Separation of variables for the Dirac equation in an extended class of Lorentzian metrics with local rotational symmetry

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The question of the separability of the Dirac equation in metrics with local rotational symmetry is reexamined by adapting the analysis of Kamran and McLenaghan [J. Math. Phys. 25, 1019 (1984)] for the metrics admitting a two-dimensional Abelian local isometry group acting orthogonally transitively. This generalized treatment, which involves the choice of a suitable system of local coordinates and spinor frame, allows one to establish the separability of the Dirac equation within the class of metrics for which the previous analysis of Iyer and Vishveshwara [J. Math. Phys. **26**, 1034 (1985)] had left the question of separability open.

I. INTRODUCTION

Chandrasekhar's[1] proof of the separability of the Dirac equation in the Kerr metric was not only the culmination of a long list[2,3] of separability properties which—to quote Chandrasekhar's own words[4]—have the "aura of the miraculous," but also the starting point of a series of new investigations on the separability properties of the Dirac equation in curved space-time. Reviews of these contributions can be found in Refs. 5 and 6.

A deeper understanding of the geometrical aspects of Chandrasekhar's result was achieved through the work of Carter and McLenaghan[7] who constructed a first-order matrix differential operator—a generalization of the total angular momentum operator of flat space-time—commuting with the Dirac operator in the Kerr metric and admitting the separable solution of the Dirac equation as an eigenspinor with the corresponding separation constant as an eigenvalue. They showed that this operator is constructed from the rank-two Killing–Yano tensor field whose associated Killing tensor field arises from the separability of the Hamilton–Jacobi equation for the geodesic flow in the case of nonisotropic geodesics.[2] Subsequently, Kamran and McLenaghan[8] investigated in detail the question of the separability of the Dirac equation in the class of metrics admitting a two-dimensional Abelian local isometry group acting orthogonally transitively and a pair of shear-free geodesic congruences of isotropic curves. They could treat this case in detail because of the earlier results of Debever, Kamran, and McLenaghan,[9,10] which provide a canonical form for this class of metrics and a single expression for the general solution of the Einstein–Maxwell and Einstein vacuum field equations with cosmological constant assuming a type D Weyl tensor and a nonsingular aligned Maxwell field. Around the same time, Iyer and Vishveshwara[11] investigated the separability properties of the Dirac equation in space-times with local rotational symmetry.[12,13] These space-times form a subclass of the generalized Goldberg–Sachs class and include a number of well-known solutions like the Friedmann, Gödel, Kasner, Kantowski–Sachs, and Taub–NUT geometries as special cases. A direct adaptation of Chandrasekhar's treatment yielded separation in only a subclass and left open the issue of separability in the remaining cases. This is the issue we address in this paper. Adapting the treatment of Ref. 8—which involves and appropriate choice of coordinates and spinor frame—to the case of metrics with local rotational symmetry, we show explicitly in Propositions 1 and 2 below how the separability of the Dirac equation can be achieved within the class of metrics for which the question was left open by Iyer and Vishveshwara in Ref. 11. All the separable systems we obtain are examples of systems Miller[6] calls factorizable separable systems. As such, we know from Theorem 7 of Miller's paper[6] that for each of our separable systems, there exist three linearly independent first-order matrix differential operators commuting amongst themselves and with the Dirac operator and admitting the separable solutions as eigenspinors with the separation constants as eigenvalues. These operators are constructed explicitly in Proposition 3 below and characterized invariantly in terms of the two commuting Killing vector fields and the rank-two Killing–Yano tensor field admitted by the locally rotationally symmetric metrics studied in this paper.

II. THE DIRAC EQUATION IN SPACE-TIMES WITH LOCAL ROTATIONAL SYMMETRY

We first recall from Ref. 12, some general facts about space-times with local rotational symmetry that are needed in order to study the separability of the Dirac equation in these space-times.

