A combinatorial approach for studying local operations and classical communication transformations of multipartite states

Sudhir Kumar Singh[a)]
Department of Electrical Engineering, University of California, Los Angeles, California 90095

Sudebkumar Prasant Pal[b)]
Department of Computer Science and Engineering, Indian Institute of Technology, Kharagpur-721302, India

Somesh Kumar[c)]
Department of Mathematics, Indian Institute of Technology, Kharagpur-721302, India

R. Srikanth[d)]
Raman Research Institute, Sadashiva Nagar, Bangalore-560080, Karnataka, India

We develop graph theoretic methods for analyzing maximally entangled pure states distributed between a number of different parties. We introduce a technique called bicolored merging, based on the monotonicity feature of entanglement measures, for determining combinatorial conditions that must be satisfied for any two distinct multiparticle states to be comparable under local operations and classical communication. We present several results based on the possibility or impossibility of comparability of pure multipartite states. We show that there are exponentially many such entangled multipartite states among n agents. Further, we discuss a new graph theoretic metric on a class of multipartite states, and its implications. © 2005 American Institute of Physics. [DOI: 10.1063/1.2142840]

I. INTRODUCTION

Given the extensive use of quantum entanglement as a resource for quantum information processing,[6,19,23] the theory of entanglement, in particular, entanglement quantification, is a topic important to quantum information theory. However, apart from a limited number of cases like low dimension Hilbert spaces and for pure states, the mathematical structure of entanglement is not yet fully understood. The entanglement properties of bipartite states have been widely explored (see Refs. 8 and 12 for a comprehensive review). This has been aided by the fact that bipartite states possess the nice mathematical property in the form of the Schmidt decomposition,[19] the Schmidt coefficients encompassing all their nonlocal properties. No such simplifying structure is known in the case of larger systems. Approaches using certain generalizations of Schmidt decomposition[4,14,20] and group theoretic or algebraic methods,[15–17] have been taken in this direction. A number of methods for comparing or quantifying or qualifying entanglement have been proposed for bipartite systems and/or pure states such as entanglement of formation,[3] entanglement cost,[3,26] distillable entanglement,[3,21] relative entropy of entanglement,[11] negativity,[27]

[a)]Electronic mail: suds@ee.ucla.edu
[b)]Electronic mail: spp@cse.iitkgp.ernet.in
[c)]Electronic mail: smsh@maths.iitkgp.ernet.in
[d)]Electronic mail: srik@rri.res.in

    46, 122105-1

concurrence,[28] and entanglement witnesses.[13] However, these quantifications do not always lend themselves to being computed, except in some restricted situations. As such, a general formulation is still an open problem.

It is known that state transformations under local operations and classical communication (LOCC) are very important to quantifying entanglement because LOCC can at the best increase only classical correlations. Therefore a good measure of entanglement is expected not to increase under LOCC. A necessary and sufficient condition for the possibility of such transformations in the case of bipartite states was given by Nielsen.[18] An immediate consequence of his result was the existence of *incomparable* states (the states that cannot be obtained by LOCC from one another). Bennett *et al.*,[4] formalized the notions of reducibility, equivalence and incomparability to multipartite states and gave a sufficient condition for incomparability based on *partial* entropic criteria.

In this work, our principal aim is not to quantify entanglement, but to develop graph theoretic techniques to analyze the comparability of maximally entangled multipartite states of several qubits distributed between a number of different parties. We obtain various qualitative results concerning reversibility of operations and comparability of states by observing the combinatorics of multiparitite entanglement. For our purpose, it is sufficient to consider the graph theoretic representation of various maximally entangled states (represented by specific graphs built from EPR, GHZ, and so on). Although this might at first seem overly restrictive, we will in fact be able to demonstrate a number of new results. Furthermore, being based only on the monotonicity principle, it can be adapted to any specific quantification of entanglement. Therefore, our approach is quite generic, in principle applicable to all entanglement measures. Since the entanglement of maximally entangled states is usually represented by integer values, it turns out that we can analyze entangled systems simply by studying the combinatorial properties of graphs and set systems representing the states. The basic definitions and concepts are introduced through the framework set in Sec. II. We introduce a technique called *bicolored merging* in Sec. III, which is essentially a combinatorial way of quantifying maximal entanglment between two parts of the system, and inferring transformation properties to be satisfied by the states.

In Sec. IV, we present our first result: the impossibility of obtaining two Einstein-Podolsky-Rosen (EPR) pairs among three players starting from a Greenberger-Horne-Zeilinger (GHZ) state (Theorem 2). We then show that this can be used to establish the impossibility of implementing a two-pronged teleportation (called *selective teleportation*) given preshared entanglement in the form of a GHZ state. We then demonstrate various classes of incomparable multipartite states in Sec. V. Finally, we discuss the minimum number of copies of a state required to prepare another state by LOCC and present bounds on this number in terms of the *quantum distance* between the two states in Sec. VII.

We believe that our combinatorial approach vastly simplifies the study of entanglement in very complex systems. Moreover, it opens up the road for further analysis, for example, to interpret entanglement topologically. In future works, we intend to apply and extend these insights to nonmaximal and mixed multipartite states, and to combine our approach with a suitable measure of entanglement.

II. THE COMBINATORIAL FRAMEWORK

In this section we introduce a number of basic concepts useful to describe combinatorics of entanglement. First, an *EPR graph* $G(V,E)$ is a graph whose vertices are the players ($\in V$) and edges ($\in E$) represent shared entanglement in the form of an EPR pair. Formally, we have the following.

Definition 1: EPR graph. For n agents $A_1, A_2, \ldots, A_n$ an undirected graph $G=(V,E)$ is constructed as follows: $V=\{A_i : i=1,2,\ldots,n\}$, $E=\{\{A_i,A_j\}:A_i \text{ and } A_j \text{ share an EPR pair, } 1 \leq i,j. \leq n; i \neq j\}$. The graph $G=(V,E)$ thus formed is called the EPR graph of the n agents.

A spanning tree is a graph which connects all vertices without forming cycles (i.e., loops). Accordingly, we have the following.

Definition 2: Spanning EPR tree. A spanning tree is a connected, undirected graph linking all vertices without forming cycles. An EPR graph $G=(V,E)$ is called a spanning EPR tree if the

undirected graph $G=(V,E)$ is a spanning tree.

The above notions are generalized to more general multipartite entanglement by means of the concept of a *hypergraph*. A usual graph is built up from edges, where a normal edge links precisely two vertices. A hyperedge is a generalization that links r vertices, where $r \geqslant 2$. A graph endowed with at least one hyperedge is called a hypergraph. From the combinatorial viewpoint, a simple and interesting connection can be made between entanglement and hyperedges, an n-cat state (also sometimes called an n-GHZ state) corresponds to a hyperedge of size n. In particular, an EPR state corresponds to a simple edge connecting only two vertices. Formally, we have the following.

Definition 3: Entangled hypergraph. Let S be the set of n agents and $F=\{E_1, E_2, \ldots, E_m\}$, where $E_i \subseteq S; i=1,2,\ldots,m$ and E_i is such that its elements (agents) are in $|E_i|$-CAT state. The hypergraph (set system) $H=(S,F)$ is called an entangled hypergraph of the n agents.

A graph is connected if there is a path (having a length of one or more edges) between any two vertices. Accordingly, we have the following.

Definition 4: Connected entangled hypergraph. A sequence of j hyperedges $E_1, E_2, \ldots, E_j$ in a hypergraph $H=(S,F)$ is called a hyperpath (path) from a vertex a to a vertex b if

(1)   *E_i and E_{i+1} have a common vertex for all $1 \leqslant i \leqslant j-1$,*
(2)   *a and b are agents in S,*
(3)   *$a \in E_1$, and*
(4)   *$b \in E_j$.*

If there is a hyperpath between every pair of vertices of S in the hypergraph H, we say that H is connected.

Analogous to a spanning EPR tree we have the following.

Definition 5: Entangled hypertree. A connected entangled hypergraph $H=(S,F)$ is called an entangled hypertree if it contains no cycles, that is, there do not exist any pair of vertices from S such that there are two distinct paths between them.

Further, we have the following.

Definition 6: r-uniform entangled hypertree: An entangled hypertree is called an r-uniform entangled hypertree if all of its hyperedges are of size r for $r \geqslant 2$.

In ordinary graphs, a vertex that terminates, i.e., has precisely a single edge linked to it is called a terminal or pendent vertex. This concept is extended to the case of hypergraphs.

Definition 7: Pendant vertex. A vertex of a hypergraph $H=(S,F)$ such that it belongs to only one hyperedge of F is called a pendant vertex in H. Vertices which belong to more than one hyperedge of H are called nonpendant.

In the paper we use polygons for pictorially representing an entangled hypergraph of multipartite states. (There should be no confusion with a closed loop of EPR pairs because we consider only tree structured states.) A hyperedge representing an n-CAT amongst the parties $\{i_1, i_2, \ldots, i_n\}$ is pictorially represented by an n-gon with vertices distinctly numbered by $i_1, i_2, \ldots, i_n$. We write these vertices $i_1, i_2, \ldots, i_n$ corresponding to the n vertices of the n-gon in the pictorial representation in arbitrary order. This only means that out of n qubits of the n-CAT, one qubit is with each of the n parties.

A result we will require frequently is that there exist teleportation[2] protocols to produce n-partite entanglement starting from pairwise entanglement shared along any spanning tree connecting the n parties. That is, there exist LOCC protocols to turn a n-party spanning EPR tree into an n-regular hypergraph consisting of a single hyperedge of size n. The protocol is detailed in Refs. 24 and 23, but the basic idea is readily described. It is essentially a scheme to deterministically create a maximally entangled n-cat state from $n-1$ EPR pairs shared along a spanning tree. Briefly, the protocol consists in teleporting entanglement along a spanning tree. Players not on terminal vertices along the tree execute the following subroutine. Suppose player Alice shares an m-cat with $(m-1)$ preceding players along the tree and wishes to create an $(m+1)$-cat state including Bob, the next player down the tree. First she entangles an auxiliary particle with her
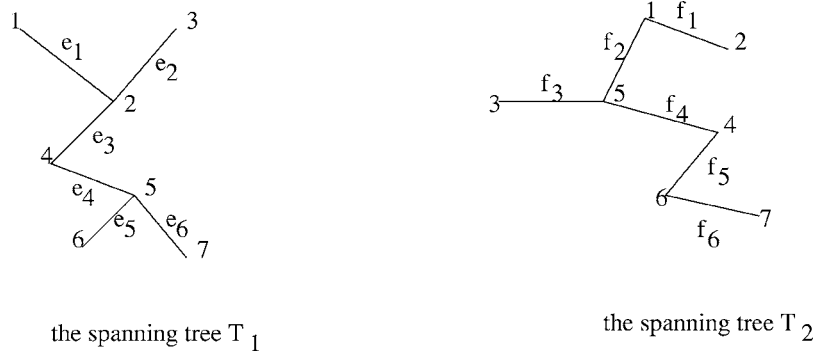
FIG. 1. Spanning EPR trees T_1 and T_2.

particle in the m-cat state by means of local operation. She then uses her EPR pair shared with Bob to teleport the state of the auxiliary particle to Bob. The $(m+1)$ players, including Alice and Bob, now share an $(m+1)$-cat state, as desired.

