Scalar waves in the Witten bubble spacetime

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(Received 18 April 1990)

Massless scalar waves in the Witten bubble spacetime are studied. The timelike and angular parts of the separated Klein-Gordon equation are written in terms of hyperbolic harmonics characterized by the generalized frequency ω. The radial equation is cast into the Schrödinger form. The above mathematical formulation is applied to study the scattering problem, the bound states, and the corresponding stability criteria. The results confirm the concept of a bubble wall as a perfectly reflecting expanding sphere.

I. INTRODUCTION

The ground state of the five-dimensional Kaluza-Klein theory is assumed to be $M^4\times S^1$, the product of four-dimensional Minkowski space and a circle S of characteristic radius $R\sim m_P^{-1}$, where m_P is the Planck mass. Witten[1] studied whether or not a ground state of this form is a reasonable candidate of such a theory. He concluded that $M^4\times S^1$ is unstable against a process of semiclassical barrier penetration. The consequence of this semiclassical decay process is a classical, time-dependent, source-free solution of the Kaluza-Klein field equations. This results in an expanding bubble with unusual properties.

In certain nonlinear field theories, the expansion of a bubble corresponds to the decay of a "false" vacuum into a "true" vacuum. However, rather than containing the "true" vacuum, the Witten bubble has no interior at all. The spacetime is terminated at its wall which is a perfectly reflecting, expanding sphere. The expansion rate increases with time.

Recently, the nature of timelike and null geodesics in this spacetime has been studied by Brill and Matlin.[2] In this paper, we investigate the behavior of scalar waves in the background of the Witten bubble geometry.

As in the case of the geodesics, the study of the behavior of scalar waves also probes into the geometry of the spacetime. The scattering phenomenon throws light on the nature of the bubble as well as on its effect on the surrounding spacetime. Further it provides us with valuable information about the bound states and the stability of the spacetime. Also, the investigation of scalar waves in an exact solution such as the Witten metric offers insight into the propagation of waves in strong gravitational fields.

The present paper is organized as follows. In Sec. II, we describe the main features of the Witten bubble metric. In Sec. III, the Klein-Gordon equation has been separated, and the different functions appearing in the solution have been discussed. The formalism developed in Sec. III is employed in Sec. IV to study the scattering of scalar waves and the bound states. The higher-mode solutions are discussed in Sec. V. Section VI provides a summary of the salient results. We also give an alternative form of solutions of τ, and radial equation in Appendix A, while Appendix B discusses the significance of a coordinate transformation generally employed in scattering problems.

II. THE WITTEN BUBBLE METRIC

The four-dimensional Minkowskian subspace of the ground state of five-dimensional Kaluza-Klein theory is being described here in terms of "spherical Rindler" coordinates. A spherical array of uniformly accelerated observers uses such type of "hyperbolic" coordinates. These are related to the Minkowskian coordinates in the following way:

$$\begin{aligned} t&=r\sinh\tau\ ,\\ x^1&=r\cosh\tau\cos\phi\sin\theta\ ,\\ x^2&=r\cosh\tau\sin\phi\sin\theta\ ,\\ x^3&=r\cosh\tau\cos\theta\ . \end{aligned} \tag{2.1}$$

Here, we are using these coordinates to make the vacuum metric comparable with the metric of the semiclassical vacuum decay which starts at $t=0$. Therefore, the five-dimensional Kaluza-Klein vacuum metric can be written as

$$ds^2=-r^2d\tau^2+dr^2+r^2\cosh^2\tau(d\theta^2+\sin^2\theta\, d\phi^2)+d\chi^2\ , \tag{2.2}$$

where χ is the coordinate of the compactified fifth dimension. This metric is valid for $\tau<0$. For $\tau>0$, the decay state of the Kaluza-Klein vacuum can be described by the metric

$$ds^2=-r^2d\tau^2+\left[1-\frac{R^2}{r^2}\right]^{-1}dr^2+r^2\cosh^2\tau(d\theta^2+\sin^2\theta\,d\phi^2)+\left[1-\frac{R^2}{r^2}\right]d\chi^2\;, \tag{2.3}$$

where r has now the range $R\le r<\infty$. Therefore, as a result of this decay, a microscopic hole of radius R will be spontaneously formed in space. Like the bubble wall in conventional vacuum decay, this hole will start expanding to infinity with a uniform acceleration. The bubble surface is a two-sphere of area $4\pi R^2\cosh^2\tau$. So at any time its radius is $r(t)=(R^2+t^2)^{1/2}$. For very large r, the metric (2.3) asymptotically approaches the $M^4\times S^1$ spacetime described by (2.2).

