Post-Newtonian gravitational radiation reaction for two-body systems: Nonspinning bodies

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We study gravitational radiation reaction in the equations of motion for binary systems of nonspinning point masses to post-Newtonian order $[O((v/c)^2)]$ beyond the quadrupole approximation, corresponding to post$^{7/2}$-Newtonian order corrections to Newtonian motion. One method uses post-Newtonian expressions for energy and angular momentum flux to infinity, and an assumption of energy and angular momentum balance. The equations of motion so derived are valid for general binary orbits, and for a class of coordinate gauges. Another method uses explicit formulas for near-zone reaction potentials, valid to post-Newtonian order, derived by an asymptotic matching procedure in a fixed gauge. The two methods give equivalent results.

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I. INTRODUCTION AND SUMMARY

During the past 20 years, gravitational-radiation damping has been recognized as a process with important observational consequences. Observations of the binary pulsar PSR1913+16 have yielded a verification of the "quadrupole formula" for radiation damping to a precision of better than 0.4% [1]. Gravitational-wave damping is a central evolutionary mechanism in massive cataclysmic binary systems [2], and a potentially important capture mechanism in dense stellar systems [3].

Laser interferometric gravitational-wave observatories such as the U.S. Laser Interferometric Gravitational Wave Observatory (LIGO) [4] and the European VIRGO [5] projects are expected to have the capability to detect waves from the final inspiral and coalescence of two compact objects (neutron stars or black holes), a process dominated by gravitational radiation damping. The detection and study of the characteristic "chirp" wave form emitted by such systems involves a matched filtering technique using a theoretical template that is a function of the parameters of the source [6,7]. In order to extract useful astrophysical information about inspiraling binaries from observed gravitational-wave signals, one must use templates that are very accurate, especially in their treatment of the evolution of the orbital phase, which evolves nonlinearly in time because of radiation damping [8–10].

Several approaches have attempted to derive accurate formulas for the evolution of the orbital phase (or frequency). In the post-Newtonian or post-Minkowskian framework, gravitational wave forms and fluxes of energy and angular momentum have been calculated accurately to two full orders $[O((v/c)^4)]$ beyond the Newtonian for systems of arbitrary mass ratio, and including the effects of spin [11–20] (for an overview of the post-Newtonian program, see [21]). Numerical computations of test-body perturbations of black holes have been carried to the equivalent of $O((v/c)^8)$ [22–24].

In obtaining the evolution of the orbital phase of inspiralling binaries from formulas for energy flux, it is generally assumed that the orbit is circular, and that energy is globally conserved, i.e., that the energy radiated to infinity is balanced by an equivalent loss of energy of a circular orbit. With the energy of a circular orbit written explicitly as a function of the orbital frequency, one arrives at a formula for the evolution of the orbital frequency, and then of the phase.

In reality, the orbital motion is determined by local equations that include damping terms that reflect the radiation of gravitational waves to infinity. The history of gravitational-wave damping effects in the equations of motion is long and contentious. Numerous authors have attempted to obtain, from first principles, approximate solutions of Einstein's equations that incorporate into "near-zone" gravitational fields, the back reaction from radiation to infinity. These methods were characterized by varying levels of rigor, often involving the presence of divergent integrals or uncontrolled approximations. Some even led to different quantitative results for the lowest-order, "Newtonian" radiation reaction terms (for a critical review up to 1980, see [25]; for a later survey, see [26]). However, recent refinements of a "post-Minkowskian" method for calculating gravitational radiation from weak-field slow-motion systems, combined with asymptotic matching techniques for connecting far-zone solutions with near-zone solutions [27] have resulted in the ability to address "near-zone" gravitational-radiation damping more systematically, without the problems that plagued other methods, and to orders of approximation beyond the Newtonian order [28].

The other method for discussing radiation reaction, popular in textbooks (see, for example, [29]), is to assume energy balance, and to derive a radiation-reaction term in the equation of motion sufficient to balance the energy radiated. Crudely the argument goes as follows: the energy flux at infinity is given by $\dot{E} \sim (M_{ij}^{(3)})^2$ where M_{ij} is the quadrupole moment of the source and (n) denotes the number of time derivatives. Integrating twice by parts and moving the total time derivatives to the left-hand side into a redefinition of E, one finds $\dot{E} \sim \dot{M}_{ij} M_{ij}^{(5)} \sim m v^i (x^j M_{ij}^{(5)})$. Since $\dot{E} \sim m\mathbf{v} \cdot \mathbf{a}$, one reads off the appropriate radiation-reaction contribution to the acceleration, related to five time-derivatives of the quadrupole moment.

In view of the importance to matched templates of highly accurate determinations of orbital dissipation of inspiraling binaries, this paper addresses the question of extending radiation-reaction formulas beyond the Newtonian order. Our main result is a formula, suitable for determining the evolution of general binary orbits (not just circular orbits), that includes the first post-Newtonian $[O((v/c)^2)]$ corrections to the dominant Newtonian radiation-damping terms. This result is obtained by two methods. One is a refinement of the "balance" method, which makes use of *both* energy and angular momentum balance, and extends the argument to post-Newtonian order. The method was summarized in [30]. The other method applies the post-Minkowskian approach of Blanchet, Damour, and collaborators [27], to derive directly the requisite terms in the equations of motion (see [28,31] for background and details). We show that the two methods give physically equivalent results.

A post-Newtonian (PN) approximation is an expansion of corrections to Newtonian gravitational theory in terms of a small parameter $\epsilon \sim (v/c)^2 \sim Gm/rc^2$, where m, v, and r are the total mass, orbital velocity, and separation of the binary system. The three key elements are the equations of motion, the gravitational wave form, and formulas for energy and angular-momentum flux. Schematically the equations of motion for spinless bodies are given as

$$\mathbf{a} \equiv d^2\mathbf{x}/dt^2$$
$$\approx -(m\mathbf{x}/r^3)[1 + O(\epsilon) + O(\epsilon^2) + O(\epsilon^{5/2}) + \cdots] , \quad (1.1)$$

where $\mathbf{x}$ and $r = |\mathbf{x}|$ denote the separation vector and distance between the bodies, and $m = m_1 + m_2$ denotes the total mass. The symbols $O(\epsilon)$ and $O(\epsilon^2)$ denote post-Newtonian (PN) and post-post-Newtonian (2PN) corrections. Gravitational radiation reaction first appears at $O(\epsilon^{5/2})$ beyond Newtonian gravitation, or at 5/2PN order. We call this "Newtonian" radiation reaction. In this paper we obtain the terms at 7/2PN order, $O(\epsilon^{7/2})$, or "post-Newtonian" radiation reaction.

The quantity relevant for gravitational-wave detectors is the gravitational wave form, the transverse-traceless (TT) part of the far-zone field, denoted h^{ij}. In terms of an expansion beyond the quadrupole formula, it has the schematic form

$$h^{ij} \approx \frac{2G}{Rc^4}\Big\{Q^{ij}[1 + O(\epsilon^{1/2}) + O(\epsilon) + O(\epsilon^{3/2}) + \cdots]\Big\}_{\mathrm{TT}} , \quad (1.2)$$

where Q^{ij} represents the usual quadrupole term (two time derivatives of the mass quadrupole moment tensor), R is the distance between source and detector, and TT denotes the transverse-traceless part. Finally, the fluxes of energy and angular momentum, which can be calculated from h^{ij}, are intimately related to the gravitational radiation reaction terms in the equations of motion. Schematically, they can be written

$$dE/dt \approx (dE/dt)_Q[1 + O(\epsilon) + \cdots] , \quad (1.3a)$$
$$d\mathbf{J}/dt \approx (d\mathbf{J}/dt)_Q[1 + O(\epsilon) + \cdots] . \quad (1.3b)$$