Another result we will require in some of our proofs, given as the theorem below, is that the spanning EPR tree mentioned above is also a necessary condition to prepare an n-CAT state starting from shared EPR pairs.

Theorem 1: *Given a communication network of n agents with only EPR pairs permitted for pairwise entanglement between agents, a necessary condition for creation of a n-CAT state is that the EPR graph of the n agents must be connected.*

Proof of the theorem is given in Appendix A using our method of bicolored merging developed in Sec. III.

III. BICOLORED MERGING

Monotonicity is easily the most natural characteristic that should be satisfied by all entanglement measures.[12] It requires that any appropriate measure of entanglement must not change under local unitary operations and more generally, the expected entanglement must not increase under LOCC. We should note here that in LOCC, LO involves unitary transformations, additions of ancillas (that is, enlarging the Hilbert space), measurements, and throwing away parts of the system, each of these actions performed by one party on his or her subsystem. CC between the parties allows local actions by one party to be conditioned on the outcomes of the earlier measurements performed by the other parties.

Apart from monotonicity, there are certain other characteristics required to be satisfied by entanglement measures. However, monotonicity itself vastly restricts the choice of entanglement measures (for example, marginal entropy as a measure of entanglement for bipartite pure states or entanglement of formation for mixed states). In the present work, we find that monotonicity, where proven for a particular entanglement measure candidate, restricts a large number of state transformations and gives rise to several classes of incomparable (multipartite) states. So, in order to study the possible state transformations of (multipartite) states under LOCC, it would be interesting to look at the kind of state transforms under LOCC which monotonicity does not allow. We can observe that monotonicity does not allow the preparation of $n+1$ or more EPR pairs between two parties starting from only n EPR pairs between them. In particular, it is not possible to prepare two or more EPR pairs between two parties starting only with a single EPR pair and only LOCC. This is an example of impossible state transformation in the bipartite case as dictated by the monotonicity postulate. We anticipate that a large class of multipartite states could also be shown to be incomparable by using impossibility results for the bipartite case through suitable reductions. For instance, consider transforming (under LOCC) the state represented by a spanning EPR tree, say T_1, to that of the state represented by another spanning EPR tree, say T_2 (see Fig. 1). This transformation can be shown to be impossible by reducing to the bipartite case as follows: We
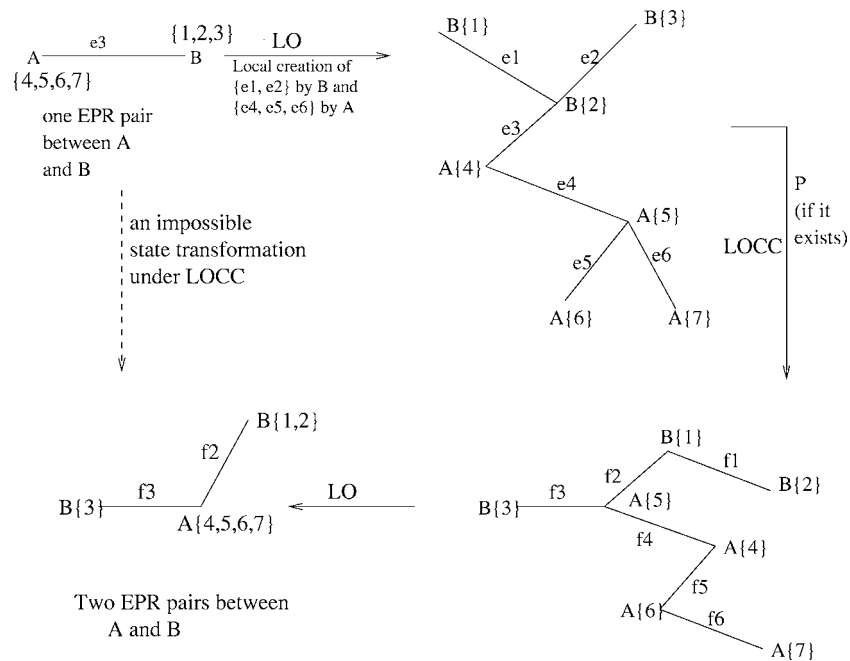
FIG. 2. Converting T_1 to T_2 under LOCC.

assume for the sake of contradiction that there exists a protocol P which can perform the required transformation. It is easy to see that the protocol P is also applicable in the case when a party A possesses all the qubits of parties 4, 5, 6, and 7 and another party B possesses all the qubits of the parties 1, 2, and 3. This means that party A is playing the role of parties 4, 5, 6, and 7 and B is playing the role of parties 1, 2, and 3. Clearly, any LOCC actions done within group {1, 2, 3} ({4, 5, 6, 7}) is a subset of LO available to B (A) and any CC done between one party from {1, 2, 3} and the other from {4, 5, 6, 7} is managed by CC between B and A.

Therefore, starting only with one edge (e_3) they eventually construct T_1 just by LO (by local creation of EPR pairs representing the edges e_1, e_2, e_4, e_5, and e_6 ({e_1,e_2} by B and {e_4,e_5,e_6} by A). They then apply protocol P to obtain T_2 with the edges f_1, f_2, f_3, f_4, f_5, and f_6 (refer to Fig. 2). All edges except f_2 and f_3 are local EPR pairs (that is, both qubits are with the same party, A or B). Now the parties A and B share two EPR pairs in the form of the edges f_2 and f_3, even though they started sharing only one EPR pair. But this is in contradiction with monotonicity, that expected entanglement should not increase under LOCC. Hence, we can conclude that such a protocol P cannot exist.

The approach we took in the above example could also be motivated from the marginal entropic criterion (noting that this criterion in essence is also a direct implication of monotonicity). As clear from the above example, the above scheme aims to create a bipartition among the n players in such a way that the marginal entropy of each partition is different for the two states. In many cases, this difference will simply correspond to different number of EPR pairs shared between the two partitions. Given two multipartite states, the relevant question is, "is there a bipartition such that the marginal entropy for the two states is different?" If yes, then the state (configuration of entanglement) corresponding to the higher entropy cannot be obtained from that to lower entropy by means of LOCC. It is convenient to imagine the two partitions being colored distinctly to identify the partitions which they make up.

In general, suppose we want to show that the multipartite state $|\psi\rangle$ cannot be converted to the multipartite state $|\phi\rangle$ by LOCC. This can be done by showing an assignment of the qubits (of all parties) to only two parties such that $|\psi\rangle$ can be obtained from n ($n=0,1,2,\dots$) EPR pairs between the two parties by LOCC while $|\phi\rangle$ can be converted to more than n EPR pairs between

the two parties by LOCC. This is equivalent to saying that each party is given either of two colors (say A or B). Finally all qubits with parties colored with color A are assigned to the first party (say A) and that with parties colored with the second color to the second party (say B). This coloring is done in such a way that the state $|\psi\rangle$ can be obtained by LOCC from less number of EPR pairs between A and B than that obtained from $|\phi\rangle$ by LOCC. Local preparation (or throwing away) of EPR pairs is what we call merging in combinatorial sense. Keeping this idea in mind, we now formally introduce the idea of bicolored merging for such reductions in the case of the multipartite states represented by EPR graphs and entangled hypergraphs.

Suppose that there are two EPR graphs $G_1=(V,E_1)$ and $G_2=(V,E_2)$ on the same vertex set V (this means that these two multipartite states are shared amongst the same set of parties) and we want to show the impossibility of transforming G_1 to G_2 under LOCC, then this is reduced to a bipartite LOCC transformation which violates monotonicity, as follows:

(1)   Bicoloring, assign either of the two colors A or B to every vertex, that is, each element of V.
(2)   Merging, for each element $\{v_i,v_j\}$ of E_1, merge the two vertices v_i and v_j if and only if they have been assigned the same color during the bicoloring stage and assign the same color to the merged vertex. Call this graph obtained from G_1 as BCM (bicolored-merged) EPR graph of G_1 and denote it by G_1^{bcm}. Similarly, obtain the BCM EPR graph G_2^{bcm} of G_2.
(3)   The bicoloring and merging is done in such a way that the graph G_2^{bcm} has more number of edges than that of G_1^{bcm}.
(4)   Give all the qubits possessed by the vertices with color A to the first party (say, party A) and all the qubits possessed by the vertices with color B to the second party (say, party B). Combining this with the previous steps, it is ensured that in the bipartite reduction of the multipartite state represented by G_2, the two parties A and B share more number of EPR pairs (say, state $|\psi_2\rangle$) than that for G_1 (say, state $|\psi_1\rangle$).

We denote this reduction as $G_1 \not> G_2$. Now if there exits a protocol P which can transform G_1 to G_2 by LOCC, then P can also transform $|\psi_1\rangle$ to $|\psi_2\rangle$ just by LOCC as follows: A (B) will play the role of all vertices in V which were colored as A (B). The edges which were removed due to merging can easily be created by local operations (local preparation of EPR pairs) by the party A (B) if the color of the merged end vertices of the edge was assigned color A (B). This means that starting from $|\psi_1\rangle$ and only LO, G_1 can be created. This graph is virtually amongst $|V|$ parties even though there are only two parties. The protocol P then, can be applied to G_1 to obtain G_2 by LOCC. Subsequently $|\psi_2\rangle$ can be obtained by the necessary merging of vertices by LO, that is by throwing away the local EPR pair represented by the edges between the vertices being merged. Since the preparation of $|\psi_2\rangle$ from $|\psi_1\rangle$ by LOCC violates the monotonocity postulate, such a protocol P cannot exist. An example of bicolored merging for EPR graphs has been illustrated in Fig. 3.

