Duality and conformal structure

Tevian Dray[a)]
Raman Research Institute, Bangalore 560080, India
and Department of Mathematics, Oregon State University, Corvallis, Oregon 97331
Ravi Kulkarni[b)] and Joseph Samuel[c)]
Raman Research Institute, Bangalore 560080, India

(Received 22 November 1988; accepted for publication 15 February 1989)

In four dimensions, two metrics that are conformally related define the same Hodge dual operator on the space of two-forms. The converse, namely, that two metrics that have the same Hodge dual are conformally related, is established. This is true for metrics of arbitrary (nondegenerate) signature. For Euclidean signature a stronger result, namely, that the conformal class of the metric is completely determined by choosing a dual operator on two-forms satisfying certain conditions, is proved.

I. INTRODUCTION

Self-dual fields have played a major role in many of the recent developments in general relativity. Foremost among these is Penrose's twistor program (see, e.g., Refs. 1 and 2). Such fields seem to be fundamentally involved in attempts to quantize gravity (see, e.g. Ref. 3), notably Ashtekar's new variables (for a review see Ref. 4). The material discussed here was motivated by the attempts of one of us (JS) to better understand Ashtekar's new variables, but the presentation is entirely self-contained.

In four dimensions the Hodge dual operator on two-forms is manifestly conformally invariant. Thus, if two metrics are conformally related, they have the same Hodge dual. We show that the converse is also true: Two metrics of arbitrary (nondegenerate) signature that have the same Hodge dual are conformally related. For Riemannian manifolds (signature $++++$), we are able to establish a much stronger result: Any choice of a three-dimensional, positive-definite subspace of the space of two-forms determines a dual operator, which in turn determines a conformal class of metrics whose Hodge dual agrees with the original choice. Thus there is a one–one correspondence between conformal classes of metrics and dual operators.

Our presentation is organized as follows. After establishing the notation in Sec. II we show that the equality of Hodge duals implies that the metrics are conformally related. In Sec. III we consider the Riemannian case and establish the stronger result described above. Finally, in Sec. IV we discuss our results.

II. CONVERSE

Let M be an oriented four-dimensional manifold with (nondegenerate) metric g_{ab}. The volume element (Levi-Civita tensor) is the four-form $\epsilon_{abcd} = \epsilon_{[abcd]}$, which agrees with the orientation and whose nonzero components are $\pm\sqrt{|g|}$, where $g = \det(g_{ab})$. Denote by Λ^2, the space of two-forms $F_{ab} = F_{[ab]}$ on M. Then the Hodge dual operator $*$, defined by g_{ab}, is a map from $\Lambda^2 \to \Lambda^2$ given by

$$(*F)_{ab} = \tfrac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}. \quad (1)$$

It is straightforward to check that

$$\hat{g}_{ab} = \Omega g_{ab} \Rightarrow \hat{*} \equiv *. \quad (2)$$

One also has

$$** = \pm I, \quad (3)$$

where I is the identity operator and

$$\epsilon_{abmn}\epsilon_{cd}{}^{mn} = \pm 4 g_{a[c}g_{d]b}, \quad (4)$$

where the $-$ sign holds for the Lorentzian signature $[(-+++)$ or $(+---)]$ and the $+$ sign holds for all other signatures. We are now ready to prove the converse to (2).

Theorem 1: Let g_{ab} and $\hat{g}_{ab}$ be (real, nondegenerate) metrics of arbitrary signature on a four-dimensional manifold M, such that for all two-forms F on M,

$$*F \equiv \hat{*}F. \quad (5)$$

Then

$$\hat{g}_{ab} = \pm\Omega g_{ab}, \quad (6)$$

where $\Omega = |\hat{g}/g|^{1/4}$.