III. THE KLEIN-GORDON EQUATION

The Klein-Gordon equation for a massive scalar field is written as

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\mu\nu}\tilde{\Phi}_{,\nu})_{,\mu}-M^2\tilde{\Phi}=0\;, \tag{3.1}$$

where M is the mass.

The metric (2.3) is independent of the fifth coordinate χ and, therefore, there is a Killing symmetry in the fifth dimension. The solution of the Klein-Gordon equation is, therefore, found to be

$$\tilde{\Phi}=\tilde{R}(r)\tilde{H}^{\,lm}_{\,i\omega}(\tau,\theta,\phi)e^{im_1\chi}\;, \tag{3.2}$$

$$\tilde{\Phi}=\tilde{R}(r)\tilde{T}^{\,l}_{\,i\omega}(\tau)Y_l^m(\theta,\phi)e^{im_1\chi}\;, \tag{3.3}$$

where $Y_l^m(\theta,\phi)$ are the spherical harmonics. Other functions appearing in this solution will be discussed in the following subsections. Scalar waves for which the fifth-dimensional component vanishes ($m_1=0$) represent the propagation of ordinary scalar waves in such a spacetime. The case of nonzero m_1 cannot readily be interpreted in terms of realistic scalar waves.[3] In this work, we shall consider $m_1=0$ throughout.

A. τ equation

The wave field represented by the solution (3.2) does not oscillate in a simple harmonic way, but in a more complicated way as given by the hyperbolic harmonics $\tilde{H}^{\,lm}_{\,i\omega}(\tau,\theta,\phi)$. These hyperbolic harmonics are characterized by the generalized frequency ω, which labels the representation of the Lorentz group SO(3,1). These are actually the eigenfunctions of the D'Alembertian on the unit timelike hyperboloid:

$$\left[-\frac{1}{\cosh^2\tau}\frac{\partial}{\partial\tau}\cosh^2\tau\frac{\partial}{\partial\tau}+\frac{1}{\cosh^2\tau}\nabla^2_{\theta,\phi}\right]\tilde{H}^{\,lm}_{\,i\omega}=(\omega^2+1)\tilde{H}^{\,lm}_{\,i\omega}\;. \tag{3.4}$$

These form a complete orthonormal set with respect to the Lorentz-invariant volume on the timelike hyperboloid:

$$\int_0^\infty\int_0^{2\pi}\int_0^\pi\tilde{H}^{\,lm}_{\,i\omega}(\tau,\theta,\phi)\tilde{H}^{\,*l'm'}_{\,i\omega'}(\tau,\theta,\phi)\cosh^2\tau\sin\theta\,d\theta\,d\phi\,d\tau=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\;. \tag{3.5}$$

A detailed construction of these hyperbolic harmonics has been discussed by Gerlach[4] in the Appendix of his paper. Some inadvertent errors seem to have crept into his constructions, probably stemming from the original sources used. Nevertheless, these errors do not affect the results of that paper. In our work, we shall use explicit solutions, thereby avoiding any possible ambiguities.

Also,

$$\tilde{H}^{\,lm}_{\,i\omega}(\tau,\theta,\phi)=\tilde{T}^{\,l}_{\,i\omega}(\tau)Y_l^m(\theta,\phi)\;, \tag{3.6}$$

where $\tilde{T}^{\,l}_{\,i\omega}(\tau)$ satisfies the equation

$$\left[-\frac{1}{\cosh^2\tau}\frac{d}{d\tau}\cosh^2\tau\frac{d}{d\tau}-\frac{l(l+1)}{\cosh^2\tau}\right]\tilde{T}^{\,l}_{\,i\omega}(\tau)=(\omega^2+1)\tilde{T}^{\,l}_{\,i\omega}(\tau)\;. \tag{3.7}$$

Introducing the function

$$\tilde{u}^{\,l}_{\,i\omega}(\tau)=\cosh\tau\,\tilde{T}^{\,l}_{\,i\omega}(\tau)\;, \tag{3.8}$$

we can write Eq. (3.7) in the Schrödinger form

$$\left[-\frac{d^2}{d\tau^2}-\frac{l(l+1)}{\cosh^2\tau}\right]\tilde{u}^{\,l}_{\,i\omega}(\tau)=\omega^2\tilde{u}^{\,l}_{\,i\omega}(\tau)\;. \tag{3.9}$$

In this section, we are confining ourselves only to the lowest mode ($l=0$). The higher-mode solutions ($l>0$) will be discussed in Sec. V.