The space-times with local rotational symmetry admit

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local coordinates (x^0,x^1,x^2,x^3) in which the metric is of the form

$$ds^2 = (1/F^2) dx^0 \odot dx^0 - A^2 dx^1 \odot dx^1 - B^2 (dx^2 \odot dx^2 + t^2 dx^3 \odot dx^3) - (y/F^2)(2 dx^0 - y dx^3) \odot dx^3 + hA^2 (2 dx^1 - h dx^3) \odot dx^3, \quad (1)$$

where F, A, and B are functions of x^0 and x^1 only and t, y, and h are functions of x^2 only. The metric function t has one of the following expressions

$$t = a, \quad t = x^2, \quad t = \sin x^2, \quad t = \sinh x^2, \quad (2)$$

where a is a constant and the metric function t is related to the metric functions h and y by the relations

$$\frac{dh}{dx^2} = ct, \quad \frac{dy}{dx^2} = c't, \quad (3)$$

where c and c' are real constants.

The locally rotationally symmetric space-times fall into three classes defined by the following conditions.

Class 1:

$$A = 1, \quad \frac{\partial B}{\partial x^0} = 0, \quad \frac{\partial F}{\partial x^0} = 0, \quad h = 0. \quad (4)$$

Class 2:

$$h = 0, \quad y = 0. \quad (5)$$

Class 3:

$$F = 1, \quad \frac{\partial A}{\partial x^1} = 0, \quad \frac{\partial B}{\partial x^1} = 0, \quad y = 0. \quad (6)$$

It was shown in Ref. 11 that the Dirac equation is separable for the space-times in Class 1 if $c' = 0$, that is y is a constant. For Class 2, it was established that while the angular variable can always be separated, the temporal and radial variables only separate in the following two special cases:

(a)

$$\frac{\partial A}{\partial x^0} = 0, \quad \frac{\partial B}{\partial x^0} = 0, \quad \frac{\partial F}{\partial x^1} = 0; \quad (7)$$

(b)

$$\frac{\partial A}{\partial x^0} = 0, \quad \frac{\partial B}{\partial x^1} = 0, \quad \frac{\partial F}{\partial x^1} = 0. \quad (8)$$

Some exact solutions corresponding to these two cases are given in Refs. 14 and 15.

The starting point of our paper is the observation that since the metrics in Class 1 and Class 3 admit a two-dimensional Abelian local isometry group acting orthogonally transitively, the general results of Ref. 8 should be adaptable to the study of the separation of variables problem for the Dirac equation in these locally rotationally symmetric space-times.

The metrics in Class 1 are given by

$$ds_1^2 = F(x^1)^{-2}(dx^0 - y(x^2)dx^3) \odot (dx^0 - y(x^2)dx^3) - dx^1 \odot dx^1 - B(x^1)^2 dx^2 \odot dx^2 - B(x^1)^2 t(x^2)^2 dx^3 \odot dx^3, \quad (9)$$

while for Class 3 we have

$$ds_3^2 = -A(x^0)^2 (dx^1 - h(x^2)dx^3) \odot (dx^1 - h(x^2)dx^3) + dx^0 \odot dx^0 - B(x^0)^2 dx^2 \odot dx^2 - B(x^0)^2 t(x^2)^2 dx^3 \odot dx^3. \quad (10)$$

We observe that for the metrics (9), the two-dimensional Abelian local isometry group generated by the Killing vector fields $\partial/\partial x^0$ and $\partial/\partial x^3$ acts on timelike orbits while for the metrics (10) the orbits of the local isometry group generated by $\partial/\partial x^1$ and $\partial/\partial x^3$ are spacelike.

To study the separability of the Dirac equation in the metrics (9) and (10), it is convenient to use a slightly different set of local coordinates. These are precisely the coordinates that appear at the starting point of the study of the Dirac equation made in Ref. 8.

