PHYSICAL REVIEW LETTERS

VOLUME 76 | 29 JANUARY 1996 | NUMBER 5

Merons and Thomas Precession

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(Received 28 November 1994)

In a recent study of Thomas precession of electron spin in the context of the Dirac equation, Shankar and Mathur identify the non-Abelian Berry potential (arising from Kramers degeneracy) with the meron. We point out that there is a global mathematical subtlety which prevents such an identification. We go on to clarify the physical context in which merons *do* arise as Berry potentials.

PACS numbers: 03.65.–w, 03.30.+p

In a very interesting recent Letter, Shankar and Mathur [1] (SM) claim that the non-Abelian gauge field [2,3] arising from Thomas precession (TP) is a meron. The discussion of [1] is set in the context of the Dirac electron and the non-Abelian Berry phase [3] that arises when energy levels are pairwise degenerate. Merons are singular, globally nontrivial gauge field configurations which have half-integer topological charge. As the reader may recall, globally nontrivial, nonsingular configurations which have integer topological charge are called instantons. Roughly speaking, a meron [4] is "half an instanton." These gauge field configurations (instantons and merons) are relevant to the functional integral in QCD. For certain values of the QCD coupling constant, meron pairs bind together to form instantons. This phase transition was studied [5] in the 1970s in relation to confinement in QCD. Attempts such as [1] to elucidate the properties of merons in simple and familiar physical situations are therefore well motivated and welcome.

Globally nontrivial gauge fields have also appeared in the context of Berry's phase. The simplest system exhibiting Berry's phase is a two-state system. As is well known, the Berry potential [2] for this system describes a magnetic monopole. The correct global description of this gauge field needs the notion of a fiber bundle or twisted product. As was first pointed out by Avron *et al.* [6] and subsequently discussed in [7], other globally nontrivial gauge field configurations such as instantons also emerge as non-Abelian Berry phases. SM now claim that merons appear as Berry potentials in Thomas precession of the Dirac electron.

The purpose of this Letter is to point out a gap in the reasoning presented by SM in identifying the TP gauge field with the meron. In order to make our main point clearly, we first give an elementary account of Thomas precession as a gauge field and show that the connection that emerges has no globally nontrivial features. We then review Abelian merons in the context of Berry's phase. This simple and familiar situation is a useful aid to understanding non-Abelian merons by analogy. We go on to place the interesting work of SM in its proper context by drawing upon earlier work [6,7]. Combining insights gained from [1,6,7], we conclude that meronic gauge fields *do* occur as Berry potentials, *but not in the context of TP.* This is the main conclusion of this paper.

Thomas Precession.—Shankar and Mathur [1] discuss TP in the context of Kramers degeneracy in the Dirac equation. It is somewhat simpler, however, to consider Thomas precession in the purely classical context of special relativistic kinematics. This gives an abstract, geometric picture of TP which can then be viewed in any representation of the rotation group. Our discussion in quite general and applied equally to the spin of an electron, a gyroscope, or a star.

Consider the family of inertial observers in special relativity. Each observer is described by her four-velocity u^μ, $u^\mu u_\mu = 1$. [We use a metric of signature $(+,-,-,-)$ and $\mu = 0,1,2,3$.] The space $\mathcal{H}^+$ of inertial observers is given by $\mathcal{H}^+ = \{u^\mu | u^\mu u_\mu = 1,\ u^0 > 0\}$. Each observer regards four-vectors orthogonal to u^μ as "space:"

$S = \{v | v \cdot u = 0\}$. Given two observers with four-velocities u_1 and u_2, there is a "pure boost" Lorentz transformation

$$\Lambda^{\mu}_{21\nu} = \delta^{\mu}_{\nu} - (1 + u_1 \cdot u_2)^{-1}(u_1^{\mu} + u_2^{\mu})(u_{1\nu} + u_{2\nu}) + 2u_2^{\mu} u_{1\nu},$$

