Kruskal coordinates as canonical variables for Schwarzschild black holes

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We derive a transformation from the usual ADM metric-extrinsic curvature variables on the phase space of Schwarzschild black holes to new canonical variables which have the interpretation of Kruskal coordinates. We explicitly show that this transformation is non-singular, even at the horizon. The constraints of the theory simplify in terms of the new canonical variables and are equivalent to the vanishing of the canonical momenta. Our work is based on earlier seminal work by Kuchař in which he reconstructed curvature coordinates and a mass function from spherically symmetric canonical data. The key feature in our construction of a nonsingular canonical transformation to Kruskal variables is the scaling of the curvature coordinate variables by the mass function rather than by the mass at left spatial infinity.

I. INTRODUCTION

This work is devoted to an extension and improvement of Kuchař's beautiful analysis of the Hamiltonian description of Schwarzschild black holes. In [1] Kuchař reconstructed the curvature coordinates (i.e. the areal radius and the Killing time) as well as a mass function[1] from spherically symmetric Arnowitt-Deser-Misner (ADM) canonical data on a Cauchy slice. The curvature coordinates were turned into canonical variables and the constraints in the Hamiltonian description simplified when expressed in terms of the new canonical variables. Since the curvature coordinates are not good spacetime coordinates on the horizon, Kuchař's canonical transformation is singular on the horizon. Nevertheless, it was argued in [1] that, with sufficient care near the horizon, the curvature coordinate variables could be used to simplify the Hamiltonian description and that the imposition of the constraints was equivalent to the vanishing of the momenta conjugate to the curvature coordinate variables.

In this work we improve upon the treatment of [1] by constructing a transformation to new canonical variables which have the interpretation of Kruskal coordinates. This transformation is free from the singularities of the canonical transformation to curvature coordinate variables.[2] The con-

straints simplify in terms of the new Kruskal variables and their imposition is equivalent to the vanishing of the new canonical momenta. As in [1], the true degrees of freedom are the mass at left infinity and the difference between Killing time and parametrization time at left infinity.

The layout of the paper is as follows. In Sec. II we quickly review the relevant parts of [1]. The purpose of this review is to establish notation and to collect the set of equations from [1] which we shall use to establish our results. The reader may consult [1] for more details and we shall assume familiarity with that work. In Sec. III we derive the canonical transformation to the Kruskal variables from the curvature coordinate variables of [1] and express the constraints in terms of the Kruskal variables. In Sec. IV, we express the ADM variables in terms of the new canonical variables and note that the transformation is manifestly nonsingular at the horizon.

We describe our choice of asymptotic behavior for the canonical variables in Sec. V. In Sec. VI we invert the transformation of Sec. IV and express the Kruskal variables in terms of the ADM variables. Section VII contains our concluding remarks.

II. REVIEW OF KUCHAŘ'S RESULTS

In this section we briefly review the results of [1]. As mentioned in the introduction, the purpose of this section is to establish notation and collect the set of equations from [1] which we shall use to establish our results. The reader may consult [1] for more details.

Spherically symmetric Cauchy slices in the Hamiltonian description of spherically symmetric gravity are diffeomorphic to $S^2 \times \mathbf{R}$. The spatial metric induced on such a slice is

$$d\sigma^2 = \Lambda^2(r)(dr)^2 + R^2(r)(d\Omega)^2 \qquad (1)$$

where r is a radial coordinate and $d\Omega$ is the line element on the unit sphere. $r = \infty$ labels right spatial infinity and $r = -\infty$ labels left spatial infinity. $P_\Lambda(r)$ and $P_R(r)$ are the mo-

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[1]In this regard, note that Eqs. (11) and (12) in this work were discovered prior to the work of Kuchař by Louis-Martinez, Gegenberg and Kunstatter [2] in the context of (1+1)-dilatonic gravity models of which spherically symmetric gravity is a special case. For a brief review of work on canonical spherically symmetric gravity see, for example, the introduction of [1].

[2]In [1], a canonical transformation on the curvature coordinate variables involving scaling these variables by the mass at left spatial infinity yielded new canonical variables which, on shell, also had the interpretation of Kruskal coordinates. However, as discussed in Sec. VII, the transformation from ADM variables to these Kruskal variables is still not free from singularities. In contrast, our canonical transformation involves scaling of the curvature coordinate variables by the *mass function*. It is this feature which enables a nonsingular description.

menta conjugate to $\Lambda(r)$ and $R(r)$. After parametrizing the times at the two spatial infinities, the action takes the form[3]

$$
\begin{aligned}
S(\Lambda, & P_\Lambda, R, P_R, N, N^r, \tau_+, \tau_-) \\
&= \int dt \int_{-\infty}^{\infty} dr (P_\Lambda \dot{\Lambda} + P_R \dot{R} - NH - N^r H_r) \\
&\quad - \int dt (M_+ \dot{\tau}_+ - M_- \dot{\tau}_-).
\end{aligned}
\tag{2}
$$