The lowest-mode solution is given by

$$\tilde{u}^{\,0}_{\,i\omega}(\tau)=\frac{e^{i\omega\tau}}{\sqrt{4\pi}}\;, \tag{3.10}$$

where $1/\sqrt{4\pi}$ factor has been taken to normalize the function:

$$\int_0^\infty \tilde{u}\,^0_{i\omega}(\tau)\tilde{u}\,^0_{i\omega'}(\tau)d\tau=\delta(\omega-\omega') \ . \tag{3.11}$$

Therefore,

$$\tilde{H}\,^{0m}_{i\omega}(\tau,\theta,\phi)=\frac{1}{\sqrt{4\pi}}\frac{e^{i\omega\tau}}{\cosh\tau}e^{im\phi} \ . \tag{3.12}$$

An alternative solution for τ equation (3.7) is discussed in Appendix A.

B. Radial equation

The radial function $\tilde{\mathcal{R}}(r)$ satisfies the equation

$$\frac{1}{r^3}\left[1-\frac{R^2}{r^2}\right]\left[r^3\left[1-\frac{R^2}{r^2}\right]\tilde{\mathcal{R}}_{,r}\right]_{,r}-M^2\left[1-\frac{R^2}{r^2}\right]\tilde{\mathcal{R}}-m_1^2\tilde{R}+\frac{\omega^2+1}{r^2}\left[1-\frac{R^2}{r^2}\right]\tilde{\mathcal{R}}=0 \ . \tag{3.13}$$

As discussed in the beginning of this section, we shall take $m_1=0$. Then for massless ($M=0$) scalar waves, Eq. (3.13) turns out to be

$$\frac{1}{r}[r(r^2-R^2)\tilde{R}_{,r}]_{,r}+(\omega^2+1)\tilde{R}=0 \ . \tag{3.14}$$

Now, we make a change of variable such that

$$\frac{dr}{dx}=(r^2-R^2)^{1/2} \tag{3.15}$$

or

$$x=\operatorname{arccosh}\left[\frac{r}{R}\right] \ . \tag{3.16}$$

As $r\rightarrow R$, $x\rightarrow 0$. As $r\rightarrow+\infty$, x goes to both $\pm\infty$. Here, we are choosing the limit to be $x\rightarrow+\infty$. Use of such a coordinate transformation has a natural significance which we shall discuss in Appendix B.

After this transformation, Eq. (3.14) becomes

$$\tilde{\mathcal{R}}_{,x,x}+f(r)\tilde{\mathcal{R}}_{,x}+(\omega^2+1)\tilde{\mathcal{R}}=0 \ , \tag{3.17}$$

where

$$f(r)=\frac{r}{(r^2-R^2)^{1/2}}+\frac{(r^2-R^2)^{1/2}}{r} \ . \tag{3.18}$$

The radial equation is still not free of first-derivative terms. Now, if we define

$$\tilde{\mathcal{R}}(r)=\frac{\tilde{\Psi}(r)}{(r\sqrt{r^2-R^2})^{1/2}} \ , \tag{3.19}$$

then it brings Eq. (3.17) to the form of a Schrödinger equation:

$$\tilde{\Psi}_{,x,x}+\left[\omega^2+\frac{1}{\sinh^2 2x}\right]\tilde{\Psi}=0 \ , \tag{3.20}$$

with an effective potential

$$V_{\text{eff}}=-\frac{1}{\sinh^2 2x} \ . \tag{3.21}$$

This Schrödinger equation has a very simple effective potential whose behavior can be readily visualized. Qualitative features of the wave behavior can also be easily discussed. To solve this equation, let us introduce a new variable

$$y=-\sinh^2 2x \ , \tag{3.22}$$

so that Eq. (3.20) becomes

$$y(1-y)\tilde{\Psi}_{,y,y}+(\tfrac{1}{2}-y)\tilde{\Psi}_{,y}+\left[\frac{1}{16y}-\frac{\omega^2}{16}\right]\tilde{\Psi}=0 \ . \tag{3.23}$$

Then we define a new function

$$\tilde{W}=y^{-1/4}\tilde{\Psi} \tag{3.24}$$

which now satisfies the equation

$$y(1-y)\tilde{W}_{,y,y}+(1-\tfrac{3}{2}y)\tilde{W}_{,y}-\frac{\omega^2+1}{16}\tilde{W}=0 \ . \tag{3.25}$$

This is in the form of a hypergeometric equation. Therefore, the analytical solutions of this equation near $y=0$ could be found in terms of hypergeometric series. These are

$$\tilde{W}_1=F(a,b;1;y) \tag{3.26}$$

and

$$\tilde{W}_2=\ln y F(a,b;1;y)+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}y^n S(n) \tag{3.27}$$

for $|y|<1$, where

$$a=\tfrac{1}{4}(1+i\omega) \ ,$$

$$b=\tfrac{1}{4}(1-i\omega) \ ,$$

$$F(a,b;1;y)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}y^n \ ,$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)} \ ,$$

$$S(n)=\psi(a+n)-\psi(a)+\psi(b+n)-\psi(b)-2\psi(n+1)+2\psi(1) \ .$$

ψ is the logarithmic derivative of the gamma function.

