Fock representations from $U(1)$ holonomy algebras

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We examine the quantization of $U(1)$ holonomy algebras using the Abelian C^* algebra based techniques which form the mathematical underpinnings of current efforts to construct loop quantum gravity. In particular, we clarify the role of ''smeared loops'' and of Poincaré invariance in the construction of Fock representations of these algebras. This enables us to critically reexamine early pioneering efforts to construct Fock space representations of linearized gravity and free Maxwell theory from holonomy algebras through an application of the (then current) techniques of loop quantum gravity.

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I. INTRODUCTION

In the early 1990s [1–3] linearized gravity in terms of connection variables and free Maxwell theory on flat spacetime were treated as useful toy models on which to test techniques being developed for loop quantum gravity [4]. Significant progress has been made in the field of loop quantum gravity since then [5]. Hence, it is useful to reexamine these systems using current techniques to clarify certain questions which arise in the context of those pioneering but necessarily nonrigorous efforts.

Two important (and related) questions are the following.

(I) How did similar techniques for the quantization of general relativity and for its linearization about flat space result in a non-Fock representation for the (kinematic sector) of the former and a Fock representation for the latter? In particular, what is the role of Poincaré invariance in obtaining the Fock representation? (This last point was a puzzle to the authors themselves [1].)

(II) What is the role of ''smeared'' loops in [1] in obtaining a Fock representation?

In this work, we use the Abelian C^* algebra techniques [6,8], which constitute the mathematically rigorous framework of the loop quantum gravity program today, to investigate (I) and (II) above. It is also our aim to clarify the role of the different mathematical structures in the quantization procedure which determines whether a Fock or non-Fock representation results. Although we restrict attention to $U(1)$ theory on a flat spacetime, we believe that our results should be of some relevance to the case of linearized gravity.

This work is motivated by the following question in loop quantum gravity: how do Fock space gravitons on flat spacetime arise from the non-Fock structure of the Hilbert space which serves as the kinematical arena for loop quantum gravity? Admittedly, the answer to this question must await the construction of the full physical state space (i.e., the kernel of all the constraints) of quantum gravity. Nevertheless, this work may illuminate some facets of the issues involved.

The starting point for our analysis is the Abelian Poisson brackets algebra of U(1) holonomies around loops on a spatial slice. This algebra is completed to the Abelian C^* algebra, $\overline{\mathcal{HA}}$ of [6,8]. Hilbert space representations of $\overline{\mathcal{HA}}$ are determined by continuous positive linear functions (PLF's) on $\overline{\mathcal{HA}}$. We review the construction of $\overline{\mathcal{HA}}$ and of the PLF introduced in [6,8] (which we shall call the Haar PLF) in Sec. II. The resulting representation is a non-Fock representation in which the electric flux is quantized [7].

In Sec. III we construct an Abelian C^* algebra $\overline{\mathcal{HA}_r}$, based on the Poisson bracket algebra of holonomies around the ''Gaussian smeared '' loops of [1].[1] Next, we derive the key result of this work, namely that there exists a natural C^* algebraic isomorphism, $I_r:\overline{\mathcal{HA}}\rightarrow\overline{\mathcal{HA}_r}$ with the property that $I_r(\mathcal{HA})=\mathcal{HA}_r$.

The standard flat spacetime Fock vacuum expectation value restricts to a positive linear function on $\mathcal{HA}_r$. We are unable to show the continuity or lack thereof, of this Fock PLF on $\mathcal{HA}_r$. Nevertheless, since the Gel'fand-Naimark-Segal (GNS) construction needs only a * algebra (as opposed to a C^* algebra), we can use the Fock PLF to construct a representation of the * algebra $\mathcal{HA}_r$. In Sec. IV we show that this representation is indeed the standard Fock representation even though $\mathcal{HA}_r$ is a proper subalgebra of the standard Weyl algebra for $U(1)$ theory.

Using the map, I_r, we can define a Haar PLF on $\overline{\mathcal{HA}_r}$. We construct the resulting representation in Sec. V A. Finally, we use I_r to define a Fock PLF on $\mathcal{HA}$. The resulting representation is, in a precise sense, an approximation to the standard Fock representation. We study it in Sec. V B.

Section VI is devoted to a discussion of our results in the context of the questions (I) and (II). Some useful lemmas are proved in the Appendix.

In this work the spacetime of interest is flat R^4 and we use global Cartesian coordinates (t,x^i), $i=1,2,3$. The spatial slice of interest is the initial $t=0$ slice and all calculations are done in the spatial Cartesian coordinate chart (x^i). We use units in which both the velocity of light and Planck's constant $\hbar$ are equal to 1. We freely raise and lower indices with the flat spatial metric. The Poisson bracket between the $U(1)$ connection $A_a(\vec{x}),a=1,2,3$ and its conjugate electric

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[1] r is a small length which characterises the width of the Gaussian smearing function in [1].

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field $E^b(\vec{y})$ is $\{A_a(\vec{x}),E^b(\vec{y})\}=e\,\delta_a^b\delta(\vec{x},\vec{y})$ where e is a constant with units of electric charge.

II. REVIEW OF THE CONSTRUCTION AND REPRESENTATION THEORY OF $\bar{\mathcal{H}\mathcal{A}}$

We quickly review the relevant contents of [6,8]. We refer the reader to [6,8], especially Appendix A2 of [8] for details.

The mathematical structures of interest are as follows.

$\mathcal{A}$ is the space of smooth $U(1)$ connections on the trivial $U(1)$ bundle on R^3.[2] We restrict attention to connections $A_a(x)$ whose Cartesian components are functions of a rapid decrease at infinity.

$\mathcal{L}_{x_0}$ is the space of unparametrized, oriented, piecewise analytic loops[3] on R^3 with base point $\vec{x}_0$. Composition of a loop α with a loop β is denoted by $\alpha\circ\beta$. Given a loop $\alpha \in \mathcal{L}_{x_0}$, the holonomy of $A_a(x)$ around α is $H_\alpha(A) := \exp(i\oint_\alpha A_a dx^a)$.

$\tilde{\alpha}$ is the holonomy equivalence class (hoop class) of α, i.e., α,β define the same hoop if $H_\alpha(A)=H_\beta(A)$ for every $A_a(x)\in\mathcal{A}$. $\mathcal{H}G$ is the group generated by all hoops $\tilde{\alpha}$, where group multiplication is hoop composition, i.e., $\tilde{\alpha}\circ\tilde{\beta} := \widetilde{\alpha\circ\beta}$. $\mathcal{H}\mathcal{A}$ is the Abelian Poisson bracket algebra of $U(1)$ holonomies.

$\mathcal{F}\mathcal{L}_{x_0}$ is the free algebra generated by elements of $\mathcal{L}_{x_0}$, with product law $\alpha\beta := \alpha\circ\beta$. With this product, all elements of $\mathcal{F}\mathcal{L}_{x_0}$ are expressible as complex linear combinations of elements of $\mathcal{L}_{x_0}$.

K is a two-sided ideal of $\mathcal{F}\mathcal{L}_{x_0}$, such that

$$\sum_{i=1}^{N} a_i\alpha_i \in K \text{ if } \sum_{i=1}^{N} a_i H_{\alpha_i}(A)=0 \text{ for every } A_a(x)\in\mathcal{A}, \tag{1}$$

where a_i are complex numbers.

$\mathcal{F}\mathcal{L}_{x_0}$ is quotiented by K to give the algebra $\mathcal{F}\mathcal{L}_{x_0}/K$. The K equivalence class of α is denoted by $[\alpha]$. As abstract algebras, $\mathcal{H}\mathcal{A}$ and $\mathcal{F}\mathcal{L}_{x_0}/K$ are isomorphic.

$$\left(\sum_{i=1}^{N} a_i[\alpha_i]\right)^* := \sum_{i=1}^{N} a_i^*[\alpha_i^{-1}] \tag{2}$$

defines a $*$ relation on $\mathcal{H}\mathcal{A}$.

$$\left\|\sum_{i=1}^{N} a_i[\alpha_i]\right\| := \sup_{A\in\mathcal{A}}\left|\sum_{i=1}^{N} a_i H_{\alpha_i}(A)\right| \tag{3}$$

defines a norm on $\mathcal{H}\mathcal{A}$. $\overline{\mathcal{H}\mathcal{A}}$ is the Abelian C^* algebra obtained by defining $*$ on $\mathcal{H}\mathcal{A}$ and completing the resulting $*$ algebra with respect to $\|\ \|$.

Δ is the spectrum of $\overline{\mathcal{H}\mathcal{A}}$. Δ is also denoted by $\overline{\mathcal{A}/\mathcal{G}}$ where $\mathcal{G}$ denotes the $U(1)$ gauge group and is a suitable completion of the space of connections of modulo gauge $\mathcal{A}/\mathcal{G}$. From Gel'fand theory, Δ is the space of continuous, linear, multiplicative $*$ homeomorphisms h, from $\overline{\mathcal{H}\mathcal{A}}$ to the (C^* algebra of) complex numbers **C**. From [8] the elements of Δ are also in 1-1 correspondence with homeomorphisms from $\mathcal{H}G$ to $U(1)$.

Given $X\in\overline{\mathcal{H}\mathcal{A}}$, $h(X)$ is a complex function on Δ. Δ is endowed with the weakest topology in which $h(X)$ for all $X\in\overline{\mathcal{H}\mathcal{A}}$ are continuous functions on Δ. In this topology, Δ is a compact, Hausdorff space and the functions $h([\alpha]),\alpha \in\mathcal{L}_{x_0}$ are dense in the C^* algebra $C(\Delta)$, of continuous functions on Δ. Further, $C(\Delta)$ is isomorphic to $\overline{\mathcal{H}\mathcal{A}}$. Every continuous cyclic representation of $\overline{\mathcal{H}\mathcal{A}}$ is in 1-1 correspondence with a continuous positive linear functional (PLF) on $\overline{\mathcal{H}\mathcal{A}}$. Since $\overline{\mathcal{H}\mathcal{A}}\cong C(\Delta)$, every continuous PLF so defined on $C(\Delta)$ is in correspondence, by the Riesz lemma, with some regular measure $d\mu$ on Δ and $\hat{H}_\alpha$ is represented on $\psi \in L^2(\Delta,d\mu)$ as a unitary operator through $(\hat{H}_\alpha\psi)(h) = h([\alpha])\psi(h)$.