H_r is the radial diffeomorphism constraint and H is the scalar constraint. They are given by

$$
H_r = P_R R' - \Lambda P_\Lambda'
\tag{3}
$$

and

$$
\begin{aligned}
H = & -R^{-1} P_R P_\Lambda + \frac{1}{2} R^{-2} \Lambda P_\Lambda^2 + \Lambda^{-1} RR'' - \Lambda^{-2} \Lambda' RR' \\
& + \frac{1}{2} \Lambda^{-1} R'^2 - \frac{1}{2} \Lambda.
\end{aligned}
\tag{4}
$$

The parameters τ_+ and τ_- label parametrization clocks at right and left spatial infinity and are to be freely varied in the action as are the lapse function, N, and the radial shift vector field, N^r. $M_\pm$ are parameters which appear in the asymptotic fall off of Λ at right and left spatial infinity.

To leading order in r, the asymptotic behavior of the canonical coordinates as $r \rightarrow \pm\infty$ on any Cauchy slice of constant '''t'' is

$$
\Lambda = 1 + M_\pm |r|^{-1}, \quad R = |r|.
\tag{5}
$$

By parametrizing the standard form of the Schwarzschild line element in curvature coordinates in terms of arbitrary parameters (t, r), comparing the result with the standard ADM form of the line element and using the relation between canonical momenta and the velocities of canonical coordinates, Kuchař was able to reconstruct the Schwarzschild mass and the rate of change of Killing time from the canonical data. The expression for the Schwarzschild mass function is

$$
M = \frac{1}{2} R^{-1} P_\Lambda^2 - \frac{1}{2} \Lambda^{-2} RR'^2 + \frac{1}{2} R.
\tag{6}
$$

It is useful to define the quantity ''F'' by

$$
F = 1 - \frac{2M}{R}
\tag{7}
$$

$$
= \left(\frac{R'}{\Lambda}\right)^2 - \left(\frac{P_\Lambda}{R}\right)^2.
\tag{8}
$$

When the constraints are satisfied, the vanishing of F determines the location of horizons of the (extended) Schwarzschild spacetime. We shall continue to use the word ''horizon'' when $F = 0$ even when the constraints are not imposed.

Denoting the Killing time by T, its rate of change along the slice is given by

$$
T' = -R^{-1} F^{-1} \Lambda P_\Lambda.
\tag{9}
$$

T is turned into a canonical coordinate by making a transformation from $[\Lambda(r), P_\Lambda(r), R(r), P_R(r), \tau_\pm]$ to the variables $[T(r), P_T(R), R(r), \Pi_R(r), m, p]$[4] where

$$
T(r) = \tau_+ + \int_{\infty}^{r} T'(\bar{r}) d\bar{r},
\tag{10}
$$

$$
P_T(r) = -M'(r),
\tag{11}
$$

$$
= \Lambda^{-1}(R'H + R^{-1} P_\Lambda H_r),
\tag{12}
$$

$$
\begin{aligned}
\Pi_R = & P_R - \frac{1}{2} R^{-1} \Lambda P_\Lambda - \frac{1}{2} R^{-1} F^{-1} \Lambda P_\Lambda \\
& - R^{-1} \Lambda^{-2} F^{-1}((\Lambda P_\Lambda)'(RR') - \Lambda P_\Lambda(RR')')
\end{aligned}
\tag{13}
$$

$$
= F^{-1}(R^{-1} P_\Lambda H + R' \Lambda^{-2} H_r),
\tag{14}
$$

$$
p = T(-\infty) - \tau_-,
\tag{15}
$$

$$
m = M_-.
\tag{16}
$$

The action in Eq. (2) is replaced by

$$
\begin{aligned}
S(T, & P_T, R, \Pi_R, N, N^r, m, p) \\
&= \int dt \left(p\dot{m} + \int_{-\infty}^{\infty} dr (P_T \dot{T} + \Pi_R \dot{R}) \right) \\
&\quad - \int dt \int_{-\infty}^{\infty} dr (NH - N^r H_r).
\end{aligned}
\tag{17}
$$

The constraints take the form

$$
H_r = \Pi_R R' + P_T T',
\tag{18}
$$

$$
\Lambda H = F^{-1} P_T R' + FT' \Pi_R,
\tag{19}
$$

with

$$\Lambda = (F^{-1}R'^{2} - FT'^{2})^{1/2}. \tag{20}$$

From Eqs. (9), (14) and (20), it can be seen that when $F \rightarrow 0$, the transformation becomes singular.