The bicolored merging in the case of entangled hypergraphs is essentially the same as that for EPR graphs. For the sake of completeness, we present it here. Suppose there are two entangled hypergraphs $H_1=(S,F_1)$ and $H_2=(S,F_2)$ on the same vertex set S (that is, the two multipartite states are shared amongst the same set of parties) and we want to show the impossibility of transforming H_1 to H_2 under LOCC. Transformation of H_1 to H_2 can be reduced to a bipartite LOCC transformation which violates monotonicity thus proving the impossibility. The reduction is done as follows:

(1)   Bicoloring, assign either of the two colors A or B to every vertex, that is, each element of S.
(2)   Merging, for each element $E=\{v_{i1},v_{i2},\ldots,v_{ij}\}$ of $F_1(F_2)$, merge all vertices with color A to one vertex and those with color B to another vertex and give them colors A and B, respectively. This merging collapses each hyperedge to either a simple edge or a vertex and thus the hypergraph reduces to a simple graph with vertices assigned with either of the two colors A or B. Call this graph obtained from H_1 as BCM EPR graph of H_1 and denote it by H_1^{bcm}. Similarly obtain the BCM EPR graph H_2^{bcm} of H_2.
(3)   The bicoloring and merging is done in such a way that the graph H_2^{bcm} has more number of edges than that of H_1^{bcm}.
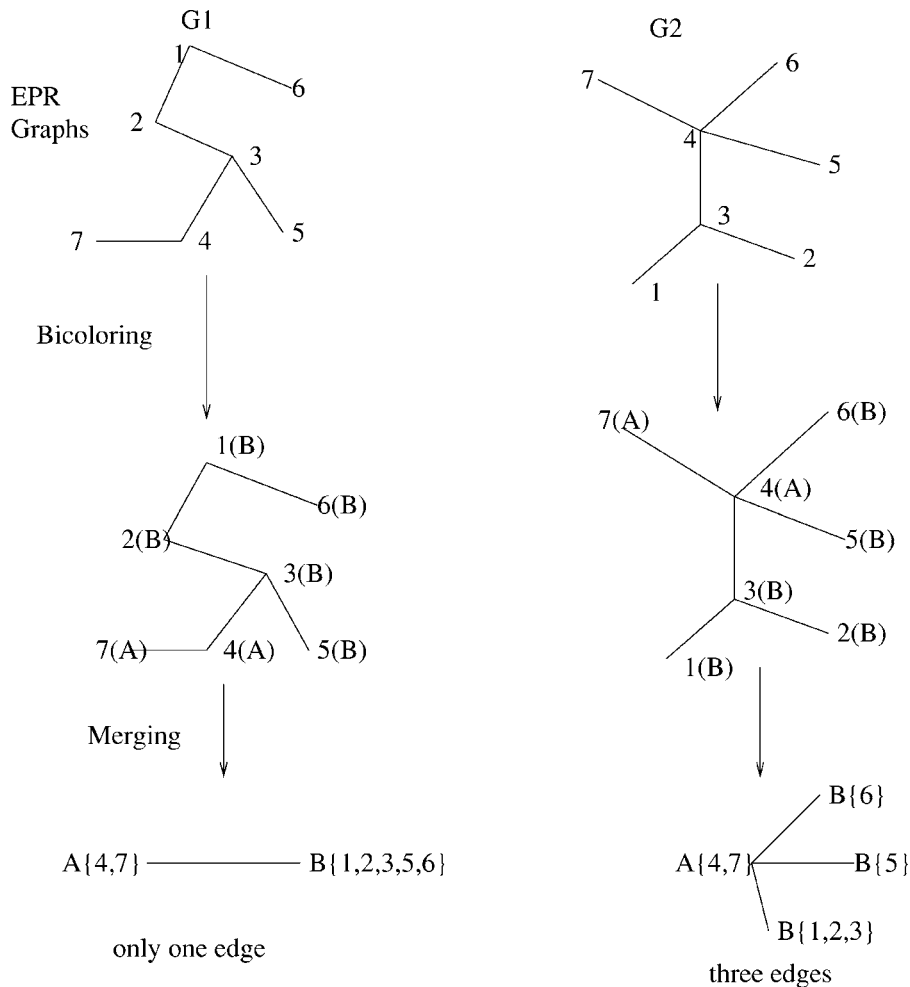
FIG. 3. Bicolored merging of EPR graphs.

(4)   Give all the qubits possessed by the vertices with color A to the party one (say party A) and all the qubits possessed by the vertices with color B to the second party (say party B).

We denote the above reduction as $H_1 \succ H_2$. The rest of the discussion is similar to that for the case of EPR graphs given before. In Fig. 4, we demostrate the bicolored merging of entangled hypergraphs. Note that the two entangled hypergraphs H_1 and H_2 are LOCC comparable only if one of $H_1 \succ H_2$ and $H_2 \succ H_1$ is not true. Equivalently, if both of $H_1 \succ H_2$ and $H_2 \succ H_1$ hold, then the entangled hypergraphs H_1 and H_2 are incomparable.

It is also interesting to note at this point that LOCC incomparability shown by using the method of bicolored merging is in fact *strong incomparability* as defined in Ref. 1. We would also like to stress that any kind of reduction (in particular, various possible extensions of bicolored merging) which leads to the violation of *any* of the properties of a potential entanglement measure, is pertinent to show the impossibility of many multipartite state transformations under LOCC. Since the bipartite case has been extensively studied, such reductions can potentially provide many ideas about multipartite case by just exploiting the results from the bipartite case. In particular, the definitions of EPR graphs and entangled hypergraphs could also be suitably extended to capture more types of multipartite pure states and even mixed states and a generalization of the idea of bicolored merging as a suitable reduction for this case could also be worked out.
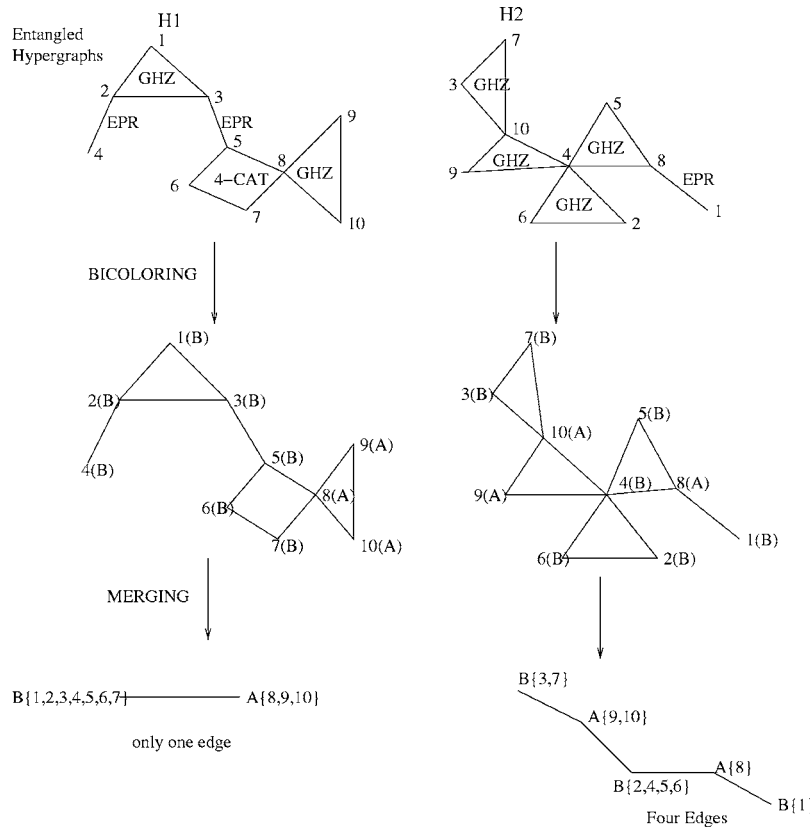
FIG. 4. Bicolored merging of entangled hypergraphs.

IV. LOCC INCOMPARABILITY AND SELECTIVE TELEPORTATION

We know that a GHZ state amongst three agents A, B, and C can be prepared from EPR pairs shared between any two pairs of the three agents using only LOCC.[24,7,29,30] We consider the problem of *reversing* this operation, that is, whether it is possible to construct two EPR pairs between any two pairs of the three agents from a GHZ state amongst the three agents, using only LOCC. By using the method of bicolored merging, we answer this question in the negative by establishing the following theorem.

Theorem 2: *Starting from a GHZ state shared amongst three parties in a communication network, two EPR pairs cannot be created between any two sets of two parties using only LOCC.*

Proof: Suppose there exists a protocol P for *reversing* a GHZ state into two EPR pairs using only LOCC. In particular, suppose protocol P starts with a GHZ state amongst the agents A, B, and C, and prepares EPR pairs between any two pairs of A, B, and C (say, $\{A,C\}$, and $\{B,C\}$, corresponding to configuration G_1 as shown in Fig. 5). Since we can prepare the GHZ state from EPR pairs between any two pairs of the three agents, we can prepare the GHZ state starting from EPR pairs between A and B, and A and C. Once the GHZ state is prepared, we can apply protocol P to construct EPR pairs between A and C and between B and C using only LOCC (i.e., configuration $G_2 \equiv \{\{A,C\},\{B,C\}\}$). So, we can use only LOCC to convert a configuration where EPR pairs exist between A and C and between A and B, to a configuration where EPR pairs are shared between A and C and between B and C. The possibility of P means that the marginal entropy of C can be increased using only LOCC, which is known to be impossible.   □

The same result could also be achieved by similar bicolored merging directly applied on the GHZ state and any of G_1 or G_2 but we prefer the above proof for stressing the argument on the symmetry of G_1 and G_2 with respect to the GHZ. Moreover, this proof gives an intuition about
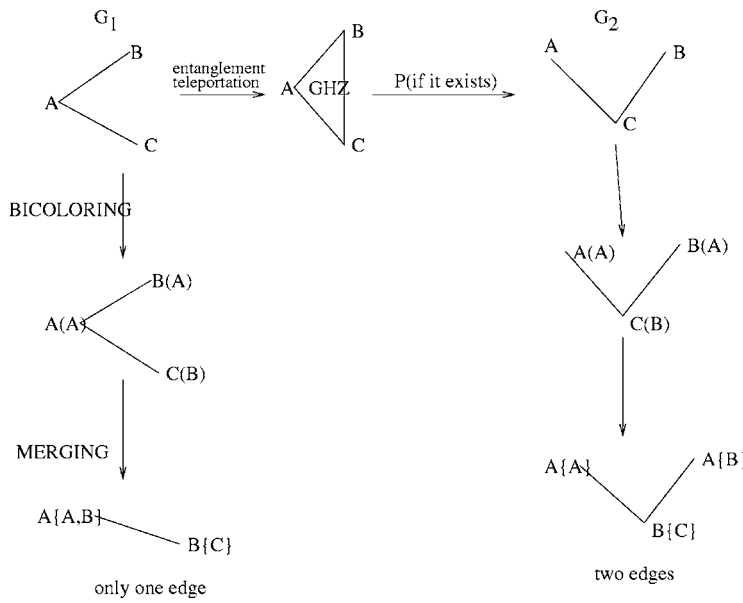
FIG. 5. LOCC irreversibility of the process [2 EPR→GHZ].

possibility of incomparability amongst spanning EPR trees as G_1 and G_2 are two distinct spanning EPR trees on three vertices. We prove this general result in the Theorem 8.