In particular, the continuous Haar PLF [8]

$$\Gamma(\alpha)=1 \text{ if } \tilde{\alpha}=\tilde{o}, \quad =0 \quad \text{otherwise}, \tag{4}$$

(where o is the trivial loop), corresponds to the Haar measure on Δ.

$\Delta=\overline{\mathcal{A}/\mathcal{G}}$ can also be constructed as the projective limit space [9] of certain finite dimensional spaces. Each of these spaces is isomorphic to n copies of $U(1)$ and is labeled by n strongly independent hoops. Recall from [8] that $\tilde{\alpha}_i$ $i=1\dots n$ are strongly independent hoops if $\alpha_i\in\mathcal{L}_{x_0}$ are strongly independent loops; α_i, $i=1\dots n$ are strongly independent loops if each α_i has at least one segment which intersects $\alpha_{j\neq i}$ at most at a finite number of points. The Haar measure on Δ is the projective limit measure of the Haar measures on each of the finite dimensional spaces.[4] Then the considerations of [10] show that the electric flux $\int_S E^a ds_a$ through a surface S can be realized as an essentially self-adjoint operator on the dense domain of cylindrical functions[5] as

[2]Thus a minor change of notation from A2 of [8] is that we denote $\mathcal{A}_0$ of that reference by $\mathcal{A}$.

[3]This is in contrast to the C^1 loops of A2 of [8].

[4]Note that the proof of continuity of the Haar PLF in [8] is incomplete in that it applies only if the loops α_j of A.7 of [8] are holonomically independent. Nevertheless, if as in this work, we restrict attention to piecewise analytic loops, continuity of the Haar PLF can immediately be inferred from its definition through the Haar measure.

[5]Cylindrical functions on Δ are of the form $\psi_{\{[\alpha_i]\}} := \psi(h([\alpha_1])\dots h([\alpha_n]))$, where α_i, $i=1\dots n$, are a finite number of strongly indendent loops and ψ is a complex function on $U(1)^n$.

$$\int_S \hat{E}^a ds_a \psi_{\{[\alpha_i]\}} = e\sum_i N(S,\alpha_i) h([\alpha_i]) \frac{\partial \psi_{\{[\alpha_i]\}}}{\partial h([\alpha_i])}, \tag{5}$$

where $N(S,\alpha_i)$ is the number of intersections between α_i and S.

III. $\overline{\mathcal{HA}_r}$ AND THE ISOMORPHISM I_r

In Sec. III A we recall the definition of ''smeared'' loops and their holonomies from [1] and construct the ''smeared'' loop related structures $\tilde{\alpha}_r$, K_r, $\mathcal{HA}_r$, $\overline{\mathcal{HA}_r}$, and Δ_r. In Sec. III B, using the Appendix, we show that an isomorphism exists between the structures $\tilde{\alpha}$, K, $\mathcal{HA}$, $\overline{\mathcal{HA}}$, Δ, and their smeared versions.

A. The construction of $\bar{\mathcal{H}}\bar{\mathcal{A}}_r$

In the notation of [1],

$$H_\alpha(A) = \exp i \int_{R^3} X^a_\gamma(\vec{x}) A_a(\vec{x}) d^3x, \tag{6}$$

$$X^a_\gamma(\vec{x}) := \oint_\gamma ds\, \delta^3(\vec{\gamma}(s),\vec{x})\, \dot{\gamma}^a, \tag{7}$$

where s is a parametrization of the loop γ, $s\in[0,2\pi]$. $X^a_\gamma(\vec{x})$ is called the form factor of γ. Its Fourier transform is

$$X^a_\gamma(\vec{k}) := \frac{1}{2\pi^{3/2}} \int_{R^3} d^3x X^a_\gamma(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}$$

$$= \frac{1}{2\pi^{3/2}} \oint_\gamma ds\, \dot{\gamma}^a(s) e^{-i\vec{k}\cdot\vec{\gamma}(s)}. \tag{8}$$

The Gaussian smeared form factor [1] is defined as

$$X^a_{\gamma_{(r)}}(\vec{x}) := \int_{R^3} d^3y f_r(\vec{y}-\vec{x}) X^a_\gamma(\vec{y}) = \oint_\gamma ds f_r(\vec{\gamma}(s)-\vec{x})\, \dot{\gamma}^a(s) \tag{9}$$

where

$$f_r(\vec{x}) = \frac{1}{2\pi^{3/2} r^3} e^{-x^2/2r^2} \quad x := |\vec{x}| \tag{10}$$

approximates the Dirac δ function for small r. The Fourier transform of the smeared form factor is

$$X^a_{\gamma_{(r)}}(\vec{k}) = e^{-k^2r^2/2} X^a_\gamma(\vec{k}), \tag{11}$$

and the smeared holonomy is defined as

$$H_{\gamma_{(r)}}(A) = \exp i \int_{R^3} X^a_{\gamma_{(r)}}(\vec{x}) A_a(\vec{x}) d^3x$$

$$= \exp i \int_{R^3} X^a_{\gamma_{(r)}}(-\vec{k}) A_a(\vec{k}) d^3k, \tag{12}$$

where $A_a(\vec{k})$ is the Fourier transform of $A_a(\vec{x})$.

We define $\tilde{\alpha}_r$, K_r, $\mathcal{HA}_r$, $\overline{\mathcal{HA}_r}$, Δ_r as follows. $\tilde{\alpha}_r$ is the r-hoop class of α, i.e., α,β define the same r hoop if $H_{\alpha_{(r)}}(A) = H_{\beta_{(r)}}(A)$ for every $A_a(x)\in\mathcal{A}$. $\mathcal{H}G_r$ is the group generated by all r hoops $\tilde{\alpha}_r$ where group multiplication is the r-hoop composition, i.e.,

$$\tilde{\alpha}_r \circ \tilde{\beta}_r := (\widetilde{\alpha\circ\beta})_r. \tag{13}$$

Note that the above definition is consistent because, from Eq. (12) and the definition of r-hoop equivalence, it follows that

$$H_{\alpha_{(r)}}(A) H_{\beta_{(r)}}(A) = H_{(\alpha\cdot\beta)_{(r)}}(A). \tag{14}$$

Note that from Eq. (13), it follows that the identity element of $\mathcal{H}G_r$ is $\tilde{o}_r$ and that $(\tilde{\alpha}_r)^{-1} = \widetilde{\alpha^{-1}}_r$.

$\mathcal{H}A_r$ is the Abelian Poissons bracket algebra of the r holonomies, $H_{\alpha_{(r)}}(A)$, $A_a\in\mathcal{A}$, $\alpha\in\mathcal{L}_{x_0}$.

Recall that with the product law defined in Sec. II, all elements of $\mathcal{FL}_{x_0}$ are expressible as complex linear combinations of elements of $\mathcal{L}_{x_0}$. We define the two-sided ideal of $K_r\in\mathcal{FL}_{x_0}$, through

$$\sum_{i=1}^N a_i\alpha_i \in K_r \text{ if } \sum_{i=1}^N a_i H_{\alpha_{i(r)}}(A) = 0 \text{ for every } A_a(x)\in\mathcal{A}, \tag{15}$$

where a_i are complex numbers. The K_r equivalence class of α is denoted by $[\alpha]_r$. It can be seen that, as abstract algebras, $\mathcal{HA}_r$ and $\mathcal{FL}_{x_0}/K_r$ are isomorphic.

It can be checked that the relation $*_r$ defined on $\mathcal{HA}_r$ by

$$\left(\sum_{i=1}^N a_i[\alpha_i]_r\right)^{*_r} := \sum_{i=1}^N a_i^*[\alpha_i^{-1}]_r \tag{16}$$

is a * relation. Note that from Eq. (12), the complex conjugate of $H_{\alpha_{(r)}}(A)$ is $H_{\alpha^{-1}{}_{(r)}}(A)$ and hence the abstract $*_r$ relation just encodes the operation of complex conjugation on the algebra $\mathcal{H}A_r$.

Next we define the norm $\|\,\|_r$ as

$$\left\|\sum_{i=1}^N a_i[\alpha_i]_r\right\|_r := \sup_{A\in\mathcal{A}} \left|\sum_{i=1}^N a_i H_{\alpha_{i(r)}}(A)\right|. \tag{17}$$

It is easily verified that $\|\,\|_r$ is indeed a norm on the * algebra $\mathcal{HA}_r$ with * relation defined by Eq. (16). Completion of $\mathcal{HA}_r$ with respect to $\|\,\|_r$ gives the Abelian C^* algebra $\overline{\mathcal{HA}}_r$.

Next, we characterize the spectrum Δ_r of $\overline{\mathcal{HA}_r}$ as the space of all homomorphisms from $\mathcal{H}G_r$ to $U(1)$.

Let $h\in\Delta_r$. Thus h is a linear, multiplicative, continuous * homorphism from $\overline{\mathcal{HA}_r}$ to $\mathbf{C}$:

$$\Rightarrow h([\alpha]_r) h([\alpha^{-1}]_r) = h([o]_r), \tag{18}$$

choosing $\alpha=o$, $\Rightarrow h([o]_r)^2=h([o]_r)\Rightarrow h([o]_r)=1$, (19)

$$\Rightarrow h([\alpha^{-1}]_r)=\frac{1}{h([\alpha]_r)}=h^*([\alpha]_r). \tag{20}$$

Equation (20) implies that $|h([\alpha]_r)|=1$ and this, coupled with the fact that $\mathcal{H}G_r$ is commutative, shows that every $h\in\Delta_r$ defines a homomorphism from $\mathcal{H}G_r$ to $U(1)$.

Conversely, let h be a homomorphism from $\mathcal{H}G_r$ to $U(1)$. Its action can be extended by linearity to elements of $\mathcal{H}A_r$ so that $h(\Sigma_{i=1}^N a_i[\alpha_i]_r):=\Sigma_{i=1}^N a_i h([\alpha_i]_r)$.[6] It is also easy to see that $h([\alpha^{-1}]_r)=h^*([\alpha]_r)$. These properties and the fact that h is a homomorphism from $\mathcal{H}G_r$ to $U(1)\subset\mathbf{C}$, imply that h is a linear, multiplicative, * homomorphism from $\mathcal{H}A_r$ to $\mathbf{C}$.

