Cyclic statistics in three dimensions

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(Received 7 October 2003; accepted 10 March 2004; published online 18 May 2004)

The existence of anyons in two-dimensional systems is a well-known example of nonpermutation group statistics. In higher dimensions, however, it is expected that statistics is dictated solely by representations of the permutation group. Using basic elements from representation theory we show that this expectation is false in three-dimensions for a certain nongravitational system. Namely, we demonstrate the existence of "cyclic," or $\mathbb{Z}_n$, *nonpermutation group* statistics for a system of $n>2$ identical, unknotted rings embedded in $\mathbb{R}^3$. We make crucial use of a theorem due to Goldsmith in conjunction with the Fuchs–Rabinovitch relations for the automorphisms of the free product group on n elements. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1738189]

I. INTRODUCTION

It is a well-established fact that the topology of the configuration space of a classical system can have a nontrivial effect on its quantization. A simple illustration of this is found in the sum-over-histories quantization of a particle on a circle wherein the set of paths with fixed initial and final positions fall into classes labeled by the winding number m.[1] The full partition function is expressed as a sum of partitions over these different classes of paths, each multiplied by an overall phase $e^{im\theta}$, where $\theta\in[0,2\pi]$ labels the unitary irreducible representations of the fundamental group $\mathbb{Z}$ of the circle. Each choice of θ thus leads to an inequivalent quantization of the system. In general, inequivalent quantizations of a classical system are labeled by the unitary irreducible representations of the fundamental group of the configuration space. Indeed, several phenomena of physical interest ranging from the quantum statistics of point particles, to a Hamiltonian interpretation of the QCD theta angle, to spinorial states in quantum gravity, can be attributed to such inequivalent quantizations.[2–6]

The particular phenomenon of interest to us in this paper is the emergence of quantum statistics in systems of n identical objects. For spatial dimensions $d>2$, the fundamental group of the configuration space of such systems contains the permutation group on n elements S_n as a subgroup. For typical systems, the unitary irreducible representations of S_n and its permutation subgroups are sufficient to determine quantum statistics. For $n=2$, $d>2$, for example, the permutation group S_2 generated by the exchange operation $\mathcal{E}$ has two inequivalent unitary irreducible representations: The trivial one ($\mathcal{E}\rightarrow 1$) corresponds to bose statistics and the nontrivial one ($\mathcal{E}\rightarrow -1$) corresponds to fermi statistics. For $n>2$, $d>2$, S_n has nonabelian unitary irreducible representations which give rise to parastatistics. In dimension $d=2$, however, statistics is dictated by a nonpermutation infinite discrete group called the braid group B_n, rather than the finite group S_n. The resulting statistics is referred to as "anyonic" and plays a central role in the study of two-dimensional systems.[5,6]

Since the permutation group S_n is always a subset of the fundamental group of the configuration space for $d>2$, it is generally believed that quantum statistics is dictated by a nonpermutation group only in two-dimensions. However, for $d>2$ quantum statistics does not merely depend on the existence of the permutation group S_n as a subgroup of the fundamental group

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$\pi_1(\mathcal{Q}_n)$ of the configuration space $\mathcal{Q}_n$, but more crucially on how it "sits" in $\pi_1(\mathcal{Q}_n)$.[7] Typically, for a system of n identical objects, $\pi_1(\mathcal{Q}_n)$ has the semidirect product structure $P \ltimes S_n$, with $\ltimes$ the semidirect product and P a normal subgroup. Standard representation theory[7,8] then tells us that quantum statistics is determined not by unitary irreducible representations of S_n, but rather those of the little groups (or stability subgroups) $\mathcal{R} \subseteq S_n$ with respect to the action of S_n on the space of representations of P.

For most systems the little groups are themselves permutation subgroups S_m of S_n, with $m \leqslant n$. This can be traced to the fact that the normal subgroup P is generated only by the "internal" symmetry groups K of each object and is simply the product of n copies of K. Representation theory then tells us that the little group $\mathcal{R}$ must be a permutation subgroup of S_n.[7,8] For example, consider a system of 3 identical extended solitons which are allowed to possess spin, i.e., a 2π rotation of the soliton is nontrivial (see Ref. 9 for an example). The permutation group on 3 elements, S_3 is a subgroup of $\pi_1(\mathcal{Q}_3) = P \ltimes S_3$, where for concreteness, P can be identified with the spin subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, each $\mathbb{Z}_2$ factor representing the spin subgroup of a single soliton. Even though the solitons are classically identical, one can construct the representation $\{1/2,1/2,0\}$ of P in which two of the solitons are spin half and the third one is spin zero, thus rendering it quantum mechanically distinguishable from the others. As one might expect, the little group $\mathcal{R} \subseteq S_3$ associated with P is S_2 and not S_3, thus implying 2 rather than 3 particle quantum statistics.

In this paper, we will explicitly construct a nongravitational example in which the normal subgroup P does not have a simple product structure, and thus does indeed admit nonpermutation little groups $\mathcal{R} \subset S_n$. Namely, we will construct quantum sectors for a system of n closed, identical unknotted rings embedded in $\mathbb{R}^3$, for which $\mathcal{R}$ is a cyclic group, so that the associated quantum statistics is *cyclic*. An analogy was made between this system and that of n $\mathbb{R}P^3$ geons in $3+1$ canonical quantum gravity by the authors of Ref. 10; drawing on earlier results of Ref. 2 they demonstrated the existence of quantum sectors exhibiting indeterminate statistics when $n=2$. [A reanalysis of these sectors for 2 $\mathbb{R}P^3$ geons shows that this ambiguity is due to the lack of a canonical exchange operator (Ref. 7). Such indeterminate statistics have also been found for a system of two particles on $\mathbb{R}P^3$.] In Ref. 7 a rigorous analysis of the quantum sectors for a system of n topological geons in $3+1$ canonical quantum gravity was carried out and the existence of sectors obeying cyclic, or $\mathbb{Z}_n$ statistics was demonstrated for a system of n $\mathbb{R}P^3$ geons. Here, we will employ techniques developed in Ref. 7 to demonstrate the existence of cyclic statistics (an analogue of cyclic statistics in five-dimensions has been constructed in Ref. 12) for the system of $n \geqslant 3$ closed rings embedded in $\mathbb{R}^3$. Rings can appear in a wide class of physical systems, ranging from closed string theory, to cosmic strings, to closed superconducting flux tubes, to name a few. Recently, the existence of ring-like solitons was shown for certain nonlinear sigma models.[11] The existence of kinematical sectors exhibiting novel statistics in such systems may therefore have nontrivial physical implications.