which takes u_1 to u_2: $\Lambda^{\mu}_{21\nu} u_1^{\nu} = u_2^{\mu}$. It follows that Λ_{21} maps S_1 to S_2, and moreover takes orthonormal frames in S_1 to orthonormal frames in S_2. It is easy to see that Λ_{11} is the identity and $\Lambda_{12} = \Lambda_{21}^{-1}$. However, if u_1, u_2, and u_3 are *three* observers, one finds that the composite Lorentz transformation $\Lambda_{123} = \Lambda_{13}\Lambda_{32}\Lambda_{21}$ is not the identity. Λ_{123} leaves u_1 invariant ($\Lambda_{123}u_1 = \Lambda_{13}\Lambda_{32}\Lambda_{21}u_1 = \Lambda_{13}\Lambda_{32}u_2 = \Lambda_{13}u_3 = u_1$). Further, the spacelike vector $w^{\mu} := \epsilon^{\mu\nu\alpha\beta} u_{1\nu} u_{2\alpha} u_{3\beta}$ is orthogonal to u_1, u_2, u_3 and is therefore unchanged by Λ_{123}: $\Lambda_{123} w = w$. It follows that Λ_{123} must be a rotation in the spacelike plane orthogonal to u_1 and w. To find the angle of rotation, one need only work out the trace of Λ_{123}. A straightforward calculation shows that the angle of Thomas rotation [8] is the deficit angle [9] of the geodesic triangle on the unit hyperboloid $\mathcal{H}^+$ with vertices u_1, u_2, u_3. Thus TP is a direct consequence of the curvature of $\mathcal{H}^+$.

Thomas precession arises because a system which is not in uniform motion changes inertial frames. A closed loop in the space of inertial frames can result in a nontrivial rotation. Thus a gyroscope whose four-velocity executes a loop in $\mathcal{H}^+$ *will* precess, even if it is not acted on by external torques in its rest frame. This can also be expressed as Fermi transport of the spin vector $s^{\mu} \in S$ along the world line [10]:

$$ds^{\mu}/d\tau = -(u \wedge \dot{u})^{\mu\nu} s_{\nu} = -u^{\mu} \dot{u}^{\nu} s_{\nu} \tag{1}$$

(since $s \cdot u = 0$). Choosing an orthonormal frame $e^{i}_{\mu} \in S_u$, $i = 1, 2, 3$, at every $u \in \mathcal{H}^+$, one can rewrite (1) as $ds^i + A^i_j s^j = 0$, where $A^i_j = e^i \cdot de_j$ and $s^i = s^{\mu} e^i_{\mu}$. $A^{ij} = -A^{ji}$ is the Thomas "angular velocity." Under local changes of frame $[e^i(u) \to R^i_j(u) e^j(u)]$, A transforms inhomogeneously like a gauge field. Thus Thomas precession can be viewed as due to a gauge field, which lives on the unit hyperboloid $\mathcal{H}^+$ and takes values in the Lie algebra of the rotation group. The purpose of the preceding rather elementary discussion is to obtain an explicit expression for this gauge field so that its global structure (or lack thereof) can be investigated. Fix a fiducial observer N with four-velocity u_N^{μ}($u_N^0 = 1$, $u_N^i = 0$, $i = 1, 2, 3$) and an orthonormal frame $\mathring{e}_i^{\mu}$ in S_N($\mathring{e}_i \cdot u_N = 0, \mathring{e}_i \cdot \mathring{e}_j = -\delta_{ij}$). Transport this frame to all of $\mathcal{H}^+$ following the earlier discussion: Define $\Lambda(u) = \Lambda_{21}(u_2, u_1)\|_{u_1 = u_N, u_2 = u}$ and $e^i(u) := \Lambda(u)\mathring{e}^i$. This is a global choice of gauge on $\mathcal{H}^+$. The vector potential describing Thomas precession is now easily written down: $A^i_j = e^i \cdot de_j = \mathring{e}^i \Lambda^{-1} d\Lambda \mathring{e}_j = (1 + u^0)^{-1}(du^i u_j - du_j u^i)$. Note that u^i are global coordinates over $\mathcal{H}^+$ and the gauge we pick is globally well defined all over $\mathcal{H}^+$. A small calculation shows that the Chern-Simons three-form

$$K(A) := \mathrm{Tr}[A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A] \tag{2}$$