III. CANONICAL TRANSFORMATION TO KRUSKAL VARIABLES

In this section we construct a canonical tranformation from the curvature coordinate variables (R, Π_R, T, P_T) to Kruskal variables (U, P_U, V, P_V). The transformation is performed in two steps. The curvature coordinate variables are scaled with the mass function, $M(r)$, in the first step and in the second, a point transformation is made from the scaled variables to the Kruskal variables.

In what follows, we shall make use of the identity[5]

$$\int_{-\infty}^{\infty} dr f(r) \int_{-\infty}^{r} d\bar{r} g(\bar{r}) = -\int_{-\infty}^{\infty} dr g(r) \int_{\infty}^{r} d\bar{r} f(\bar{r}). \tag{21}$$

The Liouville form, ω, in terms of the curvature coordinate variables is

$$\omega = p\dot{m} + \int_{-\infty}^{\infty} dr (P_T \dot{T} + \Pi_R \dot{R}). \tag{22}$$

Using Eqs. (21) and (11), and ignoring total time derivatives, it can be shown that

$$\int_{-\infty}^{\infty} dr \Pi_R \dot{R} = \int_{-\infty}^{\infty} dr (2M\Pi_R) \frac{d}{dt} \left(\frac{R}{2M} \right) + \int_{-\infty}^{\infty} dr \frac{\Pi_R R}{M^2} m\dot{m}$$
$$- \int_{-\infty}^{\infty} dr (2MP_T) \frac{d}{dt} \left(\int_{\infty}^{r} d\bar{r} \frac{\Pi_R R}{2M^2} \right), \tag{23}$$

and that

$$\int_{-\infty}^{\infty} dr P_T \dot{T} = \int_{-\infty}^{\infty} dr (2MP_T) \frac{d}{dt} \left(\frac{T}{2M} \right) + \int_{-\infty}^{\infty} dr \frac{P_T T}{M^2} m\dot{m}$$
$$- \int_{-\infty}^{\infty} dr (2MP_T) \frac{d}{dt} \left(\int_{\infty}^{r} d\bar{r} \frac{P_T T}{2M^2} \right). \tag{24}$$

Equations (23) and (24) imply that, up to a total time derivative,

$$\omega = \bar{p}\dot{m} + \int_{-\infty}^{\infty} dr (P_{\bar{T}} \dot{\bar{T}} + \Pi_{\bar{R}} \dot{\bar{R}}), \tag{25}$$

where the new canonical variables are given by

$$\bar{p} = p + m \int_{-\infty}^{\infty} dr \left(\frac{P_T T + \Pi_R R}{M^2} \right), \tag{26}$$

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[5]For a proof of the identity, see [1].

$$\bar{R} = \frac{R}{2M}, \tag{27}$$

$$\Pi_{\bar{R}} = 2M\Pi_R, \tag{28}$$

$$\bar{T} = \frac{T}{2M} - \int_{\infty}^{r} d\bar{r} \left(\frac{P_T T + \Pi_R R}{2M^2} \right), \tag{29}$$

$$P_{\bar{T}} = 2M P_T. \tag{30}$$

Since the variable canonically conjugate to the new momentum, $\bar{p}$, is still the left mass, we have continued to denote it by m.

This completes the first step of the transformation to Kruskal variables. In the second step, we define the Kruskal variables through the following point transformation on the scaled variables:

$$(\bar{R} - 1)e^{\bar{R}} = -UV, \tag{31}$$

$$\bar{T} = \ln \left| \frac{V}{U} \right|. \tag{32}$$

It follows that

$$P_{\bar{T}} = \frac{VP_V - UP_U}{2}, \tag{33}$$

and

$$\Pi_{\bar{R}} = \frac{VP_V + UP_U}{2F}, \tag{34}$$

where F is a function of UV through Eqs. (7) and (31). Thus, the new set of canonical variables is (U, P_U, V, P_V) as well as the canonically conjugate parameters $(m, \bar{p})$. This completes our presentation of the canonical transformation from the curvature coordinate variables to the Kruskal variables.

Next, we present expressions for the constraints in terms of the new canonical variables. It is easy to check that in terms of the Kruskal variables the diffeomorphism constraint is

$$H_r = P_U U' + P_V V'. \tag{35}$$

It is easier to express the rescaled scalar constraint, ΛH, in terms of the Kruskal variables, rather than H. To do this, we use Eqs. (11), (14) and (27)–(32) in Eq. (19). Then it follows that ΛH takes the form

$$\Lambda H = P_V V' - P_U U' - \frac{\bar{R}^2}{2M^2} e^{\bar{R}} P_U P_V. \tag{36}$$

Note that from Eq. (31), the vanishing of F implies the vanishing of at least one of U or V. It follows from Eqs. (32) and (34), that the transformation between curvature coordinate variables and Kruskal variables is singular on the hori-

zon. Note, however, that the expressions for the constraints in terms of the Kruskal variables remain non-singular on the horizon.