The above theorem motivates us to propose some kind of comparison between a GHZ state and two pairs of EPR pairs in terms of the nonlocal correlations they possess. In this sense, therefore, a GHZ state may be viewed as less than two EPR pairs. It is easy to see that an EPR pair between any two parties can be obtained starting only from a GHZ state shared amongst the three parties and LOCC. The third party will just do a measurement in the diagonal basis and send the result to the other two. By applying the corresponding suitable operations they get the required EPR pair. From Theorem 1, we observe that a single EPR pair, between any two of the three parties, is not sufficient for preparing a GHZ state amongst the three parties using only LOCC. These arguments can be summarized in the following theorem.

Theorem 3: *1-EPR pair $<_{LOCC}$ a GHZ state $<_{LOCC}$ 2-EPR pairs.*

An interesting problem in quantum information theory is that of *selective teleportation.*[22] Given three agents A, B, and C, and two qubits of unknown quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ with A, the problem is to send $|\psi_1\rangle$ to B and $|\psi_2\rangle$ to C selectively, using only LOCC and *a priori* entanglement between the three agents. A simple solution to this problem is applying standard teleportation,[2] in the case where A shares EPR pairs with both B and C. An interesting question is whether any other form of *a priori* entanglement can help achieving selective teleportation. In particular, is it possible to perform selective teleportation where the *a priori* entanglement is in the form of a GHZ state amongst the three agents? The following theorem answers this question using the result of the Theorem 2.

Theorem 4: *With a priori entanglement given in the form of a GHZ state shared amongst three agents, two qubits cannot be selectively teleported by ane of the three parties to the other two parties.*

Proof: Suppose there exists a protocol P which can enable one of the three parties (say A) to teleport two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ selectively to the other two parties (say B and C). Now A takes four qubits; she prepares two EPR pairs one from the first and second qubits and the other from the third and fourth qubits. He then teleports the first and third qubits selectively to B and C using P (consider first qubit as $|\psi_1\rangle$ and the third qubit as $|\psi_2\rangle$). We can note here that in this way A is able to share one EPR pair each with B and C. But this is impossible because it allows A to prepare two EPR pairs starting from a GHZ state and only using LOCC. This contradicts Theorem 2. Hence follows the result.   □

V. COMBINATORIAL CONDITIONS FOR LOCC INCOMPARABILITY OF EPR GRAPHS

An immediate result comparing an n-CAT state with EPR pairs follows from noting that, given a spanning EPR tree among n parties, an n-CAT state can be constructed using only LOCC using the teleportation protocol described in Sec. II. The result we present below generalizes Theorem 3.

Theorem 5: *1-EPR pair $<_{LOCC}$ n-CAT $<_{LOCC}$ $(n-1)$-spanning EPR tree.*

We can argue in a similar manner that an n-CAT state amongst n-parties cannot be converted by just using LOCC to any form of entanglement structure which possesses EPR pairs between any two or more different sets of two parties. Assume this is possible for the sake of contradiction. Then the two edges could be in either of the two forms: (1) $\{i_1, i_2\}$ and $\{j_1, j_2\}$ and (2) $\{i_1, i_2\}$ and $\{i_2, j_2\}$, where i_1, i_2, j_1, j_2 are all distinct. In bicolored-merging assign the colors as follows. In case (1), give color A to i_2 and j_2 and give the color B to the rest of the vertices. In case (2), give color A to i_2 and color B to the rest of the vertices. Since both the cases are contrary to our assumption, the assertion follows. Moreover, from Theorem 1 (see Appendix A for proof), no disconnected EPR graph would be able to yield n-CAT just by LOCC. These two observations combined together lead to the following theorem which signifies the fact that these two multipartite states cannot be compared.

Theorem 6: *A CAT state amongst n agents in a communication network is LOCC incomparable to any disconnected EPR graph associated with the n agents having more than one edge.*

The above result indicates that there are many possible forms of entanglement structures (multipartite states) which cannot be compared at all in terms of nonlocal correlations they may have. This simple result is just an implication of the necessary combinatorics required for the preparation of CAT states. One more interesting question with respect to this combinatorics is to compare a spanning EPR tree and a CAT state. A spanning EPR tree is combinatorially sufficient for preparing the CAT state and thus seems to entail more nonlocal correlations than in a CAT state. The question whether this ordering is strict needs to be further investigated. It is easy to see that an EPR pair between any two parites can be obtained starting from a CAT state shared amongst the n agents just by LOCC (Theorem 5). Therefore, given $n-1$ copies of the CAT state we can build all the $n-1$ edges of any spanning EPR tree just by LOCC. But whether this is the lower bound on the number of copies of n-CAT required to obtain a spanning EPR tree is even more interesting. The following theorem shows that this is indeed the lower bound.

Theorem 7: *Starting with only $n-2$ copies of n-CAT state shared amongst its n agents, no spanning EPR tree of the n agents can be obtained just by LOCC.*

Proof: Suppose it is possible to create a spanning EPR tree T from $(n-2)$ copies of n-CAT states. As we know, an n-CAT state can be prepared from any spanning EPR tree by LOCC.[24,7,29,30] Thus, if $(n-2)$ copies of n-CAT can be converted to T, then $(n-2)$ copies of any spanning EPR tree can be converted to T just by LOCC. In particular, $(n-2)$ copies of a chain EPR graph (which is clearly a spanning EPR tree, Fig. 6) can be converted to T just by LOCC. Now, we know that any tree is a bipartite connected graph with $n-1$ edges across the two parts. Let vertices $i_1, i_2, \ldots, i_m$ be the members of the first group and the rest be in the other group. Construct a chain EPR graph where the first m vertices are $i_1, i_2, \ldots, i_m$ in the sequence, and the rest of the vertices are from the other group in the sequence (Fig. 6). In bicolored merging, we give the color A to the parties $\{i_1, i_2, \ldots, i_m\}$ and the rest of the parties are given the color B. This way we are able to create $(n-1)$ EPR pairs (note that there are $n-1$ edges in T across the two groups) between A and B starting from only $(n-2)$ EPR pairs (considering the $n-2$ chainlike spanning EPR trees). So, we conclude that $(n-2)$ copies of n-CAT cannot be converted to any spanning EPR tree just by LOCC. See Fig. 6 for illustration of the required bicolored merging. The proof could also be acheived by similar kind of bicolored merging directly applied on n-CAT and T.   □

In the preceding results we have compared spanning EPR trees with CAT states. We discuss the comparability/incomparability of two distinct spanning EPR trees in the next theorem and corollary.
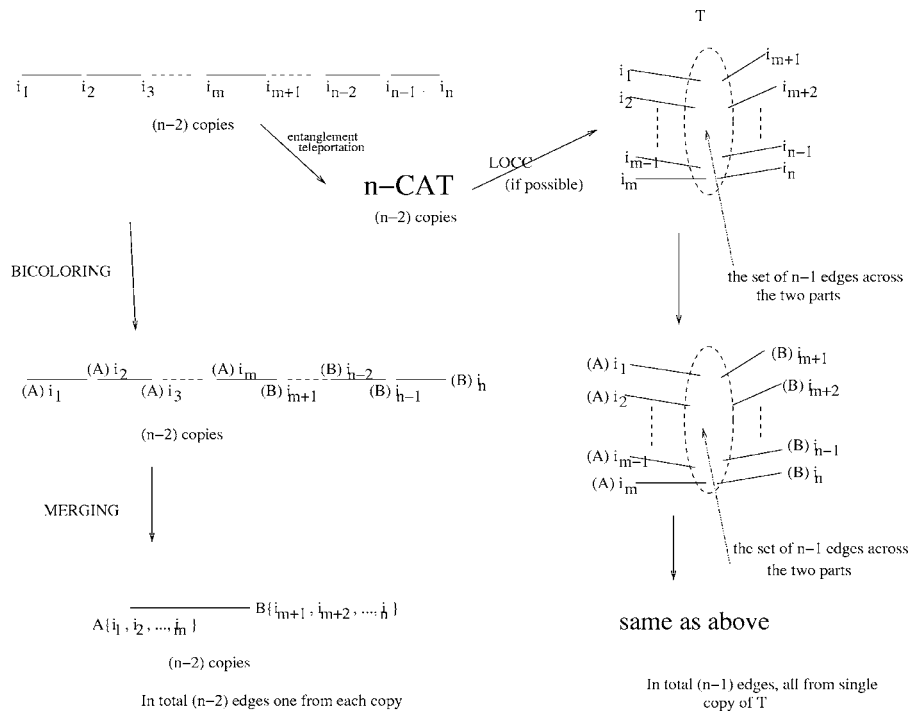
FIG. 6. $n-2$ copies of n-CAT are not sufficient to prepare a spanning EPR tree.

Theorem 8: *Any two distinct spanning EPR trees are LOCC incomparable.*

Proof: Let T_1 and T_2 be the two distinct spanning EPR trees on the same n vertices. Clearly, there exist two vertices (say i and j) which are connected by an edge in T_2 but not in T_1. Also by virtue of connectedness of spanning trees, there will be a path between i and j in T_1. Let this path be $ik_1k_2\cdots k_mj$ with $m>0$ (see Fig. 7). Since $m>0$, k_1 must exist.

Let $T_i^1 \equiv$ subtree in T_1 rooted at i except for the branch which contains the edge $\{i,k_1\}$, $T_j^1 \equiv$ subtree in T_1 rooted at j except for the branch which contains the edge $\{j,k_m\}$, $T_{k_r} \equiv$ subtree in T_1 rooted at k_r except for the branches which contain either of the edges $\{k_{r-1},k_r\}$ and $\{k_r,k_{r+1}\}$ $(k_0=i, k_{m+1}=j)$, $T_i^2 \equiv$ subtree in T_2 rooted at i except for the branch which contains the edge $\{i,j\}$, and $T_j^2 \equiv$ subtree in T_2 rooted at j except for the branch which contains the edge $\{i,j\}$.

It is easy to see that the set $T_i^2 \cup T_j^2$ is nonempty as T_1 and T_2, being distinct, must contain more than two vertices. Also T_i^2 and T_j^2 must be disjoint; for, otherwise there will be a path between i and j in T_2 which does not contain the edge $\{i,j\}$. Thus there will be two paths between i and j in T_2 contradicting the fact that T_2 is a spanning EPR tree (Fig. 7). With these two charactistics of T_i^2 and T_j^2, it is clear that k_1 will lie either in T_i^2 or in T_j^2. Without loss of generality, let us assume that $k_1 \in T_i^2$. Now we do bicolored merging where the color A is assigned to i and all vertices in T_i^1 and the color B is assigned to the rest of the vertices (refer to Fig. 7 for illustration). Since T_1 and T_2 were chosen arbitrarily, the same arguments also imply that there cannot exist a method which converts T_2 to T_1. This leads to the conclusion that any two distinct spanning EPR trees are LOCC incomparable.

Corollary 1: There are at least exponentially many LOCC-incomparable classes of pure multipartite entangled states.