The method for deriving the post-Newtonian radiation reaction terms in Eq. (1.1) proceeds as follows. We write down a general form for the Newtonian ($\epsilon^{5/2}$) and post-Newtonian ($\epsilon^{7/2}$) radiation-reaction terms in the equations of motion for two bodies, ignoring tidal and spin effects [32]. For the relative acceleration $\mathbf{a} \equiv \mathbf{a}_1 - \mathbf{a}_2$, this has the provisional form ($G = c = 1$)

$$\mathbf{a} = -\frac{8}{5}\eta(m/r^2)(m/r)\big[-(A_{5/2} + A_{7/2})\dot{r}\mathbf{n} + (B_{5/2} + B_{7/2})\mathbf{v}\big] , \quad (1.4)$$

where $\mu \equiv m_1 m_2/m$ is the reduced mass, with $\eta = \mu/m$ and $\mathbf{n} = \mathbf{x}/r$. The form of Eq. (1.4) is dictated by the fact that it must be a correction to the Newtonian acceleration (i.e., be proportional to m/r^2), must vanish in the test body limit when gravitational radiation vanishes (i.e., be proportional to η), must be related to the emission of gravitational radiation or be nonlinear in Newton's constant G (i.e., contain another factor m/r), and must be dissipative, or odd in velocities (i.e., contain the factors $\dot{r}\mathbf{n}$ and $\mathbf{v}$ linearly). For spinless, structureless bodies, the acceleration must lie in the orbital plane (i.e., depend only on the vectors $\mathbf{n}$ and $\mathbf{v}$). The prefactor 8/5 is chosen for convenience. Then to make the leading term of $O(\epsilon^{5/2})$ beyond Newtonian order, $A_{5/2}$ and $B_{5/2}$ must be of $O(\epsilon)$. The only variables in the problem of this order are v^2, m/r, and $\dot{r}^2$. Thus $A_{5/2}$ and $B_{5/2}$ each can consist at most of a linear combination of these three terms; to those terms we assign six "Newtonian" parameters. By the same reasoning, $A_{7/2}$ and $B_{7/2}$ must be of $O(\epsilon^2)$, hence must each be a linear combination of the six terms v^4, $v^2 m/r$, $v^2\dot{r}^2$, $\dot{r}^2 m/r$, $\dot{r}^4$, and $(m/r)^2$. To these we assign 12 PN parameters. We ignore terms in the equations of motion of $O(\epsilon^3)$ beyond Newtonian order, because they are nondissipative. [There is a clean split between integer-order nondissipative and odd-half-integer-order dissipative terms in this procedure, at least through $O(\epsilon^{7/2})$. At $O(\epsilon^4)$ and beyond, the split is no longer clean, because of the appearance of "tail" contributions [33].] Our goal is to evaluate these 18 parameters.

A. Energy and angular momentum balance

In the "balance" approach, we take 2PN expressions for orbital energy and angular momentum (per unit reduced mass), $\tilde{E} \equiv E/\mu = \frac{1}{2}v^2 - m/r + O(\epsilon^2) + O(\epsilon^3)$, $\tilde{\mathbf{J}} = \mathbf{x} \times \mathbf{v}[1 + O(\epsilon) + O(\epsilon^2)]$, and calculate $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ using 2PN two-body equations of motion [34–36] supplemented by the radiation-reaction terms of Eq. (1.4). Through 2PN order, $\tilde{E}$ and $\tilde{\mathbf{J}}$ are constant, and correspond to asymptotically measured quantities, but the radiation reaction terms lead to nonvanishing expressions for $d\tilde{E}/dt$ and $d\tilde{\mathbf{J}}/dt$ containing the 18 undetermined parameters. However, at orders of approximation beyond those at which they are strictly conserved (and thus well defined), $\tilde{E}$ and $\tilde{\mathbf{J}}$ can be ambiguous. Consequently, we have the freedom to add to $\tilde{E}$ and $\tilde{\mathbf{J}}$ arbitrary terms of order $\epsilon^{5/2}$ and $\epsilon^{7/2}$ beyond the Newtonian expressions without affecting their conservation at 2PN order. There are six such terms of the appropriate general form at $O(\epsilon^{5/2})$ in $\tilde{E}$ and $\tilde{\mathbf{J}}$ and 12 at $O(\epsilon^{7/2})$, resulting in six additional Newtonian parameters and 12 additional PN parameters.

We now equate time derivatives of the resulting generalized energy and angular momentum expressions to the negatives of the corresponding far-zone flux formulas, calculated to PN order [11,37,38], and compare them term by term. The result of the comparison is 12 constraints on the 12 Newtonian parameters and 20 constraints on the 24 PN parameters. Of the 12 constraints at Newtonian order, two are not linearly independent, resulting in ten constraints on the 12 parameters. Solving these constraints results in the form

$$A_{5/2} = 3(1+\beta)v^2 + \frac{1}{3}(23 + 6\alpha - 9\beta)m/r - 5\beta\dot{r}^2 \,, \tag{1.5a}$$

$$B_{5/2} = (2+\alpha)v^2 + (2-\alpha)m/r - 3(1+\alpha)\dot{r}^2 \,, \tag{1.5b}$$

where α and β represent the remaining two unconstrained degrees of freedom. The choice $\alpha = -1$, $\beta = 0$ leads to the Damour-Deruelle two-body radiation-reaction formula [39] used in [36]; the choice $\alpha = 4$, $\beta = 5$ leads to the form obtained from the "Burke-Thorne" radiation-reaction potential

$$V_{\text{Burke-Thorne}} \approx -\frac{1}{5}x^i x^j \frac{d^5}{dt^5}M_{ij} \,, \tag{1.6}$$

where M_{ij} is the trace-free moment-of-inertia tensor of the system [40]. In fact, it is straightforward to show that the arbitrariness represented by α and β is a consequence of the freedom to make coordinate transformations whose resultant effect on the two-body separation vector is $\mathbf{x} \to \mathbf{x} + \frac{8}{15}\eta(m/r)^2[\beta\dot{r}\mathbf{x} + (2\beta - 3\alpha)r\mathbf{v}]$. The two degrees of freedom correspond to the possible functional forms of such transformations at $O(\epsilon^{5/2})$. Thus they represent the residue of a gauge freedom that has not been fixed by the energy balance method, but that has no physical consequences.

At PN order, of the 20 constraints, two are again linearly dependent on the others, resulting in 18 constraints on 24 parameters. The remaining six degrees of freedom can also be shown to correspond to residual gauge freedom at PN order. The explicit results are given in Sec. II, Eq. (2.18).

B. Near-zone radiation-reaction potentials

The other method for deriving PN radiation-reaction terms in the binary system equations of motion makes use of general "near-zone" reaction potentials valid to 7/2PN order, recently derived using a "post-Minkowskian" expansion method and a matching procedure that connects fields in the far radiative zone to fields in the local zone where the sources reside [28,31]. In this formalism, the local metric, appropriate for deriving equations of motion, can be expressed in terms of linear combinations of time-symmetric, Newtonian-like potentials and time-asymmetric, reactive potentials that depend on time derivatives of source multipole moments. To the order needed, the metric has the form

$$g_{00} = -1 + 2(V + V_{\text{react}}) - 2(V + V_{\text{react}})^2 \,, \tag{1.7a}$$

$$g_{0i} = -4(V^i + V^i_{\text{react}}) \,, \tag{1.7b}$$

$$g_{ij} = \delta_{ij}[1 + 2(V + V_{\text{react}})] \,, \tag{1.7c}$$

where $V = O(\epsilon)$ is a Newtonian-like potential (which includes PN corrections) generated by mass densities, $V^i = O(\epsilon^{3/2})$ is a potential generated by mass currents, and $V_{\text{react}} = O(\epsilon^{7/2})$ and $V^i_{\text{react}} = O(\epsilon^4)$ are "time-odd" potentials (see explicit definitions in Sec. III A). The "time-symmetric," nondissipative part of the metric is valid through post-Newtonian order (2PN effects are ignored). At lowest order, V_{react} is given by the Burke-Thorne potential, Eq. (1.6).

We carefully evaluate both the Newtonian-like potentials and the reactive potentials, including their PN corrections, using PN equations of motion where needed to evaluate the effects of time derivatives, keeping all terms that will give contributions to the equations of motion at $O(\epsilon^{5/2})$ and $O(\epsilon^{7/2})$. From the metric, we determine the connection coefficients, apply the geodesic equation for each body, and then reduce the problem to a relative acceleration $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$. Comparing the resulting radiation reaction terms with Eq. (1.4), we obtain

$$A_{5/2} = 18v^2 + \frac{2}{3}m/r - 25\dot{r}^2 \,, \tag{1.8a}$$

$$B_{5/2} = 6v^2 - 2m/r - 15\dot{r}^2 \,, \tag{1.8b}$$

$$\begin{aligned} A_{7/2} = {} & \left(\frac{87}{14} - 48\eta\right)v^4 - \left(\frac{5379}{28} + \frac{136}{3}\eta\right)v^2\frac{m}{r} \\ & + \frac{25}{2}(1+5\eta)v^2\dot{r}^2 + \left(\frac{1353}{4} + 133\eta\right)\dot{r}^2\frac{m}{r} \\ & - \frac{35}{2}(1-\eta)\dot{r}^4 + \left(\frac{160}{7} + \frac{55}{3}\eta\right)\left(\frac{m}{r}\right)^2 \,, \end{aligned} \tag{1.8c}$$

$$\begin{aligned} B_{7/2} = {} & -\frac{27}{14}v^4 - \left(\frac{4861}{84} + \frac{58}{3}\eta\right)v^2\frac{m}{r} \\ & + \frac{3}{2}(13 - 37\eta)v^2\dot{r}^2 \\ & + \left(\frac{2591}{12} + 97\eta\right)\dot{r}^2\frac{m}{r} - \frac{25}{2}(1 - 7\eta)\dot{r}^4 \\ & + \frac{1}{3}\left(\frac{776}{7} + 55\eta\right)\left(\frac{m}{r}\right)^2 \,. \end{aligned} \tag{1.8d}$$

The coefficients in $A_{5/2}$ and $B_{5/2}$ correspond, as expected, to the "Burke-Thorne" choice of parameters in

Eq. (1.5), $\alpha = 4$ and $\beta = 5$. It can also be shown that the 12 coefficents in $A_{7/2}$ and $B_{7/2}$ correspond to a unique and consistent choice of the six arbitrary parameters of the energy balance method that were left unspecified.