Finally, we show that h extends to a continuous homomorphism on $\overline{\mathcal{HA}_r}$. From [6] it follows that for $\alpha_i\in\mathcal{L}_{x_0}$, $i=1\dots n$, there exist strongly independent β_j, $j=1\dots m$ such that each α_i is the composition of some of the $\{\beta_j\}$. From this fact and Lemma 2 of the Appendix, it can be shown that, for a given $\Sigma_{i=1}^N a_i[\alpha_i]_r\in\mathcal{HA}_r$ and any $\delta>0$, there exists $A_a^{(a_i,\delta,r)}\in\mathcal{A}$ such that

$$\left|\sum_{i=1}^N a_i\{h([\alpha_i]_r)-H_{\alpha_{i(r)}}(A^{(a_i,\delta,r)})\}\right|<\delta. \tag{21}$$

From Eq. (21), it is straightforward to show that

$$\left|\sum_{i=1}^N a_i h([\alpha_i]_r)\right|\leqslant\sup_{A\in\mathcal{A}}\left|\sum_{i=1}^N a_i H_{\alpha_{i(r)}}(A)\right| =\left|\left|\sum_{i=1}^N a_i[\alpha_i]_r\right|\right|_r. \tag{22}$$

Since $\mathcal{HA}_r$ is dense in $\overline{\mathcal{HA}_r}$, Eq. (22) implies that h can be extended to a continuous (linear, multiplicative) homorphism from $\overline{\mathcal{HA}_r}$ to $\mathbf{C}$.

Thus Δ_r can be identified with the set of all homomorphisms from $\mathcal{H}G_r$ to $U(1)$.

B. The isomorphism I_r

We show that (i) $K=K_r$: Let

$$\sum_{i=1}^N a_i H_{\alpha_i}(A)=0 \text{ for every } A_a(x)\in\mathcal{A}. \tag{23}$$

From Lemma 3 of the Appendix, given $A_a\in\mathcal{A}$, there exists $A_{a(r)}\in\mathcal{A}$ such that

$$\sum_{i=1}^N a_i H_{\alpha_{i(r)}}(A)=\sum_{i=1}^N a_i H_{\alpha_i}(A_{(r)}). \tag{24}$$

Equations (23) and (24) imply that $K\subset K_r$.

Let

$$\sum_{i=1}^N a_i H_{\alpha_{i(r)}}(A)=0 \text{ for every } A_a(x)\in\mathcal{A}. \tag{25}$$

$\Rightarrow$ Given $A_a, B_a\in\mathcal{A}$,

$$\left|\sum_{i=1}^N a_i H_{\alpha_i}(A)\right|=\left|\sum_{i=1}^N a_i H_{\alpha_i}(A)-\sum_{i=1}^N a_i H_{\alpha_{i(r)}}(B)\right|. \tag{26}$$

Choose, $B_a=A_a^\epsilon$ where A_a^ϵ is defined in Lemma 1 of the Appendix.

Then $|\Sigma_{i=1}^N a_i H_{\alpha_i}(A)|\leqslant\Sigma_{i=1}^N|a_i|\epsilon$ for every $\epsilon>0$. $\Rightarrow\Sigma_{i=1}^N a_i H_{\alpha_i}(A)=0$ and hence, $K_r\subset K$. Thus $K=K_r$, $[\alpha]=[\alpha]_r$ and $\tilde{\alpha}=\tilde{\alpha}_r$.

(ii) $||\Sigma_{i=1}^N a_i[\alpha_i]||=||\Sigma_{i=1}^N a_i[\alpha_i]_r||_r$: Let

$$\left|\left|\sum_{i=1}^N a_i[\alpha_i]_r\right|\right|_r=c_r.$$

Then $c_r\geqslant|\Sigma_{i=1}^N a_i H_{\alpha_{i(r)}}(A)|$ for every $A_a\in\mathcal{A}$. Further, for every $\tau>0$ there exists ${}^{(\tau)}A_a\in\mathcal{A}$ such that $c_r-|\Sigma_{i=1}^N a_i H_{\alpha_{i(r)}}({}^{(\tau)}A)|\leqslant\tau$. Then, from Lemma 3 of the Appendix, there exists ${}^{(\tau)}A_{a(r)}\in\mathcal{A}$ such that

$$0\leqslant c_r-\sum_{i=1}^N a_i H_{\alpha_i}({}^{(\tau)}A_{(r)})\leqslant\tau, \tag{27}$$

$$\Rightarrow\sup_{A\in\mathcal{A}}\left|\sum_{i=1}^N a_i H_{\alpha_i}(A)\right|\geqslant c_r\Rightarrow\left|\left|\sum_{i=1}^N a_i[\alpha_i]\right|\right| \geqslant||\sum_{i=1}^N a_i[\alpha_i]_r||_r. \tag{28}$$

Let $||\Sigma_{i=1}^N a_i[\alpha_i]||=c$. Then for every $\tau>0$ there exists ${}^{(\tau/2)}A_a\in\mathcal{A}$ such that

$$c-\left|\sum_{i=1}^N a_i H_{\alpha_i}({}^{(\tau/2)}A)\right|\leqslant\frac{\tau}{2}. \tag{29}$$

From Lemma 1, there exists ${}^{(\tau/2)}A_a^\epsilon\in\mathcal{A}$ such that

$$|H_{\alpha_{i(r)}}({}^{(\tau/2)}A^\epsilon)-H_{\alpha_i}({}^{(\tau/2)}A)|\leqslant\epsilon,$$

$$\Rightarrow\left|\sum_{i=1}^N a_i[H_{\alpha_{i(r)}}({}^{(\tau/2)}A^\epsilon)-H_{\alpha_i}({}^{(\tau/2)}A)]\right|\leqslant\sum_{i=1}^N|a_i|\epsilon. \tag{30}$$

Choose $0<\epsilon<\tau/2\Sigma_{i=1}^N|a_i|$. From Eqs. (29) and (30) it follows that, for every $\tau>0$,

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[6] h can be defined on $[\alpha]_r$ because K_r equivalence subsumes r-hoop equivalence.

$$c-\left|\sum_{i=1}^{N} a_i H_{\alpha_{i(r)}}(^{(\tau/2)}A^{\epsilon})\right|<\tau, \quad (31)$$

$$\Rightarrow c\leq \sup_{A\in\mathcal{A}}\left|\sum_{i=1}^{N} a_i H_{\alpha_{i(r)}}(A)\right|$$

$$\Rightarrow\left|\left|\sum_{i=1}^{N} a_i[\alpha_i]\right|\right|\leq\left|\left|\sum_{i=1}^{N} a_i[\alpha_i]_r\right|\right|_r. \quad (32)$$

Thus, for any finite N, $||\Sigma_{i=1}^N a_i[\alpha_i]||=||\Sigma_{i=1}^N a_i[\alpha_i]_r||_r$.

From (i) and (ii) it follows that the structures K_r, $[\alpha_r]$, $\tilde{\alpha}_r$, $\mathcal{H}G_r$, $*_r$, $\mathcal{HA}_r$, $\overline{\mathcal{HA}_r}$, Δ_r are isomorphic to K, $[\alpha]$, $\tilde{\alpha}$, $\mathcal{H}G$, $*$, $\mathcal{HA}$, $\overline{\mathcal{HA}}$, Δ.

Thus a C^* isomorphism $I_r:\overline{\mathcal{HA}}\rightarrow\overline{\mathcal{HA}_r}$ exists such that

$$I_r\left(\sum_{i=1}^{N} a_i[\alpha_i]\right)=\sum_{i=1}^{N} a_i[\alpha_i]_r. \quad (33)$$

I_r defines a natural 1-1 map from Δ to Δ_r (which we shall also call I_r). Given the * isomorphism $h\in\Delta$, from $\overline{\mathcal{HA}}$ to **C**, its image is $h_r\in\Delta_r$ where

$$h_r\left(\sum_{i=1}^{N} a_i[\alpha_i]_r\right):=h\left(\sum_{i=1}^{N} a_i[\alpha_i]_r\right). \quad (34)$$

Note that if h_r is defined by some smooth, nonflat $A_a\in\mathcal{A}$ then *it is not true* that h is associated with (the gauge equivalence class of) the same connection.

IV. FOCK REPRESENTATION FROM $\overline{\mathcal{HA}_r}$

The standard Fock space vacuum expectation value restricted to $\mathcal{H}A_r$ defines the Fock PLF on $\mathcal{H}A_r$ as

$$\Gamma_F\left(\sum_{i=1}^{N} a_i[\alpha_i]_r\right):=\sum_{i=1}^{N} a_i \exp-\left(\int\frac{d^3k}{k}|X^a_{\alpha_{i(r)}}(\vec{k})|^2\right). \quad (35)$$

Since $\mathcal{HA}_r$ is a proper subalgebra of the standard Weyl algebra for $U(1)$ theory, it is not clear that its quantization (through the GNS construction based on the Fock PLF) reproduces the full Fock space. We prove that the full Fock space is indeed obtained.

Let the GNS Hilbert space (based on Γ_F) be $\mathcal{H}$. Let $\mathcal{D}$ be the linear subspace of $\mathcal{H}$ spanned by elements of the form $\hat{H}_{\alpha_{(r)}}\Omega$, $\alpha\in\mathcal{L}_{x_0}$ where Ω is the GNS vacuum. It can be seen that $\mathcal{D}$ is dense in $\mathcal{H}$. $\mathcal{D}$ is naturally embedded in the Fock space, $\mathcal{F}$, through the map $U:\mathcal{D}\rightarrow\mathcal{F}$ defined by

$$U(\Omega)=|0\rangle \text{ and } U\left(\sum_{i=1}^{N} a_i\hat{H}_{\alpha_{i(r)}}\Omega\right)$$
$$=\sum_{i=1}^{N} a_i \exp i\int_{R^3} X^a_{\alpha_{i(r)}}(\vec{x})\hat{A}_a(\vec{x})d^3x|0\rangle. \quad (36)$$

Here $\hat{A}_a$ is the standard Fock space operator valued distribution at $t=0$

$$\hat{A}_a(\vec{x})=\frac{1}{2\pi^{3/2}}\int\frac{d^3k}{\sqrt{k}}[e^{i\vec{k}\cdot\vec{x}}\hat{a}_a(\vec{k})+e^{-i\vec{k}\cdot\vec{x}}\hat{a}_a^\dagger(\vec{k})], \quad (37)$$

where

$$\hat{a}_a(\vec{k})k^a=0,\ [\hat{a}_a(\vec{k}),\hat{a}_b^\dagger(\vec{l})]=\delta_{ab}\delta(\vec{k},\vec{l}). \quad (38)$$

By construction, U is a unitary map and can be uniquely extended to $\mathcal{H}$ so that it embeds $\mathcal{H}$ in $\mathcal{F}$. We show that Cauchy limits of states in $U(\mathcal{D})$ span a dense set in $\mathcal{F}$—this suffices to show that the entire Fock space is indeed obtained, i.e., that $U(\mathcal{H})=\mathcal{F}$.