The inequivalent quantizations for this system of rings are determined by the unitary irreducible representations of the so-called *motion group* $\mathcal{G}$ which we present in Sec. II. Using a theorem due to Goldsmith,[13] combined with the Fuchs–Rabinovitch relations for the automorphisms of the free product group on n elements,[14] we show that $\mathcal{G}$ has a nested semidirect product structure. In Sec. III, using Mackey's theory of induced representations,[8] we construct quantum sectors which exhibit cyclic statistics in a system of $n>2$ rings. We end with some brief remarks in Sec. IV on the question of modeling cyclic statistics using string Lagrangians with topological terms.

Since the spin of the rings we consider is trivial, the sectors obeying cyclic statistics clearly violate the spin-statistics connection. In Ref. 15 a spin-statistics correlation was shown to hold when the configuration space is expanded to allow the creation and annihilation of rings, thus excluding nonpermutation group statistics. However, first quantized systems with ring-like structures could very well occur in condensed matter systems; as suggested in Ref. 10, the rings can be stabilized against creation and annihilation by carrying conserved charges. Whether sectors obeying cyclic statistics are physically realized or not is, of course, ultimately a question for experiment to decide.

II. THE MOTION GROUP FOR A SYSTEM OF *n* RINGS

We consider the system of n identical, nonintersecting, infinitely thin, unknotted, unlinked, unoriented rings, $C=C_1\cup C_2\cup\cdots\cup C_n$ in $\mathbb{R}^3$, which cannot be destroyed or created. The configuration space $\mathcal{Q}_n$ for this system of rings is the space of embeddings of C in $\mathbb{R}^3$ quotiented by an appropriate group of symmetries called the *motion group* $\mathcal{G}$ which we will define below. An obvious example of a symmetry is the exchange of a pair of identical rings. The fundamental group of $\mathcal{Q}_n$ for this system is isomorphic to the motion group $\mathcal{G}$. This group is nontrivial for all $n\geqslant 1$, and has been extensively studied by Dahm and Goldsmith.[13]

Since the configuration space $\mathcal{Q}_n$ is multiply connected, on quantization, the Hilbert space splits into inequivalent quantum sectors. A systematic study of such quantum sectors can be found in Ref. 4. The wave functions $\psi:\tilde{\mathcal{Q}}_n\to\mathbb{C}$, where $\tilde{\mathcal{Q}}_n$ is the universal cover of $\mathcal{Q}_n$, so that $\pi_1(\mathcal{Q}_n)$ acts nontrivially on ψ. Since physically measurable quantities like inner products should only be functions on the classical configuration space $\mathcal{Q}_n$, the action of $\pi_1(\mathcal{Q}_n)$ on ψ must be represented as a "phase," which can be nonabelian for $n\geqslant 2$. Thus, at every point $q\in\mathcal{Q}_n$, ψ is valued in the carrier spaces of the unitary irreducible representations of $\pi_1(\mathcal{Q}_n)$. The inequivalent unitary irreducible representations of $\pi_1(\mathcal{Q}_n)$ then correspond to inequivalent quantum sectors.

The motion group $\mathcal{G}$ for this system of rings is defined as follows.[13] Let $H(\mathbb{R}^3)$ denote the space of continuous maps or homeomorphisms of $\mathbb{R}^3$ into itself and $H(\mathbb{R}^3,C)$ the subspace of homeomorphisms which leave C invariant. Let $H_\infty(\mathbb{R}^3)$ and $H_\infty(\mathbb{R}^3,C)$ be subspaces of $H(\mathbb{R}^3)$ and $H(\mathbb{R}^3,C)$, respectively, consisting of homeomorphisms with compact support. A *motion* is then defined as a path h_t in $H_\infty(\mathbb{R}^3)$ such that h_0 is the identity map from $\mathbb{R}^3$ to itself and $h_1=H_\infty(\mathbb{R}^3,C)$. The product of two motions can then be defined and the inverse g^{-1} of the motion g is the path $g_{(1-t)}\circ g_1^{-1}$.[13] Two motions h,h' are taken to be equivalent if $h'^{-1}h$ is homotopic to a path which lies entirely in $H_\infty(\mathbb{R}^3,C)$. The motion group $\mathcal{G}$ is then the set of equivalence classes of motions of C in $\mathbb{R}^3$ with multiplication induced by "$\circ$" (for brevity of expression we will henceforth refer to an equivalence class of motions as a motion).

We will use Hendricks' definition of a rotation[16] to describe the generators of the motion group. A 3-ball $\mathbb{B}^3\subset\mathbb{R}^3$ will be said to be rotated by an angle α in the following sense: take a collar neighborhood $S^2\times[0,1]$ of $\partial\mathbb{B}^3\approx S^2$ and let the S^2's be differentially rotated from 0 to α with $S^2\times\{0\}=\partial\mathbb{B}^3$ rotated by α and $S^2\times\{1\}$ not rotated at all. The rotation by an angle α of a solid torus $U=\mathbb{B}^2\times S^1$ in the direction of its noncontractible circle S^1 is similarly defined as a differential rotation of a collar neighborhood $T^2\times[0,1]$ of $\partial U\approx T^2$, with $T^2\times\{0\}=\partial U$ rotated by α and $T^2\times\{1\}$ not rotated at all.

$\mathcal{G}$ is generated by three types of motions which are quite easily visualized.[13] The first is the flip motion f_i which corresponds to "flipping" the ith ring (in the case of oriented rings, this motion yields a configuration distinct from the first and is not a symmetry). This motion corresponds to a rotation by π of an open ball in $\mathbb{R}^3$ containing C_i, about an axis lying in the plane of C_i. Since the rings are embedded in three dimensions, $f_i^2=e$, so that each flip generates a $\mathbb{Z}_2$ subgroup. Next is the exchange motion e_i which exchanges the ith ring with the $(i+1)$th ring. This can be thought of as a π rotation of a solid torus in $\mathbb{R}^3$ containing both C_i and C_{i+1} (but no others). These motions generate the permutation group S_n. Finally, one has the slide motion s_{ij} which requires a slightly more detailed description. A point in the configuration space (i.e., $\mathbb{R}^3-C$ modulo the action of the motion group) is itself a multiply connected space with $\pi_1(\mathbb{R}^3-C)$ isomorphic to the free product group on n generators $F(x_1,x_2\ldots,x_n)\approx\mathbb{Z}*\mathbb{Z}*\cdots*\mathbb{Z}$, each factor of $\mathbb{Z}$ isomorphic to the fundamental group of a single ring in $\mathbb{R}^3$. s_{ij} is then the motion of C_i along one of these $\mathbb{Z}$ factors, specifically, the generator of $\mathbb{Z}\subset\pi_1(\mathbb{R}^3-C)$ passing through C_j. Again, one can define the slide using a rotation: Consider a solid torus containing C_i and "threading" C_j, without intersecting it. A slide is then a 2π rotation of this solid torus. The existence of slide motions is key to the present analysis, and is what makes the analogy with the system of topological geons explicit.