We consider the metrics

$$d\sigma_f^2 = f\frac{p(w)}{W(w)^2} dw \odot dw - \frac{p(w)}{X(x)^2} dx \odot dx - X(x)^2 p(w) dv \odot dv - f\frac{W(w)^2}{p(w)}(du + m(x)dv) \odot (du + m(x)dv), \quad (11)$$

where f is a parameter taking the values $+1$ or -1 according to whether the orbits of the local isometry group generated by $\partial/\partial u$ and $\partial/\partial v$ are spacelike or timelike.

Proposition 1: (i) Consider the local diffeomorphism Φ defined by

$$w \circ \Phi = \int^{x^1} \frac{d\xi}{F(\xi)}, \quad x \circ \Phi = \int^{x^2} t(y) dy, \quad (12)$$

$$u \circ \Phi = x^0, \quad v \circ \Phi = x^3. \quad (13)$$

If we choose the metric functions so as to have

$$(p \circ \Phi)(x^1) = B(x^1)^2, \quad (X \circ \Phi)(x^2) = t(x^2), \quad (14)$$

$$(m \circ \Phi)(x^2) = -y(x^2), \quad (W \circ \Phi)(x^1) = B(x^1)/F(x^1), \quad (15)$$

then

$$\Phi^* d\sigma_{-1}^2 = ds_1^2. \quad (16)$$

(ii) Consider the local diffeomorphism Ψ defined by

$$w \circ \Psi = \int^{x^0} A(\xi) d\xi, \quad x \circ \Psi = \int^{x^2} t(y) dy, \quad (17)$$

$$u \circ \Psi = x^1, \quad v \circ \Psi = x^3. \quad (18)$$

If we choose the metric functions so as to have

$$(p \circ \Psi)(x^0) = B(x^0)^2, \quad (X \circ \Psi)(x^2) = t(x^2), \quad (19)$$

$$(m \circ \Psi)(x^2) = -h(x^2), \quad (W \circ \Psi)(x^0) = A(x^0) B(x^0), \quad (20)$$

Then we have

$$\Psi^* d\sigma_1^2 = ds_3^2.$$

The proof of Proposition 1 is straightforward. We now show how the Dirac equation can be solved by separation of variables for the locally rotationally symmetric metrics (9) and (10) using the coordinates (u,v,w,x) introduced in Proposition 1 and an appropriately chosen coframe.

First, we consider the isotropic coframe

$$\theta^1 = 2^{-1/2}[fWp^{-1/2}(du + m\,dv) + p^{1/2}W^{-1}\,dw], \quad (21)$$

$$\theta^2 = 2^{-1/2}[Wp^{-1/2}(du + m\,dv) - fp^{1/2}W^{-1}\,dw], \quad (22)$$

$$\theta^3 = 2^{-1/2}[Xp^{1/2}\,dv + ip^{1/2}X^{-1}\,dx], \quad (23)$$

$$\theta^4 = 2^{-1/2}[Xp^{1/2}\,dv - ip^{1/2}X^{-1}\,dx], \quad (24)$$

in which we have

$$d\sigma_f^2 = 2(\theta^1 \odot \theta^2 - \theta^3 \odot \theta^4). \quad (25)$$

We shall express the Dirac equation

$$\mathbf{H}_D\psi := (i\gamma^k\nabla_k - \sqrt{2}\mu_e I)\psi = 0, \quad (26)$$

where ∇_k denotes the covariant differentiation operator four-spinors and I is the 4×4 identity matrix, for the metrics (11) using the Weyl representation for the Dirac matrices and the Newman–Penrose spin coefficient formalism. Thus letting

$$\theta^1 = n_i\,dx^i, \quad \theta^2 = l_i\,dx^i,$$
$$\theta^3 = -\bar{m}_i\,dx^i, \quad \theta^4 = -m_i\,dx^i, \quad (27)$$