Proof:

Step 1: Equation (1) implies

$$\hat{\epsilon}_{cd}{}^{mn}F_{mn} = \epsilon_{cd}{}^{mn}F_{mn}, \quad \forall F \in \Lambda^2, \quad (7a)$$

which implies

$$\hat{\epsilon}_{cd}{}^{mn} = \epsilon_{cd}{}^{mn}. \quad (7b)$$

But from the definition of the volume element,

$$\hat{\epsilon}_{abmn} = \Omega^2\epsilon_{abmn}. \quad (8)$$

Contracting (8) with (7b), using (4), yields

$$\hat{g}_{a[c}\hat{g}_{d]b} = \Omega^2 g_{a[c}g_{d]b}. \quad (9)$$

Step 2: It is sufficient to establish (6) at each point $p \in M$. Choose coordinates x^i on a neighborhood of p so that $\hat{g}_{ij}|_p$ is diagonal. Then

$$\hat{g}_{i[j}\hat{g}_{k]l} = 0, \quad \text{unless } (i,k) = (j,l) \text{ or } (i,j) = (k,l). \quad (10)$$

[a)] Permanent address: Department of Mathematics, Oregon State University, Corvallis, Oregon 97331.
[b)] Permanent address: Department of Mathematics, University of Poona, Pune 411007, India.
[c)] Present address: Department of Physics, University of Utah, Salt Lake City, Utah 84112.

In particular, (10) holds if j, k, and l are all different. (Here and for the remainder of the proof, all quantities are to be evaluated at p.)

Using (9) and (10) we have

$$(g_{ii}g_{jj} - g_{ij}^2)g_{kl}^2 = (g_{ii}g_{kl})(g_{jj}g_{kl}) - (g_{ij}g_{kl})(g_{ij}g_{lk})$$
$$= (g_{ik}g_{il})(g_{jk}g_{jl}) - (g_{ik}g_{jl})(g_{il}g_{jk})$$
$$= 0 \quad (\text{for } \epsilon_{ijkl} \neq 0). \tag{11}$$

But since $\hat{g}_{ii}\hat{g}_{jj} - \hat{g}_{ij}^2 \neq 0$ by assumption, one final use of (9) yields

$$g_{kl} = 0, \tag{12}$$

so that g_{ij} is also diagonal at p.

Step 3: Inserting the diagonality of both $\hat{g}_{ij}$ and g_{ij} into (9) yields

$$\hat{g}_{ii}\hat{g}_{jj} = \Omega^2 g_{ii}g_{jj}, \quad \text{for } i \neq j, \tag{13}$$

which implies the result (6). Q.E.D.

III. EUCLIDEAN SIGNATURE

We now turn to the special case of an oriented Riemannian manifold [signature $(++++)$ or $(----)$] with volume element ϵ_{abcd}. First we need some results about the vector space Λ_p^2 of two-forms at a point $p \in M$.

There is a natural product (symmetric bilinear form) on Λ^2, given by the wedge product of forms, namely,

$$\langle F_{ab}, G_{cd} \rangle = \epsilon^{abcd} F_{ab} G_{cd}, \tag{14a}$$

or equivalently

$$\langle F, G \rangle \epsilon = F \wedge G, \tag{14b}$$

where $\epsilon^{abcd} = \epsilon^{[abcd]}$ is defined by

$$\epsilon_{abcd}\epsilon^{abcd} = 4! \ . \tag{15}$$

Note that the metric has not been used in defining (14) and that the inner product is not positive definite. If one chooses a basis α^i of the space Λ_p^1 of one-forms at p, then the independent $\alpha^i \wedge \alpha^j$ form a basis for Λ_p^2. In four dimensions there are six such two-forms, so dim $\Lambda_p^2 = 6$. Furthermore, by choosing appropriate linear combinations, one easily sees that the signature of the wedge product (14) on Λ_p^2 is $(+++---)$.

Lemma 1: Given a vector space V with a symmetric bilinear form $\omega: V \times V \to \mathbb{R}$ and a subspace $W^+ \subset V$, such that (W^+, ω) is an inner product space (i.e., $\omega|_{W^+}$ is positive definite), there exists an operator $\#: V \to V$, such that

$$V = W^+ \oplus W^-, \tag{16a}$$

with

$$W^\pm = \tfrac{1}{2}(I \pm \#)V. \tag{16b}$$

Proof: Pick an orthonormal basis w_i of W^+. Then the projection operator P from V to W^+ is given by

$$P: V \to W^+,$$
$$v \mapsto \langle v, w_i \rangle w_i. \tag{17}$$

Define $\#$ by

$$\# v = 2Pv - v. \tag{18}$$

Then

$$P = \tfrac{1}{2}(I + \#), \tag{19}$$

and the result follows. Q.E.D.