We denote the three subgroups generated by the flips, the exchanges and the slides as $\mathcal{F}$, S_n and $\mathcal{S}$, respectively. We will also need to identify the subgroup $\tilde{\mathrm{G}}$ generated by only the flips and the exchanges. The structure of S_n is known: It is simply the permutation group on n elements. However, the structures of $\mathcal{F}$ and $\mathcal{S}$ need to be deduced, as does information on how these groups sit in $\mathcal{G}$. While the generators of $\mathcal{G}$ have been known for some years, its explicit structure in terms of these subgroups has not been obtained until now.

Definition: A group G is said to be a semidirect group $P \ltimes K$ if (a) $\forall g \in G$, $\exists p \in P$ and $k \in K$ such that $g = pk$ (b) P is normal in G and (c) $P \cap K = e$. For every fixed $k \in K$, $p \rightarrow kpk^{-1}$ generates an automorphism α_k of P. G is said to have a nested semidirect product structure if further, either K or P or both, themselves are semidirect product groups.

We now show that $\mathcal{G}$ has the nested semidirect product structure

$$\mathcal{G} = \mathcal{S} \ltimes (\mathcal{F} \ltimes S_n). \tag{1}$$

We also show that $\mathcal{S}$ is the nonabelian group made up of the free product group on $n(n-1)$ generators

$$\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n(n-1)}, \tag{2}$$

subject to the conditions

$$s_{ij}s_{kl} = s_{kl}s_{ij}, \quad s_{ij}s_{kj} = s_{kj}s_{ij}, \quad s_{ik}s_{jk}s_{ij} = s_{ij}s_{ik}s_{jk}. \tag{3}$$

$\mathcal{F}$, on the other hand, can be shown to be the abelian group isomorphic to the direct product group of the $\mathbb{Z}_2$ flips of each ring

$$\mathcal{F} = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n}. \tag{4}$$

For brevity of notation we define the exchange action π_i on the set of n integers labeling the n rings as follows: For $1 \leqslant j \leqslant n$, $\pi_i : j \rightarrow \pi_i(j)$ where $\pi_i(j) = j$ for $j \neq i, i+1$, $\pi_i(i) = i+1$ and $\pi_i(i+1) = i$. Here when $i = n$, $i+1$ is identified with 1.

Lemma: $\mathcal{G}$ has the nested semidirect product structure (1). Thus, $\mathcal{S}$ is normal in $\mathcal{G}$ and $\mathcal{F}$ is normal in the subgroup $\tilde{\mathrm{G}} \subset \mathcal{G}$ generated by the flips and the permutations. The automorphisms of $\mathcal{S}$ generated by $\tilde{\mathrm{G}}$ and those of $\mathcal{F}$ generated by S_n are given by the Fuchs–Rabinovitch relations, induced by the Dahm isomorphism $D: \mathcal{G} \rightarrow \overline{\mathcal{G}} \subseteq Aut(F(x_1, \ldots, x_n))$ where $Aut(F(x_1, \ldots, x_n))$ is the group of automorphisms of $F(x_1, \ldots, x_n)$. Moreover, $\mathcal{S}$ is isomorphic to the group (2) subject to the conditions (3), and $\mathcal{F}$ is isomorphic to the group (4).

Proof: The induced action of the motion group on $\pi_1(\mathbb{R}^3 - C)$ has been examined by Goldsmith,[13] and provides us with a crucial step in deducing the structure of $\mathcal{G}$. As noted earlier, $\pi_1(\mathbb{R}^3 - C)$ is isomorphic to $F(x_1, \ldots, x_n)$, the free product group on n-generators, x_i, $i = 1, \ldots, n$. In Ref. 13 the "Dahm" homomorphism $D: \mathcal{G} \rightarrow Aut(F(x_1, \ldots, x_n))$ is defined where $Aut(F(x_1, \ldots, x_n))$ is the group of automorphisms of $F(x_1, \ldots, x_n)$. For each motion $g \in \mathcal{G}$, D induces an automorphism of $F(x_1, \ldots, x_n)$. The following theorem then states:

Goldsmith's Theorem:[13] The group of motions $\mathcal{G}$ of the trivial n-component link C in $\mathbb{R}^3$ is generated by the following types of motions:

(1) f_i or flips. Turn the ith ring over. This induces the automorphism $\phi_i : x_i \rightarrow x_i^{-1}$, $x_k \rightarrow x_k$, $k \neq i$, of $F(x_1, \ldots, x_n)$.
(2) e_i or exchange. Interchange the ith and the $(i+1)$th rings. The induced automorphism of $F(x_1, \ldots, x_n)$ is $\epsilon_i : x_i \rightarrow x_{i+1}, x_{i+1} \rightarrow x_i$ and $x_k = x_k$ for $k \neq i, i+1$.
(3) s_{ij} or slides. Pull the ith ring through the jth ring. This induces the automorphism $\sigma_{ij} : x_i \rightarrow x_j x_i x_j^{-1}$, $x_k \rightarrow x_k$, $k \neq i$, of $F(x_1, \ldots, x_n)$.

Moreover, the Dahm homomorphism, $D: \mathcal{G} \rightarrow Aut(F(x_1, \ldots, x_n))$ is an isomorphism onto the subgroup $\overline{\mathcal{G}}$ of $Aut(F(x_1, \ldots, x_n))$ generated by ϕ_i, ϵ_i and σ_{ij}, where $1 \leqslant i, j \leqslant n, i \neq j$.