The remainder of this paper provides the details underlying these conclusions. In Sec. II, we obtain the radiation-reaction terms using the balance method. Section III applies the near-zone formalism to derive two-body equations of motion with the post-Newtonian reaction terms. Section IV makes concluding remarks.

II. GRAVITATIONAL RADIATION REACTION VIA ENERGY AND ANGULAR MOMENTUM BALANCE

A. Post-Newtonian two-body equations of motion

We begin by displaying the basic equations for two-body systems in the post-Newtonian approximation to general relativity. We restrict attention to two-body systems containing objects that are sufficiently small that finite-size effects, such as spin-orbit, spin-spin, or tidal interactions can be ignored [32]. We write the two-body equation of motion explicitly in the form

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}^{(1)}_{\rm PN} + \mathbf{a}^{(2)}_{\rm 2PN} + \mathbf{a}^{(5/2)}_{\rm RR} + \mathbf{a}^{(3)}_{\rm 3PN} + \mathbf{a}^{(7/2)}_{\rm RR} + O(\epsilon^4)\,, \tag{2.1}$$

where the subscripts denote the nature of the term, post-Newtonian (PN), post-post-Newtonian (2PN), radiation reaction (RR), and so on; and the superscripts denote the order in ϵ. Through 2PN order, the individual terms are given by [34–36]

$$\mathbf{a}_N = -\frac{m}{r^2}\hat{\mathbf{n}}\,, \tag{2.2a}$$

$$\mathbf{a}^{(1)}_{\rm PN} = -\frac{m}{r^2}\left\{\hat{\mathbf{n}}\left[-2(2+\eta)\frac{m}{r} + (1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2\right] - 2(2-\eta)\dot{r}\mathbf{v}\right\}\,, \tag{2.2b}$$

$$\begin{aligned}\mathbf{a}^{(2)}_{\rm 2PN} = -\frac{m}{r^2}\Bigg\{\hat{\mathbf{n}}\Bigg[&\frac{3}{4}(12+29\eta)\left(\frac{m}{r}\right)^2 + \eta(3-4\eta)v^4 + \frac{15}{8}\eta(1-3\eta)\dot{r}^4 \\ &-\frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2 - \frac{1}{2}\eta(13-4\eta)\frac{m}{r}v^2 - (2+25\eta+2\eta^2)\frac{m}{r}\dot{r}^2\Bigg] \\ &-\frac{1}{2}\dot{r}\mathbf{v}\left[\eta(15+4\eta)v^2 - (4+41\eta+8\eta^2)\frac{m}{r} - 3\eta(3+2\eta)\dot{r}^2\right]\Bigg\}\,.\end{aligned} \tag{2.2c}$$

Through 2PN order, the motion is conservative, that is, can be characterized by conserved total energy and angular momentum. At 5/2PN order, the first dissipative terms arise resulting from gravitational radiation reaction. At this order and beyond, conserved energy and angular momentum can no longer be defined. The 3PN terms in $\mathbf{a}$ are formally conservative, while the 7/2PN terms represent the post-Newtonian corrections to radiation reaction that are the goal of this work. Through 2PN order, the conserved energy and angular momentum are given by [34,35,41]

$$E = E_N + E_{\rm PN} + E_{\rm 2PN}\,, \tag{2.3a}$$

$$\mathbf{J} = \mathbf{J}_N + \mathbf{J}_{\rm PN} + \mathbf{J}_{\rm 2PN}\,, \tag{2.3b}$$

where

$$E_N = \mu\left(\frac{1}{2}v^2 - \frac{m}{r}\right)\,, \tag{2.4a}$$

$$E_{\rm PN} = \mu\left\{\frac{3}{8}(1-3\eta)v^4 + \frac{1}{2}(3+\eta)v^2\frac{m}{r} + \frac{1}{2}\eta\frac{m}{r}\dot{r}^2 + \frac{1}{2}\left(\frac{m}{r}\right)^2\right\}\,, \tag{2.4b}$$

$$\begin{aligned}E_{\rm 2PN} = \mu\Bigg\{&\frac{5}{16}(1-7\eta+13\eta^2)v^6 + \frac{1}{8}(21-23\eta-27\eta^2)\frac{m}{r}v^4 \\ &+\frac{1}{4}\eta(1-15\eta)\frac{m}{r}v^2\dot{r}^2 - \frac{3}{8}\eta(1-3\eta)\frac{m}{r}\dot{r}^4 - \frac{1}{4}(2+15\eta)\left(\frac{m}{r}\right)^3 \\ &+\frac{1}{8}(14-55\eta+4\eta^2)\left(\frac{m}{r}\right)^2 v^2 + \frac{1}{8}(4+69\eta+12\eta^2)\left(\frac{m}{r}\right)^2\dot{r}^2\Bigg\}\,,\end{aligned} \tag{2.4c}$$

$$\mathbf{J}_N = \mathbf{L}_N\,, \tag{2.4d}$$

$$\mathbf{J}_{\rm PN} = \mathbf{L}_N\left\{\frac{1}{2}v^2(1-3\eta) + (3+\eta)\frac{m}{r}\right\}\,, \tag{2.4e}$$

$$\mathbf{J}_{\rm 2PN} = \mathbf{L}_N\left\{\frac{1}{2}(7-10\eta-9\eta^2)\frac{m}{r}v^2 - \frac{1}{2}\eta(2+5\eta)\frac{m}{r}\dot{r}^2 + \frac{1}{4}(14-41\eta+4\eta^2)\left(\frac{m}{r}\right)^2 + \frac{3}{8}(1-7\eta+13\eta^2)v^4\right\}\,, \tag{2.4f}$$

where $\mathbf{L}_N \equiv \mu\mathbf{x}\times\mathbf{v}$. These, together with the equations of motion, can be derived from a generalized Lagrangian and Euler-Lagrange equations [34,35]. Alternatively, E and $\mathbf{J}$ can be derived directly from the equations of motion (2.1) and (2.2) by constructing $\frac{1}{2}dv^2/dt \equiv \mathbf{v}\cdot\mathbf{a}$ and $d(\mathbf{x}\times\mathbf{v})/dt \equiv \mathbf{x}\times\mathbf{a}$ and showing that, after substitution for $\mathbf{a}$, the right-hand sides can also be expressed explicitly as total time derivatives.

Because we will be deriving post-Newtonian corrections to radiation reaction, it will be necessary to use the post-Newtonian equations consistently in lowest order or "Newtonian" expressions. However, we will only need to make use of the first post-Newtonian correction, Eq. (2.2b).

B. Radiation reaction to 5/2PN order

We begin by obtaining the radiation reaction terms in Eq. (2.1) to 5/2PN or Newtonian order. We propose the general form of the 5/2PN coefficients in Eq. (1.4):

$$A_{5/2} = a_1 v^2 + a_2 m/r + a_3 \dot{r}^2 \,, \tag{2.5a}$$

$$B_{5/2} = b_1 v^2 + b_2 m/r + b_3 \dot{r}^2 \,. \tag{2.5b}$$

Because the equation of motion now has dissipative terms, and a Lagrangian can no longer be defined to the necessary order, the energy and angular momentum are no longer conserved explicitly. Furthermore, they are ambiguous, because one can now add arbitrary terms to $\tilde{E}$ and $\tilde{\mathbf{J}}$ at 5/2PN order that do not affect their conservation through 2PN order. One way to see the irrelevance of such terms is to consider the special case of a bound two-body orbit. At lowest order, $\tilde{E} \approx (v^2/2 - m/r)$ decreases from one orbit to the next because of the effects of damping on v and r at 5/2PN order. However, in any of the arbitrary 5/2PN terms added to the energy, only the Newtonian behavior of v and r is needed to the order considered. But since that behavior is periodic, there will be no secular change in that added 5/2PN term from one orbit to the next, to lowest order.