Proof: We know from results in graph theory[9] that on a labelled graph on n vertices, there are n^{n-2} posible distinct spanning trees. Hence there are n^{n-2} distinct spanning EPR trees in a network of n agents. From Theorem 8 all these spanning EPR trees are LOCC incomparable. It can be noted here that the most general local operation of n qubits is an element of the group $U(2)^n$ (local unitary rotations on each qubit alone). So, if two states are found incomparable, this means that there are actually two incomparable equivalence classes of states [where members in a class are
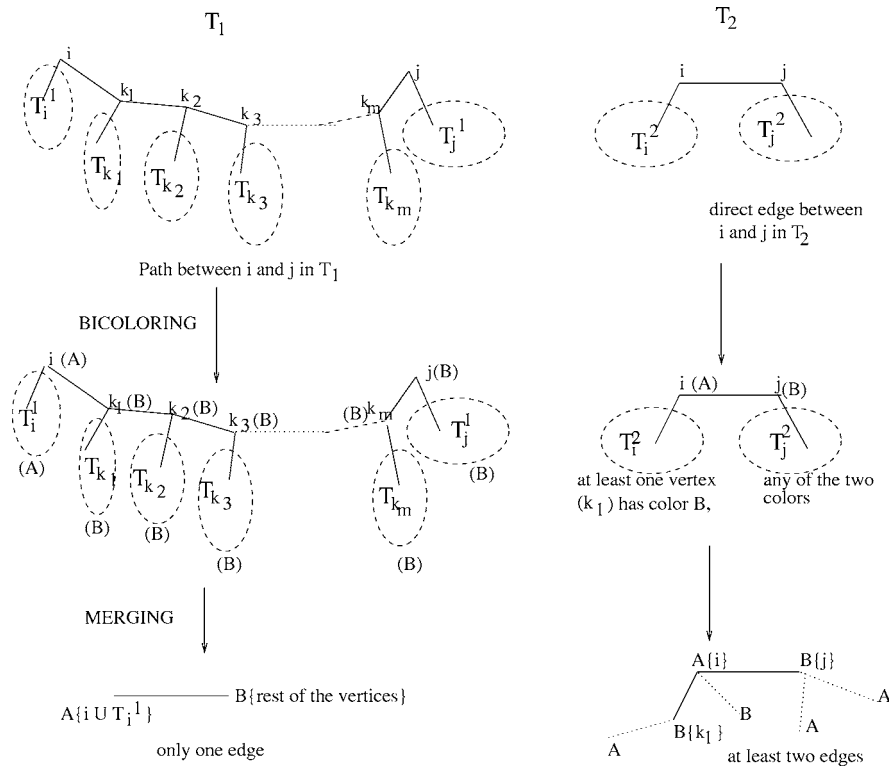
FIG. 7. Spanning EPR trees are LOCC incomparable.

related by a $U(2)^n$ transformation]. Thus we have at least exponentially many LOCC-incomparable classes of multipartite entangled states. □

VI. COMBINATORIAL CONDITIONS FOR LOCC INCOMPARABILITY OF ENTANGLED HYPERGRAPHS

Since entangled hypergraphs represent more general entanglement structures than those represented by the EPR graphs (in particular spanning EPR trees are nothing but 2-uniform entangled hypertrees), it is likely that there will be even more classes of incomparable multipartite states and this motivates us to generalize Theorem 8 for entangled hypertrees. However, remarkably this intuition does not work directly and there are entangled hypertrees which are not incomparable. But there are a large number of entangled hypertrees which do not fall under any such partial ordering and thus remain incomparable. To this end we present our first imcomparability result on entangled hypergraphs.

Theorem 9: *Let $H_1 = (S, F_1)$ and $H_2 = (S, F_2)$ be two entangled hypertrees. Let P_1 and P_2 be the set of pendant vertices of H_1 and H_2, respectively. If the sets $P_1 \backslash P_2$ and $P_2 \backslash P_1$ are both nonempty then the multipartite states represented by H_1 and H_2 are necessarily LOCC incomparable.*

Proof: Using bicolored merging we first show that H_1 cannot be converted to H_2 under LOCC. Impossibility of the reverse conversion will also be immediate. Since $P_1 \backslash P_2$ is nonempty, there exists $u \in S$ such that $u \in P_1 \backslash P_2$. That is, u is pendant in H_1 but nonpendant in H_2 (Fig. 8).

In the bicolored merging assign the color A to the vertex u and the color B to all other vertices. This reduces H_1 to a single EPR pair shared between the two parties A and B whereas H_2 reduces to at least two EPR pairs shared between A and B. The complete bicolored merging is shown in Fig. 8. □
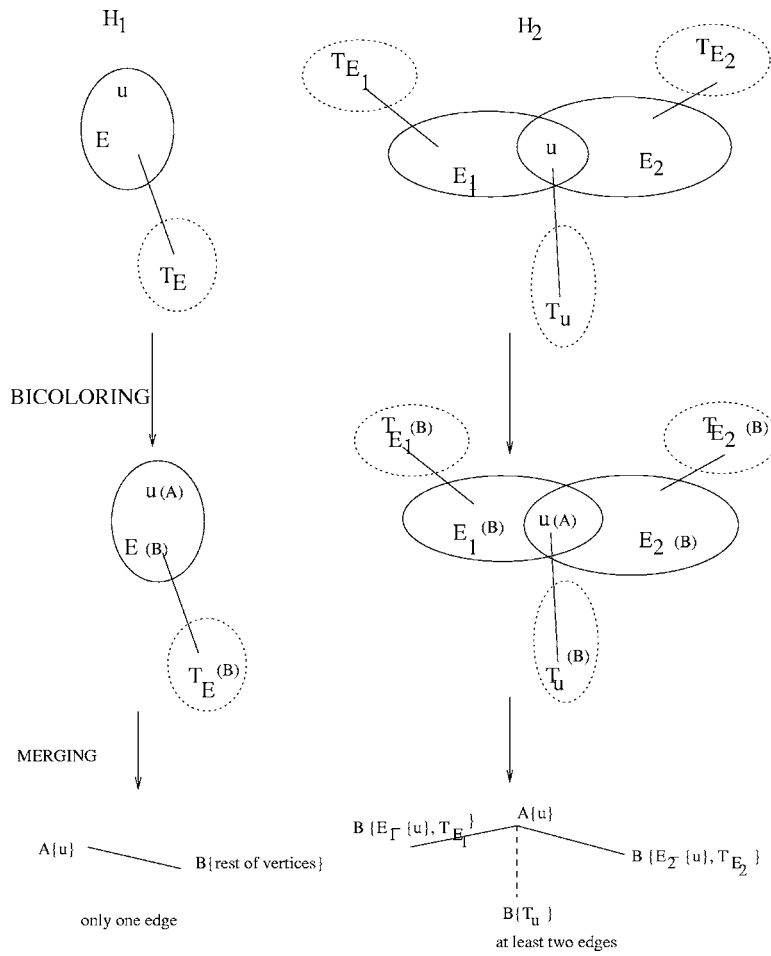
FIG. 8. Entangled hypergraphs with $P_1 \backslash P_2$ nonempty.

We note that this proof does not utilize the fact that H_1 and H_2 are entangled hypertrees, and thus the theorem is indeed true even for entangled hypergraphs satisfying the conditions specified on the set of pendant vertices.

The conditions specified on the set of pendant vertices in Theorem 9 cover a very small fraction of the entangled hypergraphs. However, these conditions are not necessary and it may be possible to find further characterizations of incomparable classes of entangled hypergraphs. We present two examples where the conditions of Theorem 9 are not satisfied.

Example 1: (Figs. 9 and 10) $P_1 \neq P_2$ but either $P_1 \subset P_2$ or $P_2 \subset P_1$.

Example 2: (Fig. 11) $P_1 = P_2$.

In the first example, the entangled hypergraphs H_1 and H_2 staisfy $P_1 \neq P_2$ and $P_1 \subset P_2$. H_1 and H_2 are comparable in Fig. 9 but incomparable in Fig. 10. In Fig. 10, the incomparability has been proved by showing that H_1 is not convertible to H_2 under LOCC because the impossibility of reverse conversion follows from the proof of Theorem 9 ($P_2 \backslash P_1 \neq \phi$). Figure 11 gives examples of comparable and incomparable entangled hypergraphs with condition $P_1 = P_2$.

Theorem 8 shows that two distinct EPR spanning trees are LOCC incomparable and the spanning EPR trees are nothing but 2-uniform entangled hypertrees. Therefore, a natural generalization of this theorem would be to r-uniform entangled hypertrees for any $r \geqslant 3$. As we show below, the generalization indeed holds. It should be noted that Theorem 9 does not necessarily capture such entanglement structures (multipartite states) (Fig. 12). However, in order to prove
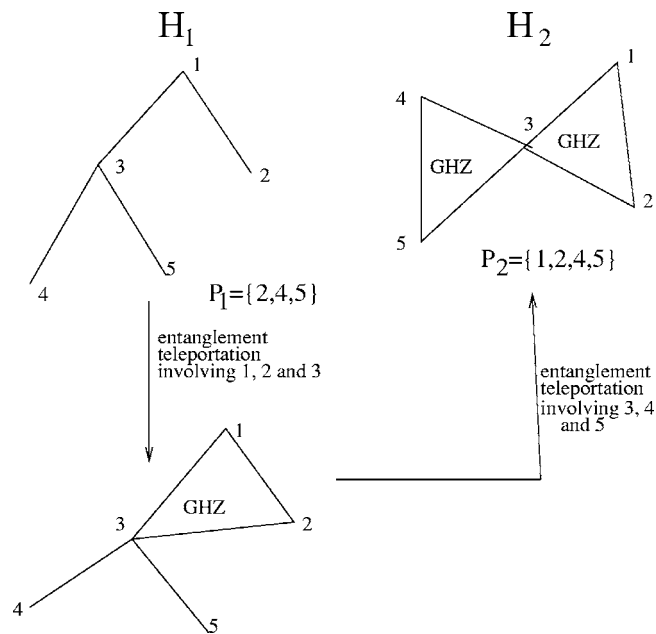
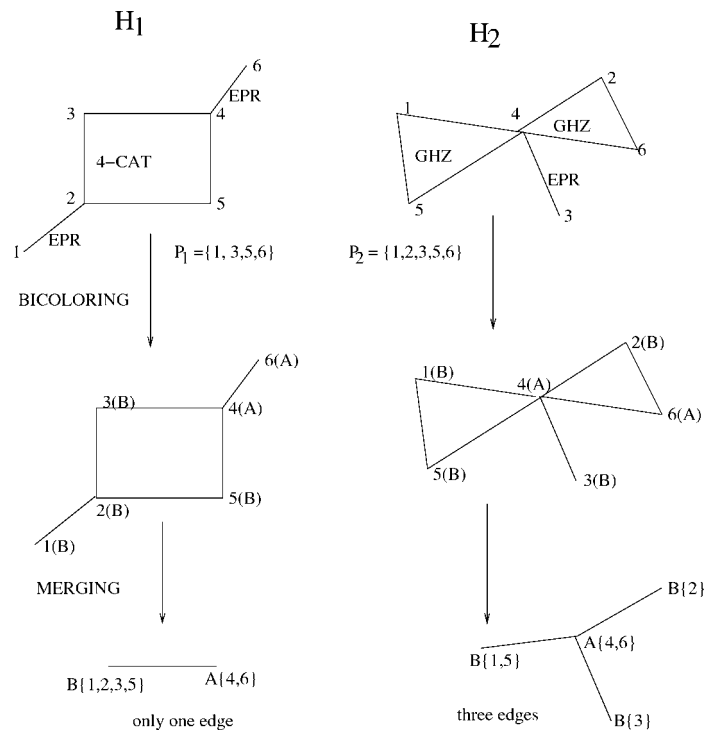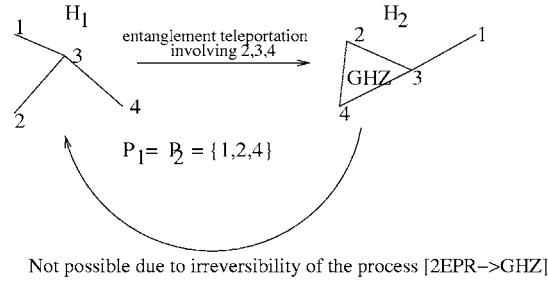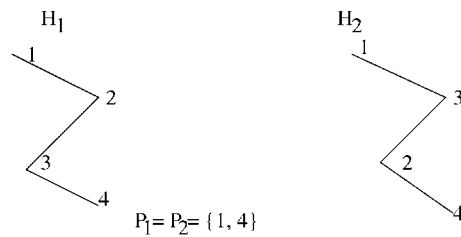FIG. 9. Comparable with $P_1 \neq P_2$ and $P_1 \subset P_2$.