We define the ''occupation number'' states

$$|\phi,p\rangle:=\int d^3k_1\ldots d^3k_p\phi^{a_1\cdots a_p}(\vec{k}_1\ldots\vec{k}_p)$$
$$\times\hat{a}^\dagger_{a_1}(\vec{k}_1)\ldots\hat{a}^\dagger_{a_p}(\vec{k}_p)|0\rangle. \quad (39)$$

$\phi^{a_1\cdots a_p}(\vec{k}_1\ldots\vec{k}_p)$ (with p a positive integer) is such that (a) $\int d^3k_i|\phi^{a_1\cdots a_p}(\vec{k}_1,\ldots,\vec{k}_i,\ldots,\vec{k}_p)|^2<\infty$ and $\phi^{a_1\cdots a_p}(\vec{k}_1\ldots\vec{k}_p)$ falls of faster than any inverse power of k_i as $k_i\rightarrow\infty$, $\vec{k}_{j\neq i}$ fixed, (b) $\phi^{a_1\cdots a_i\cdots a_p}(\vec{k}_1,\ldots,\vec{k}_i,\ldots,\vec{k}_p)(k_i)_{a_i}=0$, i.e., it is transverse, and (c) it is symmetric under interchange of $(a_i,\vec{k}_i)$ with $(a_j,\vec{k}_j)$ for all $i,j=1\ldots p$. $|\phi,p\rangle$ for all p together with $|0\rangle$, span a dense set, $\mathcal{D}_0\in\mathcal{F}$.

Given two vectors $\vec{x},\vec{v}$, define the operator

$$\hat{O}_{\vec{x},\vec{v}}:=\frac{i}{2\pi^{3/2}}\int\frac{d^3k}{\sqrt{2k}}e^{i\vec{k}\cdot\vec{x}}e^{-k^2r^2/2}(\vec{v}\times\vec{k})^a[\hat{a}_a(\vec{k})+\hat{a}_a^\dagger(-\vec{k})]. \quad (40)$$

As argued in the Appendix, states of the form $|\psi_{\{\vec{x}_i,\vec{v}_i\}}\rangle:=\Pi_{i=1}^p\hat{O}_{(\vec{x}_i,\vec{v}_i)}|0\rangle$, $p=1,2\ldots$ together with $|0\rangle$ span $\mathcal{D}_0$.

Our proof that $\psi_{\{\vec{x}_i,\vec{v}_i\}}\in U(\mathcal{H})$ is as follows. (i) Note that $\psi_{\{\vec{x}_i,\vec{v}_i\}}\in\mathcal{D}_0$ (ii) Let $\gamma^{\{m,\vec{x},\vec{n}\}}$ be a circular loop of radius $\epsilon_m:=1/2^m$ (m is a positive integer), centered at $\vec{x}$ and let its plane have unit normal $\vec{n}$.[7] The image of $\hat{H}_{\gamma^{\{m,\vec{x},\vec{n}\}}_{(r)}}$ on $U(\mathcal{D})$ is $\exp[i\int X^a_{\gamma^{\{m,\vec{x},\vec{n}\}}_{(r)}}(\vec{y})\hat{A}_a(\vec{y})d^3y]$. Define

$$\hat{O}_{\vec{x},\vec{n},m}:=\frac{e^{i\int X^a_{\gamma^{\{m,\vec{x},\vec{n}\}}_{(r)}}(\vec{y})\hat{A}_a(\vec{y})d^3y}-1}{i\pi\epsilon_m^2}. \quad (41)$$

[7]Although $\gamma^{\{m,\vec{x},\vec{n}\}}$ is not in $\mathcal{L}_{x_0}$, the loop formed by joining $\gamma^{\{m,\vec{x},\vec{n}\}}$ to the base point x_0 and retracing, is. We shall continue to denote this loop, which represents the same hoop, by $\gamma^{\{m,\vec{x},\vec{n}\}}$.

The formal limit of $\hat{O}_{\vec{x},\vec{n},m}$ as $m\rightarrow\infty$ is $\hat{O}_{\vec{x},\vec{n}}$. We show below that $\hat{O}_{\vec{x},\vec{n},m}|\psi\rangle$, $|\psi\rangle\in\mathcal{D}_0$,[8] form a Cauchy sequence with limit $\hat{O}_{\vec{x},\vec{n}}|\psi\rangle$. Then, choosing $|\psi\rangle=|0\rangle$, we see that $\hat{O}_{\vec{x},\vec{n}}|0\rangle$ is in the completion of $U(\mathcal{D})$.

(iii) From (i) above, $\hat{O}_{\vec{x},\vec{n}}|0\rangle\in\mathcal{D}_0$. We can repeat the argument in (ii) above to conclude that $\hat{O}_{\vec{x}_2,\vec{n}_2}\hat{O}_{\vec{x}_1,\vec{n}_1}|0\rangle$ is obtained as the Cauchy limit of the states $\hat{O}_{\vec{x}_2,\vec{n}_2,m}\hat{O}_{\vec{x}_1,\vec{n}_1}|0\rangle$. Iterating this argument we see that $|\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle$, $|\vec{n}_i|=1$ is in the completion of $U(\mathcal{D})$. Finally, set $\vec{v}_i:=v_i\vec{n}_i$, where v_i are real numbers. Then it follows that $|\psi_{\{\vec{x}_i,\vec{v}_i\}}\rangle=(\Pi_{i=1}^{p}v_i)|\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle$ and hence that $|\psi_{\{\vec{x}_i,\vec{v}_i\}}\rangle\in U(\mathcal{H})$.

Thus, it remains to show [see (ii) above] that given $\psi\in\mathcal{D}_0$, $\hat{O}_{\vec{x},\vec{n}}$ as defined in Eq. (40) and $\hat{O}_{\vec{x},\vec{n},m}$ as defined in Eq. (41),

$$\lim_{m\rightarrow\infty}||\hat{O}_{\vec{x},\vec{n}}-\hat{O}_{\vec{x},\vec{n},m}|\psi\rangle||=0. \tag{42}$$

Proof: Let

$$\hat{D}:=\int X^{a}_{\gamma^{\{m,\vec{x},\vec{n}\}}_{(r)}}(\vec{y})\hat{A}_a(\vec{y})d^3y-\pi\epsilon_m^2\hat{O}_{(\vec{x},\vec{n})}. \tag{43}$$

Thus

$$\hat{O}_{\vec{x},\vec{n},m}=\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}+i\hat{D}}-1}{i\pi\epsilon_m^2}. \tag{44}$$

Since both $e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}$ and $e^{i\hat{D}}$ are commuting elements of the standard Weyl algebra,

$$\hat{O}_{\vec{x},\vec{n},m}=\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}e^{i\hat{D}}-1}{i\pi\epsilon_m^2} \tag{45}$$

$$\begin{aligned}\Rightarrow&||\hat{O}_{\vec{x},\vec{n}}-\hat{O}_{\vec{x},\vec{n},m}|\psi\rangle||\\ =&\left|\left|\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}(e^{i\hat{D}}-1)}{i\pi\epsilon_m^2}|\psi\rangle\right.\right.\\ &\left.\left.+\left(\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}-1}{i\pi\epsilon_m^2}-\hat{O}_{\vec{x},\vec{n}}\right)|\psi\rangle\right|\right|\\ \leqslant&\left|\left|\left(\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}-1}{i\pi\epsilon_m^2}-\hat{O}_{\vec{x},\vec{n}}\right)|\psi\rangle\right|\right|\\ &+\left|\left|\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}(e^{i\hat{D}}-1)}{i\pi\epsilon_m^2}|\psi\rangle\right|\right|.\end{aligned} \tag{46}$$

[8]Note that since $\hat{O}_{\vec{x},\vec{n},m}$ are bounded operators defined on the entire Fock space, $\hat{O}_{\vec{x},\vec{n},m}$ are well defined on $\mathcal{D}_0$.

From Lemma 2 of the Appendix, $\hat{O}_{\vec{x},\vec{n}}$ is a densely defined symmetric operator on $\mathcal{D}_0$ and admits self-adjoint extensions. Hence, from [11], the first term in Eq. (46) vanishes in the $\epsilon_m\rightarrow0$ limit. Further, since $e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}$ is a unitary operator, we have

$$\left|\left|\frac{e^{i\pi\epsilon_m^2\hat{O}_{\vec{x},\vec{n}}}(e^{i\hat{D}}-1)}{i\pi\epsilon_m^2}|\psi\rangle\right|\right|=\left|\left|\frac{(e^{i\hat{D}}-1)}{i\pi\epsilon_m^2}|\psi\rangle\right|\right|. \tag{47}$$

But

$$\begin{aligned}\left|\left|\frac{(e^{i\hat{D}}-1)}{i\pi\epsilon_m^2}|\psi\rangle\right|\right|^2=&-\left(\left\langle\psi\left|\frac{(e^{i\hat{D}}-1)}{\pi^2\epsilon_m^4}\right|\psi\right\rangle\right.\\ &\left.+\left\langle\psi\left|\frac{(e^{-i\hat{D}}-1)}{\pi^2\epsilon_m^4}\right|\psi\right\rangle\right).\end{aligned} \tag{48}$$

From Lemma 3 and Eq. (48), $||(e^{i\hat{D}}-1)/i\pi\epsilon_m^2|\psi\rangle||\rightarrow0$ as $\epsilon_m\rightarrow0$ and then Eq. (46) implies Eq. (42).