IV. ADM VARIABLES IN TERMS OF KRUSKAL VARIABLES

As noted in previous sections, both the transformation from the ADM variables to the curvature coordinate variables as well the transformation from curvature coordinate variables to Kruskal variables, are singular when $F=0$. In this section we present expressions for the ADM variables in terms of the Kruskal variables and see that this transformation is manifestly *non-singular* at $F=0$.

To express Λ in terms of the Kruskal variables we start from Eq. (20) and use Eqs. (7), (11) and (27)–(32). Then it is straightforward to show that

$$\Lambda^2 = \frac{16M^2}{\bar{R}e^{\bar{R}}}\left(U'+\frac{P_V\bar{R}^2e^{\bar{R}}}{4M^2}\right)\left(-V'+\frac{P_U\bar{R}^2e^{\bar{R}}}{4M^2}\right). \quad (37)$$

In the above expression, $\bar{R}$ and M are to be thought of as functions of the Kruskal variables. $\bar{R}$ is a function of UV through Eq. (31). To express M in terms of the Kruskal variables, we use Eqs. (11), (30), (33) and (16). We obtain

$$M^2 = m^2 + \int_{-\infty}^{r} d\bar{r}\frac{UP_U - VP_V}{2}. \quad (38)$$

To express ΛP_Λ in terms of the Kruskal variables we start from Eq. (9) and use Eqs. (11), (27), (29) and Eqs. (31), (32), (34). We obtain

$$\Lambda P_\Lambda = -4M^2 e^{-\bar{R}}\left(V\left(U'+\frac{P_V\bar{R}^2e^{\bar{R}}}{4M^2}\right)\right.$$
$$\left. +U\left(-V'+\frac{P_U\bar{R}^2e^{\bar{R}}}{4M^2}\right)\right). \quad (39)$$

Next, R can be expressed as

$$R = 2M\bar{R} \quad (40)$$

with M given by Eq. (38) and $\bar{R}$ by Eq. (31).

The calculation of P_R in terms of the Kruskal variables is fairly involved and we sketch the main steps here. We evaluate P_R through Eq. (13). We first evaluate the expression $[(\Lambda P_\Lambda)'(RR')-\Lambda P_\Lambda(RR')']$ which occurs in Eq. (13). To this end, it is useful to define

$$g_1 = U\left(-V'+\frac{P_U\bar{R}^2e^{\bar{R}}}{4M^2}\right), \quad (41)$$

$$g_2 = V\left(U'+\frac{P_V\bar{R}^2e^{\bar{R}}}{4M^2}\right). \quad (42)$$

Then from Eqs. (11), (30), (27) and (39) it follows that

$$(\Lambda P_\Lambda)'(RR')-\Lambda P_\Lambda(RR')'$$
$$=4M^2\left(\frac{R'}{2M}\right)(-\Lambda P_\Lambda\bar{R}'(\bar{R}+1)-4M^2e^{-\bar{R}}\bar{R}(g_1'+g_2'))$$
$$-4M^2\bar{R}\left(\frac{R'}{2M}\right)'\Lambda P_\Lambda. \quad (43)$$

Equations (27), (30), (11), (31) and (33) imply that

$$\frac{R'}{2M} = \bar{R}^{-1}e^{-\bar{R}}(g_1-g_2). \quad (44)$$

Using Eqs. (44) and (39) in Eq. (43) we get

$$(\Lambda P_\Lambda)'(RR')-\Lambda P_\Lambda(RR')' = 32M^4 e^{-2\bar{R}}(g_2g_1'-g_2'g_1). \quad (45)$$

By substituting this expression in Eq. (13) and using Eqs. (7), (34) and (39), we obtain

$$P_R = -\bar{R}\left(\frac{VP_V+UP_U}{4M}\right)-Me^{-\bar{R}}(UV'-VU')+\frac{\Lambda P_\Lambda}{4M\bar{R}}$$
$$-16\frac{M^3e^{-\bar{R}}}{\Lambda^2}\mathcal{H} \quad (46)$$

where

$$\mathcal{H} = \left(U'+\frac{P_V\bar{R}^2e^{\bar{R}}}{4M^2}\right)\left(-V'+\frac{P_U\bar{R}^2e^{\bar{R}}}{4M^2}\right)'$$
$$-\left(U'+\frac{P_V\bar{R}^2e^{\bar{R}}}{4M^2}\right)'\left(-V'+\frac{P_U\bar{R}^2e^{\bar{R}}}{4M^2}\right). \quad (47)$$

In Eq. (46), ΛP_Λ and Λ^2 are given by Eqs. (37) and (39) and M by Eq. (38) and $\bar{R}$ through Eq. (31).