Using (3.24), we can now get the solutions of the Schrödinger equation (3.20) to be

$$\tilde{\Psi}_1=y^{1/4}F(a,b;1;y) \ , \tag{3.28}$$

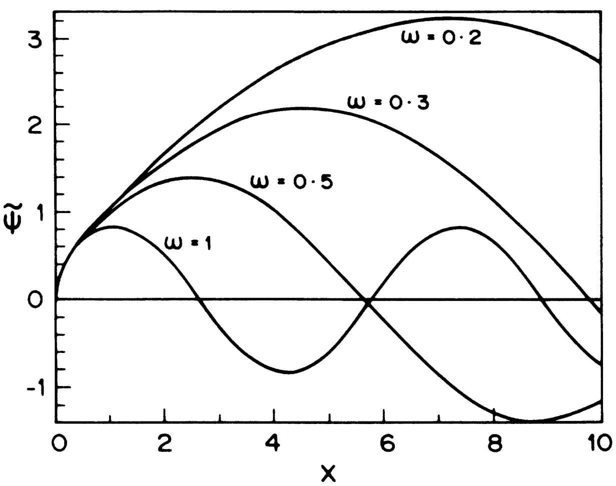

FIG. 1. The solution $\tilde{\Psi}_1(x)$ for different frequencies ω.

$$\tilde{\Psi}_2=y^{1/4}\left[\ln y F(a,b;1;y)+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}y^nS(n)\right] . \tag{3.29}$$

Both these solutions go to zero as $y\to 0$.

We have also solved Eq. (3.20) numerically and plotted the solutions in Fig. 1 for different frequencies ω. We observe that starting from $x=0$, the solution rises very rapidly to a maximum value and then starts oscillating like a cosine wave. As ω increases, the influence of the spacetime on the waves reduces and the solution starts oscillating very close to the bubble wall.

However, if we look at the corresponding solutions for $\tilde{\mathcal{R}}$ equation by using Eq. (3.19),

$$\tilde{\mathcal{R}}_1=\frac{1+i}{R}F(a,b;1;y) , \tag{3.30}$$

$$\tilde{\mathcal{R}}_2=\frac{1+i}{R}\left[\ln y F(a,b;1;y)+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}y^nS(n)\right] . \tag{3.31}$$

We observe that, at $r=R$,

$$\tilde{\mathcal{R}}_1=\frac{1+i}{R} , \tag{3.32}$$

whereas $\tilde{\mathcal{R}}_2\to-\infty$.

Therefore, as far as the Schrödinger equation is concerned, the second solution behaves properly in that coordinates system. But when we consider the actual radial equation, the corresponding solution blows up at $r=R$. We are, therefore, discarding the second solution throughout our further calculations.

From Eq. (3.20), we can readily obtain the asymptotic behavior of its solution as $x\to\infty$:

$$\tilde{\Psi}\underset{x\to+\infty}{\sim} Ae^{-i\omega x}+Be^{+i\omega x} , \tag{3.33}$$

where A and B are arbitrary constants. Then, using (3.19), we get

$$\tilde{\mathcal{R}}\underset{r\to+\infty}{\sim}\frac{Ae^{-i\omega x}+Be^{+i\omega x}}{r} , \tag{3.34}$$

since as $r\to+\infty$, $(r\sqrt{r^2-R^2})^{1/2}\to r$. We shall now apply these considerations to the wave scattering by the bubble.

IV. SCATTERING AND BOUND STATES

The total scalar wave solution in its lowest mode can now be written in the asymptotic limit to be

$$\tilde{\Phi}(r\to+\infty)_{m_1,l=0}=\frac{e^{im\phi}e^{i\omega\tau}}{\sqrt{4\pi}r\cosh\tau}\times(Ae^{-i\omega x}+Be^{+i\omega x}) . \tag{4.1}$$

The factor $(r\cosh\tau)$ in the denominator ensures that the total flux of energy passing through a unit solid angle $d\Omega$ does not depend on r or τ for very large r.