Thus we have shown above that the GNS representation of $\mathcal{HA}_r$ on the GNS Hilbert space $\mathcal{H}$, is unitarily equivalent to the standard Fock representation on $\mathcal{F}=L^2(\mathcal{S}',d\mu_G)$ [$\mathcal{S}'$ denotes the appropriate space of tempered distributions and μ_G is the standard Gaussian measure with covariance $\frac{1}{2}(-\nabla^2)^{-1/2}$ [14]] via the unitary map U.

The action of the smeared electric-field operator, $\hat{E}(\vec{f}):=\int d^3x f_a(\vec{x})\hat{E}^a(\vec{x})$, on $\psi\in\mathcal{C}_F\subset L^2(\mathcal{S}',d\mu_G)$ is written in the standard way [14] as

$$\begin{aligned}\hat{E}(\vec{f})\psi=e\int d^3x\bigg(&-if_a(\vec{x})\frac{\delta}{\delta A_a(\vec{x})}\\ &+i[(-\nabla^2)^{1/2}f^a(\vec{x})]A_a(\vec{x})\bigg)\psi.\end{aligned} \tag{49}$$

Here $f_a(\vec{x})$ is real, divergence free, smooth, and of rapid decrease, and $\mathcal{C}_F\subset L^2(\mathcal{S}',d\mu_G)$ is the standard dense domain of cylindrical functions appropriate to Fock space. The smeared electric-field operator on $\mathcal{H}$ is defined as the unitary image of $\hat{E}(\vec{f})$ by U^{-1}, i.e., for $\psi\in U^{-1}(\mathcal{C}_F)\subset\mathcal{H}$

$$\begin{aligned}\hat{E}(\vec{f})\psi=U^{-1}e\int d^3x\bigg(&-if_a(\vec{x})\frac{\delta}{\delta A_a(\vec{x})}\\ &+i[(-\nabla^2)^{\frac{1}{2}}f^a](\vec{x})A_a(\vec{x})\bigg)U\psi.\end{aligned} \tag{50}$$

With this action, $\hat{E}(\vec{f})$ is densely defined on the dense domain $U^{-1}(\mathcal{C}_F)\subset\mathcal{H}$, and just like its unitary image on $\mathcal{C}_F$, admits a unique self-adjoint extension.

V. INDUCED REPRESENTATIONS THROUGH I_r

It can be verified that I_r is a topological homorphism from Δ to Δ_r (where Δ and Δ_r are equipped with their Gel'fand topologies). Hence, I_r defines a measurable isomorphism

$I_r:\mathcal{B}\rightarrow\mathcal{B}_r$ where $\mathcal{B}$ and $\mathcal{B}_r$ are the Borel sigma algebras associated with Δ and Δ_r, respectively. Any regular Borel measure μ on Δ induces a regular Borel measure μ_r on Δ_r, with $\mu_r := \mu I_r^{-1}$. It follows that I_r defines a unitary map U_r from $L^2(\Delta,d\mu)$ to $L^2(\Delta_r,d\mu_r)$.

U_r can be explicitly defined through its action on the dense set $\mathcal{C}\in L^2(\Delta,d\mu)$, of cylindrical functions (cylindrical functions in the context of $\overline{\mathcal{HA}}$ have been defined in Sec. II). Denote the dense set of cylindrical functions in $L^2(\Delta_r,d\mu_r)$ by $\mathcal{C}_r$.[9] U_r maps $\mathcal{C}$ to $\mathcal{C}_r$ through

$$U_r(\psi_{\{[\alpha_i]\}})=\psi_{\{[\alpha_i]_r\}}. \tag{51}$$

It also follows that

$$U_r\hat{H}_\alpha U_r^{-1}=\hat{H}_{\alpha_{(r)}}. \tag{52}$$

Thus I_r induces a representation of $\overline{\mathcal{HA}_r}$ from a representation of $\overline{\mathcal{HA}}$. In Sec. V A, we induce a Haar-like representation of $\overline{\mathcal{HA}_r}$ from the Haar representation of $\overline{\mathcal{HA}}$.

Since the image of I_r restricted to $\mathcal{HA}$ is $\mathcal{HA}_r$, I_r (or I_r^{-1}) can also be used to induce representations of $\mathcal{H}A$ from those of $\mathcal{H}A_r$ and vice versa. In Sec. V B, we induce a Fock-like representation of $\mathcal{HA}$ from the Fock representation of $\mathcal{HA}_r$. The elements of $\mathcal{HA}_r$ define a dense subspace of $\mathcal{H}$ through the GNS construction and a map U_r is defined through Eqs. (51) and (52). U_r^{-1} induces a Fock-like representation of $\mathcal{HA}$.

A. Haar representation of $\bar{\mathcal{H}}\bar{\mathcal{A}}_r$

We denote both the Haar measure on Δ as well as its image on Δ_r by $d\mu_H$. The induced PLF (corresponding to $d\mu_H$) on $\overline{\mathcal{HA}_r}$ is defined by

$$\Gamma(\alpha)=1 \text{ if } \tilde{\alpha}_r=\tilde{o}_r=0 \quad \text{otherwise}. \tag{53}$$

From Eqs. (51) and (52) it follows that $\hat{H}_{\alpha_{(r)}}$ are represented by unitary operators on $\psi\in L^2(\Delta_r,d\mu_H)$ by

$$(\hat{H}_{\alpha_{(r)}}\psi)(h)=h([\alpha]_r)\psi(h), \quad h\in\Delta_r. \tag{54}$$

We construct electric-field operators on $L^2(\Delta_r,d\mu_H)$ as unitary images of appropriate electric-field operators on $L^2(\Delta,d\mu_H)$ as follows. Define the classical Gaussian smeared electric field as

$$E_r^a(\vec{x}):=\int d^3y f_r(\vec{y}-\vec{x})E^a(\vec{y}), \tag{55}$$

where f_r has been defined in Sec. III. Given $\psi_{\{[\alpha_i]\}}\in\mathcal{C}\subset L^2(\Delta,d\mu_H)$ it can be checked that

$$[\hat{E}_r^a(\vec{x})\psi_{\{[\alpha_i]\}}](h)=e\sum_{i=1}^{n}X^a_{\alpha_{i(r)}}(\vec{x})h([\alpha_i])\frac{\partial\psi_{\{[\alpha_i]\}}}{\partial h([\alpha_i])}. \tag{56}$$

The methods of [10] can be used to show that $\hat{E}_r^a(\vec{x})$ is essentially self-adjoint on $\mathcal{C}$. Note that Eq. (56) implies that

$$[\hat{E}_r^a(\vec{x}),\hat{H}_\alpha]=eX^a_{\alpha_{(r)}}(\vec{x})\hat{H}_\alpha. \tag{57}$$

The unitary image of Eq. (57) is

$$[U_r\hat{E}_r^a(\vec{x})U_r^{-1},\hat{H}_{\alpha_{(r)}}]=eX^a_{\alpha_{(r)}}(\vec{x})\hat{H}_{\alpha_{(r)}}. \tag{58}$$

Denote the classical counterpart of $U_r\hat{E}_r^a(\vec{x})U_r^{-1}$ by $F^a(\vec{E})$. Then Eq. (58) provides a quantum representation of the classical Poisson bracket,

$$\{F^a(\vec{E}),H_{\alpha_{(r)}}(A)\}=-ieX^a_{\alpha_{(r)}}(\vec{x})H_{\alpha_{(r)}}(A). \tag{59}$$

Note that $\{E^a(\vec{x}),H_{\alpha_{(r)}}(A)\}=-ieX^a_{\alpha_{(r)}}(\vec{x})H_{\alpha_{(r)}}(A)$. Hence, we can consistently identify $F^a(\vec{E})$ with $E^a(\vec{x})$. Thus, $U_r\hat{E}_r^a(\vec{x})U_r^{-1}=\hat{E}^a(\vec{x})$ and from Eq. (56),

$$[\hat{E}^a(\vec{x})\psi_{\{[\alpha_i]_r\}}](h)=e\sum_{i=1}^{n}X^a_{\alpha_{i(r)}}(\vec{x})h([\alpha_i]_r)\frac{\partial\psi_{\{[\alpha_i]_r\}}(h)}{\partial h([\alpha_i]_r)}. \tag{60}$$

Since U_r is unitary, $\hat{E}^a(\vec{x})$ is essentially self-adjoint on $\mathcal{C}_r\subset L^2(\Delta_r,d\mu_H)$.

To summarize: The induced Haar representation of $\overline{\mathcal{HA}_r}$ provides a quantum representation of the classical Poisson bracket algebra of smeared holonomies $H_{\alpha_{(r)}}(A)$ and (divergence free) electric field $E^a(\vec{x})$. $\hat{H}_{\alpha_r}$ are represented by unitary operators through Eq. (54) and the *unsmeared* electric-field operator, $\hat{E}^a(\vec{x})$, is represented through Eq. (60) as an essentially self-adjoint operator on the dense domain of cylindrical functions, $\mathcal{C}_r\subset L^2(\Delta_r,d\mu_H)$. Note that $\hat{E}^a(\vec{x})$ is a genuine operator as opposed to an operator valued distribution.

B. Fock representation of $\mathcal{HA}$

We denote the Fock PLF on $\mathcal{HA}_r$ as well as its image on $\mathcal{HA}$ by Γ_F. Note that the induced PLF on $\mathcal{HA}$ is defined by

$$\Gamma_F\left(\sum_{i=1}^{N}a_i[\alpha_i]\right):=\sum_{i=1}^{N}a_i\exp-\left(\int\frac{d^3k}{k}|X^a_{\alpha_{i(r)}}(\vec{k})|^2\right). \tag{61}$$

Since $\hat{H}_{\alpha_{(r)}}$ are represented as unitary operators on $\mathcal{H}$, it follows that

$$\hat{H}_\alpha:=U_r^{-1}\hat{H}_{\alpha_{(r)}}U_r \tag{62}$$

are represented as unitary operators on $U_r^{-1}(\mathcal{H})$.

[9] Cylindrical functions are of the form $\psi_{\{[\alpha_i]_r\}}(h):=\psi(h([\alpha_1]_r)\dots h([\alpha_n]_r))$, for $\alpha_i\in\mathcal{L}_{x_0}, i=1\dots n$, $h\in\Delta_r$, and they span $\mathcal{C}_r$.