Corollary: It is an immediate consequence of (18) that

$$\#\# = I. \tag{20}$$

Furthermore, since $(I + \#)(I - \#) = 0$, we have

$$\langle w^+, w^- \rangle = 0, \quad \forall w^\pm \in W^\pm, \tag{21a}$$

or equivalently

$$\langle v_1, \# v_2 \rangle = \langle \# v_1, v_2 \rangle, \quad \forall v \in V. \tag{21b}$$

Let T_pM denote the tangent space to M at p and for $T \in T_pM$ let $\text{Ker}(T) = \{\alpha \in \Lambda_p^1: \alpha(T) := \alpha_a T^a = 0\}$ denote the kernel.

Lemma 2: Let Λ^+ be a three-dimensional subspace of Λ_p^2, such that

$$0 \neq F \in \Lambda^+ \Rightarrow F \wedge F \neq 0. \tag{22}$$

Then for each vector $T \in T_pM$, the map

$$\phi_T: \Lambda^+ \to \text{Ker } T,$$
$$F \mapsto F(T) := F_{ab} T^b, \tag{23}$$

is an ismorphism.

Proof: Since dim(Ker T) = 3 is suffices to show that ϕ_T is one–one, i.e., that $F(T) = 0$ implies $F = 0$. But by choosing a basis of Ker T and extending it to a basis α^i of Λ_p^1 and then forming the associated basis of Λ_p^2, one sees that

$$F(T) = 0 \Rightarrow F \wedge F = 0. \tag{24}$$

Using (22) now proves the assertion. Q.E.D.

For the remainder of this section, we assume that Λ^+ is as in Lemma 2, and that the wedge product (14) is positive definite on Λ^+. Using Lemma 1, this is equivalent to giving a dual operator $\#$ on Λ_p^2; Λ^+ is the space of *self-dual* two-forms, i.e.,

$$F \in \Lambda^+ \Leftrightarrow \# F = F. \tag{25}$$

We now show how to construct a conformal metric $h_\#$ from Λ^+. Fix any $\eta \in \Lambda_p^1$. Then for any $\alpha, \beta \in \Lambda_p^1$, choose $T \in \text{Ker } \alpha \cap \text{Ker } \beta \cap \text{Ker } \eta$. By Lemma 2, there exist unique $F_\alpha, F_\beta, F_\eta \in \Lambda^+$, so that $F_\alpha(T) = \alpha$, $F_\beta(T) = \beta$, and $F_\eta(T) = \eta$. Define

$$h_\#(\alpha,\beta)/h_\#(\eta,\eta) = \langle F_\alpha, F_\beta \rangle / \langle F_\eta, F_\eta \rangle. \tag{26}$$

This defines $h_\#$ up to the single choice of the scale $h_\#(\eta,\eta)$, i.e., $h_\#$ is determined up to a conformal factor.

We now establish that $h_\#$ is well defined, i.e., that the right-hand side of (26) is unchanged under the transformation $T \mapsto T'$ with both $T, T' \in \text{Ker } \alpha \cap \text{Ker } \beta \cap \text{Ker } \eta$. This is obvious if T' is a multiple of T, so we will assume that T and T' are linearly independent. First we introduce some notation.

Extend T and T' to a basis $\{e_0 = T, e_1 = T', e_2, e_3\}$ of T_pM and let $\{\omega^0, \omega^1, \omega^2, \omega^3\}$ be the dual basis of Λ_p^1. Let F_2, F_3, F_2', and F_3' be the unique elements of Λ^+ obtained using Lemma 2 that satisfy