As mentioned earlier, Eqs. (7) and (31) imply that $F=0$ corresponds to the vanishing of at least one of U or V. As can be explicitly verified, the expressions for the ADM variables $(\Lambda,P_\Lambda,R,P_R)$ are all non-singular when this happens and hence, at the horizon, the ADM variables continue to be smooth functions of the Kruskal variables.

For the transformation to be defined, it is necessary to impose the condition $M\neq 0$. Since we are interested in black holes rather than naked singularities, we shall impose $M>0$. Further, in the ADM description, the conditions $\Lambda>0$ and $R>0$ hold and these conditions must be imposed in the description in terms of the Kruskal variables. We shall comment further on these points in Sec. VI where we invert the transformation and express the Kruskal variables in terms of the ADM variables.

Finally, note that we have yet to reconstruct the remaining ADM variables, namely the parametrization times $(\tau_\pm)$, from the Kruskal variables. Since this reconstruction requires not only the variable $\bar{p}$ in the Kruskal description but also the

asymptotic behavior of the Kruskal variables, we shall return to this point in Sec. V, after we have analyzed the asymptotics.

V. ASYMPTOTICS

In this section we describe our choice of asymptotic conditions for the ADM variables and the Kruskal variables. These conditions ensure that the various integrals encountered in the canonical transformation of Sec. III are convergent near spatial infinity.

In future work we would like our framwework to admit matter couplings. To this end, we expect that our choice of boundary conditions is general enough to handle couplings to a large class of matter fields, namely those for which the matter fields fall off faster than any power of $|r|^{-1}$ at infinity.[6]

In the Schwarzschild spacetime, the Kruskal coordinates are related to the curvature coordinates near right and left infinity by

$$U = \mp \sqrt{\frac{R}{2M} - 1} \, e^{(R-T)/4M}, \tag{48}$$

$$V = \pm \sqrt{\frac{R}{2M} - 1} \, e^{(R+T)/4M}. \tag{49}$$

Therefore, as $r \to \pm\infty$ we impose

$$U = \mp \sqrt{\frac{|r|}{2M_\pm} - 1} \, e^{|r|/4M_\pm} e^{-\bar{T}(\pm\infty)/2}(1 + \Theta(r)), \tag{50}$$

$$V = \pm \sqrt{\frac{|r|}{2M_\pm} - 1} \, e^{|r|/4M_\pm} e^{\bar{T}(\pm\infty)/2}(1 + \Theta(r)), \tag{51}$$

$$U P_U = \Theta(r), \tag{52}$$

$$V P_V = \Theta(r). \tag{53}$$

Here $\bar{T}(\pm\infty) := \bar{T}(r)|_{r=\pm\infty}$ and $M_\pm = M(r)|_{r=\pm\infty}$, with $M(r)$ given by Eq. (38). $\Theta(r)$ denotes smooth fall off faster than $|r|^{-n}$, n arbitrarily large.

By virtue of the relation between Kruskal variables and ADM variables derived in Sec. IV, the conditions (50)–(53) induce the following asymptotic behavior for the ADM variables[7] as $r \to \pm\infty$:

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[6]Note that we have not looked for the weakest possible asymptotic conditions on the gravitational variables but simply for ones which provide an elegant, consistent description and which admit coupling to a fairly large class of matter fields.

[7]Note that Eq. (54) admits the Schwarzscild solution. In contrast (41) of [1] admits the Schwarzschild solution only if $\epsilon < 1$ in that equation, whereas (58) of [1] does not admit the Schwarzschild solution at all.

$$\Lambda = \frac{1}{\sqrt{1 - \frac{2M_\pm}{|r|}}} + \Theta(r), \tag{54}$$

$$R = |r| + \Theta(r), \tag{55}$$

$$P_\Lambda = \Theta(r), \tag{56}$$

$$P_R = \Theta(r). \tag{57}$$

Alternatively, we can start from Eqs. (54)–(57) and following the transformations described in Sec. III, we can induce boundary conditions on the curvature coordinate variables, the scaled curvature coordinate variables and finally, the Kruskal variables. Following this procedure, it can be checked that Eqs. (50)–(53) are obtained.

Next, we analyze the asymptotic behavior of the constraints, the lapse function and the shift vector field. It can be checked that the constraints fall off faster than any inverse power of $|r|$: i.e.,

$$H, H_r = \Theta(r). \tag{58}$$

The behavior of the lapse and shift should be such that the smeared scalar constraint, $\int_{-\infty}^\infty dr N H$ and the smeared diffeomorphism constraint, $\int_{-\infty}^\infty dr N^r H_r$ be well defined, differentiable functions on the phase space and that the motions they generate preserve the boundary conditions (54)–(57).

