. The first player who is unable to choose a card loses.

There is one final rule to make the game worth playing. Recall that a prime number has no divisors other than itself and 1. It so happens that if a player picks a prime larger than 50, then the next player loses. Suppose Alice plays against Bob, with Alice going first. She plays 97; Bob must play 1. Now Alice plays another big prime—say, 89. At this point Bob has used up card 1 and is stuck. To prevent this spoiling strategy, we have:

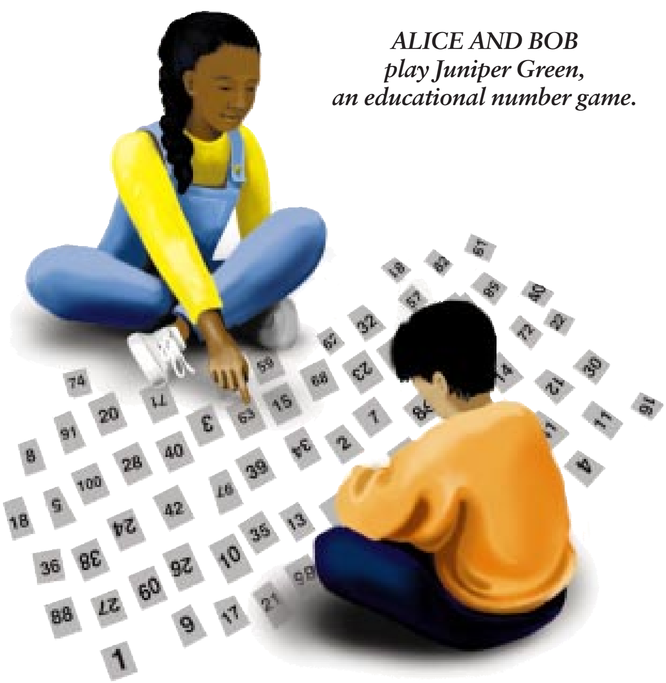
4. The opening move in the game must be an even number.

Even though the game starts with an even number, big primes still influence play. In particular, if any player picks card 1, then he or she loses, assuming the opponent is awake. Say Bob chooses 1, and Alice responds with a big prime—97. (Note that 97 must be available, because it can be chosen only if the previous player chooses 1.) Then Bob has nowhere to go. So the game effectively ends when a player is forced to choose card 1.

The chart below shows a sample game, played without much regard for good tactics. I would suggest that at this point you stop reading, make a set of cards and play the game for a while. Although I'm not going to give away the winning strategy—I'll put it in a subsequent Feedback section so as not to spoil your fun—I will analyze the same game when there are only 40 cards, numbered 1 to 40. The analysis will give you some broad hints on the 100-card game as well. Very young children might use a pack numbered 1 to 20. For brevity, I will call the Juniper Green *n*-card "JG-*n*" and find a winning strategy for JG-40.

Some opening moves, of course, lose rapidly. For example:

MOVE	ALICE	BOB
1	38	
2		19
3	1	
4		37
5	LOSES	



ALICE AND BOB play Juniper Green, an educational number game.

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MOVE	ALICE	BOB	COMMENTS
1	48		Even number, as required by rule 4
2		96	Doubles Alice's choice
3	32		One third of Bob's choice
4		64	Bob is forced to choose a power of 2
5	16		So is Alice
6		80	
7	10		
8		70	
9	35		Halve
10		5	Only choices are 7, 5 (or 1 and lose)
11	25		
12		75	Only 50 and 75 available
13	3		
14		81	
15	9		Only 27 and 9 available
16		27	Bad move!
17	54		Forced because 1 loses
18		2	Forced
19	62		Inspired variant on big prime tactic
20		31	Forced
21	93		Only choice but a good one
22		1	Forced, and loses, because . . .
23	97		Big prime tactic

The same goes for an opening move of 34. Some other numbers are also best avoided. For instance, suppose that Alice is unwise enough to play 5. Then Bob strikes back with a vengeance by picking 25. Alice has no choice but to play 1; however, this move is bound to lose. (Note that 25 must still be available, because it can be chosen only if the previous player plays 1 or 5.)

Alice's obvious tactic is to force Bob to play 5 instead. Can she do this? Well, if Bob plays 7, then she can play 35, and Bob has to play 1 or 5, both of which lose. Good, but can she force Bob to play 7? Yes: if Bob has chosen 3, then Alice can play 21, and that forces a reply of 7. Fine, but how does she make Bob play 3? Well, if he plays 13, then Alice plays 39. Alice can go on in this manner, building hypothetical sequences that force Bob's reply at every stage and lead to his inevitable defeat.