In keeping with our philosophy for postulating terms in $\mathbf{a}$, we propose adding 5/2PN terms to $\tilde{E}$ and $\tilde{\mathbf{J}}$ that are proportional to η (vanish in the test-body limit when radiation vanishes), m/r (nonlinear effects), and $\dot{r}$ (odd order). We also require that $\tilde{\mathbf{J}}$ remain a pseudovector. There are three possible terms each for $\tilde{E}$ and $\tilde{\mathbf{J}}$; we add these with an arbitrary coefficient per term. We write $\tilde{E}^*$ and $\tilde{\mathbf{J}}^*$ to distinguish the ambiguous, 5/2PN quantities from the conserved 2PN quantities, and write

$$\begin{aligned}\tilde{E}^* &\equiv \tilde{E}_N + \tilde{E}_{5/2}\\ &= \tilde{E}_N - \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\dot{r}\left(\gamma v^2 + \delta\frac{m}{r} + \beta\dot{r}^2\right)\,,\end{aligned} \tag{2.6a}$$

$$\begin{aligned}\tilde{\mathbf{J}}^* &\equiv \tilde{\mathbf{L}}_N + \tilde{\mathbf{J}}_{5/2}\\ &= \tilde{\mathbf{L}}_N + \frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r}\dot{r}\left(\kappa v^2 + \alpha\frac{m}{r} + \varepsilon\dot{r}^2\right)\,,\end{aligned} \tag{2.6b}$$

where $\tilde{\mathbf{L}}_N = \mathbf{x}\times\mathbf{v}$, and where α, β, γ, δ, ε, and κ are arbitrary [42]. The signs and the 8/5 factor are chosen for convenience [43].

We now calculate $d\tilde{E}^*/dt$ and $d\tilde{\mathbf{J}}^*/dt$, where in the lowest-order, Newtonian expressions, E_N and $\tilde{\mathbf{L}}_N$, we substitute the Newtonian and the provisional 5/2PN terms in the equation of motion, while in $\tilde{E}_{5/2}$ and $\tilde{\mathbf{J}}_{5/2}$, we substitute only the Newtonian equations of motion. Schematically, the result is

$$\frac{d\tilde{E}^*}{dt} = \mathbf{v}\cdot\mathbf{a}_{\rm RR}^{(5/2)} + \frac{d\tilde{E}_{5/2}}{dt}\,, \tag{2.7a}$$

$$\frac{d\tilde{\mathbf{J}}^*}{dt} = \mathbf{x}\times\mathbf{a}_{\rm RR}^{(5/2)} + \frac{d\tilde{\mathbf{J}}_{5/2}}{dt}\,. \tag{2.7b}$$

In evaluating the derivatives of the added 5/2PN terms, we use the identity, derived from the Newtonian equations of motion:

$$\begin{aligned}\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}\right) = \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\Big(&pv^4 - pv^2\frac{m}{r}\\ &-(p+q)v^2\dot{r}^2 - 2s\frac{m}{r}\dot{r}^2\Big)\,.\end{aligned} \tag{2.8}$$

The result is

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} = -\frac{8}{5}\eta\frac{m^2}{r^3}\Big[&(b_2-\gamma+\delta)v^2\frac{m}{r} - (a_2+2\gamma+3\beta+4\delta)\dot{r}^2\frac{m}{r} + (b_3-a_1-3\gamma+3\beta)v^2\dot{r}^2\\ &+(b_1+\gamma)v^4 - (a_3+5\beta)\dot{r}^4 - \delta\left(\frac{m}{r}\right)^2\Big]\,,\end{aligned} \tag{2.9a}$$

$$\frac{d\tilde{\mathbf{J}}^*}{dt} = -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\left[(b_1-\alpha+\kappa)v^2\frac{m}{r} + (b_3+3\alpha+3\varepsilon+2\kappa)\dot{r}^2\frac{m}{r} - (3\varepsilon-2\kappa)v^2\dot{r}^2 - \kappa v^4 + 4\varepsilon\dot{r}^4 + (b_2+\alpha)\left(\frac{m}{r}\right)^2\right]. \tag{2.9b}$$

We now use the assumption of energy and angular momentum balance to equate the rate of orbital loss in these quantities to the corresponding far-zone fluxes, evaluated to the lowest, Newtonian order [44]: namely,

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} &= -\left(\frac{d\tilde{E}}{dt}\right)_{\text{far zone}}\\ &= -\frac{8}{5}\eta\frac{m^2}{r^3}\frac{m}{r}\left(4v^2 - \frac{11}{3}\dot{r}^2\right)\,,\end{aligned} \tag{2.10a}$$

$$\begin{aligned}\frac{d\tilde{\mathbf{J}}^*}{dt} &= -\left(\frac{d\tilde{\mathbf{J}}}{dt}\right)_{\text{far zone}}\\ &= -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\frac{m}{r}\left(2v^2 + 2\frac{m}{r} - 3\dot{r}^2\right)\,.\end{aligned} \tag{2.10b}$$

Comparing Eqs. (2.9) and (2.10) term by term, we see that there are 12 constraints on the 12 parameters, for example, $\delta = \varepsilon = \kappa = 0$. However, we note that two of these constraints are redundant: $\varepsilon = \kappa = 0$ makes

$3\varepsilon - 2\kappa = 0$ redundant, and the constraint $b_1 + b_2 + \delta = 4$ from the energy and $b_1 + b_2 + \kappa = 4$ from the angular momentum are equivalent. As a consequence, the system is underdetermined; the solution for the parameters will have two arbitrary degrees of freedom. A convenient form of the solution is

$$a_1 = 3 + 3\beta\,, \quad a_2 = 23/3 + 2\alpha - 3\beta\,, \quad a_3 = -5\beta\,, \tag{2.11a}$$

$$b_1 = 2 + \alpha\,, \quad b_2 = 2 - \alpha\,, \quad b_3 = -3 - 3\alpha\,. \tag{2.11b}$$

The final constraint is $\gamma = -(\alpha + 2)$. Thus, the 5/2PN, radiation-reaction terms in the equation of motion, required to balance energy and angular momentum fluxes are given by Eqs. (1.4) and (1.5). We shall discuss the significance of the freedom represented by the parameters α and β in Sec. II D.

C. Radiation reaction to 7/2PN order

We henceforth adopt the 5/2PN constraints established above and the forms of $A_{5/2}$ and $B_{5/2}$ given in Eq. (1.5). For the 7/2PN terms, we write

$$A_{7/2} = c_1 v^4 + c_2 v^2 \frac{m}{r} + c_3 v^2 \dot{r}^2 + c_4 \dot{r}^2 \frac{m}{r} + c_5 \dot{r}^4 + c_6 \left(\frac{m}{r}\right)^2 , \tag{2.12a}$$

$$B_{7/2} = d_1 v^4 + d_2 v^2 \frac{m}{r} + d_3 v^2 \dot{r}^2 + d_4 \dot{r}^2 \frac{m}{r} + d_5 \dot{r}^4 + d_6 \left(\frac{m}{r}\right)^2 . \tag{2.12b}$$

We now take PN expressions for $\tilde{E}$ and $\tilde{\mathbf{J}}$, Eqs. (2.4a), (2.4b), (2.4d), and (2.4e), and add the 5/2PN terms with coefficients already determined, Eqs. (2.6), as well as arbitrary 7/2PN terms. This last step introduces 12 additional parameters. The result is

$$\begin{aligned}\tilde{E}^* &\equiv \tilde{E}_N + \tilde{E}_{\rm PN} + \tilde{E}_{5/2} + \tilde{E}_{7/2} \\ &= \tilde{E}_N + \tilde{E}_{\rm PN} + \frac{8}{5}\eta \left(\frac{m}{r}\right)^2 \dot{r}[(2+\alpha)v^2 - \beta \dot{r}^2] \\ &\quad -\frac{8}{5}\eta \left(\frac{m}{r}\right)^2 \dot{r} \left[\delta_1 v^4 + \delta_2 v^2 \dot{r}^2 + \delta_3 v^2 \frac{m}{r} + \delta_4 \dot{r}^4 \right.\\ &\quad \left. + \delta_5 \dot{r}^2 \frac{m}{r} + \delta_6 \left(\frac{m}{r}\right)^2 \right] ,\end{aligned} \tag{2.13a}$$

$$\begin{aligned}\tilde{\mathbf{J}}^* &\equiv \tilde{\mathbf{J}}_N + \tilde{\mathbf{J}}_{\rm PN} + \tilde{\mathbf{J}}_{5/2} + \tilde{\mathbf{J}}_{7/2} \\ &= \tilde{\mathbf{J}}_N + \tilde{\mathbf{J}}_{\rm PN} + \frac{8}{5}\eta \tilde{\mathbf{L}}_N \frac{m}{r}\dot{r}\left(\alpha \frac{m}{r}\right) \\ &\quad -\frac{8}{5}\eta \tilde{\mathbf{L}}_N \frac{m}{r} \dot{r} \left[\varepsilon_1 v^4 + \varepsilon_2 v^2 \dot{r}^2 + \varepsilon_3 v^2 \frac{m}{r} + \varepsilon_4 \dot{r}^4 \right.\\ &\quad \left. + \varepsilon_5 \dot{r}^2 \frac{m}{r} + \varepsilon_6 \left(\frac{m}{r}\right)^2 \right] .\end{aligned} \tag{2.13b}$$