It remains to construct, following the strategy of Sec. V A, electric-field operators on $U_r^{-1}(\mathcal{H})$ as unitary images of the appropriate electric-field operators on $\mathcal{H}$. On $U^{-1}(\mathcal{C}_F)\subset\mathcal{H}$,

$$[\hat{E}(\vec{f}),\hat{H}_{\alpha_{(r)}}]=e\int d^3x f_a(\vec{x})X^a_{\alpha_{(r)}}(\vec{x})\hat{H}_{\alpha_{(r)}}, \tag{63}$$

$$\Rightarrow[U_r^{-1}\hat{E}(\vec{f})U_r,\hat{H}_\alpha]$$
$$=e\int d^3x f_a(\vec{x})X^a_{\alpha_{(r)}}(\vec{x})\hat{H}_\alpha. \tag{64}$$

We define the classical function

$$E_r(\vec{f}):=\int d^3x f_a(\vec{x})E^a_r(\vec{x}), \tag{65}$$

where $E^a_r(\vec{x})$ is defined by Eq. (55). Since

$$\{E_r(\vec{f}),H_\alpha(A)\}=-ie\int d^3x f_a(\vec{x})X^a_{\alpha_{(r)}}(\vec{x})H_\alpha(A), \tag{66}$$

we identify

$$\hat{E}^a_r(\vec{f}):=U_r^{-1}\hat{E}(\vec{f})U_r. \tag{67}$$

To summarize: The induced Fock representation of $\mathcal{HA}$ provides a quantum representation of the classical Poisson bracket algebra of holonomies $H_\alpha(A)$ and ''Gaussian-smeared, smeared'' electric fields $E_r(\vec{f})$. The ''*unsmeared*'' holonomy operators $\hat{H}_\alpha$ are represented by unitary operators through Eq. (62) and $\hat{E}_r(\vec{f})$ is represented as a self-adjoint operator through Eq. (67). Note that the Gaussian-smeared object $\hat{E}_r(\vec{x})$ is represented as an *operator valued distribution* (as opposed to a genuine operator) on $U_r^{-1}(\mathcal{H})$.

VI. DISCUSSION

Preliminary remarks: In this paper, representations of the Poisson algebra of $U(1)$ theory were constructed in two steps. First, Hilbert space representations of the Abelian-Poisson algebra of configuration functions (i.e., functions of A_a) were constructed by specifying a PLF. Second, real functions of the conjugate electric field were represented by self-adjoint operators on this Hilbert space. The Haar representation of $\overline{\mathcal{HA}}$ and its image on $\overline{\mathcal{HA}}_r$ support a representation of the electric field wherein, formally,

$$\hat{E}^a(\vec{x})=-i\frac{\delta}{\delta A_a(\vec{x})}. \tag{68}$$

This action is not connected with Poincaré invariance and it is not surprising that the resulting representations of Secs. II and V A, are non-Fock representations. On the Fock representation of $\mathcal{HA}_r$[10] and its image on $\mathcal{HA}$, Eq. (68) is incompatible with the requirement of self-adjointness of the electric-field operators. Their action necessarily contains a term dependent on the Gaussian measure [see Eq. (50)] to ensure self-adjointness. The choice of Gaussian measure is intimately associated with the properties of the D'Alembertian, $\partial^2/\partial t^2-\nabla^2$, and hence with Poincaré invariance.

A rephrasing of the above remarks which brings them closer to the strategy of [1,3] is as follows. Given a representation in which (smeared or unsmeared) holonomies are represented by multiplication by unitary operators and the electric field acts, as in Eq. (68), purely by functional differentiation, the requirement of self-adjointness of the electric-field operator determines the Hilbert space measure to be the Haar measure. The self-adjointness of electric-field operators results in the Gaussian measure only if their action has a contribution dependent on the Gaussian measure. Thus, to obtain the standard Fock representation or the induced one of Sec. V B, the Gaussian measure and hence, Poincaré invariance, plays an essential and explicit role.

Note that this work concerns the ''connection'' representation of a theory of a real U(1) connection. In contrast [1] constructs the *loop* representation of a description of linearized gravity based on a *self-dual* connection.[11] Despite these differences, there is also a certain amount of shared mathematical structure in our work and [1]. Therefore, the delineation of the structures involved in the construction of U(1) theory as spelled out in this paper, allows us to identify the role of the key structures in [1].

Discussion of (I) and (II): We use the notation of [1] and [3] when discussing those papers. We first discuss (I). In [1] the action of the linearized metric variable in the loop representation is deduced, ultimately, from its action of the form $-i(\delta/\delta A^i_a)$ in the connection representation. The loop representation then becomes an electric-field-type representation in which the magnetic field operator acts purely by functional differentiation with respect to the loop form factor. Yet a Fock representation (of the positive and negative helicity gravitons) results in apparent contradiction to our claims that such a representation cannot result without using Poincaré invariance explicitly.

The resolution of this apparent contradiction for the positive helicity graviton sector seems to lie, in what appears at first sight, to be a mere mathematical nicety. In [1] the Gaussian measure contribution to the $\hat{B}^+$ operator is absorbed (and hidden) in the rescaling of the wave function. Such a rescaling is permissible for finite dimensional systems but results in a mathematically ill-defined measure for

[10]We remind the reader that we displayed a fairly rigorous argument that the entire Fock space is obtained in such a representation through the constructions of Sec. IV and the Appendix. We reiterate our belief that the formal Eq. (A21) can be rendered mathematically well-defined in a more careful treatment.

[11]Note, however, that the descripton reduces to one in terms of a triplet of Abelian connections.

the field theory in question. In spite of the fact that for most applications this formal treatment suffices, it is crucial to realize, in the context of (I), that it hides the role of Poincaré invariance in constructing the Fock representation. To obtain a well-defined (Gaussian) measure, the wave functions [Eq. (101) of [1]] need to be rescaled and a Gaussian measure term needs to be added to the action of the $\hat{B}^{+}$—this, of course, feeds explicit Poincaré invariance back into the construction.

Note that this argument does not apply to the negative helicity sector. There, the choice of self-dual connection results in the negative helicity magnetic-field operator, $\hat{B}^{-}$ being the same as the negative helicity annihilation operator. $\hat{B}^{-}$ is naturally represented as a functional derivative [Eq. (70) of [1]] and, in this aspect, matches the standard Fock representation of the annihilation operator as a pure functional derivative term. The resulting representation is the Fock representation for negative helicity gravitons and indeed, for this sector, it seems that explicit Poincare invariance is not invoked.

Thus the Fock representation of linearized gravity seems to result partly due to explicit Poincaré invariance (which is suppressed in [1] by a mathematically ill-defined operation) and partly due to the use of self-dual connections.

Considerations similar to those for the positive helicity gravitons also apply to the treatment of free Maxwell theory in [3]. There, it is shown that the extended loop representation coincides, formally, with the electric-field representation. Again, an ill-defined measure is used and a proper mathematical treatment restores the explicit role of Poincaré invariance.

We turn now, to a discussion of (II). In the loop representation of [1] the two sets of important operators are the magnetic field, $\hat{B}^{\pm}$, and the linearized metric, $\hat{h}^{\pm}$. They are represented by functional differentiation and multiplication, on the representation space of functionals of loop form factors. This representation space supports the holonomies as operators. Indeed, the action of the $\hat{B}^{\pm}$ operators is deduced from the fact that the classical magnetic flux is the lowest nontrivial term in the expansion of the holonomy of a small loop [Eq. (57) of [1]]. However, all these constructions are rendered formal because of the distributional nature of the loop form factor and the resulting divergence of the ground-state functional. Therefore, a regularization procedure is adopted wherein the loop form factors are replaced by their Gaussian smeared, r versions [see Eq. (9)] and $\hat{B}^{\pm}$, $\hat{h}^{\pm}$ are represented as functional differentiation and multiplication operators on the space of functionals of r-loop form factors. An important question is: Are the holonomy operators or some regularized version thereof, represented in this space?

One may choose to ignore this question and simply postulate the action of $\hat{B}^{\pm}$, $\hat{h}^{\pm}$ in terms of r-loop form factors. Then the primary configuration variables of the theory are $B^{\pm}$ and the construction does not seem to have much to do with loops and holonomies. Since holonomies and loops are the primary objects in the loop approach to full-blown quantum gravity, an interpretation of the regularization which allows for the representation of holonomy operators is of interest. It seems to us that such an interpretation must regard the r-form factor representation as an *approximation* to the standard Fock representation, which becomes better as $r \to 0$.

A precise formulation of such an interpretation in the context of $U(1)$ theory is provided by the induced Fock representation of Sec. V B. There, the holonomy operators are represented on the Hilbert space and the magnetic-field operators can be constructed by a ''shrinking of loop'' limit, as the image of $U^{-1}\hat{O}_{\vec{x},\vec{n}}U$ via U_r^{-1}. That representation, although *not* the standard Fock representation, is a good approximation to it for small r. The nature of the approximation is as follows. For sufficiently small r, the holonomies $H_{\gamma}(A)$, the electric field $E^a(\vec{x})$, and their Gaussian-smeared counterparts, $H_{\gamma_{(r)}}(A)$, $E^a_r(\vec{x})$ approximate each other well. An approximate Fock representation can be constructed in which the operators corresponding to $H_{\gamma}(A)$, $E^a_r(\vec{x})$ act in the same way as the operators corresponding to $H_{\gamma_{(r)}}(A)$, $E^a(\vec{x})$ in the standard Fock representation. This approximate Fock representation is the induced Fock representation of Sec. V B.

To summarize: The standard Fock representation for U(1) theory is obtained only when the algebra of smeared holonomies is used *and* explict Poincaré invariance is invoked. However, the role of Poincaré invariance (or equivalently, the choice of PLF) seems to be more important than that of smeared loops. If the requirement of smeared loops is dropped, it is still possible to construct an approximate Fock representation; but dropping Poincaré invariance results in the non-Fock representations of Secs. II and V A.

Comments: (i) The ''area derivative'' plays an important role in some approaches to loop quantum gravity [4,3]. Our construction of $\hat{O}_{\vec{x},\vec{n}}|\psi\rangle$ (or its image in the induced Fock representation of Sec. V B) as a Cauchy limit is a rigorous realization of the area derivative in the context of Fock-like representations. Note that the required limits do not exist in the Haar representation and hence the area derivative is ill-defined there.