the Dirac equation (2.26) reads

$$\begin{pmatrix} -i\mu_e & 0 & D+\bar{\epsilon}-\bar{\rho} & \delta+\bar{\pi}-\bar{\alpha} \\ 0 & -i\mu_e & \bar{\delta}+\bar{\beta}-\bar{\tau} & \Delta+\bar{\mu}-\bar{\gamma} \\ \Delta+\mu-\gamma & -(\delta+\beta-\tau) & -i\mu_e & 0 \\ -(\bar{\delta}+\pi-\alpha) & D+\epsilon-\rho & 0 & -i\mu_e \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \bar{Q}^0 \\ \bar{Q}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (28)$$

where

$$D+\epsilon-\rho = 2^{-1/2}p^{-1/2}\Big[W\frac{\partial}{\partial w} + fW^{-1}p\frac{\partial}{\partial u} - \frac{1}{4}Wp^{-1}(-p' + im') + \frac{1}{2}W'\Big], \quad (29)$$

$$\Delta+\mu-\gamma = 2^{-1/2}p^{-1/2}\Big[-fW\frac{\partial}{\partial w} + W^{-1}p\frac{\partial}{\partial u} + \frac{1}{4}fWp^{-1}(-p' + im') - \frac{1}{2}fW'\Big], \quad (30)$$

$$\delta+\beta-\tau = 2^{-1/2}p^{-1/2}\Big[-iX\frac{\partial}{\partial x} + X^{-1}\Big(\frac{\partial}{\partial v} - m\frac{\partial}{\partial u}\Big) - \frac{1}{2}iX'\Big], \quad (31)$$

$$\bar{\delta}+\pi-\alpha = 2^{-1/2}p^{-1/2}\Big[iX\frac{\partial}{\partial x} + X^{-1}\Big(\frac{\partial}{\partial v} - m\frac{\partial}{\partial u}\Big) + \frac{1}{2}iX'\Big]. \quad (32)$$

Proposition 2: The Dirac equation (26) expressed in the isotropic coframe ${}^t(\theta^1\theta^2\theta^3\theta^4)$ for the metrics $d\sigma_f^2$ given by Eqs. (21) to (24) admits in the Weyl representation an R-separable solution

$$\psi(u,v,w,x) = e^{\lambda_1 u + \lambda_2 v}p(w)^{-1/4} \times \begin{pmatrix} \exp\Big(\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big)R_1(x)S_2(w) \\ \exp\Big(\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big)R_2(x)S_1(w) \\ \exp\Big(-\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big)R_1(x)S_1(w) \\ \exp\Big(-\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big)R_2(x)S_2(w) \end{pmatrix}, \quad (33)$$

where λ_1 and λ_2 are arbitrary constants, if the metrics $d\sigma_f^2$ are of Petrov type D, that is $m(x) = 2lx$, l an arbitrary constant, and the coordinate w is chosen so as to have $p(w) = w^2 + l^2$.

Proof: From Eq. (4.2) of Ref. 9, we know that the metrics $d\sigma_f^2$ will be of Petrov type D if and only if

$$m(x) = 2lx + n, \quad (34)$$

where l and n are arbitrary real constants. The constant n may be set to zero by an appropriate translation on x, so we have $m(x) = 2lx$ with no loss of generality.

We now transform the Dirac spinor ψ according to

$$\psi = S\psi', \quad (35)$$

where

$$S := p^{-1/4}\,\mathrm{diag}\Big(\exp\Big(\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big), \exp\Big(\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big), \exp\Big(-\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big), \exp\Big(-\frac{il}{2}\int^w \frac{d\xi}{p(\xi)}\Big)\Big), \quad (36)$$