We calculate $d\tilde{E}^*/dt$ and $d\tilde{\mathbf{J}}^*/dt$, substituting PN equations of motion in all lowest-order expressions. Schematically, the result is

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} &= \mathbf{v}\cdot(\mathbf{a}_{\rm RR}^{(5/2)} + \mathbf{a}_{\rm RR}^{(7/2)}) + \mathbf{v}\cdot \mathbf{a}_{\rm RR}^{(5/2)} \\ &\quad \times \left(\frac{3}{2}(1-3\eta)v^2 + (3+\eta)\frac{m}{r}\right) \\ &\quad + \eta \frac{m}{r}\dot{r}\mathbf{n}\cdot \mathbf{a}_{\rm RR}^{(5/2)} + \frac{d\tilde{E}_{5/2}}{dt} + \frac{d\tilde{E}_{7/2}}{dt} ,\end{aligned} \tag{2.14a}$$

$$\begin{aligned}\frac{d\tilde{\mathbf{J}}^*}{dt} &= \mathbf{x}\times(\mathbf{a}_{\rm RR}^{(5/2)} + \mathbf{a}_{\rm RR}^{(7/2)}) + \mathbf{x}\times \mathbf{a}_{\rm RR}^{(5/2)} \\ &\quad \times \left(\frac{1}{2}(1-3\eta)v^2 + (3+\eta)\frac{m}{r}\right) \\ &\quad + (\mathbf{x}\times\mathbf{v})\mathbf{v}\cdot \mathbf{a}_{\rm RR}^{(5/2)}(1-3\eta) + \frac{d\tilde{\mathbf{J}}_{5/2}}{dt} + \frac{d\tilde{\mathbf{J}}_{7/2}}{dt} .\end{aligned} \tag{2.14b}$$

In evaluating the derivatives of the added 5/2PN terms to 7/2PN order, we must use the identity, derived from the PN equations of motion:

$$\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}\right) = \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\left(pv^4 - pv^2\frac{m}{r} - (p+q)v^2\dot{r}^2 - 2s\frac{m}{r}\dot{r}^2 - \frac{m}{r}(pv^2 + 2s\dot{r}^2)A + \frac{m}{r}v^2\dot{r}^2(p+2s)(4-2\eta)\right) , \tag{2.15}$$

where $A = (1+3\eta)v^2 - 2(2+\eta)m/r - \frac{3}{2}\eta\dot{r}^2$. The result is

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} = -\frac{8}{5}\eta\frac{m^2}{r^3}\Bigg[&\left(4v^2\frac{m}{r} - \frac{11}{3}\frac{m}{r}\right) + \mathcal{R}_1 v^6 + \mathcal{R}_2 v^4 \frac{m}{r} + \mathcal{R}_3 v^4 \dot{r}^2 + \mathcal{R}_4 v^2 \dot{r}^4 + \mathcal{R}_5 v^2 \dot{r}^2 \frac{m}{r} \\ &+ \mathcal{R}_6 v^2 \left(\frac{m}{r}\right)^2 + \mathcal{R}_7 \dot{r}^4 \frac{m}{r} + \mathcal{R}_8 \dot{r}^2 \left(\frac{m}{r}\right)^2 + \mathcal{R}_9 \left(\frac{m}{r}\right)^3 + \mathcal{R}_{10}\dot{r}^6\Bigg] ,\end{aligned} \tag{2.16a}$$

$$\begin{aligned}\frac{d\tilde{\mathbf{J}}^*}{dt} = -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\Bigg\{&\left[2v^2\frac{m}{r} + 2\left(\frac{m}{r}\right)^2 - 3\dot{r}^2\frac{m}{r}\right] + \mathcal{S}_1 v^6 + \mathcal{S}_2 v^4 \frac{m}{r} + \mathcal{S}_3 v^4 \dot{r}^2 + \mathcal{S}_4 v^2 \dot{r}^4 \\ &+ \mathcal{S}_5 v^2 \dot{r}^2 \frac{m}{r} + \mathcal{S}_6 v^2 \left(\frac{m}{r}\right)^2 + \mathcal{S}_7 \dot{r}^4 \frac{m}{r} + \mathcal{S}_8 \dot{r}^2 \left(\frac{m}{r}\right)^2 + \mathcal{S}_9 \left(\frac{m}{r}\right)^3 + \mathcal{S}_{10}\dot{r}^6\Bigg\} ,\end{aligned} \tag{2.16b}$$

where $\mathcal{R}_i$ and $\mathcal{S}_i$ consist of combinations of the parameters c_i and d_i from $A_{7/2}$ and $B_{7/2}$, α and β combined with functions of η from PN corrections of 5/2PN terms, and the parameters δ_i and ε_i from $E_{7/2}$ and $\mathbf{J}_{7/2}$. Notice that by choosing the 5/2PN terms in the form determined in the previous subsection, with arbitrary parameters α and β, we automatically reproduce 5/2PN energy and angular momentum losses matching the corresponding far-zone fluxes.

We equate these expressions to PN formulas for the far-zone fluxes [11,37,38]:

$$-\left(\frac{d\tilde{E}}{dt}\right)_{\text{far zone}} = -\frac{8}{5}\eta\frac{m^2}{r^3}\frac{m}{r}\left[\left(4v^2-\frac{11}{3}\dot{r}^2\right)+\frac{785-852\eta}{84}v^4-\frac{40(17-\eta)}{21}v^2\frac{m}{r}-\frac{1487-1392\eta}{42}v^2\dot{r}^2\right.$$
$$\left.+\frac{2(367-15\eta)}{21}\dot{r}^2\frac{m}{r}+\frac{687-620\eta}{28}\dot{r}^4+\frac{4(1-4\eta)}{21}\left(\frac{m}{r}\right)^2\right], \tag{2.17a}$$

$$-\left(\frac{d\tilde{\mathbf{J}}}{dt}\right)_{\text{far zone}} = -\frac{8}{5}\eta\tilde{\mathbf{L}}_N\frac{m}{r^2}\frac{m}{r}\left[\left(2v^2+2\frac{m}{r}-3\dot{r}^2\right)+\frac{307-548\eta}{84}v^4-\frac{58+95\eta}{21}v^2\frac{m}{r}-\frac{74-277\eta}{14}v^2\dot{r}^2\right.$$
$$\left.+\frac{372+197\eta}{42}\dot{r}^2\frac{m}{r}+\frac{5(19-72\eta)}{28}\dot{r}^4-\frac{745-2\eta}{42}\left(\frac{m}{r}\right)^2\right]. \tag{2.17b}$$

This results in 20 constraints on the 20 coefficients $\mathcal{R}_i$ and $\mathcal{S}_i$, involving the 24 unknown parameters c_i, d_i, δ_i, and ε_i. However, two of these 20 constraint equations are linear combinations of others, hence the solution is underdetermined by six degrees of freedom. The most convenient parameters to choose to express this freedom are δ_1, δ_2, δ_3, δ_4, δ_5, and ε_5 [45]. In terms of these six parameters, together with the free parameters α and β from the 5/2PN solution, we solve for the c_i and d_i coefficients, with the result

$$c_1 = \frac{1}{28}(117+132\eta)-\frac{3}{2}\beta(1-3\eta)+3\delta_2-3\varepsilon_5\,, \tag{2.18a}$$

$$c_2 = -\frac{1}{42}(297-310\eta)-3\alpha(1-4\eta)-\frac{3}{2}\beta(7+13\eta)$$
$$-2\delta_1-3\delta_2+3\delta_5+3\varepsilon_5\,, \tag{2.18b}$$

$$c_3 = \frac{5}{28}(19-72\eta)+\frac{5}{2}\beta(1-3\eta)-5\delta_2+5\delta_4+5\varepsilon_5\,, \tag{2.18c}$$

$$c_4 = -\frac{1}{28}(687-368\eta)-6\alpha\eta+\frac{1}{2}\beta(54+17\eta)$$
$$-2\delta_2-5\delta_4-6\delta_5\,, \tag{2.18d}$$

$$c_5 = -7\delta_4\,, \tag{2.18e}$$

$$c_6 = -\frac{1}{21}(1533+498\eta)-\alpha(14+9\eta)+3\beta(7+4\eta)$$
$$-2\delta_3-3\delta_5\,, \tag{2.18f}$$

$$d_1 = -3(1-3\eta)-\frac{3}{2}\alpha(1-3\eta)-\delta_1\,, \tag{2.18g}$$

$$d_2 = -\frac{1}{84}(139+768\eta)-\frac{1}{2}\alpha(5+17\eta)+\delta_1-\delta_3\,, \tag{2.18h}$$

$$d_3 = \frac{1}{28}(369-624\eta)+\frac{3}{2}(3\alpha+2\beta)(1-3\eta)+3\delta_1-3\varepsilon_5\,, \tag{2.18i}$$

$$d_4 = \frac{1}{42}(295-335\eta)+\frac{1}{2}\alpha(38-11\eta)-3\beta(1-3\eta)$$
$$+2\delta_1+4\delta_3+3\varepsilon_5\,, \tag{2.18j}$$

$$d_5 = \frac{5}{28}(19-72\eta)-5\beta(1-3\eta)+5\varepsilon_5\,, \tag{2.18k}$$

$$d_6 = -\frac{1}{21}(634-66\eta)+\alpha(7+3\eta)+\delta_3\,. \tag{2.18l}$$

Equations (2.12) and (2.18), together with (2.5) and (2.11), substituted into Eq. (1.4), yield the required two-body radiation-reaction terms, necessary to balance energy and angular momentum fluxes to 7/2PN order.