vanishes identically on $\mathcal{H}^+$. To make contact with the parameter space of [1], we now consider $\mathcal{P}^+$, the space of future pointing timelike momenta $\mathcal{P}^+ = \{p_{\mu} | m^2 := p \cdot p > 0,\ p_0 > 0\}$. [SM use coordinates $(\vec{p}, m)$ on $\mathcal{P}^+$, but any set will do. The complete parameter space includes $\mathcal{P}^-$, the space of past pointing timelike momenta, which are needed to describe positrons. Our remarks also apply to $\mathcal{P}^-$ with suitable modifications.] $\mathcal{P}^+$ naturally projects down to $\mathcal{H}^+$: $\pi_1(p_{\mu}) = u^{\mu} = p^{\mu}/m$. It is clear that the connection describing Thomas precession on $\mathcal{P}^+$ is just the pullback $\pi_1^* A$ of the connection A on $\mathcal{H}^+$. The integral of the Chern-Simons form over *any* three-surface $\mathcal{M}$ in $\mathcal{P}^+$ is identically zero: $\int_{\mathcal{M}} \pi_1^* K = \int_{\pi_1(\mathcal{M})} K = 0$. Since merons are characterized by a fractional surface integral for $K(A)$, we conclude that the gauge field describing Thomas precession is *not* a meron. This appears to be in direct conflict with the claim made by SM. The rest of this Letter is devoted to resolving this apparent contradiction.

Monopoles and Berry's phase.—Let us begin with a simple system, in which a globally nontrivial Berry potential arises—a spin-$\frac{1}{2}$ system in an external magnetic field. We regard the three components of the magnetic field as parameters x^i, $i = 1, 2, 3$, which can be varied, and write the Hamiltonian as $H = x^i \sigma_i$, where σ_i are the three Pauli matrices. It is enough to restrict attention to the unit sphere $S^2 = \{x^i \in \mathbb{R}^3 | x^i x_i = 1\}$ in the parameter space. At each point of S^2, there is a two-dimensional complex vector space $\mathbb{C}^2$ vector space of spin states (whose elements we write $|\rangle$) on which the Hamiltonian acts. Since H squares to unity, its eigenvalues are ± 1. The subspace of positive energy states $\{|v(x)\rangle \in \mathbb{C}^2 | H(x)|v(x)\rangle = |v(x)\rangle\}$ defines a line bundle over S^2. Normalizing the state ($\langle v(x)|v(x)\rangle = 1$) produces an S^1 bundle over S^2 (the Hopf bundle). To compute the Berry potential which arises when the parameters x^i are varied, note that $H(x) = h(x)\sigma_3 h^{-1}(x)$, where $h(x)$ is defined by $h(x) := [2(1 + x^3)]^{-1/2}[1 + H(x)\sigma_3]$ at all points except the south pole, where $x^3 = -1$. If we pick a normalized positive energy state $|\vartheta\rangle$ at the north pole, which satisfies $\sigma_3|\vartheta\rangle = |\vartheta\rangle$, $|v(x)\rangle := h(x)|\vartheta\rangle$ is a normalized positive energy state all over the sphere (except the south pole). (Similar considerations also apply to the south polar patch, which excludes the north pole.) The Berry potential is $A = \langle v(x)|d|v(x)\rangle = \langle\vartheta|h^{-1} dh|\vartheta\rangle$ and its field strength is $F = dA$. It is easily seen that this field strength describes a magnetic monopole and that $1/(2\pi)\int_{S^2} F = 1$.

The Abelian meron.—Let us now regard A as living on $\mathbb{R}^3 - \{0\}$ (pull back the connection defined on S^2 by the natural map $\vec{x} \to \vec{x}/|x|$). To see the gauge field describing an Abelian meron, one just slices this parameter space along the equatorial $x^3 = 0$ plane. The Berry potential then becomes $A = (i/2)d\phi$ where ϕ is the azimuthal angle on the plane. Evidently, the field strength of this connection is identically zero on the punctured plane. Nevertheless, the integral $\kappa_1 := 1/(2\pi i)\int A = \frac{1}{2}$ for a loop that encircles the origin anticlockwise. This can be seen either by explicit computation or by noticing by Stokes' theorem that κ_1 can be converted to a surface integral of F over the northern hemisphere in the three-dimensional parameter space. Since the integral of F over the entire sphere is unity, we get half this answer. One could view κ_1 as just the holonomy of the flat connection A on $\mathbb{R}^2 - \{0\}$ (the "Aharonov-Bohm effect"). Or one could take a more lofty point of view (which carries over to the non-Abelian case) and describe this as a secondary characteristic class [11]. Unlike the primary characteristic classes (Chern classes) (which are integers, like $\int_{S^2} F$ above), the secondary classes (Chern-Simons invariants) are fractions. The integer part of κ_1 can be altered by (large) gauge transformations. But the fractional part is a gauge-invariant quantity and describes a global property of the connection.