Let us denote the subgroups of $\overline{\mathcal{G}}$ generated by the automorphisms σ_{ij}, ϕ_i and ϵ_i as $\overline{\mathcal{S}}$, $\overline{\mathcal{F}}$ and $\overline{\mathrm{S}_n}$, respectively. We may now employ the Fuchs–Rabinovitch relations for $Aut(F(x_1,\ldots,x_n))$ which provides a complete set of relations for the generators of $\overline{\mathcal{G}}$.[14] For $\pi_1(\mathbb{R}^3-C)=\mathbb{Z}*\mathbb{Z}*\cdots*\mathbb{Z}$, in particular, these relations are simple and imply that $\overline{\mathcal{G}}\subset Aut(F(x_1,\ldots,x_n))$ has the nested semidirect product structure

$$\overline{\mathcal{G}}=\overline{\mathcal{S}}\ltimes(\overline{\mathcal{F}}\ltimes\overline{\mathrm{S}_n})=\overline{\mathcal{S}}\ltimes\bar{\tilde{\mathrm{G}}}, \tag{5}$$

where $\bar{\tilde{\mathrm{G}}}=\overline{\mathcal{F}}\ltimes\overline{\mathrm{S}_n}$. In particular, the generators ϵ_i of $\overline{\mathrm{S}_n}$ generate the following automorphisms of $\overline{\mathcal{F}}$:

$$\epsilon_i\phi_j\epsilon_i^{-1}=\phi_{\pi_i(j)}, \tag{6}$$

and the generators ϵ_i and ϕ_i of $\bar{\tilde{\mathrm{G}}}$ generate the following automorphisms of $\overline{\mathcal{S}}$:

$$\begin{gathered}\epsilon_i\sigma_{jk}\epsilon_i^{-1}=\sigma_{\pi_i(j)\pi_i(k)},\\ \phi_i\sigma_{jk}\phi_i^{-1}=\sigma_{jk},\forall k\neq i,\\ \phi_i\sigma_{ji}\phi_i^{-1}=\sigma_{ji}^{-1}.\end{gathered} \tag{7}$$

Moreover, these relations imply that $\overline{\mathcal{S}}$ is the free product group on $n(n-1)$ generators $\mathbb{Z}*\mathbb{Z}*\cdots*\mathbb{Z}$ subject to the conditions $\sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}$, $\sigma_{ij}\sigma_{kj}=\sigma_{kj}\sigma_{ij}$, and $\sigma_{ik}\sigma_{jk}\sigma_{ij}=\sigma_{ij}\sigma_{ik}\sigma_{jk}$ while $\overline{\mathcal{F}}$ is the abelian direct product group made up of n factors of $\mathbb{Z}_2$, $\overline{\mathcal{F}}=\mathbb{Z}_2\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$. Since D is an isomorphism with $D(\mathcal{S})\subseteq\overline{\mathcal{S}}$, $D(\mathcal{F})\subseteq\overline{\mathcal{F}}$ and $D(S_n)\subseteq\overline{\mathrm{S}_n}$, this means that $\mathcal{S}\approx\overline{\mathcal{S}}$, $\mathcal{F}\approx\overline{\mathcal{F}}$ and $S_n\approx\overline{\mathrm{S}_\mathrm{n}}$. From (5), it is then obvious that $\mathcal{G}$ itself has the nested semidirect product structure (1). The automorphisms of S_n on $\mathcal{F}$ and of $\tilde{\mathrm{G}}$ on $\mathcal{S}$, respectively, are given by (6) and (7) and induced by the isomorphism $D:\mathcal{G}\to\overline{\mathcal{G}}$. Moreover, $\mathcal{S}$ is the free product group on $n(n-1)$ generators $\mathbb{Z}*\mathbb{Z}*\cdots*\mathbb{Z}$ subject to the relations (3) and $\mathcal{F}$ is given by (4). □

While the structure of the motion group can be completely deduced from the Dahm homomorphism and the Fuchs–Rabinovitch relations, it is instructive to examine this group without recourse to $Aut(F(x_1,\ldots,x_n))$. Using just the definition of the motion group we now illustrate the following properties of $\mathcal{G}$: (a) $\mathcal{S}$ is normal in $\mathcal{G}$ and satisfies the relations (3) and (b) that $\mathcal{F}$ is normal in $\tilde{\mathrm{G}}$.

By definition, an element of the motion group is a homotopy equivalence class of paths in the space of homeomorphisms with compact support. Two homeomorphisms h_1 and h_2 with compact support on the regions U_1 and U_2 commute if $U_1\cap U_2=\phi$ and hence so do the corresponding motions. It is, therefore, useful to isolate the "minimal" neighborhoods in which homeomorphisms representing the generators of the motion group act so as to determine which two motions commute.

Let U_i denote an open ball neighborhood of C_i in $\mathbb{R}^3$ which contains no other C_j, $j\neq i$, and let U_{ij} denote an open ball neighborhood of $C_i\cup C_j$ containing no other C_k, $k\neq i,j$, etc. We will refer to the U_i as "exclusive" neighborhoods and the $U_{ij},U_{ijk},\ldots$, etc. as "common" neighborhoods. The flip motion f_i is then defined by a homotopy equivalence class of paths in $H_\infty(\mathbb{R}^3)$ which include a "model" path made up of homeomorphisms with support only on U_i, i.e., a path in $H_\infty(\mathbb{R}^3)$ along which C_i is flipped without disturbing any of the other rings. Next, the exchange motion e_i is defined by a homotopy equivalence class of paths including a model path made up of homeomorphisms with support only on $U_{i(i+1)}$, i.e., the ith and the $(i+1)$th ring are exchanged without disturbing the other rings. Finally, the slides s_{ij} are defined by a homotopy equivalence class of paths including a model path with support only on U_{ij}, i.e., a path in which the other rings are not disturbed.

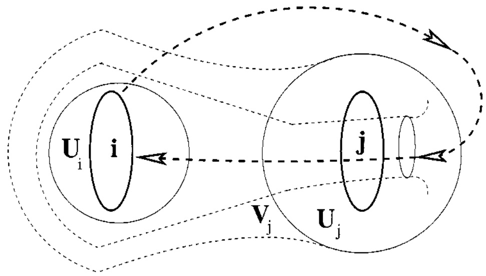

FIG. 1. Under the slide s_{ij} C_i "tunnels" through the neighborhood U_j of C_j and maps it onto the region V_j, shown by dashed lines. V_j, therefore, "encloses" C_i without containing it, i.e., $U_i \cap V_j = \phi$.

Now, the set of exclusive neighborhoods $\{U_i\}$ remains invariant when acted upon by the subgroup $\tilde{\mathrm{G}}$ generated by the flips and by the exchanges. This is obvious for $\mathcal{F}$, since each flip f_i acts within an exclusive neighborhood. For S_n, while the exchange e_i has compact support on $U_{i(i+1)}$, its action can be considered as a pure exchange of U_i with U_{i+1}. Thus, one can consider as a model path for the exchange, a localized π rotation in $U_{i(i+1)}$ which exchanges U_i with U_{i+1}. This, however, is not the case with the slides s_{ij}. While the set $\{U_k\}$ for $k \neq j$ remains invariant under the slide s_{ij} of C_i through C_j, the exclusive neighborhood U_j does not. The nonlocal action of the slide takes U_j into a set V_j which "encloses" C_i even though it does not contain it, i.e., there exists a U_i such that $U_i \cap V_j = \phi$ (see Fig. 1). Thus, V_j is not an exclusive neighborhood of C_j. This feature leads to subtleties in what follows.