$$F_2(T) = \omega^2 = F_2'(T'), \quad F_3(T) = \omega^3 = F_3'(T'). \tag{27}$$

But since (27) implies

$$F_2(T',T) = 0 = F_2(T',T'), \tag{28}$$

we have

$$F_2(T') = A_{22}\omega^2 + A_{23}\omega^3, \tag{29a}$$

for some constants A_{22} and A_{23} so that, again using Lemma 2,

$$F_2 \equiv A_{22}F_2' + A_{23}F_3'. \tag{29b}$$

Similarly,

$$F_3 \equiv A_{32}F_2' + A_{33}F_3'. \tag{29c}$$

Since this argument can be reversed to express F_2', and F_3' in terms of F_2 and F_3, we must have

$$\lambda := \det A \equiv A_{22}A_{33} - A_{23}A_{32} \neq 0. \tag{30}$$

Now consider the transformation $T_pM \to T_pM$, defined by

$$\begin{aligned} &e_0 \mapsto e_1, \quad e_1 \mapsto \lambda_0 + be_1, \\ &e_2 \mapsto e_2 + ce_1, \quad e_3 \mapsto e_3 + de_1, \end{aligned} \tag{31a}$$

where

$$b := \operatorname{tr} A = A_{22} + A_{33}, \tag{31b}$$

and c,d are constants to be determined. The induced transformation on Λ_p^1 is

$$\begin{aligned} &\omega^0 \mapsto -(b/\lambda)\omega^0 + \omega^1 - c\omega^2 - d\omega^3, \\ &\omega^1 \mapsto (1/\lambda)\omega^0, \quad \omega^2 \mapsto \omega^2, \quad \omega^3 \mapsto \omega^3, \end{aligned} \tag{32a}$$

which we will also write as

$$\omega^i \mapsto B^i_j\omega^j. \tag{32b}$$

Lemma 3: Let $\gamma \in \Lambda_p^1$ satisfy

$$\gamma(T) = 0 = \gamma(T') \tag{33}$$

and let $F,F' \in \Lambda^+$ be determined by Lemma 2, so that

$$F(T) = \gamma = F'(T'). \tag{34}$$

Then (for an appropriate, γ-independent choice of c,d),

$$F' \equiv B'FB, \tag{35a}$$

i.e.,

$$F'_{ab} \equiv F_{mn}B^m{}_aB^n{}_b. \tag{35b}$$

Proof: One has immediately that

$$B'FB(T') = B'F(T) = B'\gamma = \gamma = F'(T). \tag{36}$$

In order to invoke Lemma 2 to conclude that (35) holds by uniqueness we must show that c,d can be chosen, so that $B'FB$ is in Λ^+.

But since

$$\gamma \equiv \gamma_2\omega^2 + \gamma_3\omega^3, \tag{37a}$$

we see that

$$F = \gamma_2F_2 + \gamma_3F_3, \quad F' = \gamma_2F_2' + \gamma_3F_3', \tag{37b}$$

so that it is enough to show that $B'F_2B$ and $B'F_3B$ are in Λ^+. Direct calculation using (29), (34), and (37) shows that the first of these reduces to a linear equation involving d only, while the second determines c. We note in passing that (32) is *not* the only linear transformation that satisfies (35). Q.E.D.

Lemma 4: $h_\#$ is well-defined.

Proof: Assume, as above, that T and T' are linearly independent and let F_α', F_β', and F_η' be the unique elements of Λ^+ determined by Lemma 2 which satisfy $F_\alpha'(T') = \alpha$, $F_\beta'(T') = \beta, F_\eta'(T') = \eta$. Then

$$\begin{aligned} \langle F_\alpha', F_\beta' \rangle &= \epsilon^{abcd}F_\alpha{}'_{ab}F_\beta{}'_{cd} \\ &= \epsilon^{abcd}F_{\alpha mn}F_{\beta pq}B^m{}_aB^n{}_bB^p{}_cB^q{}_d \\ &\equiv \epsilon^{mnpq}F_{\alpha mn}F_{\beta pq}\,(\det B)f \\ &= \langle F_\alpha, F_\beta \rangle (\det B) f, \end{aligned}$$

where f is a constant that depends on the volume element. Therefore the two factors of $(f \det B)$ in the primed version of (26) cancel so (26) is independent of the choice of T.

Theorem 2: Let $*$ be the Hodge dual defined by the metric g_{ab}. Then h_* and g are conformally related.