Actually, this "Abelian meron" has already played a part in the historical development of Berry's phase. Herzberg and Longuet-Higgins [12] (HL) were studying the quantum mechanics of polyatomic molecules when they noticed a curious sign change in the wave function around a degeneracy. The result is easy to see for 2×2 matrices. As θ goes from 0 to 2π, the real symmetric matrix $[\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1]$ returns to itself, but its parallel transported eigenvector $[\cos(\theta/2), \sin(\theta/2)]$ reverses sign. As HL were interested in time-reversal-invariant systems, they restricted themselves to real Hamiltonians. Consequently, they were restricted to a "slice" of the Poincaré sphere and discovered a poor cousin of Berry's phase [2]. From the present perspective, what HL saw was an Abelian meron. Historically, this sign mentioned above was an important stepping stone that led Berry to the discovery [2] of the phase.

Instantons from Berry's phase.—We now draw upon the work of [6,7] to see how instantons appear as Berry potentials. The Pauli matrices can be viewed either as generators of the rotation group obeying commutation relations or as generators of the Clifford algebra, obeying *anticommutation* relations. If one takes the first viewpoint, generalizing to higher spin generators yields monopoles of higher strength. Following [7], we choose the second viewpoint for generalization. One considers higher-dimensional Euclidean Clifford algebras. Consider the five-dimensional Clifford algebra generated by Γ_i, $i = 1, \ldots, 5$, $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$. These generators can be realized as 4×4 matrices: $\Gamma_1 = \sigma_1 \otimes \sigma_1$, $\Gamma_2 = \sigma_1 \otimes \sigma_2$, $\Gamma_3 = \sigma_1 \otimes \sigma_3$, $\Gamma_4 = \sigma_2 \otimes I$, $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4 = -\sigma_3 \otimes I$, where I is the 2×2 unit matrix. We consider the system described by the Hamiltonian

$$H = x^i \Gamma_i \,, \tag{3}$$

where the x^i now span a five-dimensional parameter space $\mathbb{R}^5 - \{0\}$. This might appear as a mathematically motivated generalization, as indeed it is, but such a system can be physically realized [6] by a spin-$\frac{3}{2}$ system in an external quadrupole electric field and is relevant to nuclear quadrupole resonance (NQR). As before, we need only restrict our attention to the unit sphere S^4 in parameter space. The positive energy subspace of H defines a $\mathbb{C}^2$ bundle over S^4. Choosing an orthonormal frame in the fiber gives an U(2) bundle over S^4. Just as before we notice that $H(x) = h(x)\Gamma_5 h^{-1}(x)$, where $h(x)$ is now defined by $h(x) := [2(1 + x^5)]^{-1/2}[1 + H(x)\Gamma_5]$ at all points of S^4 except the south pole, where $x^5 = -1$. If we pick an orthonormal pair of positive energy states $|\vartheta_\alpha\rangle$ $(\alpha = 1, 2)$ at the north pole, which satisfy $\Gamma_5|\vartheta_\alpha\rangle = |\vartheta_\alpha\rangle$, the states $|v_\alpha(x)\rangle := h(x)|\vartheta_\alpha\rangle$ are orthonormal positive energy states all over the sphere [except the south pole, where $h(x)$ is ill defined]. The Berry potential is now a 2×2 anti-Hermitian matrix

$$A_{\alpha\beta} = \langle v_\alpha(x)\,|d|v_\beta(x)\rangle = \langle\vartheta_\alpha|h^{-1}dh|\vartheta_\beta\rangle \,. \tag{4}$$

A is in fact traceless and so is really an SU(2) connection. Its field strength is given by $F = dA + A \wedge A$, in the notation of matrix-valued differential forms. As explained at length in [6,7], this A describes an instanton and $1/\mathcal{N}\int_{S^4} \mathrm{Tr}[F \wedge F] = 1$, where $\mathcal{N}$ is a normalization factor.