Since the exchanges and the flips leave the set $\{U_i\}$ invariant, model paths are sufficient to see that $\mathcal{F}$ is normal in $\tilde{\mathrm{G}}$, i.e., for all $\tilde{g} \in \tilde{\mathrm{G}}$, $i \leqslant n$, $\tilde{g} f_i \tilde{g}^{-1} \in \mathcal{F}$. To show this, it is sufficient to take $\tilde{g}$ to be an exchange. For the motion $e_i f_j e_i^{-1}$ with $j \neq i, i+1$, the model paths for e_i and f_j have compact support on $U_{i(i+1)}$ and U_j, respectively, where $U_{i(i+1)} \cap U_j = \phi$. Hence the motions commute, so that $e_i f_j e_i^{-1} = f_j$. Now consider the motion $e_i f_i e_i^{-1}$. The model paths for e_i^{-1} exchange U_i with U_{i+1}. One can then use a model path for the motion f_i which acts on some $U'_{i+1} \subset U_{i+1}$ so that the final exchange e_i which exchanges U_{i+1} with U_i does not disturb the action of f_i on U'_{i+1}. Thus, $e_i f_i e_i^{-1} = f_{i+1}$. Similarly, $e_i f_{i+1} e_i^{-1} = f_i$.

However, model paths are insufficient when one wants to deal with the slides. Let us consider a motion whose model path involves homeomorphisms with support only on the compact region U. The homotopy class of paths defining this motion also includes the nonmodel, or "gregarious" paths, which involve homeomorphisms with nontrivial compact support on $\mathbb{R}^3 - U$. In other words, gregarious paths *can* disturb the other rings; they can contain homeomorphisms with nontrivial support on neighborhoods of rings left undisturbed by the model path. Consider the motion f_i for simplicity. A model path for f_i has support only on U_i and corresponds to a π rotation about an axis in the plane of C_i. A gregarious path on the other hand can be constructed piecewise as follows: (a) Rotate C_i by $\pi/3$, about an axis $\hat{x}$ in its plane; (b) flip another C_j, $j \neq i$; (c) rotate C_i by a further $\pi/3$ about $\hat{x}$ in the same sense as before; (d) flip C_j again; (e) and complete with a further $\pi/3$ rotation of C_i about $\hat{x}$ in the same sense as before. Such a path clearly corresponds to the motion f_i, but involves homeomorphisms of $\mathbb{R}^3$ in $H_\infty(\mathbb{R}^3)$ which have nontrivial support on the ring C_j, $j \neq i$.

Both model and gregarious paths are necessary to demonstrate that $\mathcal{S}$ is normal in $\mathcal{G}$. $\mathcal{S}$ is a normal subgroup of $\mathcal{G}$ if $\forall g \in \mathcal{G}$ and $\forall i,j \leqslant n$ $g s_{ij} g^{-1} \in \mathcal{S}$. It is sufficient to take g to be a generator of S_n or $\mathcal{F}$.

We begin with the exchanges. Let us examine the motion $e_k s_{ij} e_k^{-1}$ by considering only model paths in the appropriate homotopy class. For $k \neq i,j$ $e_k s_{ij} e_k^{-1} = s_{ij}$, since the homeomorphisms that make up model paths for e_k and s_{ij} have compact supports on U_k and U_{ij} with $U_k \cap U_{ij} = \phi$. Model paths are, however, insufficient to show that $e_k s_{ij} e_k^{-1}$ is also a slide for $k = i$, $i-1$, j or $j-1$. Consider the motion $e_j s_{ij} e_j^{-1}$ with $k = j$, $i \neq j, j+1$. e_j^{-1} swaps U_j with U_{j+1} by a π rotation of a torus containing both U_j and U_{j+1}. Next, s_{ij} rotates by 2π a solid torus containing

C_i and threading C_{j+1}, thus mapping U_{j+1} into a nonexclusive neighborhood V_{j+1}. A model path for the final exchange e_j would rotate by π a solid torus containing new exclusive neighborhood U'_{j+1} of C_{j+1} and U_i. $U_{i(j+1)}$ in which the slide acts, is not left invariant by this final exchange, making the resultant motion difficult to unravel. Instead, we use the following gregarious path to perform the final exchange: consider a path in $H_\infty(\mathbb{R}^3)$ where $U_{i(j+1)}$ and U_j are swapped by performing an appropriate π rotation in the common neighborhood $U_{ij(j+1)}$ of C_i, C_j and C_{j+1}. The final exchange motion is then completed by merely moving C_i back to its original position. $U_{i(j+1)}$ is thus left undisturbed so that the full motion is the slide $s_{i(j+1)}$. A use of a similar gregarious path for the final exchange shows that $e_{(j-1)}s_{ij}e_{(j-1)}^{-1}=s_{i(j-1)}$, $e_i s_{ij} e_i^{-1} = s_{(i+1)j}$ and $e_{(i-1)}s_{ij}e_{(i-1)}^{-1}=s_{(i-1)j}$.

Next, consider the flips. The motion $f_k s_{ij} f_k^{-1}$ can again be examined using only model paths for $k\neq j$, and we can see that it is s_{ij}. This is because the model path for f_k has compact support only on U_k which is undisturbed by the slide even when $k=i$. However, the use of model paths is insufficient to examine the motion $f_j s_{ij} f_j^{-1}$: not only does U_j not remain an exclusive neighborhood under the slide s_{ij}, but the f_j moves the points in U_j relative to each other. Rather than consider just a single gregarious path, following Ref. 7, we use a particular set of homotopy equivalent paths. Let κ be the generator of $\pi_1(\mathbb{R}^3 - C)$ through C_j about which the slide s_{ij} takes C_i. We define the paths γ_α as follows: (a) perform a "part" inverse flip corresponding to a $(\pi - \alpha)$ rotation of C_j about $\hat{x}$ (b) slide C_i through C_j along κ^{-1} (c) finish the inverse flip f_j^{-1} of C_j by a rotation α about $\hat{x}$ and (d) finally, perform the flip f_j of C_j about $\hat{x}$. γ_0 then corresponds to the model path for the motion $f_j s_{ij} f_j^{-1}$ while the path γ_π corresponds to the slide s_{ij}^{-1}. Since α is a continuous parameter $\alpha\in[0,\pi]$, the γ_α provide a homotopy map from γ_0 to γ_π, which implies that $f_j s_{ij} f_j^{-1} = s_{ij}^{-1}$ (it is perhaps a useful exercise for the reader to see why a similar argument cannot be used to find a set of homotopic paths between $s_{ij} f_j s_{ij}^{-1}$ and an element of $\mathcal{F}$).