For the shift vector field as $r \to \pm\infty$, we impose

$$N^r = \Theta(r). \tag{59}$$

It can be checked that with this behavior the smeared diffeomorphism constraint is a well defined, differentiable function on the phase space.

For the lapse function we impose

$$N = N_\pm + \Theta(r), \tag{60}$$

where $N_\pm$ are constants. In the description in terms of the ADM variables, the smeared scalar constraint is a differentiable function on phase space only when N_+ and N_- vanish. For non-vanishing N_+ or N_-, the boundary term $(N_+ M_+ - N_- M_-)$ has to be added to the smeared scalar constraint to render the combination differentiable on the phase space. As discussed in [1], parametrization of the times at spatial infinity leads to the action (2) in which the boundary term is replaced by the term $(\dot{\tau}_+ M_+ - \dot{\tau}_- M_-)$. In the description in terms of Kruskal variables, the smeared scalar constraint is differentiable _without_ the addition of any boundary term even for non-vanishing N_+ or N_-. As in the case of the description in terms of the curvature coordinate variables, instead of the boundary terms of the ADM description there are a pair of canonically conjugate parameters $(\bar{p}, m)$ (see Sec. II).

As mentioned earlier m is the mass at left spatial infinity. $\bar{p}$ can still be interpreted as the difference between the Killing time and the parametrization time at left infinity as we now show. From Eq. (29) we have

$$\int_{-\infty}^{\infty} \frac{\Pi_R R + P_T T}{2M^2} = \bar{T}(-\infty) - \frac{T(-\infty)}{2m}. \tag{61}$$

Then Eq. (26) implies that

$$\bar{p} = p + 2m\left(\bar{T}(-\infty) - \frac{T(-\infty)}{2m}\right). \tag{62}$$

From Eq. (15) we have that

$$p = T(-\infty) - \tau_-, \tag{63}$$

which together with Eq. (62) implies that

$$\bar{p} = 2m\bar{T}(-\infty) - \tau_-. \tag{64}$$

Since $2m\bar{T}(-\infty)$ is the Killing time at left spatial infinity, $\bar{p}$ has the interpretation of the difference between the Killing time at left infinity and the parametrization time at left infinity.[8] Note also that from Eqs. (10) and (29) we have

$$\bar{T}(\infty) = \frac{\tau_+}{2M_+}. \tag{65}$$

Now we can finally complete the reconstruction of Sec. IV of the ADM parameters $\tau_\pm$ from the Kruskal variables. Using Eqs. (64) and (65) we have

$$\tau_- = -(\bar{p} - 4m\bar{T}(-\infty)), \tag{66}$$

$$\tau_+ = 2M_+\bar{T}(\infty). \tag{67}$$

Here $M_+ = M(r)|_{r=\infty}$ is obtained from the Kruskal variables through Eq. (38) and $\bar{T}(\pm\infty)$ are obtained from the asymptotic behavior of the Kruskal variables from Eqs. (50) and (51).

This completes our discussion of the asymptotics, as well as the reconstruction, of the ADM variables from the Kruskal variables. In the next section we shall invert the expressions for the ADM variables in terms of the Kruskal variables.

VI. KRUSKAL VARIABLES IN TERMS OF ADM VARIABLES

The Kruskal variables can be expressed in terms of the ADM variables using the results of Sec. III in conjunction with Eqs. (10)–(16). However, the transformation described in Sec. III as well as the transformation from ADM variables to curvature coordinate variables are both singular at the horizon. For the transformation between the ADM variables and the Kruskal variables to be non-singular and *invertible*, it is necessary to prove that the Kruskal variables can be constructed from the ADM variables through a non-singular

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[8]Note that in general $\bar{p} \neq p$. However, on the constraint surface $p = \bar{p}$. Since the interpretation of $p, \bar{p}$ comes from their interpretation on a solution, they have identical interpretations even though they define different functions on the (unconstrained) phase space.

transformation. In this section we provide the required proof. We shall be brief and only describe the main steps.

In what follows, we shall assume that

$$\Lambda > 0, \quad M > 0, \quad R > 0. \tag{68}$$

We define

$$\Lambda_1 = -V' + \frac{\bar{R}^2}{4M^2} e^{\bar{R}} P_U, \tag{69}$$

$$\Lambda_2 = U' + \frac{\bar{R}^2}{4M^2} e^{\bar{R}} P_V. \tag{70}$$

Substitution of Eqs. (69), (70) in Eqs. (37), (39) yields

$$\Lambda^2 = \frac{16M^2}{\bar{R}e^{\bar{R}}} \Lambda_1 \Lambda_2, \tag{71}$$

$$\Lambda P_\Lambda = -4M^2 e^{-\bar{R}}(U\Lambda_1 + V\Lambda_2). \tag{72}$$

Note that Eq. (68) together with the asymptotic behavior of the Kruskal variables implies that

$$\Lambda_1 < 0, \quad \Lambda_2 < 0. \tag{73}$$

By using Eqs. (7), (27), (28), (34), (69), (70) and (39) in Eq. (46), we get

$$\mathcal{H}(r) = -\frac{\Lambda^2 e^{\bar{R}}}{16M^3}\left(P_R + \frac{F\Pi_R R(\bar{R}+1)}{2M} + \frac{\Lambda P_\Lambda F}{4M}\right). \tag{74}$$