(ii) As noted above, self-duality of the connection plays a key role in obtaining the (negative helicity) graviton Fock representation without explicit recourse to Poincaré invariance (see [15] for a detailed examination of the relation between self-duality and helicity). However, recent efforts in loop quantum gravity use real (as opposed to self-dual) connections. It would be useful to reformulate linearized gravity in terms of real connections and construct its quantization.

(iii) Note that we have mainly been concerned with the kinematics of $U(1)$ theory. The Fock representation, of course, supports the Maxwell Hamiltonian as an operator. Note that the normal ordering prescription adopted in [1] is, of course, connected with Poincaré invariance. It is an open question as to how to express (presumably an approximation of) the Hamiltonian as an operator in the Haar representation.

(iv) We have not been able to show continuity of the Fock PLF on $\mathcal{HA}_r$ or lack thereof. If the Fock PLF is continuous, a ''Fock'' measure $d\mu_F$ can be constructed on Δ_r and $\mathcal{H}$ can be identified with $L^2(\Delta_r, d\mu_F)$. The considerations of Sec. V B can then be extended to the C^* algebras $\overline{\mathcal{HA}_r}$ and $\overline{\mathcal{HA}}$.

If, however, the Fock PLF on $\mathcal{HA}_r$ turns out *not* to be continuous, then a corresponding Fock measure on Δ_r does not exist and it is incorrect to identify Δ_r with the ''quantum configuration space.'' If this is indeed the case, then the emphasis on *continuous* cyclic representations of $\overline{\mathcal{HA}}$ in loop quantum gravity [6] would seem unduly restrictive.

(v) The representation of kinematic loop quantum gravity is the SU(2) counterpart of the Haar representation for $U(1)$ theory. An important question is how the Fock space-graviton description of linearized gravity arises out of loop quantum gravity. It is possible that some insight into this issue may be obtained by considering the following (simpler) question in the context of $U(1)$ theory. Is there any way in which an approximate Fock structure can be obtained from the Haar representation of $U(1)$ theory? Since the PLF's play a key role in determining the type of representation, this work suggests that to get an approximate Fock structure, it may be a good strategy to try to approximate (in some, yet unknown way) the Fock PLF by the Haar PLF.

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APPENDIX

1. Lemma 1: Given (i) $\gamma_i \in \mathcal{L}_{x_0}$, $i=1\ldots n$, n finite, (ii) $A_a(\vec{x}) \in \mathcal{A}$, and (iii) $\epsilon > 0$, there exists a connection $A_a^{\epsilon}(\vec{x}) \in \mathcal{A}$ such that

$$|H_{\gamma_{i(r)}}(A^{\epsilon}) - H_{\gamma_i}(A)| < \epsilon \tag{A1}$$

for $i=1,\ldots,n$.

Proof: For a single loop γ, from Eq. (8)

$$|X^a_\gamma(\vec{k})| < C_\gamma \quad C_\gamma := \frac{3}{(2\pi)^{3/2}} L_\gamma \tag{A2}$$

where L_γ is the length of the loop as measured by the flat metric. Since $A_a(\vec{x})$ is Schwartz, we have, for arbitrarily large $N>0$,

$$|A_a(\vec{k})| < \frac{C_N}{k^N} \text{ for some } C_N > 0. \tag{A3}$$

From Eqs. (A2) and (A3),

$$\int_{k>\Lambda} d^3k |X^a_\gamma(-\vec{k}) A_a(\vec{k})| < \frac{C_{N,\gamma}}{\Lambda^{N-3}}, \quad C_{N,\gamma} = \frac{4\pi C_\gamma C_N}{N-1}. \tag{A4}$$

Thus, given $\delta > 0$, there exists $\Lambda(\gamma,\delta)$ such that

$$\int_{k>\Lambda(\gamma,\delta)} d^3k |X^a_\gamma(-\vec{k}) A_a(\vec{k})| < \delta. \tag{A5}$$

Let $f(k) > 0$ be a smooth function such that

$$\begin{aligned} f(k) &= e^{k^2 r^2/2} \text{ for } k < \Lambda(\gamma,\delta) \\ &< e^{k^2r^2/2} \text{ for } \Lambda(\gamma,\delta) < k < 2\Lambda(\gamma,\delta) \\ &= 1 \text{ for } k > 2\Lambda(\gamma,\delta). \end{aligned} \tag{A6}$$

Define $A^{\delta}_{a(r)}(\vec{x})$ through its Fourier transform,

$$A^{\delta}_{a(r)}(\vec{k}) := f(k) A_a(\vec{k}). \tag{A7}$$

Note that $A^{\delta}_{a(r)}(\vec{x}) \in \mathcal{A}$.

From Eqs. (11), (A7), and (A5) it follows that

$$\left| \int_{k>\Lambda(\gamma,\delta)} d^3k X^a_{\gamma_{(r)}}(-\vec{k}) A^{\delta}_{a(r)}(\vec{k}) \right| < \delta. \tag{A8}$$

From Eqs. (11) and (A7)

$$\int_{k<\Lambda(\gamma,\delta)} d^3k X^a_{\gamma_{(r)}}(-\vec{k}) A^{\delta}_{a(r)}(\vec{k}) = \int_{k<\Lambda(\gamma,\delta)} d^3k X^a_\gamma(-\vec{k}) A_a(\vec{k}). \tag{A9}$$

Using Eq. (A9)

$$\begin{aligned} &|H_{\gamma_{i(r)}}(A^{\delta}_{(r)}) - H_{\gamma_i}(A)| \\ &= \left| \exp i \left(\int_{k>\Lambda(\gamma,\delta)} d^3k X^a_{\gamma_{(r)}}(-\vec{k}) A^{\delta}_{a(r)}(\vec{k}) - X^a_\gamma(-\vec{k}) A^a(\vec{k}) \right) - 1 \right|. \end{aligned} \tag{A10}$$

From Eqs. (A5), (A8) and (A10), for small enough $\delta > 0$, it can be seen that

$$|H_{\gamma_{i(r)}}(A^{\delta}_{(r)}) - H_{\gamma_i}(A)| < 4\delta. \tag{A11}$$

For the loops γ_i, $i=1\ldots n$,

$$\overline{\Lambda}(\delta) := \max_i \Lambda(\delta, \gamma_i),$$

$$\overline{A}^{\delta}_{a(r)}(\vec{k}) := \overline{f} A_a(\vec{k}) \tag{A12}$$

with $\overline{f}(k) > 0$ a smooth function such that

$$\begin{aligned} \overline{f}(k) &= e^{k^2r^2/2}, \text{ for } k < \overline{\Lambda}(\delta) \\ &< e^{k^2r^2/2} \text{ for } \overline{\Lambda}(\delta) < k < 2\overline{\Lambda}(\delta) \\ &= 1 \text{ for } k > 2\overline{\Lambda}(\delta). \end{aligned} \tag{A13}$$

Then, given $\epsilon > 0$, choose some $\delta \leq \epsilon/4$ and set

$$A^{\epsilon}_a(\vec{k}) := \overline{A}^{4\delta}_{a(r)}(\vec{k}). \tag{A14}$$

Then Eq. (A1) holds.

Lemma 2: Given (i) strongly independent loops γ_i, $i=1\dots n$, n finite, (ii) $g_i\in U(1)$, $i=1\dots n$, and (iii) $\epsilon>0$, there exists a connection $A_a^{\epsilon}(\vec{x})\in\mathcal{A}$ such that

$$|H_{\gamma_{i(r)}}(A^{\epsilon})-g_i|<\epsilon \tag{A15}$$

for $i=1\dots n$.

Proof: From [8], $A_a\in\mathcal{A}$ exists such that $H_{\gamma_i}(A)=g_i$, $i=1\dots n$. Therefore, it suffices to construct A_a^{ϵ} such that $|H_{\gamma_{i(r)}}(A^{\epsilon})-H_{\gamma_i}(A)|<\epsilon$. But this is exactly the content of Lemma 1.

Lemma 3: Given $A_a(\vec{x})\in\mathcal{A}$, there exists $A_{a(r)}(\vec{x})\in\mathcal{A}$ such that

$$H_{\gamma_{(r)}}(A)=H_{\gamma}(A_{(r)}) \tag{A16}$$

for every $\gamma\in\mathcal{L}_{x_0}$.

Proof: From Eqs. (11) and (12) it immediately follows that the required $A_{a(r)}(\vec{x})$ is determined by its Fourier transform via $A_{a(r)}(\vec{k})=e^{-k^2r^2/2}A_a(\vec{k})$.

2. Proposition: The states $|\psi_{\{\vec{x}_i,\vec{v}_i\}}\rangle:=\Pi_{i=1}^{p}\hat{O}_{\vec{x}_i,\vec{v}_i}|0\rangle$, $(p=1,2\dots)$ together with $|0\rangle$, span $\mathcal{D}_0$.

Heuristic Proof: The argument below is a bit formal, but we expect that it can be converted to a rigorous proof.

Define

$$|\psi,p\rangle:=\int d^3k_1\dots d^3k_p\psi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)\times\left(\prod_{i=1}^{p}\hat{a}_{a_i}(\vec{k}_i)+\hat{a}^{\dagger}_{a_i}(-\vec{k}_i)\right)|0\rangle, \tag{A17}$$

where $\psi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)$ has the same properties (a)–(c) (see Sec. IV) as $\phi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)$. $|\psi,p\rangle$ along with $|0\rangle$ span $\mathcal{D}_0$. $|\psi,p\rangle$ can be generated from $|\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle$ as follows.

Note that from Eq. (40),

$$\frac{1}{2\pi^{3/2}}\int d^3x e^{-i\vec{k}\cdot\vec{x}}\hat{O}_{\vec{x},\vec{n}}=\frac{i(\vec{n}\times\vec{k})^a}{\sqrt{2}k}e^{\frac{-k^2r^2}{2}}[\hat{a}_a(\vec{k})+\hat{a}^{\dagger}_a(-\vec{k})]. \tag{A18}$$

Define

$$g^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)=\left(\prod_{i=1}^{p}e^{k_i^2r^2}\sqrt{2}k_i\right)\psi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p). \tag{A19}$$

Given $\vec{k}_i$ $i=1\dots p$, it is possible to construct a triplet of vectors $\vec{u}^{\,i}_{a_i}(\vec{k}_i)$ ($a_i=1,2,3$ for each i), such that[12]

$$\frac{(\vec{u}^{\,i}_{b_i}\times\vec{k}_i)^{a_i}}{k}=\delta^{a_i}_{b_i}-\frac{k^{a_i}k_{b_i}}{k^2}. \tag{A20}$$

Then from Eqs. (A18), (A19), and (A20),

$$|\psi,p\rangle=\int\left(\prod_{l=1}^{p}d^3k_l\prod_{m=1}^{p}d^3x_m\right)g^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)\times\left(\prod_{i=1}^{p}\frac{e^{-i\vec{k}_i\cdot\vec{x}_i}}{2\pi^{3/2}}\right)|\psi_{\{\vec{x}_i,\vec{u}^{\,i}_{a_i}\}}\rangle. \tag{A21}$$

It is in this formal sense that states of the type $|\psi_{\{\vec{x}_i,\vec{v}_i\}}\rangle$ together with $|0\rangle$ span $\mathcal{D}_0$.