The non-Abelian meron.—To see the gauge field describing the non-Abelian meron, one just slices this S^4 parameter space along the equatorial $x^5 = 0$ plane. The unit sphere S^3 in this equatorial plane defined by $x^i x_i = 1$ can be identified with SU(2): $g(x) = x^4 + i\vec{x} \cdot \vec{\sigma}$. It is also convenient to introduce $\omega = g^{-1}dg$, the Maurer-Cartan form on the SU(2) group. $|v_\alpha(x)\rangle$ provide a global choice of positive energy states all over S^3. (The meron bundle is trivial, unlike the instanton bundle.) A little computation shows that the vector potential $A_{\alpha\beta} = \frac{1}{2}\langle\vartheta_\alpha|H(x)dH(x)\,|\vartheta_\beta\rangle$ defined by (4) is just half the Maurer-Cartan form ω, $A = \frac{1}{2}\omega$, on S^3. It is now easy to work out the Chern-Simons invariant $\kappa_2(A) := 1/\mathcal{N}\int_{S^3} K(A)$, where $\mathcal{N}$ is a normalization constant (like $2\pi i$ above), which is so chosen that $\kappa_2(\omega)$ is unity. Using the fact that $d\omega + \omega \wedge \omega = 0$, we find that $\kappa_2 = \frac{1}{2}$. This result can also be seen by noticing that by Stokes' theorem κ_2 can be converted to an integral of $\mathrm{Tr}[F \wedge F]$ over the northern hemisphere of S^4. Since the (suitably normalized) integral of $\mathrm{Tr}[F \wedge F]$ over the entire sphere is unity, we get half this answer. This shows that the gauge field on S^3 has a nonvanishing secondary characteristic class. As mentioned earlier, the integer part of κ_2 can be altered by (large) gauge transformations, but the fractional part is a gauge invariant quantity and describes a global property of the connection. The meron gauge field configuration is obtained by pulling back the gauge field $(\frac{1}{2}\,\omega)$ from S^3 to $\mathbb{R}^4 - \{0\}$. By using Stokes' theorem to transform the Chern-Simons integral to a four-dimensional integral, we conclude that the meron has instanton number $\frac{1}{2}$, all of which is concentrated at the (excised) origin of $\mathbb{R}^4$.

Merons and Thomas precession.—SM claimed that the meron arises as a Berry potential in Thomas precession of the spin of the Dirac electron. This claim was based on the fact that the Dirac Hamiltonian $H = \vec{p}\cdot\vec{\alpha} + \beta m$ is of the form (3), since the Dirac matrices $\vec{\alpha}$ and β realize the Euclidean Clifford algebra. Viewing the rest mass m and spatial momenta $\vec{p}$ as parameters, SM would seem to have a physical realization of the system (3) discussed above. This is indeed true locally. In any allowed region of this parameter space, one can compute Thomas precession effects of the Dirac electron by invoking the wisdom gained from the system described by (3). What then prevents the identification of the Thomas precession gauge field with the meron? The answer lies in the global nature of the parameter space. The allowed values of the electron four-*momentum* in special relativity are restricted to lie within the light cone. It is impossible to accelerate a subluminal particle like the electron to the speed of light, since such a process would take an infinite amount of proper acceleration. As a result, the global structure of the SM parameter space $(\vec{p}, m)$ of the Dirac electron does not support a meronic gauge field. This point is explained in more detail below.

In our earlier, elementary discussion of Thomas precession we regarded the Berry connection on $\mathcal{P}^+$ as deriving from the connection on $\mathcal{H}^+$ by pullback. However, the SM parametrization associates with each four-momentum p_μ the point $\pi_2(p_\mu) = (1/p_0)(\vec{p}, m) = (\vec{x}, \sqrt{1 - \vec{x}\cdot\vec{x}})$ on the sphere S^{3+} (the northern hemisphere of S^3). Since both π_1 and π_2 satisfy $\pi(\lambda p) = \pi(p)$ (for λ real and positive), the composite map $\pi_2 \circ \pi_1^{-1}$ from $\mathcal{H}^+$ to S^{3+} is well defined. The mapping is given explicitly by