D. Significance of arbitrary parameters in equations of motion

The formulas for energy and angular momentum flux in the far-zone are gauge invariant, i.e., they are independent of changes in the coordinate system that leave the spacetime asymptotically flat. Consequently, our method has a residual gauge freedom. That coordinate freedom will be reflected in a freedom to change the relative orbital variable $\mathbf{x}$. It is important to note that $\mathbf{x}$ is not itself a coordinate, it is the difference between the centers of mass of the two bodies $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ evaluated at a moment of coordinate time t. Nevertheless, changes in spacetime coordinates $\{x^\mu\}$ will induce changes in $\mathbf{x}$. Here we demonstrate that the residual freedom in the radiation-reaction terms corresponds precisely to coordinate-change-induced variations in $\mathbf{x}$. For coordinate changes that can affect the equations of motion at 5/2PN and 7/2PN order, the resulting variations in $\mathbf{x}$ can have a limited range of forms. We assume the general form $\mathbf{x}\to\mathbf{x}+\delta\mathbf{x}$, where $\delta\mathbf{x}$ can depend only on the two vectors $\mathbf{x}$ and $\mathbf{v}$:

$$\delta\mathbf{x} = (f_{5/2}+f_{7/2})\dot{r}\mathbf{x}+(g_{5/2}+g_{7/2})r\mathbf{v}\,. \tag{2.19}$$

In order that $\delta\mathbf{x}/\mathbf{x}$ be $O(\epsilon^{5/2})$ and $O(\epsilon^{7/2})$, $f_{5/2}$ and $g_{5/2}$ must be $O(\epsilon^2)$ and $f_{7/2}$ and $g_{7/2}$ must be $O(\epsilon^3)$, and will depend on combinations of m/r, v^2, and $\dot{r}^2$. Note that the $\mathbf{v}$-dependent term in $\delta\mathbf{x}$ takes into account changes in the coordinate t, via $\mathbf{x}(t+\delta t)\approx\mathbf{x}(t)+\mathbf{v}\delta t$.

We illustrate the solution for the coordinate change at 5/2PN order, and merely sketch the results at 7/2PN order. Subjecting the Newtonian equations of motion to the variable change $\mathbf{x}'=\mathbf{x}+\delta\mathbf{x}$, we find

$$\frac{d^2\mathbf{x}'}{dt^2}+\frac{m\mathbf{x}'}{r'^3} = \frac{d^2\mathbf{x}}{dt^2}+\frac{m\mathbf{x}}{r^3}$$
$$+\left[\frac{d^2\delta\mathbf{x}}{dt^2}+\frac{m}{r^3}(\delta\mathbf{x}-3\mathbf{n}\mathbf{n}\cdot\delta\mathbf{x})\right], \tag{2.20}$$

where $\mathbf{n} = \mathbf{x}/r$. Defining the quantity in square brackets to be Q, and substituting Eq. (2.19) for $\delta\mathbf{x}$, truncated to 5/2PN order, we obtain

$$\begin{aligned} Q = \mathbf{x}[&\dddot{r} f_{5/2} + 2\ddot{r}\dot{f}_{5/2} + \dot{r}\ddot{f}_{5/2} \\ &-(m/r^3)(2\dot{r}g_{5/2} + 2r\dot{g}_{5/2} + 3\dot{r}f_{5/2})] \\ &+\mathbf{v}[2\ddot{r}f_{5/2} + 2\dot{r}\dot{f}_{5/2} + \ddot{r}g_{5/2} + 2\dot{r}\dot{g}_{5/2} + r\ddot{g}_{5/2}]\,, \end{aligned} \tag{2.21}$$

where overdots denote time derivatives. We want Q to cancel the arbitrary α- and β-dependent terms in the radiation-reaction contributions to $d^2\mathbf{x}/dt^2$, i.e., from Eqs. (1.4) and (1.5),

$$\begin{aligned} Q = -\frac{8}{5}\eta\frac{m^2}{r^3}\Big\{&\dot{r}\mathbf{n}\left[\beta\left(3v^2 - 3\frac{m}{r} - 5\dot{r}^2\right) + 2\alpha\frac{m}{r}\right] \\ &-\mathbf{v}\left[\alpha\left(v^2 - \frac{m}{r} - 3\dot{r}^2\right)\right]\Big\}\,. \end{aligned} \tag{2.22}$$

Equating the coefficients of $\mathbf{x}$ and $\mathbf{v}$ in Eqs. (2.21) and (2.22), and using the Newtonian equations of motion, it is straightforward to integrate the two differential equations, to obtain $f_{5/2} = \frac{8}{15}\beta\eta(m/r)^2$ and $g_{5/2} = \frac{8}{15}(2\beta - 3\alpha)\eta(m/r)^2$. Thus the arbitrariness in the 5/2PN radiation-reaction terms is equivalent to the variable change

$$\delta\mathbf{x} = \frac{8}{15}\eta\left(\frac{m}{r}\right)^2[\beta\dot{r}\mathbf{x} + (2\beta - 3\alpha)r\mathbf{v}]\,. \tag{2.23}$$

To determine $f_{7/2}$ and $g_{7/2}$, we now subject the PN equations of motion (2.2a) and (2.2b) to the variable change $\mathbf{x}' = \mathbf{x} + \delta\mathbf{x}$, adopt Eq. (2.19) for $\delta\mathbf{x}$ to 7/2PN order, but using the 5/2PN solution of Eq. (2.23), apply the PN equations of motion to accelerations resulting from time derivatives in terms at 5/2PN order, and demand that the resulting Q cancel the arbitrary 7/2PN terms in the radiation-reaction contributions to $d^2\mathbf{x}/dt^2$ (the 5/2PN terms now cancel automatically). It is again straightforward to integrate the resulting two differential equations to obtain

$$f_{7/2} = \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\left(\frac{1}{3}\left[\delta_2 + \frac{2}{5}\delta_4 - \varepsilon_5 - \frac{1}{2}\beta(1-3\eta)\right]v^2 - \frac{1}{6}\left[\delta_2 + \delta_4 - \frac{3}{2}\delta_5 - \varepsilon_5 - \frac{3}{2}\alpha\eta + \frac{1}{2}\beta(4+11\eta)\right]\frac{m}{r} + \frac{1}{5}\delta_4\dot{r}^2\right), \tag{2.24a}$$

$$\begin{aligned} g_{7/2} = \frac{8}{5}\eta\left(\frac{m}{r}\right)^2\Bigg(&\left[\delta_1 + \frac{2}{3}\delta_2 + \frac{8}{15}\delta_4 + \frac{1}{2}(3\alpha - 2\beta)(1-3\eta)\right]v^2 \\ &-\frac{1}{6}\left[6\delta_1 + 5\delta_2 - 3\delta_3 + 5\delta_4 - \frac{3}{2}\delta_5 + \varepsilon_5 - \frac{63}{2}\alpha\eta - \frac{1}{2}\beta(4-55\eta)\right]\frac{m}{r} + \frac{1}{3}\left[\frac{2}{5}\delta_4 + \varepsilon_5 - \beta(1-3\eta)\right]\dot{r}^2\Bigg)\,. \end{aligned} \tag{2.24b}$$

Note that the six arbitrary parameters $\delta_1, \ldots, \delta_5$ and ε_5 appear linearly independently in the six terms in Eq. (2.24).