Lemma 2: $\hat{O}_{\vec{x},\vec{n}}$ is a densely defined, symmetric operator on the dense domain $\mathcal{D}_0$, which admits self-adjoint extensions.

Proof: It is straightforward to check that

$$\begin{aligned}\hat{O}_{(\vec{x},\vec{n})}|\phi,p\rangle=&\int\left(\prod_{i=1}^{p+1}d^3k_i\right)f^{a_{p+1}}(-\vec{k}_{p+1})\\&\times\phi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)\left(\prod_{j=1}^{p+1}\hat{a}^{\dagger}_{a_j}(\vec{k}_j)\right)|0\rangle\\&+p\int\left(\prod_{i=1}^{p}d^3k_i\right)f_{a_1}(\vec{k}_1)\phi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)\\&\times\left(\prod_{j=2}^{p+1}\hat{a}^{\dagger}_{a_j}(\vec{k}_j)\right)|0\rangle,\end{aligned} \tag{A22}$$

where

$$f^a(\vec{k}):=\frac{i}{2\pi^{3/2}}e^{i\vec{k}\cdot\vec{x}}\frac{(\vec{n}\times\vec{k})^a}{\sqrt{2}k}e^{-k^2r^2/2}. \tag{A23}$$

The ultraviolet behavior of $\phi^{a_1\cdots a_p}(\vec{k}_1\dots\vec{k}_p)$, $f^a(\vec{k})$ ensures that $||\hat{O}_{(\vec{x},\vec{n})}|\phi,p\rangle||$ is finite. Thus $\hat{O}_{(\vec{x},\vec{n})}$ is densely defined on $\mathcal{D}_0$. By inspection $\hat{O}_{(\vec{x},\vec{n})}$ is also symmetric on $\mathcal{D}_0$.

To show existence of its self-adjoint extensions, it is sufficient to exhibit an antilinear operator $\hat{C}$ on $\mathcal{F}$ with $\hat{C}^2=\mathbf{1}$ which leaves $\mathcal{D}_0$ invariant and commutes with $\hat{O}_{(\vec{x},\vec{n})}$ [12,13]. As in [13], take $\hat{C}$ to be the complex conjugation operator (in the standard Schrödinger representation) on $\mathcal{F}=L^2(\mathcal{S}',d\mu)$ where $\mathcal{S}'$ is the appropriate space of tempered distributions and $d\mu$ is the standard Gaussian measure, for free Maxwell theory. It can be seen that $\hat{C}a_a(\vec{k})=a_a(-\vec{k})\hat{C}$ and $\hat{C}a^{\dagger}_a(\vec{k})=a^{\dagger}_a(-\vec{k})\hat{C}$, and hence $\hat{O}_{(\vec{x},\vec{n})}\hat{C}=\hat{C}\hat{O}_{(\vec{x},\vec{n})}$.

Lemma 3:

$$\left|\left|\frac{(e^{i\hat{D}}-1)}{i\pi\epsilon_m^2}|\psi\rangle\right|\right|\to 0, \tag{A24}$$

as $\epsilon_m\to 0$.

Proof:

[12] An explicit choice is as follows. Fix Cartesian coordinates (x,y,z) and the corresponding unit vectors $(\hat{x},\hat{y},\hat{z})$. Then for $i=1\dots p$, $\vec{u}^{\,i}_1=\vec{k}_i\times\hat{x}/k$, $\vec{u}^{\,i}_2=\vec{k}_i\times\hat{y}/k$, $\vec{u}^{\,i}_3=\vec{k}_i\times\hat{z}/k$.

$$i\hat{D}=i\int d^3k h^a(\vec{k})[\hat{a}_a(\vec{k})+\hat{a}_a^\dagger(-\vec{k})] \tag{A25}$$

with

$$h^a(\vec{k})=\frac{e^{i\vec{k}\cdot\vec{x}}e^{-k^2r^2/2}}{\sqrt{2}k2\pi^{3/2}} \times\left(\oint ds e^{i\vec{k}\cdot(\vec{\gamma}^{\{m,\vec{x},\vec{n}\}}-\vec{x})}\dot{\gamma}^{a\{m,\vec{x},\vec{n}\}}-i\pi\epsilon_m^2(\vec{n}\times\vec{k})^a\right). \tag{A26}$$

A straightforward calculation, using [16], shows that

$$h^a(\vec{k})=\frac{ie^{i\vec{k}\cdot\vec{x}}e^{-k^2r^2/2}}{\sqrt{2}k2\pi^{3/2}}(\vec{n}\times\vec{k})^a\pi\epsilon_m^2\left(\frac{2J_1(\alpha_k\epsilon_m)}{\alpha_k\epsilon_m}-1\right) \tag{A27}$$

with $\alpha_k:=|\vec{n}\times\vec{k}|$.

Now, from Eqs. (A25) and (38),

$$e^{i\hat{D}}=e^{-\int d^3k|h^a(\vec{k})h_a(\vec{k})|}e^{i\int d^3kh^a(\vec{k})\hat{a}_a^\dagger(\vec{k})}e^{i\int d^3kh^a(\vec{k})\hat{a}_a(\vec{k})}, \tag{A28}$$

$$\begin{aligned}\Rightarrow\langle\phi,p|e^{i\hat{D}}|\phi,p\rangle &=e^{-\int d^3k|h^a(\vec{k})h_a(\vec{k})|}\int\left(\prod_{i=1}^{p}d^3k_i\prod_{j=1}^{p}d^3l_j\right)\\ &\times\phi^{a_1\dots a_p}(\vec{k}_1\dots\vec{k}_p)\phi^{*b_1\dots b_p}(\vec{l}_1\dots\vec{l}_p)\\ &\times\langle 0|\left(\prod_{j=1}^{p}\hat{a}_{b_j}(\vec{l}_j)-ih^*_{b_j}(\vec{l}_j)\right)\\ &\times\left(\prod_{i=1}^{p}\hat{a}^\dagger_{a_i}(\vec{k}_i)+ih_{a_i}(\vec{k}_i)\right)|0\rangle,\end{aligned} \tag{A29}$$

$$\begin{aligned}\Rightarrow\langle\phi,p|e^{i\hat{D}}-1|\phi,p\rangle &=n!(e^{-\int d^3k|h^a(\vec{k})h_a(\vec{k})|}-1)\\ &\times\int\prod_{i=1}^{p}d^3k_i|\phi^{a_1\dots a_p}(\vec{k}_1\dots\vec{k}_p)|^2\\ &+n!ne^{-\int d^3k|h^a(\vec{k})h_a(\vec{k})|}\\ &\times\int\left(\prod_{i=2}^{p}d^3k_i\right)d^3kd^3lh_{a_1}(\vec{k})h^{*b_1}(\vec{l})\\ &\times\phi^{a_1a_2\dots a_p}(\vec{k},\vec{k}_2\dots,\vec{k}_p)\\ &\times\phi^*{}_{b_1a_2\dots a_p}(\vec{l},\vec{k}_2\dots,\vec{k}_p)\\ &+O(h^4).\end{aligned} \tag{A30}$$

Since $[2J_1(\alpha_k\epsilon_m)/\alpha_k\epsilon_m-1]$ is a bounded function, Eq. (A27) implies that the $O(h^4)$ terms do not contribute to Eq. (A24) in the $\epsilon_m\to 0$ limit.

From Lemma 4 below and Eq. (A27) the first term of Eq. (A30) is of order $\epsilon_m^{5.5}$ and the second is of order ϵ_m^5. From this it is clear that $||(e^{i\hat{D}}-1)/i\pi\epsilon_m^2|\psi>||\to 0$ as $\epsilon_m\to 0$.

Lemma 4: Let n be a positive integer and $g(\vec{k})$ be a bounded function of rapid decrease (i.e., it falls to zero as $k\to\infty$, faster than any inverse power of k). Then, as $\epsilon\to 0$,

$$I:=\left|\int d^3kg(\vec{k})\left(\frac{2J_1(\alpha_k\epsilon)}{\alpha_k\epsilon}-1\right)^n\right|<C\epsilon^{n-1/2} \tag{A31}$$

for some positive constant C which depends on n and g.

Proof:

$$I\leqslant\int_{k\leqslant\epsilon^{-1/2}}d^3k\left|g(\vec{k})\left(\frac{2J_1(\alpha_k\epsilon)}{\alpha_k\epsilon}-1\right)^n\right| +\int_{k>\epsilon^{-1/2}}d^3k\left|g(\vec{k})\left(\frac{2J_1(\alpha_k\epsilon)}{\alpha_k\epsilon}-1\right)^n\right|. \tag{A32}$$

In the first term the range of integration is such that $\alpha_k\epsilon<\epsilon^{1/2}$. A straightforward calculation shows that the small argument expansion of $J_1(\alpha_k\epsilon)$ coupled with the rapid fall off property of $g(\vec{k})$ gives the bound

$$\int_{k\leqslant\epsilon^{-\frac{1}{2}}}d^3k\left|g(\vec{k})\left(\frac{2J_1(\alpha_k\epsilon)}{\alpha_k\epsilon}-1\right)^n\right|\leqslant C_1(g,n)\epsilon^{n-1/2} \tag{A33}$$

where $C_1(g,n)$ is a positive constant dependent on both n and the properties of g.

The rapid decrease property of $g(\vec{k})$ ensures that, for small enough ϵ, the second term of Eq. (A32) falls off much faster than the first term. Hence, $I<C\epsilon^{n-1/2}$ where we have set $C:=2C_1(g,n)$.

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