What is the relation between A and B? The answer follows immediately, if we just consider the behavior of the differential equation (3.17). The hypergeometric series of Eq. (3.30) is always real, since c is just unity and a and b are complex conjugates of each other. Now, if we use initial condition (3.32) in (3.17) and study the evolution of $\tilde{\mathcal{R}}$, we shall see that the real and imaginary parts of this equation will evolve independently. However, at any point, both these parts will be equal. Considering this fact and matching the solution with (3.34) in the asymptotic limit, one can show by a very simple calculation that this is a case which corresponds to $|A|=|B|$.

The actual expressions for A and B can be obtained by analytically extending the solution (3.28) to infinitely large negative values of the argument

$$y=-\sinh^2 2x\to-2^{-2}e^{4x} . \tag{4.2}$$

Then the solution is

$$\begin{aligned}\tilde{\Psi}_{x\to+\infty}&=2^{-1/2}e^{x}\left[\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(1-a)}2^{2a}e^{-4ax}+\frac{\Gamma(a-b)}{\Gamma(a)\Gamma(1-b)}2^{2b}e^{-4bx}\right]\\ &=\frac{\Gamma\left[-i\frac{\omega}{2}\right]e^{i(\omega/2)\ln 2}}{\Gamma\left[\frac{1}{4}-i\frac{\omega}{4}\right]\Gamma\left[\frac{3}{4}-i\frac{\omega}{4}\right]}e^{-i\omega x}+\frac{\Gamma\left[+i\frac{\omega}{2}\right]e^{-i(\omega/2)\ln 2}}{\Gamma\left[\frac{1}{4}+i\frac{\omega}{4}\right]\Gamma\left[\frac{3}{4}+i\frac{\omega}{4}\right]}e^{+i\omega x} .\end{aligned} \tag{4.3}$$

Matching with (3.33), we get expressions for A and B. Since $\Gamma(\bar{z})=\overline{\Gamma(z)}$, we can easily see that A and B are complex conjugates of each other and, therefore, $|A|=|B|$.

From the foregoing discussion, we see that only one of the two independent solutions is acceptable. This solution is well behaved at infinity and consists of incoming and reflected wave components with equal amplitudes. Further, this solution goes to zero at $r=R$. Since the other solution is not well behaved at $r=R$, there is no scope for superposition of the two solutions, thereby obtaining other boundary conditions, e.g., standing waves that do not go to zero at $r=R$. On the other hand, the boundary conditions that have naturally arisen fit in well with the notion of a bubble surface enclosing a region $r<R$ that does not correspond to points in physical space. One expects the incoming wave to be totally reflected from the bubble surface. This phenomenon is, in fact, happening here. We may also note that by a similar argument, one can rule out quasinormal modes of the bubble, since waves purely incoming at $r=R$ and purely outgoing at $r\to\infty$ cannot be obtained. This indicates that the bubble surface acts as a perfectly reflecting rigid barrier.

To investigate the bound states of this problem, we have to consider imaginary frequencies. Let us replace $i\omega\to\omega_n$. Then, for τ equation (3.9), a discrete set of square-integrable wave functions can be obtained as bound states. These have been constructed in detail in Ref. 4. To obtain bound states in the radial equation (3.20), we see that the parameters a and b in solution (3.28) have now become real. Then, in the asymptotic expressions (4.3), the first term behaves as $e^{-\omega_n x}$ and the second as $e^{+\omega_n x}$. Bound states are possible, only if the coefficient of $e^{+\omega_n x}$ in the second term vanishes. However, all Γ functions in this coefficient have a positive real argument. Therefore, no Γ function in the denominator can ever blow up and make the factor vanish. Consequently, no bound state is possible. Nevertheless, we should point out here that if one performs the following integration

$$\int_0^\infty \sqrt{E-V_{\text{eff}}}\,dx$$

for $E=0$ in this case, one obtains

$$\int_0^\infty \frac{1}{\sinh 2x}dx=\tfrac{1}{2}\ln\tanh x\,|_0^{+\infty}=+\infty \ .$$

Following Ref. 5, this means the existence of an infinite number of bound states. However, our explicit calculation has shown that there is no bound state at all. This apparent contradiction is due to our discard of solution (3.29), though it was behaving well throughout the range of variable χ in Schrödinger's equation (3.20). Had we considered both solutions, we would have obtained an infinite number of bound states. But those are not realistic as far as our problem is concerned.

Now, since the same ω appears in both radial and τ equations, the nonexistence of bound states also confirms that modes exponentially growing with τ do not exist. This shows the mode stability of the bubble spacetime against scalar perturbations. Further, since the scattering modes form a complete set, the bubble spacetime is stable with respect to any arbitrary scalar perturbations.