Equation (74) is to be viewed as an expression for $\mathcal{H}$ in terms of the ADM variables. Thus in Eq. (74), M is given by Eq. (6), F by Eq. (7) and $\bar{R}$ by Eq. (27).

From Eqs. (47), (69) and (70) we have

$$\Lambda_2 \Lambda_1' - \Lambda_1 \Lambda_2' = \mathcal{H}. \tag{75}$$

At this stage it is useful to define

$$\mathcal{G}(r) := \frac{\Lambda^2 \bar{R} e^{\bar{R}}}{16M^2}. \tag{76}$$

From Eq. (71) we get

$$\Lambda_1 \Lambda_2 = \mathcal{G}, \tag{77}$$

where $\mathcal{G}$ is expressible as a function of the ADM variables through Eq. (76).

From Eqs. (75) and (77) we get a first order ordinary differential equation for Λ_1/Λ_2. The boundary conditions (50)–(53) along with Eq. (65) can be used in Eqs. (69) and (70) to fix the integration constant in the solution of the differential equation. The solution to the differential equation and Eq. (77) can be solved to obtain

$$\Lambda_1 = -\sqrt{\mathcal{G}} e^{\tau^+/4M} + e^{\int_\infty^r d\bar{r}(\mathcal{H}/\mathcal{G})}, \qquad (78)$$

$$\Lambda_2 = -\sqrt{\mathcal{G}} e^{-\tau^+/4M} + e^{-\int_\infty^r d\bar{r}(\mathcal{H}/\mathcal{G})}. \qquad (79)$$

Note that the signs of Λ_1, Λ_2 are fixed by the conditions (73). Equations (78) and (79) express Λ_1, Λ_2 in terms of the ADM variables. We shall now solve for P_U and P_V in terms of Λ_1, Λ_2 and the ADM variables. From Eqs. (12) and (14), it can be shown that

$$UP_U = -2M\Lambda^{-2}(\Lambda H - H_r)\left(R' - \frac{\Lambda P_\Lambda}{R}\right), \qquad (80)$$

$$VP_V = 2M\Lambda^{-2}(\Lambda H + H_r)\left(R' + \frac{\Lambda P_\Lambda}{R}\right). \qquad (81)$$

To evaluate $(R' \pm \Lambda P_\Lambda/R)$ we use Eqs. (72), (27), (30), (33) and (31). We obtain

$$R' + \frac{\Lambda P_\Lambda}{R} = -\frac{4M}{\bar{R}e^{\bar{R}}} V\Lambda_2, \qquad (82)$$

$$R' - \frac{\Lambda P_\Lambda}{R} = \frac{4M}{\bar{R}e^{\bar{R}}} U\Lambda_1. \qquad (83)$$

We substitute Eqs. (82), (83) in Eqs. (80), (81) and use Eqs. (69), (70) to obtain

$$P_U = -\frac{\Lambda H - H_r}{2\Lambda_2}, \qquad (84)$$

$$P_V = -\frac{\Lambda H + H_r}{2\Lambda_1}. \qquad (85)$$

Since Λ_1 and Λ_2 are given by Eqs. (78) and (79), Eqs. (84) and (85) express P_U and P_V in terms of the ADM variables.

We now show that U and V can also be determined in terms of the ADM data in a non-singular way. From Eqs. (82) and (83) we obtain

$$V = -\left(R' + \frac{\Lambda P_\Lambda}{R}\right) \frac{\bar{R}e^{\bar{R}}}{4M\Lambda_2}, \qquad (86)$$

$$U = \left(R' - \frac{\Lambda P_\Lambda}{R}\right) \frac{\bar{R}e^{\bar{R}}}{4M\Lambda_1}. \qquad (87)$$

Since Λ_1 and Λ_2 have been expressed in terms of the ADM variables, Eqs. (86) and (87) express U and V in terms of the ADM variables in a manifestly non-singular form.

Finally, it is easy to see that $(m, \bar{p})$ are also determined in terms of the ADM variables. m is trivially obtained from the asymptotic behavior of Λ at left infinity. $\bar{p}$ can be expressed as

$$\bar{p} = \left.\frac{-2mV(r)}{U(r)}\right|_{r=-\infty} - \tau_-. \qquad (88)$$

Since V and U are known in terms of the ADM variables, so is $\bar{p}$.