FIG. 10. Incomparable with $P_1 \neq P_2$ and $P_1 \subset P_2$.

Comparable entangled hypergraphs with same set of pendant vertices

$P_1 = P_2 = \{1,2,4\}$

Not possible due to irreversibility of the process [2EPR–>GHZ]

Incomparable entangled hypergraphs with same set of pendant vertices

$P_1 = P_2 = \{1, 4\}$

H_1 and H_2 being distinct spanning EPR trees are LOCC incomparable

FIG. 11. $P_1 = P_2$.

$P_1 = \{1,2,5,6,7\}$ $P_2 = \{1,2,4,5,6,7\}$

$P_1 = P_2 = \{1,2,4,5,6,7\}$
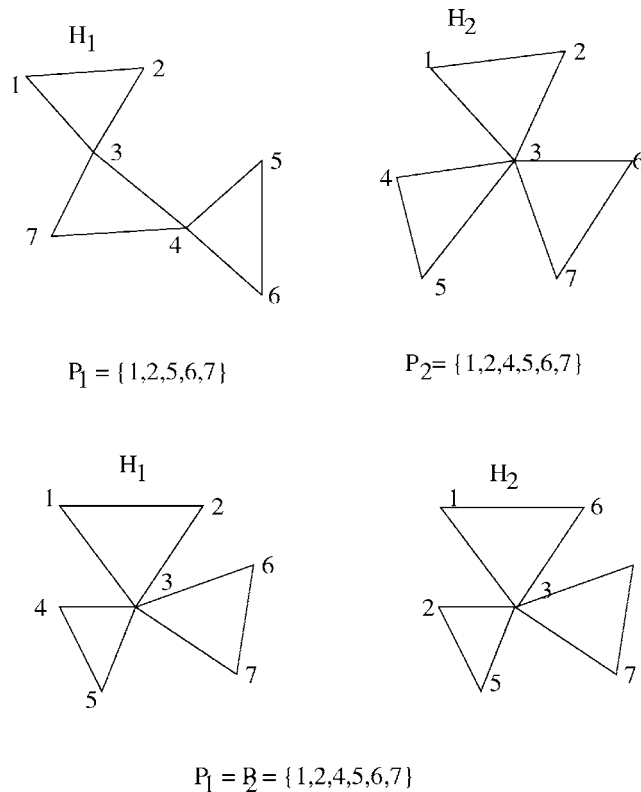
FIG. 12. r-uniform entangled hypertrees not captured in Theorem 9.
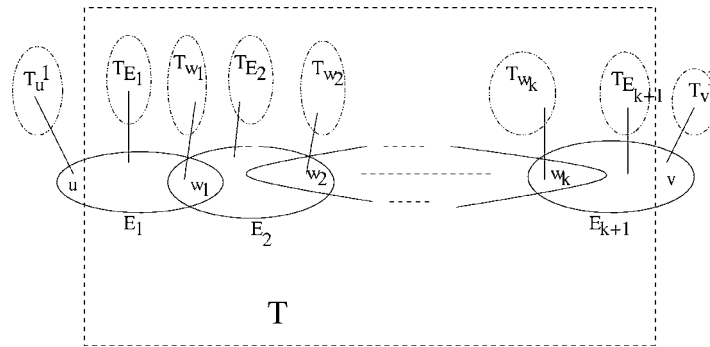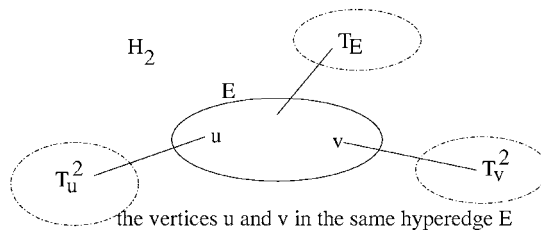
FIG. 13. Two distinct *r*-uniform entangled hypertrees.

that two distinct *r*-uniform entangled hypertrees are LOCC incomparable, we need the following important result about *r*-uniform hypertrees. See Appendix B for the proof.

Theorem 10: *Given two distinct r-uniform hypertrees $H_1=(S,F_1)$ and $H_2=(S,F_2)$ with $r \geqslant 3$, there exist vertices $u,v \in S$ such that u and v belong to the same hyperedge in H_2 but necessarily to different hyperedges in H_1.*

Now we state one of our main results on LOCC incomparability of multipartite entangled states in the following theorem.

Theorem 11: *Any two distinct r-uniform entangled hypertrees are LOCC incomparable.*

Proof: Let $H_1=(S,F_1)$ and $H_2=(S,F_2)$ be the two *r*-uniform entangled hypertrees. If $r=2$ then H_1 and H_2 happen to be two distinct spanning EPR trees and the proof follows from Theorem 8. Therefore, let $r \geqslant 3$.

Now from Theorem 10, there exist $u,v \in S$ such that u and v belong to the same hyperedge in H_2 but necessarily to different hyperedges in H_1. Let the same hyperedge in H_2 be $E \in F_2$. Also, since H_1, being hypertree, is *connected*, there exists a path between u and v in H_1. Let this path be $uE_1E_2 \cdots E_{k+1}v$. Clearly $k>0$ because u and v necessarily do not belong to the same hyperedge in H_1.

We introduce the following notations (Fig. 13).

T_u^1, subhypertree rooted at u in H_1 except the branch that contains E_1. T_v^1, subhypertree rooted at v in H_1 except the branch that contains E_{k+1}. T_{w_i}, subhypertree rooted at w_i in H_1 except branches containing E_i and E_{i+1}. T_{E_i}, Collection of all subhypertrees in H_1 rooted at some vertices in E_i other than w_{i-1} and w_i (where $w_0=u$ and $w_{k+1}=v$) except for the branches which contain E_i. $T=((E_1 \cup E_2 \cup \cdots \cup E_{k+1}) \cup (T_{E_1} \cup T_{E_2} \cup \cdots \cup T_{E_{k+1}}) \cup (T_{w_1} \cup T_{w_2} \cup \cdots \cup T_{w_k})) \backslash \{u,v\}=$set of all vertices from $S \backslash \{u,v\}$ which are not contained in $T_u \cup T_v$. T_u^2, subhypertree rooted at u in H_2 except the branch that contains E. T_v^2, subhypertree rooted at v in H_2 except the branch that contains E. T_E, collection of all subhypertrees in H_2 rooted at some vertices in $E \backslash \{u,v\}$ except for the branches which contain E.

In order to complete the proof we consider the following cases.

Case 1: $\exists w \in T$ such that $w \in (T_u^2 \cup T_v^2)$.

Without loss of generality let us take $w \in T_u^2$. Now since $w \in T$, $w \in$ exactly one of E_i, T_{w_i}, or T_{E_i} for some i. Accordingly there will be three subcases. *Case 1.1*, $w \in E_i$ for some i (take such minimum i).

Do bicolored merging where the vertex u along with all the vertices in

$$T_u^1, E_1, E_2, \ldots, E_{i-1}, T_{w_1}, T_{w_2}, \ldots, T_{w_{i-1}}, T_{E_1}, T_{E_2}, \ldots, T_{E_{i-1}}$$

are given the color A and the rest of the vertices are given the color B.

Case 1.2: $w \in T_{w_i}$ for some i.

Do the bicolored merging while assigning the colors as in the above case.

Case 1.3: $w \in T_{E_i}$ for some i.

Bicolored merging in this case is also the same as in Case 1.1.

Case 2: There does not exist any $w \in T$ such that $w \in T_u^2 \cup T_v^2$.

Clearly, $T_u^2 \cup T_v^2 \subset T_u^1 \cup T_v^1$ and $T \subset T_E \cup (E \setminus \{u, v\})$. Note that whenever we are talking of set relations like union, containment, etc., we are considering the trees, edges, etc., as sets of appropriate vertices from S which make them. First we establish the following claim.

Claim: $\exists t \in (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})$ such that $t \in T_E$.

We have $k > 0$. Therefore, both E_1 and E_2 exist and since H_1 is r-uniform $|E_1| = |E_2| = r$. Also $(E_1 \setminus \{u, w_1\}) \cap (E_2 \setminus \{w_1, w_2\})$ is empty, for, otherwise there will be a cycle in H_1 which is not possible as H_1 is a hypertree.[5,10] Therefore,

$$|(E_1 \setminus \{u, w_1\} \cup (E_2 \setminus \{w_1, w_2\})| = |(E_1 \setminus \{u, w_1\}| + |E_2 \setminus \{w_1, w_2\}| = (r-2) + (r-2) = 2r - 4.$$

Also $|E| = r$ implies that $|E \setminus \{u, v\}| = (r-2)$.

It is clear that $u, v \notin (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})$. Therefore,

$$|(E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})| - |E \setminus \{u, v\}| = (2r - 4) - (r - 2) = r - 2 \geq 1$$

since $r \geq 3$.

Also $(E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\}) \subset T \subset T_E \cup (E \setminus \{u, v\})$, and so by pigeonhole principle,[25]

$$\exists t \in (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\}) \text{ and } t \in T_E (\notin (E \setminus \{u, v\}))$$

Hence our claim is true.