Thus, the slide subgroup $\mathcal{S}$ is a normal subgroup of $\mathcal{G}$.

We can also demonstrate that the relations (3) are satisfied by $\mathcal{S}$, using just the definition of the motion group. The first of these relations is clearly satisfied by the generators of $\mathcal{S}$, since the model paths corresponding to the slides s_{ij} and s_{kl} involve homeomorphisms with compact support only on U_{ij} and U_{kl} where $U_{ij}\cap U_{kl}=\phi$. It takes a little more work to show that the other two relations are also satisfied by the generators of $\mathcal{S}$.

Consider the motion $s_{ij}s_{kj}s_{ij}^{-1}$. s_{ij} and s_{kj} are slides of the two rings C_i and C_k through a third ring C_j. These slides are obtained by 2π rotations of the solid tori $V_{ij}\approx B^2\times S^1$ and $V_{kj}\approx B^2\times S^1$ which thread through C_j, with $V_{ij}\cap V_{kj}=\phi$. Define the paths γ_α as follows: (a) A rotation by $-\alpha$ of V_{ij}; (b) a 2π rotation of V_{kj}; (c) a $-(2\pi-\alpha)$ rotation of V_{ij} and finally; (d) a 2π rotation of V_{ij}. γ_0 then defines a model path for the motion $s_{ij}s_{kj}s_{ij}^{-1}$, and $\gamma_{2\pi}$ corresponds to the slide s_{kj}. Since α is a continuous parameter, γ_0 is homotopic to $\gamma_{2\pi}$ and hence also corresponds to s_{kj}. Notice that by keeping $V_{ij}\cap V_{kj}=\phi$ we prevent a mixing of their rotations and hence the deformations of the neighborhood U_j by s_{ij} and by s_{kj}.

Next, consider the motion $s_{ij}s_{ik}s_{jk}s_{ij}^{-1}$. Although this looks considerably more complicated than the previous motion, the two elements of $\mathcal{S}$ involved, $s_{ik}s_{jk}$ and s_{ij}, have compact supports on nonintersecting neighborhoods. Namely, the element $s_{ik}s_{jk}$ corresponds to sliding C_j through a generator ρ of π_1 of C_k and then sliding C_i through the same generator. Under this action, $U_j\rightarrow U_j$ and $U_i\rightarrow U_i$, while U_k is now mapped to a region V_k which now "encloses" both C_i and C_j. Thus, there exists a path in $H_\infty(\mathbb{R}^3)$ corresponding to the motion $s_{ik}s_{jk}$ made up of homeomorphisms which leave the common neighborhood U_{ij} undisturbed. Since there is a model path corresponding to the slide s_{ij} which has compact support only on U_{ij}, this means that the two motions $s_{ik}s_{jk}$ and s_{ij} indeed commute. Thus, the generators of $\mathcal{S}$ satisfy all the relations (3).

Remark: In Ref. 10 a set of relations for the generators in the $n=2$ case was given: $f_i^2=\mathcal{E}^2=(f_i\mathcal{E})^4=(f_i\mathcal{E}s_j\mathcal{E})^2=e$ where $i=1,2$ and the slides s_i generate $\mathcal{S}$, the flips f_i generate $\mathcal{F}$ and the exchange $\mathcal{E}$ generates S_2. These follow in a straightforward manner from the relations presented above.

III. CYCLIC STATISTICS

The inequivalent quantum sectors for our system of n identical rings are labeled by the unitary irreducible representations of $\pi_1(\mathcal{Q}_n)\approx\mathcal{G}$. The group $\mathcal{G}$ represents a "gauge" symmetry and the action of the individual motions $g\in\mathcal{G}$ on $\mathbb{R}^3-C$ can be used to interpret the associated quantum phases.

Let us begin by considering the simplest case, namely a single ring for which the motion group $\pi_1(\mathcal{Q})$ is simply $\mathcal{F}=\mathbb{Z}_2$. Let ψ_1 be a wave function on $\tilde{\mathcal{Q}}$ with localized support along the fibre $\{\tilde{q}_1,\tilde{q}_2\}$ at $q\in\mathcal{Q}$, generated by the action of the flip f, i.e., $\tilde{q}_1\rightarrow\tilde{q}_2=f\circ\tilde{q}_1$. Under the action of f, $\psi_1\rightarrow\psi_2$, which is also localized on the fibre at q. Since wave functions take values in the carrier spaces of the irreducible representations of $\pi_1(\mathcal{Q})$, $\psi_2=\Delta(f)\circ\psi_1$, where Δ is a unitary irreducible representation of $\mathcal{F}=\mathbb{Z}_2$. Δ can be either the trivial representation $\Delta(f)=1$ or the nontrivial one with $\Delta(f)=-1$, the associated quantum sectors corresponding to either an "unoriented" quantum ring in which $\psi_2=\psi_1$ or an "oriented" quantum ring in which $\psi_2=-\psi_1$. (Recall that classically, the rings are unoriented, since flips are a symmetry of the classical configuration space.) When there are two rings, i.e., $n=2$, the motion group includes the permutation group $S_2=\mathbb{Z}_2$ which gives rise to nontrivial quantum statistics. Consider a wave function ψ which is localized along the fibre of a configuration where the two rings are well-separated and identical: Under an exchange operation ψ, therefore, picks up a phase of ±1 corresponding to bosonic/fermionic quantum statistics (see Refs. 2, 4, and 7 for a more detailed discussion of quantum phases and statistics for extended objects). As we will presently demonstrate, for $n>2$ the existence of the slide subgroup in $\pi_1(\mathcal{Q}_n)$ gives rise to an unexpected complexity in the structure of the phases acquired by the wave function under the action of the permutation group.