V. HIGHER-MODE ($l>0$) SOLUTIONS

As we have seen in Sec. III A, the lowest-mode ($l=0$) solution given in Eq. (3.10) is δ-function normalized. Now, to study higher-mode solutions, following Ref. 4, we can introduce the raising and lowering operators by factorization method in Eq. (3.9):

$$L_\pm^l=l\tanh\tau\mp d\tau \ . \tag{5.1}$$

Then one can write $\tilde{u}\,^l_{i\omega}(\tau)$ as an eigenfunction of $L_+^l L_-^l$, with the eigenvalue (ω^2+l^2). Now the general eigenfunction can be written in its normalized form to be

$$\tilde{u}\,^l_{i\omega}(\tau)=[(\omega^2+l^2)\cdots(\omega^2+1^2)]^{-1/2}L_+^l\cdots L_+^1\frac{e^{i\omega\tau}}{\sqrt{4\pi}} \ . \tag{5.2}$$

Also,

$$\tilde{u}\,^{l+1}_{i\omega}(\tau)=[\omega^2+(l+1)^2]^{-1/2}L_+^{l+1}\,\tilde{u}\,^l_{i\omega}(\tau) \ . \tag{5.3}$$

For $l=1$, we obtain, from Eq. (5.2),

$$\begin{aligned}\tilde{u}\,^1_{i\omega}(\tau)&=(\omega^2+1)^{-1/2}\left[\tanh\tau-\frac{d}{d\tau}\right]\frac{e^{i\omega\tau}}{\sqrt{4\pi}}\\&=A_1(\tau)e^{-i\theta_1(\tau)}\frac{e^{i\omega\tau}}{\sqrt{4\pi}} \ ,\end{aligned} \tag{5.4}$$

where

$$A_1(\tau)=\left[\frac{\omega^2+\tanh^2\tau}{\omega^2+1}\right]^{1/2} \tag{5.5}$$

and

$$\theta_1(\tau)=\arctan\frac{\omega}{\tanh\tau} \ . \tag{5.6}$$

As $\tau\to\infty$, $A_1(\tau)\to 1$, $\theta_1(\tau)\to\arctan\omega$. One can continue this process and see that any higher-mode solution can always be written in the from

$$u^l_{i\omega}(\tau)=A_l(\tau)e^{-i\theta_l(\tau)}\frac{e^{i\omega\tau}}{\sqrt{4\pi}} \ . \tag{5.7}$$

Therefore, any higher-mode solution is nothing but a phase-modulated wave function of the lowest-mode solution. However, this is a transient case of phase modulation, since, in every case, θ_l very rapidly approaches a constant value, as τ increases. The amplitude of this modulating wave is also time dependent. However, as τ increases, $A_l(\tau)$ also approaches the value l very rapidly. So, for a sufficiently large value of τ, any higher-mode solution will look like

$$u^l_{i\omega}(\tau)=\frac{e^{i(\omega\tau-\theta_c)}}{\sqrt{4\pi}} \ , \tag{5.8}$$

where θ_c is a constant.

For the sake of completeness, we are writing below the explicit expressions for A_l and θ_l for a few other higher-mode solutions.

$\underline{l=2}$:

$$A_2^2=\frac{\omega^4+(3\tanh^2\tau+2)\omega^2+(9\tanh^4\tau-6\tanh^2\tau+1)}{\omega^4+5\omega^2+4}$$

$$\tan\theta_2=\frac{3\omega\tanh\tau}{3\tanh^2\tau-(1+\omega^2)} \ .$$

$\underline{l=3}$: A_3^2

$$\text{numerator}=\omega^6+\omega^4(8+6\tanh^2\tau)+\omega^2(45\tanh^4\tau-12\tanh^2\tau+16)+(225\tanh^6\tau-270\tanh^4\tau+81\tanh^2\tau)\ ,$$

$$\text{denominator}=\omega^6+14\omega^4+49\omega^2+36\ ,$$

$$\tan\theta_l=\frac{15\omega\tanh^2\tau-4\omega-\omega^3}{15\tanh^3\tau-(6\omega^2+9)\tanh^2\tau}\ ,$$

etc.

VI. CONCLUDING REMARKS

In the preceding sections we have developed the mathematical formalism for and studied the behavior of scalar field in the Witten bubble spacetime. We have written the eigenfunctions of the temporal equation as hyperbolic harmonics which manifest wave behavior in all of its modes. By choosing the null coordinate system, we could transform the radial equation into a very simple Schrödinger form. A general basis of this operation, which can be used in similar problems in curved spacetime, is also discussed in the Appendix. Studying the scattering problem, we have observed that our results are consistent with the concept of bubble as a perfectly reflecting wall. At large enough distance, we could get both incoming and outgoing waves with the same amplitude, thus giving the value of the reflection coefficient to be unity. On the other hand near the bubble, the wave behavior gets distorted. The higher the frequency the lower is the distortion produced by spacetime. A high-frequency wave starts manifesting its wave behavior very near to the bubble wall.