is a nonsingular matrix. Letting

$$\psi' =: {}^t(F_0F_1F_2F_3), \quad (37)$$

we have using Eqs. (28) to (39) $S^{-1}\mathbf{H}_D S\psi' =$

$$\begin{pmatrix} -i\mu_e & 0 & \exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_w^+ & \exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_x^- \\ 0 & -i\mu_e & \exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_x^+ & \exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_w^- \\ \exp\left(il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_w^- & -\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_x^- & -i\mu_e & 0 \\ -\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_x^+ & \exp\left(il\int^w \frac{d\xi}{p(\xi)}\right)p^{-1/2}L_w^+ & 0 & -i\mu_e \end{pmatrix} \times \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad (38)$$

where

$$L_x^+ := 2^{-1/2}\left[iX\frac{\partial}{\partial x} + X^{-1}\left(\frac{\partial}{\partial v} - 2lx\frac{\partial}{\partial u}\right) + \frac{i}{2}X'\right], \quad (39)$$

$$L_x^- := 2^{-1/2}\left[-iX\frac{\partial}{\partial x} + X^{-1}\left(\frac{\partial}{\partial v} - 2lx\frac{\partial}{\partial u}\right) - \frac{i}{2}X'\right], \quad (40)$$

$$L_w^+ := 2^{-1/2}\left[W\frac{\partial}{\partial w} + fW^{-1}p\frac{\partial}{\partial u} + \frac{1}{2}W'\right], \quad (41)$$

$$L_w^- := 2^{-1/2}\left[-fW\frac{\partial}{\partial w} + W^{-1}p\frac{\partial}{\partial u} - \frac{1}{2}fW'\right]. \quad (42)$$

We left multiply the transformed operator $S^{-1}\mathbf{H}_D S$ by the nonsingular matrix

$$U = p^{-1/2}\operatorname{diag}\left(\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right), -\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right), -\exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right), \exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right)\right), \quad (43)$$

to obtain the equation

$$\mathbf{W}\psi' := US^{-1}\mathbf{H}_D S\psi' = 0, \quad (44)$$

which is by construction equivalent to the Dirac equation, where

$$\mathbf{W} := \begin{pmatrix} -i\mu_e p^{1/2}\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right) & 0 & L_w^+ & L_x^- \\ 0 & i\mu_e p^{1/2}\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right) & -L_x^+ & -L_w^- \\ -L_w^+ & L_x^- & i\mu_e p^{1/2}\exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right) & 0 \\ -L_x^+ & L_w^+ & 0 & -i\mu_e p^{1/2}\exp\left(-il\int^w \frac{d\xi}{p(\xi)}\right) \end{pmatrix}. \quad (45)$$

Following the statement of Proposition 2 we set the metric function $p(w)$ equal to $w^2 + l^2$ by an admissible local diffeomorphism of the form $w \mapsto k(w)$. [See Eqs. (4.24)–(4.28) of Ref. 9 for further details.] It then follows that

$$p(w)^{1/2}\exp\left(il\int^w \frac{d\xi}{p(\xi)}\right) = w - il. \quad (46)$$

We can now proceed to separate the Dirac equation. We use the existence of the two commuting Killing vector fields $\partial/\partial u$ and $\partial/\partial v$ to express ψ' in the form

$$\psi' = e^{\lambda_1 u + \lambda_2 v}\begin{pmatrix} G_0(w,x) \\ G_1(w,x) \\ G_2(w,x) \\ G_3(w,x) \end{pmatrix}, \quad (47)$$

where λ_1 and λ_2 are arbitrary real constants, in which case the transformed Dirac equation $\mathbf{W}\psi' = 0$, reduces to

$$\overset{0}{L}{}_w^+ G_2 + \overset{0}{L}{}_x^- G_3 - i\mu_e(w - il)G_0 = 0, \quad (48)$$

$$-\overset{0}{L}{}_x^+ G_2 - \overset{0}{L}{}_w^- G_3 + i\mu_e(w - il)G_1 = 0, \quad (49)$$

$$-\overset{0}{L}{}_w^- G_0 + \overset{0}{L}{}_x^- G_1 + i\mu_e(w + il)G_2 = 0, \quad (50)$$