As mentioned in the introduction, the quantum statistics of a system is not solely determined by S_n, but rather by the unitary irreducible representations of its stability subgroup $\mathcal{R}\subseteq S_N$ associated with its action on the unitary irreducible representations of the normal subgroup $\mathcal{S}\ltimes\mathcal{F}$ of $\mathcal{G}$. This follows from Mackey's theory of induced representations for semidirect product groups $P\ltimes K$.[8] In this construction, one begins with the space of unitary irreducible representations $\hat{P}$ of the normal subgroup P. The subgroup K has the (not necessarily free) action on $\hat{P}$

$$\Delta(p)\rightarrow\tilde{\Delta}(p)=\Delta(kpk^{-1}),\tag{8}$$

where $\Delta\in\hat{P}$, $p\in P$ and $k\in K$. Starting with a particular $\Delta_1\in\hat{P}$ one obtains an orbit $\mathcal{O}=\{\Delta_1,\Delta_2,\ldots,\Delta_r\}$ of the K action on $\hat{P}$, and the little group $\mathcal{R}$ associated with $\mathcal{O}$. The full unitary irreducible representation of $P\ltimes K$ is then built up by taking the direct product of (a) $[\Delta_1\oplus\Delta_2\oplus\cdots\oplus\Delta_r]$ with (b) a unitary irreducible representation of $\mathcal{R}$. For example, if one starts with the trivial representation of P, then the orbit consists of a single point and $\mathcal{R}=K$. The unitary irreducible representations of $P\ltimes K$ that can be constructed from this orbit are just the unitary irreducible representations of K. On the other hand, one may find an orbit of K in $\hat{P}$ with $\mathcal{R}=e$. The full unitary irreducible representation is then simply the sum of the unitary irreducible representations in the orbit, $\oplus_i\Delta_i$. The action of the subgroup K is then reduced to a canonical map which permutes the carrier spaces $\mathbb{H}_i$ of Δ_i[7] (as discussed in Ref. 7 for $n\geqslant4$ the possibility of projective statistics exists when $\pi_1(Q)$ has a semidirect product structure).

We now illustrate the importance of the little group in determining quantum statistics with a simple example. Because of the nested semidirect product structure of the motion group, we may begin by first representing the slides trivially. We thus need to find only the unitary irreducible representations of the subgroup $\tilde{\mathrm{G}}=\mathcal{F}\ltimes S_n$. Since $\mathcal{F}\approx\mathbb{Z}_2\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$ with n $\mathbb{Z}_2$ factors, it is trivial to list its unitary irreducible representations, i.e., $\Delta\equiv(k_1,k_2,\ldots,k_n)$, with $k_i=\pm1$. For example, for $n=3$, let us start with the unitary irreducible representation $\Delta_1=(-,-,+)$ of the normal subgroup $\mathcal{F}$ of $\tilde{\mathrm{G}}$. This choice corresponds to two of the rings being identical and oriented, while the third is unoriented and hence distinguishable from the others. The action of S_3 on Δ_1

generates the orbit $\{\Delta_1,\Delta_2,\Delta_3\}\equiv\{(-,-,+),(+,-,-),(-,+,-)\}$ in $\hat{\mathcal{F}}$ whose associated little group is S_2. The resulting unitary irreducible representation of $\tilde{\mathrm{G}}$ is then $\{\Delta_1\oplus\Delta_2\oplus\Delta_3\}\otimes\Gamma$, where Γ is a unitary irreducible representation of S_2. Under a two particle exchange Γ provides either a bosonic $(+1)$ or a fermionic (-1) phase. Since one of the three rings has been rendered quantum mechanically distinguishable from the other two, one obtains an appropriate two ring statistics. The action of the remaining elements of S_3, namely the cyclic elements, is canonical: They merely permute the carrier spaces $\mathbb{H}_i$ of the Δ_i. This general structure continues to hold for all n, and is illustrative for the case of primary interest here when the slides are nontrivially represented.

Before proceeding to construct a quantum sector exhibiting cyclic statistics for $n\geqslant 3$, let us consider the simplest case with the slides nontrivially represented, namely when $n=2$. For $n=2$, the slide subgroup is generated by the two slides s_1,s_2, the flip subgroup $\mathcal{F}$ by the two flips f_1,f_2 and the permutation group S_2 by the exchange $\mathcal{E}$. The following example demonstrates a peculiar feature which will reappear for $n>2$, whereby slides render a pair of "locally identical" rings distinguishable. Let us start with the abelian unitary irreducible representation of $\mathcal{S}$, $\Omega_1(s_1)=1,\Omega_1(s_2)=-1$. The action of f_j on Ω_1 is $\Omega_1(s_i)\rightarrow\tilde{\Omega}_1(s_i)=\Omega_1(f_js_if_j^{-1})=\Omega_1(s_i)$ and is hence contained in the little group $\mathcal{R}$ of $\tilde{\mathrm{G}}$. Under the action of $\mathcal{E}$, $\Omega_1(s_i)\rightarrow\tilde{\Omega}_1(s_i)=\Omega_1(\mathcal{E}s_i\mathcal{E}^{-1})=\Omega_1(s_j)\neq\Omega_1(s_i)$ where $j\neq i$, so that $S_2\subseteq\mathcal{R}$. Thus, the two rings are quantum mechanically distinguishable even if $\mathcal{F}$ is trivially represented. This is very unusual, since indistinguishability of a collection of objects is often thought of as a local, intrinsic property of each object. However, in this representation, it is the nonlocal action of slides which renders the two rings distinguishable: The rings slide through each other differently. Thus, there exists a wave function ψ localized along the fibre of a configuration of two well-separated identical rings such that under the action of s_1, $\psi\rightarrow\psi$ and under that of s_2, $\psi\rightarrow-\psi$. This quantum lifting of indistinguishability by slides is key to the existence of nonpermutation group statistics for $n>2$.

We are now ready to demonstrate cyclic statistics for the case of $n=3$ rings. The slide subgroup $\mathcal{S}$ is generated by the six generators s_{ij}, $i,j=1,2,3$, $i\neq j$, the flip subgroup $\mathcal{F}$ is generated by the 3 elements f_1,f_2,f_3, and the permutations form the nonabelian subgroup S_3. We start with the following abelian unitary irreducible representation Ω_1 of $\mathcal{S}$:

$$\Omega_1(s_{12})=\Omega_1(s_{23})=\Omega_1(s_{31})=-1,\quad \Omega_1(s_{21})=\Omega_1(s_{32})=\Omega_1(s_{13})=1. \tag{9}$$