Thus, we have shown that the Kruskal variables, $(U, P_U, V, P_V, m, \bar{p})$, are uniquely determined through manifestly non-singular transformations of the ADM variables, $(\Lambda, P_\Lambda, R, P_R, \tau_+, \tau_-)$.

VII. CONCLUDING REMARKS

In this work we have constructed a transformation between the ADM variables, $(\Lambda, P_\Lambda, R, P_R, \tau_+, \tau_-)$, and new canonical variables $(U, P_U, V, P_V, m, \bar{p})$. U and V are interpreted as Kruskal coordinates and P_U, P_V are their conjugate momenta. m is the mass at left infinity. $\bar{p}$ is interpreted as the difference between the Killing time at left infinity and the parametrization time, τ_-, at left infinity, with the Killing time at right infinity synchronised with the parametrization time, τ_+, at right infinity.

This transformation is manifestly non-singular and invertible. In particular, it is non-singular at the horizon. For the transformation to be well defined, we assume that the mass function, $M(r)$, and the areal radius, $R(r)$, are both strictly positive i.e. $R, M > 0$. The interpretation of Λ^2 as a metric coefficient implies that $\Lambda^2 \neq 0$. The conditions $\Lambda > 0$ and $M > 0$ lead to complicated restrictions on the Kruskal variables through Eqs. (38) and (37). The condition $R > 0$ is equivalent, through Eq. (31), to the condition $(-UV) > 1$. Note that this is exactly the condition that defines the singularity free region of the extended Schwarzschild spacetime.

In terms of the Kruskal variables, the constraints are given by Eqs. (35) and (36). Equations (84) and (85) imply that the vanishing of the constraints is equivalent to the vanishing of P_U and P_V. This equivalence does not entail the involved arguments near $F = 0$ which were used in [1] to show that the imposition of the constraints implied $\Pi_R = P_T = 0$. After the imposition of the constraints, the true degrees of freedom are $(\bar{p}, m)$ and quantization of this reduced theory is trivial. The condition $M(r) > 0$ reduces to the condition $m > 0$ on the constraint surface. It is useful to make a further point transformation on the pair $(\bar{p}, m)$ to obtain $(x = \ln m, p_x = m\bar{p})$. We can now pass to quantum theory on the Hilbert space $\{\psi(x) \in \mathcal{L}^2(\mathbf{R})\}$ by setting

$$\hat{x}\psi(x) = x\psi(x), \qquad (89)$$

$$\hat{p}_x\psi(x) = -i\frac{d}{dx}\psi(x). \qquad (90)$$

As mentioned in footnote 2, a transformation to new canonical variables which also have the interpretation of Kruskal coordinates, was constructed in [1] by rescaling the curvature coordinate variables by the left mass "m" rather than by the mass function, "$M(r)$," as is done here. It can be checked that this transformation of [1] is singular on points in the (unconstrained) phase space when $F = 1 - 2M/R \neq 0$ and $f := 1 - 2m/R = 0$. This is, of course, possible only off the constraint surface. Nevertheless, it is clear that any neighborhood of the constraint surface contains

points where $F \neq 0$, $f = 0$ and hence, where the transformation is ill defined. Since Poisson brackets involve functional *derivatives*, their definition, even on the constraint surface, requires a nonsingular structure in a neighborhood of the constraint surface. This is one reason why the transformation to Kruskal variables in [1] is unsatisfactory. In contrast the transformation described in this work is nonsingular on the entire phase space (of course, subject to the conditions $M, \Lambda^2, R > 0$).

Although this work is concerned with spherically symmetric vacuum gravity, the physically interesting problem is that of spherical matter collapse say, the collapse of a massless scalar field. In the collapse situation, the coordinate r ranges from 0 to ∞ with $r = 0$ being the fixed point under the action of the rotation isometry group. As discussed in [3], the mass function $M(r)$ can still be constructed from the ADM data. Moreover the apparent horizon is defined through $F = 0$. The canonical transformation to R, T variables can still be done but this transformation is singular on the apparent horizon. Thus, it would be of interest to construct the analog of the Kruskal coordinates for the matter collapse case. Note that the treatment in this work would have to be modified to deal with the condition $M(r)\big|_{r=0} = 0$.

Apart from the Kruskal coordinates, there exist other global coordinates for the extended Schwarzschild spacetime such as those described in [4,5]. It would be of interest to reconstruct these from the ADM data. In particular, it would be of interest to try to use these (putative) variables to construct a time variable which is a spacetime scalar [6,3] even in the presence of matter couplings.

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