At 5/2PN order, the gauge-change-induced $\delta\mathbf{x}$ induces a change in the orbital separation $\delta r/r = \frac{8}{5}\eta(m/r)^2\dot{r}(\beta - \alpha)$, and in the orbital angular frequency $\delta\dot{\phi} = -\frac{8}{5}\eta(m/r)^2\dot{r}\dot{\phi}(2\beta - 3\alpha)$. In a binary coalescence of equal-mass compact objects, for example, this will change the coordinate separation by only two parts in 10^7 at a separation $r = 20m$, and three parts in 10^4 at the innermost stable orbit around $r = 6m$ [41], for values of α and β of order unity. These are negligible compared to the corresponding PN and 2PN corrections to the orbital radius of relative order m/r and $(m/r)^2$, respectively. From the change in angular frequency, the accumulated correction in the orbital phase during the coalescence to a radius r_f is given by $\delta\Phi = \frac{16}{25}\eta(2\beta - 3\alpha)(m/r_f)^{5/2}$, which will amount to only 2×10^{-3} radians for an equal mass system at $r_f = 6m$ [46]. Only effects that contribute phase shifts of order a radian over the observed inspiral signal are expected to be important in estimating parameters of inspiraling compact binaries by matched filtering, so these effects are negligible. Consequently, in using the equations of motion to evolve inspiraling systems, one can choose α, β, δ_i, and ε_5 freely; the error made by using coordinate variables instead of invariant quantities is negligible for systems of interest. It is straightforward to show that, for a quasicircular inspiral, the physically measurable quantity $\dot{\omega}$, where ω is the orbital angular frequency, is unaffected by these arbitrary parameters.

III. NEAR-ZONE GRAVITATIONAL RADIATION REACTION FOR BINARY SYSTEMS: EXPLICIT TWO-BODY EQUATIONS OF MOTION

A. External near-zone metric for systems of compact objects

Using a post-Minkowskian expansion formalism and a procedure for matching radiation-zone and near-zone solutions, together with a careful separation of retarded gravitational potentials into time-symmetric and time-odd pieces, Blanchet [28] has derived expressions for the external near-zone metric of a radiating system that includes radiation-reaction potentials, accurate through 7/2PN order (indeed, it includes reaction contributions of "tails," at 4PN order, but we will not treat these here). Further details can be found in [31]. The metric is given by Eqs. (3.51) – (3.53) of [28], as an expansion in powers of $\epsilon \sim v^2 \sim m/r$ (in the formalism of [28], $\epsilon \sim 1/c^2$), given by

$$g_{00} = -1 + 2\epsilon(V + \epsilon^{5/2}V_{\rm react}) - 2\epsilon^2(V + \epsilon^{5/2}V_{\rm react})^2 + O(\epsilon^3)\,, \tag{3.1a}$$

$$g_{0i} = -4\epsilon^{3/2}(V^i + \epsilon^{5/2}V^i_{\rm react}) + O(\epsilon^{5/2})\,, \tag{3.1b}$$

$$g_{ij} = \delta_{ij}[1 + 2\epsilon(V + \epsilon^{5/2}V_{\rm react})] + O(\epsilon^2)\,. \tag{3.1c}$$

We expand this metric in powers of ϵ, keeping terms of Newtonian and PN order, as well as terms of 5/2PN and 7/2PN order. We do not bother with 2PN terms, because their contributions to the equations of motion are already well known [cf. Eq. (2.2c)] and are nondissipative. However, we keep terms of explicitly Newtonian and PN order because they may contain implicit, time-odd correction terms that generate 5/2PN and 7/2PN effects. These will become apparent below. We keep ϵ explicit for the time being, and display only the needed terms in the metric. These correspond to $O(\epsilon^2)$, $O(\epsilon^{3/2})$, and $O(\epsilon)$ terms in g_{00}, g_{0j}, and g_{ij} respectively, for PN effects, and to $O(\epsilon^{9/2})$, $O(\epsilon^4)$, and $O(\epsilon^{7/2})$ terms in the respective metric components, for 7/2PN effects. The result is

$$g_{00} = -1 + 2\epsilon V - 2\epsilon^2V^2 + 2\epsilon^{7/2}V_{\rm react} - 4\epsilon^{9/2}V_{\rm react}V\,, \tag{3.2a}$$

$$g_{0i} = -4\epsilon^{3/2}V^i - 4\epsilon^4V^i_{\rm react}\,, \tag{3.2b}$$

$$g_{ij} = \delta_{ij}(1 + 2\epsilon V + 2\epsilon^{7/2}V_{\rm react})\,. \tag{3.2c}$$

We have used the fact that $V_{\rm react}/V = O(\epsilon^{5/2}) + O(\epsilon^{7/2})$ and $V^i_{\rm react}/V^i = O(\epsilon^{5/2})$.

The potentials V and V^i are scalar and vector potentials of the mass and current densities, defined to be time symmetric, that is 1/2 retarded plus 1/2 advanced potentials. They are given explicitly by

$$V(\mathbf{x},t) = \frac{1}{2}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}[\sigma(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|) + \sigma(\mathbf{x}', t + |\mathbf{x}-\mathbf{x}'|)]\,, \tag{3.3a}$$

$$V^i(\mathbf{x},t) = \frac{1}{2}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}[\sigma^i(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|) + \sigma^i(\mathbf{x}', t + |\mathbf{x}-\mathbf{x}'|)]\,, \tag{3.3b}$$

where $\sigma = T^{00} + T^{ii}$, $\sigma^i = T^{0i}$, and $T^{\mu\nu}$ is the matter stress-energy tensor (for proof see Sec. IV of [31]; for further details and background see [27]).

The reactive potentials $V_{\rm react}$ and $V^i_{\rm react}$ are given by Eqs. (3.53) of [28]:

$$V_{\rm react} = -\frac{1}{5}x^ix^jM^{(5)}_{ij} + \epsilon\left(\frac{1}{189}x^ix^jx^kM^{(7)}_{ijk} - \frac{1}{70}r^2x^ix^jM^{(7)}_{ij}\right)\,, \tag{3.4a}$$

$$V^i_{\rm react} = \frac{1}{21}\hat{x}^{ijk}M^{(6)}_{jk} - \frac{4}{45}\varepsilon_{ijk}x^jx^lS^{(5)}_{kl}\,, \tag{3.4b}$$

where $M_{ij\ldots}$ and $S_{ij\ldots}$ are symmetric, trace-free moments of the source distribution (to be defined explicitly below), which are functions of retarded time relative to the center of mass. The superscript (n) denotes n derivatives with respect to retarded time; ε^{ijk} is the antisymmetric Levi-Civita symbol, and

$$\hat{x}^{ijk} \equiv x^ix^jx^k - \frac{1}{5}r^2(x^i\delta^{jk} + x^j\delta^{ik} + x^k\delta^{ij})\,. \tag{3.5}$$

Note that the $O(\epsilon)$ term in Eq. (3.4a) is an explicit PN correction to $V_{\rm react}$ which will contribute to the equations of motion at 7/2PN order; we will see that the moment M_{ij} in the leading term has implicit PN corrections of its own.

We now specialize to systems of compact objects. For our purpose, it is suitable to approximate the matter stress-energy tensor by a distribution representing "point" masses:

$$T^{\mu\nu} = \sum_A m_Av^\mu_Av^\nu_A(dt/d\tau)(-g)^{-1/2}\delta^3[\mathbf{x}-\mathbf{x}_A(\tau)]\,, \tag{3.6}$$

where $v^\mu_A = dx^\mu_A/dt$ ($v^0 = 1$), τ is proper time measured along the world line of body A, of mass m_A, and g is the determinant of the metric. The presence of τ and of the metric will result in post-Newtonian corrections in $T^{\mu\nu}$. The corrections of integer order in ϵ will lead to standard nondissipative PN effects; we are only interested in corrections that are potentially of odd half order. We find

$$\sigma \approx \sum_A m_A\delta^3(\mathbf{x}-\mathbf{x}_A)\left[1 + \epsilon\left(\frac{3}{2}v_A^2 - V(\mathbf{x}_A)\right) - \epsilon^{7/2}V_{\rm react}(\mathbf{x}_A) + \cdots\right]\,, \tag{3.7a}$$

$$\sigma^i \approx \sum_A m_Av^i_A\delta^3(\mathbf{x}-\mathbf{x}_A)[1 + \cdots]\,, \tag{3.7b}$$

where the notation $\approx$ here denotes that we keep only terms of potential interest. In addition, to the required order, we must expand the retardation of the potentials V and V^i [Eqs. (3.3)], with the result

$$V \approx \int \frac{\sigma(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' + \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}\int \sigma(\mathbf{x}',t)|\mathbf{x}-\mathbf{x}'|d^3\mathbf{x}' + O(\epsilon^2)\,, \tag{3.8a}$$

$$V^i \approx \int \frac{\sigma^i(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' + O(\epsilon)\,. \tag{3.8b}$$

Substituting Eqs. (3.7) for σ and σ^i, and keeping, apart from the Newtonian potential, only the terms that contribute to radiation reaction terms in the equations of motion, we obtain

$$V \approx \sum_A \frac{m_A}{|\mathbf{x}-\mathbf{x}_A|} - \frac{1}{2}\epsilon\sum_A \frac{m_A\mathbf{a}_A\cdot(\mathbf{x}-\mathbf{x}_A)}{|\mathbf{x}-\mathbf{x}_A|} - \epsilon^{7/2}\sum_A \frac{m_A}{|\mathbf{x}-\mathbf{x}_A|}V_{\rm react}(\mathbf{x}_A)\,, \tag{3.9a}$$

$$V^i \approx \sum_A \frac{m_Av^i_A}{|\mathbf{x}-\mathbf{x}_A|}\,, \tag{3.9b}$$

where $\mathbf{a}_A = d^2\mathbf{x}_A/dt^2$. Note that we keep the term involving $\mathbf{a}_A$ in V because it ultimately will contribute 5/2PN terms, and we keep the velocity-dependent term in V^i because a subsequent time-derivative $\dot{V}^i$ appearing in the equation of motion will also generate an acceleration $\mathbf{a}_A$, with its 5/2PN contribution. However, because these effects appear in PN corrections, they will generate 7/2PN terms in the equations of motion.