$$-\overset{0}{L}{}_x^+ G_0 + \overset{0}{L}{}_w^+ G_1 - i\mu_e(w + il)G_3 = 0, \quad (51)$$

where the superscript "0" denotes the substitution $\partial/\partial u \to \lambda_1$, $\partial/\partial v \to \lambda_2$. The separation of variables is completed by following the Chandrasekhar pattern

$$\begin{pmatrix} G_0(w,x) \\ G_1(w,x) \\ G_2(w,x) \\ G_3(w,x) \end{pmatrix} = \begin{pmatrix} R_1(x)S_2(w) \\ R_2(x)S_1(w) \\ R_1(x)S_1(w) \\ R_2(x)S_2(w) \end{pmatrix}, \quad (52)$$

which upon substitution into Eqs. (48)–(51) leads to the ordinary differential equations

$$\overset{0}{L}{}_w^+ S_1 - i\mu_e w S_2 = \lambda_3 S_2,$$

$$\overset{0}{L}{}_x^- R_2 - \mu_e l R_1 = -\lambda_3 R_1, \tag{53}$$

$$\overset{0}{L}{}_w^- S_2 - i\mu_e w S_1 = -\lambda_3 S_1,$$

$$\overset{0}{L}{}_x^+ R_1 - \mu_e l R_2 = -\lambda_3 R_2, \tag{54}$$

where λ_3 is the separation constant. This completes the proof of Proposition 2.

The separable system for the Dirac equation obtained in Proposition 2 is an example of a *factorizable separable system* in the sense of Miller.[6] From Theorem 7 of Miller's paper,[6] we know that there should exist three first-order (matrix) differential operators $\mathbf{K}_{(1)}$, $\mathbf{K}_{(2)}$, and $\mathbf{K}_{(3)}$ commuting amongst themselves and with the Dirac operator

$$[\mathbf{K}_{(i)},\mathbf{H}_D] = 0, \quad [\mathbf{K}_{(i)},\mathbf{K}_{(j)}] = 0, \tag{55}$$

and admitting the separation constants as eigenvalues, that is

$$\mathbf{K}_{(i)}\psi = \lambda_i, \quad 1 \leqslant i,j \leqslant 3, \tag{56}$$

where λ_1, λ_2, and λ_3 are the separation constants introduced in Proposition 2. The operators $\mathbf{K}_{(1)}$ and $\mathbf{K}_{(2)}$ are simply the Lie derivative operators on four-spinors corresponding to the Killing vector fields $\partial/\partial u$ and $\partial/\partial v$, namely,

$$\mathbf{K}_{(1)} = I\frac{\partial}{\partial u}, \quad \mathbf{K}_{(2)} = I\frac{\partial}{\partial v}. \tag{57}$$

The expression of $\mathbf{K}_{(3)}$ is given through the following result.

Proposition 3: The first-order matrix differential operator $\mathbf{K}_{(3)}$ defined by

$$\mathbf{K}_{(3)} = SU^{-1}(U_x \mathbf{W}_w - U_w \mathbf{W}_x)S^{-1}, \tag{58}$$

where S and U are given by Eqs. (36) and (43) and where

$$U_x = \mathrm{diag}(-il, il, -il, il),$$
$$U_w = \mathrm{diag}(w, -w, -w, w), \tag{59}$$

$$\mathbf{W}_w = \begin{pmatrix} -i\mu_e w & 0 & L_w^+ & 0 \\ 0 & i\mu_e w & 0 & -L_w^- \\ -L_w^- & 0 & i\mu_e w & 0 \\ 0 & L_w^+ & 0 & -i\mu_e w \end{pmatrix},$$

$$\mathbf{W}_x = \begin{pmatrix} -\mu_e l & 0 & 0 & L_x^- \\ 0 & \mu_e l & -L_x^+ & 0 \\ 0 & L_x^- & -\mu_e l & 0 \\ -L_x^+ & 0 & 0 & \mu_e l \end{pmatrix}, \tag{60}$$

satisfies

$$[\mathbf{K}_{(3)},\mathbf{H}] = 0, \quad \mathbf{K}_{(3)}\psi = \lambda_3\psi, \tag{61}$$

where ψ is the R-separable solution of the Dirac equation given in Proposition 2 and λ_3 is the separation constant obtained in Eqs. (53) and (54).