$$\vec{x} = \vec{u}/\sqrt{(1 + \vec{u}\cdot\vec{u})}\,. \tag{5}$$

(Similarly, $\mathcal{H}^{3-}$, the negative energy unit mass shell is mapped to the southern hemisphere S^{3-}.) As the reader can easily verify, this is just the stereographic projection of the unit hyperboloid $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ to the unit sphere S^3. This map is 1-1, but no onto. The equator of S^3 $(\vec{x}\cdot\vec{x} = 1)$ is not included in the range of (5). Consequently, the inverse map from the unit sphere S^3 to $\mathcal{H}$ does not exist. The equator of S^3 in the SM parameter space is not represented in the allowed parameter space of momenta. This gap in the parameter space is enough to spoil the identification of the Thomas precession gauge field with the meron. The meronic nature of a gauge field is a global notion and needs a global parameter space. This resolves the apparent conflict between the elementary calculation presented earlier and the claim of [1].

We have assumed throughout this discussion that the electron is acted upon by a force which produces no torque (in the electron rest frame) on the spin of the electron. One can realize this force theoretically by means of a scalar potential in the Dirac Hamiltonian. Electric fields do not produce torque-free acceleration. If an electric field is applied to the electron to accelerate it, the spin of the moving electron sees an apparent magnetic field and therefore precesses. In order to get the right physical answer, one has to add this "dynamical phase" to the "geometrical phase" that we have been concerned with in this paper. We mention this because the electron in an external electric field was the historical context in which Thomas precession was originally discovered. Readers who wish to compare the present discussions with the classical one will need to remember this.

The Chern-Simons three-form $(2)K(A)$ of the meron is closed, but not exact. $\kappa_2(A)$ measures the nontrivial cohomology of this closed form on S^3 (or equivalently $\mathbb{R}^4 - \{0\}$). It is evident that $\mathcal{P}^+$ is a contractible space and so has trivial cohomology. This is the topological basis of our objection to SM. In spite of this objection, we consider the work of [1] very interesting and for the most part valid. It is only the claimed relation to Thomas precession [1] that we dispute here. To our knowledge, SM are the first to discuss merons in the context of Berry's phase. From earlier work and the present paper, it is clear that merons *can* be realized as Berry potentials in the laboratory in NQR. This may be of interest to gauge theorists as well as the NQR and Berry phase communities.

It is a pleasure to thank R. Nityananda for several discussions on gauge theoretic aspects of Thomas precession over the years, V. Pati and K. Paranjape for their help with secondary characteristic classes, and R. Shankar and H. Mathur for an e-mail correspondence on these matters. I also thank C. S. Shukre for discussions.

Note added.—Shankar and Mathur (private communication) ascribe the discrepancy between our results and theirs to a difference in the physical effect being considered. In their view, the spin of a gyroscope is indeed described by the TP gauge field described above, but the spin of the electron is described by a different gauge field (spin-orbit coupling). We do not share this view. We hold that spin-orbit coupling and TP are identical physical effects. In other words, the spin of the Dirac electron and gyroscopes are identically affected by TP. If one were to transport an electron and a gyroscope (with initially parallel spins, applying no rest-frame torque) around the same closed loop in $\mathcal{H}^+$, their spins would remain parallel. To see this, it is enough to consider an infinitesimal loop in $\mathcal{H}^+$. (Finite loops can be built up from infinitesimal ones.) By a Lorentz transformation, one can arrange for the infinitesimal loop to be centered at u^N on $\mathcal{H}^+$. This simplifies the algebra, since we need only deal with low velocities. For low velocities Mathur has given [Eq. 9, the second of Ref. [1]] an expression for the SM gauge field in a gauge suitable for comparison with the present paper. For low velocities, the TP gauge field [see just above Eq. (2) of this paper] becomes $A^{\mathrm{TP}} = i/2\epsilon_{ijk}u^i du^j \tau^k$, where $(\tau^k)^l_m := -i\epsilon^{kl}_m$ are 3×3 matrices realizing the angular momentum commutation relations. This agrees exactly with Mathur's expression [Eq. 9, the second of Ref. [1]]. Thus the SM gauge field and the TP gauge field describe the same local physics and the motion of the gyroscope is identical to the motion of electron spin. We maintain that the discrepancy between our results and SM lies in the global problems described in this paper.

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