Now we have $t \in (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})$ such that $t \in T_E$. Since $t \in T_E$, by the definition of T_E it is clear that there must exist $w \in E \setminus \{u, v\}$ such that $t \in T_w$, the subhypertree in H_2 rooted at w except for the branch containing E. Depending on whether $t \in E_1 \setminus \{u, w_1\}$ or $t \in E_2 \setminus \{w_1, w_2\}$, we break this case into several subcases and futher in sub-subcases depending on the part in H_1 where w lies.

Case 2.1: $t \in E_1 \setminus \{u, w_1\}$ (Fig. 14).

Case 2.1.1: $w \in T_u^1$.

Do the bicolored merging where u and the vertices in T_u^1 are assigned the color A and the rest of the vertices from S are given the color B.

Case 2.1.2: $w \in T_v^1$.

Bicolored merging is done where v as well as all the vertices in T_v^1 are assigned the color B and rest of the vertices from S are given the color A.

Case 2.1.3: $w \in T$.

Here in this case, depending on whether w is in T_t or not, there can be two cases.
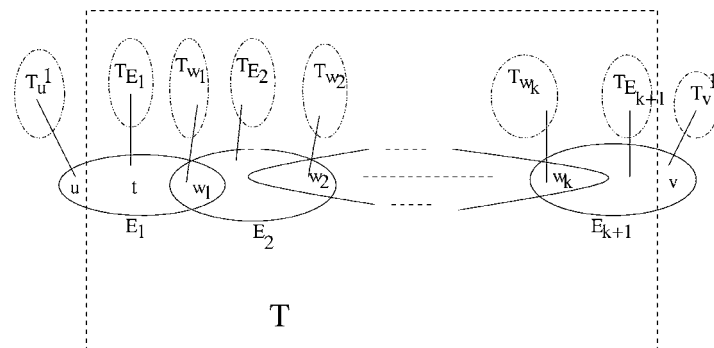
Case 2.1.3.1: $w \in T_t$.

Bicolored merging is done where all the vertices in T_t are given the color A and the rest of the vertices are assigned the color B.
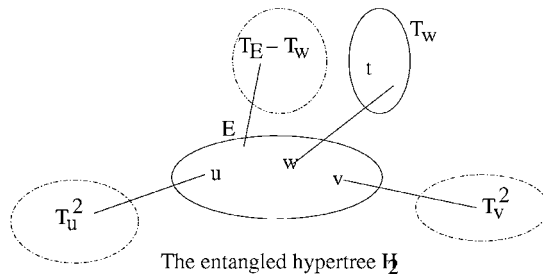
Case 2.1.3.2: $w \notin T_t$.

$w \notin T_t$ implies that either $w \in E_i$ for some i, or $w \in T_q$, where $q \in E_i$ for some i and $q \neq t$. For both of these possibilities, bicolored merging is the same and is done as follows.

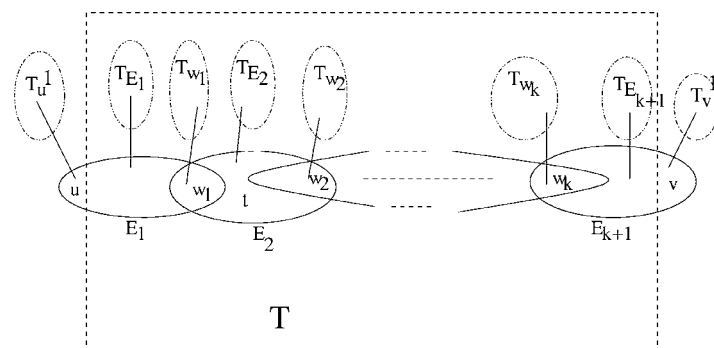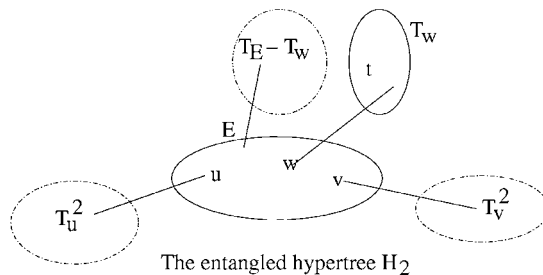Assign the color A to u as well as all vertices in

FIG. 14. Case 2.1.

FIG. 15. Case 2.2.

$$T_u^1 \cup E_1 \cup T_{E_1} \cup T_{w_1} \cup \cdots \cup E_{i-1} \cup T_{E_{i-1}} \cup T_{w_{i-1}} \cup (E_i \setminus \{q, w, w_i\}) \cup (T_{E_i} \setminus T_q)$$

and assign the color B to rest of the vertices.

Case 2.2: $t \in E_2 \setminus \{w_1, w_2\}$ (Fig. 15).

Case 2.2.1: $w \in T_u^1 \cup E_1 \cup T_{E_1} \cup T_{w_1}$.

Do the bicolored merging where all the vertices in $T_u^1 \cup E_1 \cup T_{E_1} \cup T_{w_1}$ including u are given the color A and rest of the vertices are assigned the color B.

Case 2.2.2: $w \in T_v^1 \cup T_{E_{k+1}} \cup E_{k+1} \cup T_{w_k} \cup \cdots \cup T_{E_3} \cup E_3 \cup T_{w_2}$.

In bicolored merging give the color B to all the vertices (including v) in $T_v^1 \cup T_{E_{k+1}} \cup E_{k+1} \cup T_{w_k} \cup \cdots \cup T_{E_3} \cup E_3 \cup T_{w_2}$ and color A to the rest of the vertices.

Case 2.2.3: $w \in E_2 \cup T_{E_2}$.

In this case depending on whether $w \in T_t$, or $w \notin T_t$ the bicolored merging will be different.

Case 2.2.3.1: $w \in T_t$.

Bicolored merging is done where all the vertices in T_t are given the color A and rest of the vertices are assigned the color B.

Case 2.2.3.2: $w \notin T_t$.

$w \notin T_t$ implies that either $w \in E_2$, or $w \in T_q$ for some $q (\neq t) \in E_2$. In any case do the bicolored merging where the color A is assigned to all the vertices in

$$T_u^1 \cup E_1 \cup T_{E_1} \cup T_{w_1} \cup (E_2 \setminus \{w, q, w_2\}) \cup (T_{E_2} \setminus T_q)$$

and the rest of the vertices are assigned the color B.

Now that we have exhausted all possible cases and shown by the method of bicolored merging that the r-uniform entangled hypertree H_1 cannot be LOCC converted to the r-uniform entangled hypertree H_2. The same arguments also work for showing that H_2 can not be LOCC converted to H_1 by interchanging the roles of H_1 and H_2. Hence the theorem follows. $\qquad \square$

Before ending our section on LOCC incomparability of multipartite states represented by EPR graphs and entangled hypergraphs, we note that partial entropic criteria of Bennett *et al.*[4] which gives a sufficient condition for LOCC incomparability of multipartite states, does not capture the LOCC incomparability of spanning EPR trees or entangled hypertrees in general. Consider two spanning EPR trees T_1 and T_2 on three vertices (say 1, 2, 3). T_1 is such that the vertex pairs 1, 2 and 1, 3 are forming the two edges where as in T_2 the vertex pairs 1, 3 and 2, 3 are forming the two edges. It is easy to see that T_1 and T_2 are not marginally isentropic.

VII. QUANTUM DISTANCE BETWEEN MULTIPARTITE ENTANGLED STATES

In the proof of Theorem 8, we have utilized the fact that there exist at least two vertices which are connected by an edge in T_2 but not in T_1. This follows as T_1 and T_2 are different and they also have equal number of edges (namely $n-1$, if there are n vertices). In fact, in general there may exist several such pairs of vertices depending on the structures of T_1 and T_2. Fortunately, the number of such pair of vertices has some nice features giving rise to a metric on the set of spanning (EPR) trees with fixed vertex set and thus giving a concept of distance.[9] The distance between any two spanning (EPR) trees T_1 and T_2 denoted by QD_{T_1,T_2} on the same vertex set is defined as the number of edges in T_1 which are not in T_2. Let us call this distance to be the *quantum distance* between T_1 and T_2. We have proved in Theorem 8 that obtaining T_2 from T_1 is not possible just through LOCC, so we need to do quantum communication. The minimum number of qubit required to be communicated for this purpose should be an interesting parameter related to state transformations amongst multipartite states represented by spanning EPR trees; let us denote this number by q_{T_1,T_2}. We note that $q_{T_1,T_2} \leqslant QD_{T_1,T_2}$. This is because each edge not present in T_2 can be created by only one qubit communication. The exact value of q_{T_1,T_2} will depend on the structures of T_1 and T_2 and, as we can note, on the number of edge disjoint paths in T_1 between the vertex pairs which form an edge in T_2 but not T_1.

We can say more about quantum distance. Recall Theorem 7 where we show that a lower bound on the number of copies of n-CAT to prepare a spanning EPR tree by LOCC, is $n-1$. Can
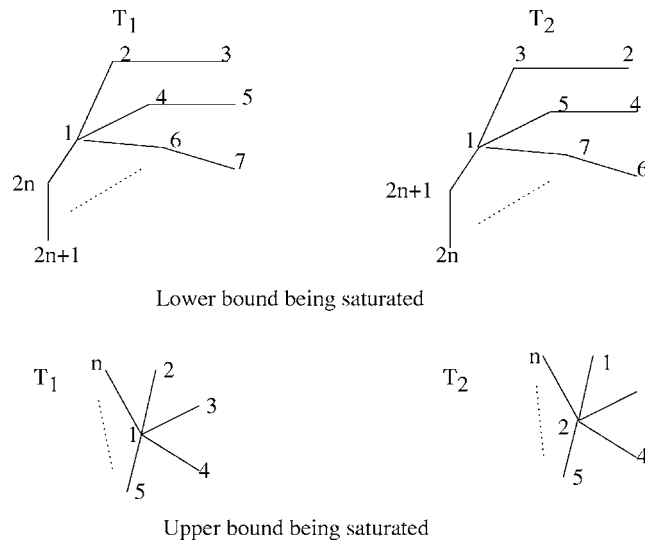
FIG. 16. Saturating bounds on copies required to go from one EPR spanning tree to another via LOCC.

we obtain a similar lower bound in the case of two spanning EPR trees and relate it to the quantum distance? The answer is indeed yes. Let C_{T_1,T_2} denote the minimum number of copies of the spanning EPR tree T_1 required to obtain T_2 just by LOCC. We claim that $2 \leq C_{T_1,T_2}$, $C_{T_2,T_1} \leq QD_{T_1,T_2}+1$. The lower bound follows from Theorem 8. The upperbound is also true because of the following reason. QD_{T_1,T_2} is the number of (EPR pairs) edges present in T_2 but not in T_1. For each such edge in T_2 (let u, v be the vertices forming the edge), while converting many copies of T_1 to T_2 by LOCC an edge between u and v must be created. Since T_1 is a spanning tree and therefore connected, there must be a path between u and v in T_1 and this path can be well converted (using entanglement swapping) to an edge between them (i.e., EPR pair between them) only using LOCC. Hence one copy each will suffice to create each such edges in T_2. Thus QD_{T_1,T_2} copies of T_1 will be sufficient to create all such QD_{T_1,T_2} edges in T_2. One more copy will supply all the edges common in T_1 and T_2. Even more interesting point is that both these bounds are saturated. This means to say that there do exist spanning EPR trees satifying these bounds (Fig. 16).