B. Equations of motion

We use the geodesic equation, written in terms of coordinate time,

$$d^2x^i/dt^2 + \Gamma^i_{\mu\nu}v^\mu v^\nu - \Gamma^0_{\mu\nu}v^\mu v^\nu v^i = 0\,, \tag{3.10}$$

where $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols computed from the metric. Taking into account that $v^0 = 1$, $v^i \sim O(\epsilon^{1/2})$, and that $d/dt = O(\epsilon^{1/2})\nabla$ (the standard slow-motion assumptions), we obtain

$$\begin{aligned} a^i = d^2x^i/dt^2 = {} & \nabla_i V + \epsilon[(v^2-4V)\nabla_i V + 4\dot V^i - 8\nabla_{[i}V_{j]}v^j - 3\dot V v^i - 4\nabla_j V v^i v^j] \\ & +\epsilon^{5/2}\nabla_i V_{\rm react} + \epsilon^{7/2}[(v^2-4V)\nabla_i V_{\rm react} - 4\nabla_i V V_{\rm react} + 4\dot V^i_{\rm react} \\ & -4\nabla_i V^j_{\rm react}v_j + 4\nabla_j V^i_{\rm react}v^j - 3\dot V_{\rm react}v^i - 4\nabla_j V_{\rm react}v^i v^j]\,, \end{aligned} \tag{3.11}$$

where here an overdot denotes $\partial/\partial t$, square brackets around indices denote antisymmetrization (parentheses around indices denote symmetrization), and spatial indices are raised and lowered using the Cartesian metric.

We now specialize the equation of motion to a binary system. We place the center of mass at the origin of coordinates, and calculate the relative acceleration $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$. We relate the individual body locations $\mathbf{x}_A$ to the relative vector $\mathbf{x}$ by

$$\mathbf{x}_1 = (m_2/m)\mathbf{x}(1+O(\epsilon))\,, \quad \mathbf{x}_2 = -(m_1/m)\mathbf{x}(1+O(\epsilon)) \tag{3.12}$$

[the post-Newtonian corrections to these expressions are known (see, e.g., [36]) but will not be needed]. We define $\delta m = m_1 - m_2$, and note the useful formulas

$$(m_1^2+m_2^2)/m^2 = 1-2\eta\,, \quad (\delta m/m)^2 = 1-4\eta\,,$$
$$(m_1^3+m_2^3)/m^3 = 1-3\eta\,. \tag{3.13}$$

At 5/2PN order, the resulting equation of motion is

$$a^i \approx -mx^i/r^3 - \frac{2}{5}\epsilon^{5/2}x^j M^{(5)}_{ij}\,. \tag{3.14}$$

Substituting the 5/2PN part of a^i wherever it appears in the post-Newtonian correction terms in Eqs. (3.9), expressing everything in terms of relative variables using Eq. (3.12), and dropping the standard, nondissipative 2PN terms, we obtain, to 7/2PN order

$$\begin{aligned} a^i \approx {} & -\frac{mx^i}{r^3} - \frac{2}{5}\epsilon^{5/2}x^j M^{(5)}_{ij} + \epsilon^{7/2}\bigg\{\frac{1}{105}(1-3\eta)(17x^ix^j - 11r^2\delta^{ij})x^k M^{(7)}_{jk} - \frac{2}{5}\left[(1-3\eta)v^2 - (4-\eta)\frac{m}{r}\right]x^k M^{(5)}_{ik} \\ & +\frac{1}{5}\left[8(1-3\eta)v^iv^j - (4-13\eta)\frac{m}{r^3}x^ix^j\right]x^k M^{(5)}_{jk} \\ & +\frac{1}{15}(1-3\eta)[9v^ix^j + 8v^jx^i - 8\mathbf{v}\cdot\mathbf{x}\delta^{ij}]x^k M^{(6)}_{jk} - \frac{1}{63}\frac{\delta m}{m}x^jx^k M^{(7)}_{ijk} \\ & +\frac{16}{45}\frac{\delta m}{m}[\varepsilon^{ijk}x^jx^l S^{(6)}_{kl} + 2\varepsilon^{ijk}x^m v^j S^{(5)}_{km} - \varepsilon^{ljk}x^jv^l S^{(5)}_{ki} + \varepsilon^{ijk}x^jv^l S^{(5)}_{kl}]\bigg\}\,. \end{aligned} \tag{3.15}$$

To the required order the symmetric trace-free (STF) moments $M_{ij\cdots}$ and $S_{ij\cdots}$ are identical to the STF moments that determine the far-zone gravitational wave form (see [31] for justification). For two-body systems, to the required order, they can be taken, for example, from Eqs. (3.6a), (3.6b), and (3.7a) of [12]:

$$M_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I_{kk}\,, \tag{3.16a}$$

$$M_{ijk} = -\mu(\delta m/m)\left[x^ix^jx^k - \frac{1}{5}r^2(\delta^{ij}x^k + \delta^{jk}x^i + \delta^{ki}x^j)\right], \tag{3.16b}$$

$$S_{ij} = -\frac{1}{2}\mu(\delta m/m)(x^i\tilde L^j_N + x^j\tilde L^i_N)\,, \tag{3.16c}$$

where

$$I_{ij} = \mu x^ix^j + \epsilon\mu\left(\left[\frac{29}{42}(1-3\eta)v^2 - \frac{1}{7}(5-8\eta)m/r\right]x^ix^j + \frac{1}{21}(1-3\eta)[11r^2v^iv^j - 12r\dot r x^{(i}v^{j)}]\right). \tag{3.17}$$

We now calculate the time derivatives of these moments, using post-Newtonian equations of motion [(2.2a) and (2.2b)] in the $M^{(5)}_{ij}$ moment that appears at 5/2PN order, but using Newtonian equations of motion in terms that are already of 7/2PN order. Note that in the moment S_{ij}, $\tilde L_N$ can be treated as constant to the order needed. Substituting into Eq. (3.15) and collecting terms, we obtain, finally, Eq. (1.4), with $A_{5/2}$, $B_{5/2}$, $A_{7/2}$, and $B_{7/2}$ given by Eq. (1.8). These time-derivatives and substitutions were implemented using MAPLE.

IV. CONCLUDING REMARKS

The calculation of near-zone gravitational radiation reaction was carried out in a specific gauge defined in [28], and thus it is no surprise that the coefficients in the radiation-reaction terms are fixed. It is a nontrivial check of our energy balance method to see whether this solution corresponds to a particular gauge choice within the balance method, i.e., to a particular choice of the parameters α, β, δ_i, and ε_5. It is nontrivial, because there are 18 constraints on these 8 parameters, so there is no guarantee *a priori* that a consistent solution will result. Comparing the near-zone-derived expressions, Eq. (1.8), term by term with the coefficients listed in Eqs. (2.11) and (2.18), we indeed find the unique, consistent solution

$$\alpha = 4\,, \quad \beta = 5\,, \tag{4.1a}$$

$$\delta_1 = -\frac{99}{14} + 27\eta\,, \quad \delta_2 = 5(1-4\eta)\,, \tag{4.1b}$$

$$\delta_3 = \frac{274}{7} + \frac{67}{21}\eta\,, \quad \delta_4 = \frac{5}{2}(1-\eta)\,, \tag{4.1c}$$

$$\delta_5 = -\frac{1}{7}(292 + 57\eta)\,, \quad \varepsilon_5 = \frac{51}{28} + \frac{71}{14}\eta\,. \tag{4.1d}$$

This strongly supports energy and angular momentum balance, both as a valid principle in gravitational radiation emission, and as a useful tool for deriving equations of motion with radiation reaction. In [31], energy and angular momentum balance are justified rigorously for general systems within the post-Minkowskian framework.

We are in the process of extending the methods of this paper to include the effects of spin-orbit coupling [47]. In principle it also should be possible to extend the radiation reaction formulas to second-post-Newtonian order, corresponding to 9/2PN order in the equations of motion, using recently derived 2PN formulas for gravitational-wave generation from binary systems [18–20].

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