Consider the action of $\tilde{\mathrm{G}}$ on Ω_1. The action of a flip f_k on Ω_1 for $k=i$ or j is: $\Omega_1(s_{ij})\rightarrow\Omega_1(f_ks_{ij}f_k^{-1})=\Omega_1^{-1}(s_{ij})=\Omega_1(s_{ij})$, while the action of f_k, $k\neq i,j$ is trivial. Thus, $\mathcal{F}$ lies in the stability subgroup of $\tilde{\mathrm{G}}$. Single exchanges e_i however, do not leave Ω_1 invariant: For example, $\Omega_1(s_{i(i+1)})\rightarrow\Omega_1(e_is_{i(i+1)}e_i^{-1})=\Omega_1(s_{(i+1)i})=-\Omega_1(s_{i(i+1)})$ (where $(i+1)$ is defined mod 3). However, a pair of exchanges does leave Ω_1 invariant! A pair of exchanges, say $z=e_2e_3$, generates the cyclic subgroup Z_3 of S_3. Under the action of e_2e_3 the slides $\{s_{12},s_{23},s_{31}\}\rightarrow\{s_{23},s_{31},s_{12}\}$, and $\{s_{21},s_{32},s_{13}\}\rightarrow\{s_{32},s_{13},s_{21}\}$, thus leaving Ω_1 invariant. Therefore, the stability subgroup associated to Ω_1 is $\mathcal{F}\ltimes\mathbb{Z}_3$. The remaining elements of $\mathcal{F}\ltimes S_3$, namely e_1,e_2, and e_3 generate the two element orbit $\mathcal{O}\equiv\{\Omega_1,\Omega_2\}$ in $\hat{\mathcal{S}}$ the space of unitary irreducible representations of $\mathcal{S}$, where

$$\Omega_2(s_{12})=\Omega_2(s_{23})=\Omega_2(s_{31})=1,\quad \Omega_2(s_{21})=\Omega_2(s_{32})=\Omega_2(s_{13})=-1. \tag{10}$$

The associated unitary irreducible representation of $\mathcal{G}$ is therefore nonabelian, and can be symbolically expressed as

$$(\Omega_1\oplus\Omega_2)\otimes\mathcal{T}, \tag{11}$$

where $\mathcal{T}$ is a unitary irreducible representation of the stability subgroup $\mathcal{F}\ltimes\mathbb{Z}_3$.

Let us for simplicity consider the case when $\mathcal{F}$ is trivially represented in $\mathcal{T}$, so that $\mathcal{T}$ is a unitary irreducible representation of $\mathbb{Z}_3$. $\mathbb{Z}_3$ has two nontrivial inequivalent unitary irreducible representations (a) $z \rightarrow e^{2\pi i/3}$ and (b) $z \rightarrow e^{4\pi i/3}$. Thus, wave functions ψ_a, ψ_b on $\tilde{\mathcal{Q}}_3$ take values in the (two-dimensional) carrier spaces $\mathbb{H}_a$ and $\mathbb{H}_b$ of the corresponding quantum sectors. If $\psi_{a,b}$ are localized along the fibre of a configuration of well separated identical rings, under the action of the cyclic permutations they pick up the respective phases $\psi_a \rightarrow e^{2\pi i/3}\psi_a$ and $\psi_b \rightarrow e^{4\pi i/3}\psi_b$. Thus, these sectors exhibit a cyclic, nonpermutation group, statistics: the rings are identical *only* when permuted by a cyclic combination, and *not* under pair-wise exchange! This is indeed a very curious behavior and is again linked to the nonlocality of slide motions: even though the flips are all trivially represented the slides render the rings pair-wise distinguishable but cyclically indistinguishable. We say that the rings obey $\mathbb{Z}_3$ *cyclic statistics*.

The case for arbitrary $n>2$ follows in a straightforward manner. Namely, we can always isolate a pair of nontrivial subsets from the set of slide generators $\{s_A\}$ and $\{s_B\}$ which are invariant under $\mathbb{Z}_n$. There is a small difference in the construction in the even $n=2m$ and odd $n=2m+1$ cases. For $n=2m$, $\mathbb{Z}_{2m}$ contains the subgroup $\mathbb{Z}_2$; if z is the generator of $\mathbb{Z}_{2m}$ with $z^{2m}=e$, then z^m generates a $\mathbb{Z}_2$ subgroup corresponding to m commuting exchanges. One can then see that the two sets of generators $\{s_A\}$ and $\{s_B\}$ which are invariant under $\mathbb{Z}_{2m}$ have cardinality $2m(m-1)$ and $2m^2$, respectively. For $n=2m+1$, Z_2 is not a subgroup of $\mathbb{Z}_{2m+1}$. Hence the two sets of generators $\{s_A\}$ and $\{s_B\}$ each have cardinality $m(2m+1)$. One can thus obtain $\mathbb{Z}_n$ cyclic statistics for arbitrary $n>2$.

We end this section by commenting on the possibility that sectors with more complicated nonpermutation group statistics may exist. To construct the above cyclic statistics sectors we started with very simple abelian unitary irreducible representations of the slide subgroup. It is conceivable that if one instead started with a nonabelian unitary irreducible representation of $\mathcal{S}$ (with certain symmetries) that the stability subgroup $\mathcal{F} \ltimes K$ associated with it is such that K is nonabelian and a nonpermutation subgroup of S_n. Such a sector would then exhibit a *nonabelian, nonpermutation* group statistics. Our current work provides a framework in which to probe such questions.

IV. REMARKS

Anyonic statistics in $2+1$ dimensions can be modeled by adding a Chern Simon's term to the n particle Lagrangian.[17] In Ref. 10 a stringy generalization of this was developed to obtain nontrivial phases from the action of the motion group, namely a $B \wedge F$ topological term made up of an abelian gauge field and an axion field was added to the n string Lagrangian along with an interaction term. Similar systems have subsequently been studied in Ref. 18. In Ref. 10 it was shown that even though the statistical phases are trivial (i.e., bosonic) the action of the slide subgroup is nontrivial, giving rise to fractional quantum phases. Since slides involve the motion of one ring through a nontrivial generator of the fundamental group of another ring, these fractional phases correspond to Aharnov–Bohm phases rather than to fractional quantum statistics. Indeed, slides can occur between nonidentical particles as well and hence the interpretation of such phases as statistics in Ref. 18 seems questionable. Since cyclic statistics occur in nonabelian sectors of the system, it would be interesting to construct appropriate nonabelian generalizations of Ref. 10 which exhibit this behavior. We leave this problem to future investigations.

ACKNOWLEDGMENTS

I would like to thank A.P. Balachandran and Rafael Sorkin for discussions and for valuable comments on an earlier draft. I would also like to thank Ajit Srivastava for discussions during the early stages of this work and the Physics Department at U.C. Davis for its hospitality while this work was being written up. This work has been supported in part by a grant from the Natural Sciences and Engineering Research Council (Canada).

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