The study of bound states confirms the stability of the spacetime against arbitrary scalar perturbations. For a complete stability analysis of such a spacetime, the study of electromagnetic and gravitational perturbations is also necessary. Our study may be able to project a clearer concept of some inherent aspects of the Witten bubble and lead to further studies related to such a spacetime. The mathematical formalism developed by us may be useful in similar problems in other curved spacetimes.

ACKNOWLEDGMENTS

It is a pleasure to thank B. R. Iyer, Sai Iyer, and H. S. Mani for helpful discussions. One of the authors (B.B.) would like to thank the Department of Science and Technology, New Delhi for financial support.

APPENDIX A: ALTERNATIVE FORM OF SOLUTIONS

Here, we are presenting alternative forms of solutions of both radial and τ equations, obtained by different procedures. Since these solutions are not convenient for formulating scattering and other problems in this spacetime, we have not used them in our work. However, for the sake of completeness and for possible use elsewhere, we are describing these here.

τ equation

The τ part of the separated Klein-Gordon equation, which we shall denote here as $\tilde{T}$, instead of $\tilde{T}^{\,l}_{i\omega}$, satisfies Eq. (3.7).

Let us do the coordinate transformation

$$\tau\to i\left[\frac{\pi}{2}-\tau'\right]\ . \tag{A1}$$

This is equivalent to the Euclidean continuation of Eq. (3.8).

Then introducing the variable

$$p=\cos\tau'\ , \tag{A2}$$

we can write Eq. (3.7) as

$$(1-p^2)\frac{d^2\tilde{T}^E}{dp^2}-3p\frac{d\tilde{T}^E}{dp}+\left[\alpha-\frac{l(l+1)}{1-p^2}\right]\tilde{T}^E=0\ , \tag{A3}$$

where by $\tilde{T}^E$ we represent the Euclidean continuation of function $\tilde{T}$. Also,

$$\alpha=-\omega^2-1\ . \tag{A4}$$

Defining

$$\tilde{z}=(1-p^2)^{-l/2}\tilde{T}^E\ , \tag{A5}$$

we get

$$(1-p^2)\frac{d^2\tilde{z}}{dp^2}-p(2l+3)\frac{d\tilde{z}}{dp}+[K(K+2)-l(l+2)]\tilde{z}=0\ , \tag{A6}$$

where we have chosen

$$\alpha=K(K+2)\ . \tag{A7}$$

Equation (A6) can now be written as

$$(1-p^2)\frac{d^2\tilde{z}}{dp^2}-p(2\mu+1)\frac{d\tilde{z}}{dp}+\lambda(\lambda+2\mu)\tilde{z}=0 \tag{A8}$$

by defining

$$\mu=l+1,\quad \lambda=K-l\ . \tag{A9}$$

This is the standard Gegenbauer equation, which has two solutions expressed in terms of hypergeometric series:

$$C_\lambda^\mu(p)=\frac{\Gamma(2\mu+\lambda)}{\Gamma(\lambda+1)\Gamma(2\mu)}F\left[-\lambda,\lambda+2\mu;\mu+\frac{1}{2};\frac{1-p}{2}\right], \quad (A10)$$

$$D_\lambda^\mu(p)=2^{-1-\lambda}\frac{\Gamma(\mu)\Gamma(2\mu+\lambda)}{\Gamma(\mu+\lambda+1)}F\left[\mu+\frac{1}{2}\lambda,\mu+\frac{\lambda}{2}+\frac{1}{2};\mu+\lambda+1;p^2\right]. \quad (A11)$$

Therefore, using (A5), we get two solutions for Eq. (A3):

$$\tilde{T}_1^E=(\sin\tau')^l C_\lambda^\mu(\cos\tau'), \quad (A12)$$

$$\tilde{T}_2^E=(\sin\tau')^l D_\lambda^\mu(\cos\tau'). \quad (A13)$$

Performing the reverse transformation of Eq. (A1), or equivalently, continuing back to the Minkowski solutions, we obtain

$$\tilde{T}_1=(\cosh\tau)^l C_\lambda^\mu(-i\sinh\tau), \quad (A14)$$

$$\tilde{T}_2=(\cosh\tau)^l D_\lambda^\mu(-i\sinh\tau). \quad (A15)$$

Radial equation

We can get a Frobenius series solution of Eq. (3.14), if we assume it first to be of the form