It is important to note that a similar concept of distance also holds in the case of r-uniform entangled hypertrees.

ACKNOWLEDGMENTS

APPENDIX A

Proof of Theorem 1: We use the method of bicolored merging to prove the fact that any disconnected EPR graph G on n-vertices cannot be converted to an n-CAT state on those vertices under LOCC. We first note that the BCM EPR graph of an n-CAT state, irrespective of the bicoloring done, is always a graph which contains exactly one edge. Now as G is disconnected it will have more than one connected components. Let these components be $C_1, C_2 : C_k$, where $k \geq 2$. The bicoloring is done as follows: assign the color A to all the vertices in the component C_1 and the color B to all other vertices, i.e., all vertices in $G \backslash C_1$. After merging, therefore, G reduces to a disconnected graph with no edges, i.e., the BCM EPR graph of G is a graph with k isolated vertices and no edges. Now if we are able to prepare an n-CAT state from G just using LOCC, we could also prepare an EPR pair between two parties who were never sharing an EPR pair just using LOCC. This violates monotonicity and hence the theorem is proved. □

APPENDIX B

Proof of Theorem 10: We first establish the following claim.

Claim: $\exists E_1 \in F_1, E_2 \in F_2$ such that $E_1 \cap E_2 \neq \phi$ and $E_2 \notin F_1 \cap F_2$.

Proof of the claim: We first show that on the same vertex set, the number of hyperedges in any r-uniform hypertree is always same. Let n and m be the number of vertices and hyperedges in a r-uniform hypertree. We show by induction on m that $n = m*(r-1)+1$.

For $m=1$, $n = 1*(r-1)+1 = r$ which is true because all possible vertices (since no one can be isolated) fall in the single hyperedge and it has exactly r vertices.

Let us assume that this relation between n and m for a fixed r holds for all values of the induction variable up to $m-1$. We show that it holds good for m.

Now take a r-uniform hypertree with m hyperedges. Remove any of the hyperedges to get another hypergraph (which may not be connected) having only $m-1$ edges. This removal may introduce k connected components (subhypertrees); $1 \leq k \leq r$. Let these components have, respectively, $m_1, m_2, \ldots, m_k$ number of hyperedges. Therefore, $\Sigma_{i=1}^{k} m_i = m-1$. The total number of vertices in the new hypergraph (with the k subhypertrees as components), $n^1 = \Sigma n_i$ where n_i is the number of vertices in the component i. Therefore, $n^1 = \Sigma n_i = \Sigma_{i=1}^{k}\{m_i(r-1)+1\} = (m-1)(r-1)+k$ (under induction assumption).

Now the number of vertices in the original hypertree, $n = n^1 + (r-k)$ because k vertices were already covered, one each in the k components. Therefore, $n = (m-1)(r-1)+k+(r-k) = (m-1)(r-1)+r = (m-1)(r-1)+(r-1)+1 = m(r-1)+1$. The result is thus true for m and hence for any number of hyperedges by induction. This result implies that any r-uniform hypertree on the same vertex set will always have the same number of hyperedges.

Let $F = F_1 \cap F_2$ and $m = |F_1| = |F_2|$. Obviously $m > |F|$ otherwise $H_1 = H_2$. This implies that $\exists E \in F_2$ such that $E \notin F$.

Take any vertex say $w \in E$. Since $w \in S$ and H_1 is a hypertree therefore connected, w can not be an isolated vertex and therefore $\exists E^1 \in F_1$ such that $w \in E^1$. Take $E_1 = E^1$ and $E_2 = E$. This proves our claim.

Now we prove the theorem. Choose E_1 and E_2 so as to satify the above claim.

Let $E_1 = \{u_1, u_2, \ldots, u_l, w_{l+1}, w_{l+2}, \ldots, w_r\}$ and $E_2 = \{u_1, u_2, \ldots, u_l, v_{l+1}, v_{l+2}, \ldots, v_r\}$.

Since $E_1 \cap E_2 \neq \phi$, $l \geq 1$ and $E_1 \neq E_2$ implies that $l \leq r-1$.

Hence $1 \leq l \leq r$.

Now based on the value of l, we have the following different cases.

Case 1: $l > 1$.

Case 1.1: $\exists v_i$ such that u_1 and v_i are not in the same hyperedge in H_1.

Take $u = u_1$ and $v = v_i$ in the statement of the theorem.

Case 1.2: Each v_i is in some hyperedge in H_1 in which u_1 also lies.

None of these v_i's can belong to the hyperedges in H_1 in which u_2 lies.

This is due to the fact that if, say, v_j happens to be in same hyperedge as of u_2 in H_1 then $u_1 u_2 v_j u_1$ will be a cycle in H_1, which is absurd as H_1 is a hypertree.

Note that at least one such v_i must exist as $l < r$. Take $u = u_2$ and $v =$ any v_i.

Case 2: $l = 1$.

Case 2.1: $\exists v_i$ such that u_1 and v_i are not in same hyperedge in H_1.

Take $u = u_1$ and $v = v_i$.

Case 2.2: Each v_i is in some hyperedge in H_1 in which u_1 also lies.

Since v_i's are $r-1$ in number and $E_2 \notin F_1 \cap F_2$, these v_i's will be distributed in at least two distinct hyperedges in H_1 in which u_1 also lies.

Therefore, $\exists v_i, v_j$ such that they are in the same hyperedge in H_2 (namely in E_2) but in necessarily different edges in H_1, otherwise (that is, if they lie in the same hyperedge in H_1) $u_1 v_i v_j u_1$ will be a cycle in H_1, which is absurd as H_1 is a hypertree.

Also note that both v_i and v_j will exist as $r \geq 3$.

Take $u = v_i$ and $v = v_j$.

Thus we have proved Theorem 10 in all possible cases. $\square$

We would like to point out that the result of Theorem 10 could follow from some standard results in combinatorics. We have however not found literature proving this result.

[1] Bandyopadhyay, S., Roychowdhury, V., and Sen, U., "Classification of nonasymptotic bipartite pure-state entanglement tansformations," Phys. Rev. A **65**, 052315 (2002).

[2] Bennett, C. H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., and Wootters, W. K., "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels," Phys. Rev. Lett. **70**, 1895 (1993).

[3] Bennett, C. H., DiVincenzo, D. P., Smolin, J. A., and Wootters, W. K., Phys. Rev. A **54**, 3824 (1996).

[4] Bennett, C. H., Popescu, S., Rohrlich, D., Smolin, J. A., and Thapliyal, A. V., "Exact and asymptotic measures of multipartite pure-state entanglement," Phys. Rev. A **63**, 012307 (2000).

[5] Berge, C., *Hypergraphs* (Elsevier Science, New Hampshire, 1989).

[6] Bouwmeester, D., Ekert, A., and Zeilinger, A., *The Physics of Quantum Information* (Springer, New York, 2000).

[7] Bose, S., Vedral, V., and Knight, P. L., "Multiparticle generalization of entanglement swapping," Phys. Rev. A **57**, 822 (1998).

[8] Bruss, D., "Characterizing entanglement," J. Math. Phys. **43**, 4237 (2002).

[9] Deo, N., *Graph Theory: With Applications to Engineering and Computational Science* (Prentice-Hall, New York, 1974).

[10] Grahmam, R. L., Grotschel, M., and Lovasz, L., *Handbook of Combinatorics* (Elsevier Science, New Hampshier, 1995).

[11] Henderson, L. and Vedral, V., "Information, Relative Entropy of Entanglement, and Irreversibility" Phys. Rev. Lett. **84**, 2263 (2000).

[12] Horodecki, M., "Entanglement measures," Quantum Inf. Comput. **1**, 3 (2001).

[13] Horodecki, M., Horodecki, P., and Horodecki, R., "Separability of Mixed States: Necessary and Sufficient Conditions" Phys. Lett. A **223**, 1 (1996).

[14] Kempe, J., "Multiparticle entanglement and its applications to cryptography," Phys. Rev. A **60**, 910 (1999).

[15] Linden, N. and Popescu, S., "On multi-partite entanglement," Fortschr. Phys. **46**, 567–578 (1998).

[16] Linden, N., Popescu, S., and Sudbery, A., "Nonlocal parameters for multiparticle density matrices," Phys. Rev. Lett. **83**, 242 (1999).

[17] Liu, D., Lu, Guoying, and Draayer, J. P., "A simple entanglement measure for multipartite pure states," Int. J. Theor. Phys. **43**, 1241 (2004).

[18] Nielsen, M. A., "Conditions for a class of entanglement transformations," Phys. Rev. Lett. **83**, 436 (1999).

[19] Nielsen, M. A. and Chuang, I. L., *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2002).

[20] Partovi, M. H., "Universal measure of entanglement," Phys. Rev. Lett. **92**, 077904 (2004).

[21] Rains, E. M., "Rigorous treatment of distillable entanglement" Phys. Rev. A **60**, 173 (1999); **63**, 173(E) (1999).

[22] Samal, M. K., (private communication).

[23] Singh, S. K., "Combinatorial approaches in quantum information theory," M.Sci. thesis, Dept. of Mathematics, IIT Kharagpur, India.

[24] Singh, S. K., Kumar, S., and Pal, S. P., quant-ph/0306049.

[25] van Lint, J. H. and Wilson, R. M., *A Course in Combinatorics* (Cambridge University Press, Cambridge, 1992).

[26] Vidal, G., Dür, W., and Cirac, J. I., "Entanglement Cost of Bipartite Mixed States" Phys. Rev. Lett. **89**, 027901 (2002).

[27] Vidal, G. and Werner, R. F., quant-ph/0102117.

[28] Wootters, W. K., "Entanglement of Formation of an Arbitrary State of Two Qubits" Phys. Rev. Lett. **80**, 2245 (1998).

[29] Zeilinger, A., Horne, M. A., Weinfurter, H., and Zukowski, M., "Three-particle entanglements from two entangled pairs," Phys. Rev. Lett. **78**, 3031 (1997).

[30] Zukowski, M., Zeilinger, A., and Weinfurter, H., "Entangling independent pulsed photon sources," Ann. N.Y. Acad. Sci. **755**, 91 (1995).