But can she maneuver Bob into such a sequence to begin with? Early in the game the moves have to involve even numbers, so the card numbered 2 is likely to play a pivotal role. Indeed, if Bob plays 2, then Alice can play 26, forcing Bob into the trap of playing 13. So now we come to the crunch. How can Alice force Bob to play 2?

If Alice opens with 22, then Bob either plays 2 and gets trapped in the long sequence of forced moves outlined above, or he plays 11. Now Alice has the choice of playing 1 and losing or going to 33. When she picks 33, 11 has already been used up, so Bob is forced to 3, and so Alice can win. The moves below summarize Alice's strategy: the two sets of columns deal with the two alternatives Bob can pick. (Assume throughout that all players avoid 1.)

MOVE	ALICE	BOB	ALICE	BOB
1	22			
2		11		2
3	33		26	
4		3		13
5	21		39	
6		7		3
7	35		21	
8		5		7
9	25		35	
10		LOSES		5
11			25	
12				LOSES

There is at least one other possible opening move for Alice that forces a win: 26. The same kind of game develops but with a few moves interchanged as in the list below.

MOVE	ALICE	BOB	ALICE	BOB
1	26			
2		13		2
3	39		22	
4		3		11
5	21		33	
6		7		3
7	35		21	
8		5		7
9	25		35	
10		LOSES		5
11			25	
12				LOSES

The crucial features here are the primes 11 and 13. If the opening move is twice such a prime (22 or 26), Bob has to reply either with 2—at which point Alice is off to a win—or with the prime. But then Alice replies with thrice the prime, forcing Bob to go to 3—and she's away again. So Alice wins because apart from two times the prime, there is exactly one other multiple that is under 40, namely, 33 or 39. These "medium primes," which amount to between one third and one quarter of the number of cards, allow Alice to win.

Does any opening choice other than 22 or 26 also lead to a win? That's for you to find out. Moreover, you are now in a good position to analyze JG-100—or even the ambitious JG-1,000. Is there a first-player strategy to force a win?

Finally, the time has come to open up the problem in its full generality. Consider JG-*n* for any whole number *n*. Because no draws are allowed, game theory implies that either Alice—who goes first—can have a winning strategy or Bob can, but not both. Suppose *n* is "primary" if Alice has a winning strategy for JG-*n* and "secondary" if Bob does. Can you characterize which *n* are primary and which are secondary?

For very small *n*, a few quick calculations indicate that 1, 3, 8 and 9 are primary, whereas 2, 4, 5, 6 and 7 are secondary. What about $n = 100$? Completely general *n*? Can anyone find any patterns? Or solve the whole thing? SA

FEEDBACK

Much of the mail I got on the interrogator's fallacy [September 1996] demonstrated how easy it is to get confused about conditional probabilities. So I'll try to clarify the points that caused the most difficulty. Most readers had trouble with the preparatory example. We were told that the Smith family has two children and that one of them is a girl. What is the probability that both are girls? (Assume boys and girls are equally likely, which may not be the case in reality. Also, when I say "one is a girl," I do not mean that only one is: I mean that at least one is.)

The big bone of contention was my ordering the children by birth. There are four types of two-child family: BB, BG, GB, GG. Each, I said, is equally likely. If one child is a girl, we are left with BG, GB and GG. Of these, only one gives two girls. So the conditional probability that if one is a girl, so is the other, is $^1/_3$. On the other hand, if we are told "the eldest child is a girl," then the conditional probability that they are both girls is now $^1/_2$.

Some of you said that I shouldn't distinguish BG and GB. Why don't we just toss two coins to check? The coins represent the sexes, with the right probabilities ($^1/_2$ each). If you're lazy, like me, you can simulate the tosses on a computer with a random-number generator. For one million simulated throws, here's what I got:

Two heads	250,025
Two tails	250,719
One of each	499,256

Try it for yourself. If BG and GB are the same, you should get 333,333 in the last category.

The other main argument was that whether or not we know that one child is G, the other is equally likely to be B or G. It is instructive to see why this reasoning is wrong. When both children are girls, there is no unique notion of "the other"—unless I specify which girl I am thinking about (for example, the elder). The specification destroys the assumed symmetry between Bs and Gs and changes the conditional probabilities. In fact, the statement "the eldest child is a girl" conveys more information than "at least one child is a girl." (The first implies the second, but the second need not imply the first.) So it ought not to be a surprise that the associated conditional probabilities are different. *—I.S.*