Proof: We have

$$\mathbf{W} = \mathbf{W}_w + \mathbf{W}_x, \quad U = U_w + U_x, \tag{62}$$
$$[\mathbf{W}_w,\mathbf{W}_x] = 0,$$
$$[\mathbf{W}_w,U_x] = 0,$$
$$[\mathbf{W}_x,U_w] = 0, \quad [U_w,U_x] = 0, \tag{63}$$

$$\mathbf{W}_w\psi' = \lambda_3\psi', \quad \mathbf{W}_x\psi' = -\lambda_3\psi'. \tag{64}$$

The eigenvalue equation $\mathbf{K}_{(3)}\psi = \lambda_3\psi$ is a consequence of Eqs. (64), (58), and (35). The proof of Proposition 3 is complete.

In analogy with Carter and McLenaghan's analysis[7] of the separability of the Dirac equation in the Kerr solution, it can be verified that the operator $\mathbf{K}_{(3)}$ obtained in Proposition 3 is constructed from the rank-two Killing–Yano tensor field admitted by the locally rotationally symmetric metrics given in Propositions 1 and 2. More precisely, $\mathbf{K}_{(3)}$ may be written as

$$\mathbf{K}_{(3)} = D^k_{\ l}\gamma^5\gamma^l\nabla_k + \tfrac{1}{3}\nabla_k(*D^k_{\ l})\gamma^l, \tag{65}$$

where the tensor field $D_{ij}\,dx^i \wedge dx^j$ defined by

$$\tfrac{1}{2}D_{ij}\,dx^i \wedge dx^j = w(\theta^1 \wedge \theta^2 - i\theta^3 \wedge \theta^4), \tag{66}$$

satisfies the Killing–Yano equation

$$\nabla_i D_{jk} + \nabla_j D_{ik} = 0. \tag{67}$$

We conclude our paper with a few brief remarks.

(i) We could have considered the problem of separating the variables for the wave equation describing a Dirac spinor field coupled to a Maxwell field, that is

$$[i\gamma^k(\nabla_k - ieA_k) - \sqrt{2}\mu_e I]\psi = 0, \tag{68}$$

where $A_k\,dx^k$ is the one-form of a vector potential. It is straightforward to show that Proposition 2 holds for Eq. (68) as well with $A_k\,dx^k$ given by

$$A_k\,dx^k = 2^{-1/2}p(w)^{-1/2}[H(w)W(w)^{-1}(\theta^1 + \theta^2) + G(x)X(x)^{-1}(\theta^3 + \theta^4)], \tag{69}$$

where G and H are arbitrary functions of their respective arguments. It should however be noted that the one-form (69) will not be a solution of the Einstein–Maxwell equations for an arbitrary choice of p, G, H, W, and X.

(ii) The parameter l introduced in the statement of Proposition 2 through the integration of the Petrov type D condition is related to the parameters c and c' which appear in Eq. (3). For Case 1 we have $l = -c'/2$ while for Case 3 we have $l = -c/2$. Likewise, X^2 will be a known function of x on account of the local rotational symmetry conditions (2) and the local diffeomorphism given in Proposition 1. The explicit form of X^2 is however not required for the Dirac equation to be separable, as shown by Proposition 2.

(iii) While the separable systems we have obtained in this paper are factorizable in the sense of Miller, it is not true that all separable systems for the Dirac equation have this property. We have recently obtained separable systems that are not factorizable and characterized by higher-order symmetry operators.[16,17]

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