$$\tilde{\mathcal{R}}\sim(r-R)^\alpha\sum_{n=0}^{\infty}a_n(r-R)^n .$$

Then, substituting this in Eq. (3.15), we obtain the equation

$$\alpha(\alpha-1)z^{\alpha-2}\sum_0^\infty a_nz^n+2\alpha z^{\alpha-1}\sum_1^\infty na_nz^{n-1}+z^\alpha\sum_2^\infty n(n-1)a_nz^{n-2}$$
$$+\left[1+\frac{3z}{2R}+\frac{5z^2}{4R^2}+\frac{27z^3}{8R^3}+\cdots\right]\left[\alpha z^{\alpha-2}\sum_0^\infty a_nz^n+z^{\alpha-1}\sum_1^\infty na_nz^{n-1}\right]+\frac{\omega^2+1}{2R}\sum_0^\infty\left[-\frac{z}{2R}\right]^n z^{\alpha-1}\sum_0^\infty a_nz^n=0, \quad (A16)$$

where $z=r-R$.

Equating the coefficients of different powers of Z, we obtain $\alpha=0$ and can determine different a_n, so that the solution turns out to be

$$\tilde{\mathcal{R}}_1=a_0\left[1-\frac{\omega^2+1}{2R}(r-R)+\frac{\omega^4+6\omega^2+5}{16R^2}(r-R)^2-\cdots\right]. \quad (A17)$$

A second solution can be found to be of the form

$$\tilde{\mathcal{R}}_2=\ln(r-R)\sum_0^\infty a_n(r-R)^2+\sum_0^\infty b_n(r-R)^n, \quad (A18)$$

where a_n,b_n are constants to be determined from Eq. (A15). The first solution behaves properly throughout the range of the variable r, whereas the second solution blows up at $r=R$.

APPENDIX B

Coordinate transformations such as Eq. (3.15) are widely used in many situations both in flat and curved spacetimes to bring the radial equation to the Schrödinger form, e.g., the "tortoise" coordinates in the Schwarzschild spacetime. However, these were considered to be just some mathematical operation. Their actual significance does not seem to have been discussed in the literature. Here, we shall attempt to give a general basis for this.

For a metric in which the Klein-Gordon equation is separable and g_{00},g_{rr} are independent of time, one can always obtain the radial equation in Schrödinger form just by choosing a null coordinate system.

In a static spacetime, if one solves the Klein-Gordon equation for a massive scalar wave, one obtains the following eigenvalue equation, after separating out the time-part which will be of the form $e^{+i\omega t}$:

$$\frac{1}{\sqrt{-g}}g_{00}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi)+g_{00}m^2\Phi=\omega^2\Phi . \quad (B1)$$

Now, if it is a two-dimensional metric

$$ds^2=-A(r)dt^2+B(r)dr^2, \quad (B2)$$

let us try to get a null-vector η_i by introducing a new coordinate r^*,

$$\eta^i=\left[1,\frac{dr}{dr^*}\right]$$

so that

$$\eta^i \eta_i = -A + \left[\frac{dr}{dr^*} \right]^2 B = 0$$

or

$$\frac{dr^*}{dr} = \left[\frac{B}{A} \right]^{1/2} . \tag{B3}$$

Then $ds^2 = A(-dt^2 + dr^{*2})$ and Eq. (B1) becomes

$$-\frac{d^2\Phi}{dr^{*2}} + m^2 A \Phi = \omega^2 \Phi . \tag{B4}$$

Only the mass term contributes to the effective potential. For $m=0$, this is just a free wave solution.

In a general dimensional spacetime, if $g_{\alpha\alpha}$, where $\alpha \neq t, r$ is r dependent, then there will be an extra first-derivative term in Eq. (B4). This first-derivative term can be easily eliminated by suitably defining a new radial function and the Schrödinger equation can be obtained. The r dependence of $g_{\alpha\alpha}$ will actually contribute to the effective potential of this equation.

If $g_{\alpha\alpha}$ is also time dependent, the eigenvalue equation of Φ will not be of the form (B1). But one can easily see that this will not create any problem in getting Schrödinger form by choosing a null coordinate.

[1]E. Witten, Nucl. Phys. **B195**, 481 (1982).
[2]D. R. Brill and M. D. Matlin, Phys. Rev. D **39**, 3151 (1989).
[3]D. Bailin and A. Love, Rep. Prog. Phys. **50** (9), 1087 (1987).
[4]U. H. Gerlach, Phys. Rev. D **28**, 761 (1983).
[5]E. Merzbacher, *Quantum Mechanics,* 2nd ed. (Wiley, New York, 1970), Sec. 7.3, p. 123.