Proof: For $F = *F$, $G = *G$, we have

$$\begin{aligned} \langle F,G \rangle &:= \epsilon^{abcd}F_{ab}G_{cd} \\ &= 2F^{ab}(*G)_{ab} = 2F^{ab}G_{ab}. \end{aligned} \tag{38}$$

But for any $T \in T_pM$, we have

$$\begin{aligned} F_{am}T^mG^{an}T_n &= \tfrac{1}{4}\epsilon_{ampq}F^{pq}T^m\epsilon^{anrs}G_{rs}T_n \\ &= \tfrac{3}{2}\delta_{[m}{}^n\delta_p{}^r\delta_{q]}{}^sF^{pq}G_{rs}T^mT_n \\ &= \tfrac{1}{2}F^{pq}G_{pq}T^mT_m + F^{pq}G_{qm}T^mT_p, \end{aligned} \tag{39}$$

so that

$$4F_{am}T^mG^{an}T_n \equiv F^{pq}G_{pq}T^mT_m, \tag{40a}$$

or in other words,

$$g(F(T),G(T)) = \langle F,G \rangle (g(T,T)/8), \tag{40b}$$

so that

$$g(F(T),G(T))/g(H(T),H(T)) = \langle F,G \rangle / \langle H,H \rangle. \tag{41}$$

Comparison with (26) shows that g is in the same conformal class as h_*. Q.E.D.

This shows that our definition (26) reproduces the given metric from its Hodge dual. We now show the converse.

Theorem 3: Let $h_\#$ be defined by (26) and denote its Hodge dual by $*$. Then $* = \#$.

Proof: Choose an orthonormal (with respect to $h_\#$) basis ω^α of Λ_p^1 satisfying

$$\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = \epsilon. \tag{42}$$

Let F^i, $i = 1, 2, 3$, be the self-dual (with respect to $\#$) two-forms defined by $F^i(X_0) = \omega^i$, where X_α is the basis of T_pM dual to ω^α. Then, e.g.,

$$F^1 = \omega^0 \wedge \omega^1 + a\omega^2 \wedge \omega^3 + b\omega^1 \wedge \omega^2 + c\omega^3 \wedge \omega^1, \tag{43}$$

with $a > 0,b,c$, to be determined. But using the definition (26), we have

$$\langle F^i, F^j \rangle = \delta^{ij}\langle F^1, F^1 \rangle, \tag{44}$$

which implies

$$F^2=\omega^0\wedge\omega^2+a\omega^3\wedge\omega^1+c\omega^2\wedge\omega^3+d\omega^1\wedge\omega^2,$$
$$F^3=\omega^0\wedge\omega^3+a\omega^1\wedge\omega^2+d\omega^3\wedge\omega^1+b\omega^2\wedge\omega^3.$$

Repeating this procedure for X_1, e.g., constructing $G^0(X_1)=\omega^0$, etc., and using the fact that the F^i form a basis for self-dual two-forms, yields a set of linear equations that can be solved to give

$$b=c=0. \quad (45)$$

Finally, using X_2 gives

$$d=0, \quad a^2=1, \quad (46)$$

so that

$$F^1=\omega^0\wedge\omega^1+\omega^2\wedge\omega^3,$$
$$F^2=\omega^0\wedge\omega^2+\omega^3\wedge\omega^1, \quad (47)$$
$$F^3=\omega^0\wedge\omega^3+\omega^1\wedge\omega^2.$$

But this is just the standard basis for self-dual two-forms with respect to $*$! Q.E.D.

IV. DISCUSSION

For Euclidean signature, let $\mathscr{M}$ denote the manifold of classes of conformal metrics at a point $p\in M$ and $\mathscr{H}$ denote the manifold of dual operators on Λ_p^2. We have the following situation:

$$\mathscr{M}\underset{B}{\overset{A}{\rightleftarrows}}\mathscr{H},$$

where A takes a metric to its Hodge dual, and B is given by (26). Theorem 2 says that $B\circ A=I$, while Theorem 3 says that $A\circ B=I$. Thus both A and B are one–one and onto, and are therefore isomorphisms. The manifold $\mathscr{M}$ is nine-dimensional (10 metric components -1 constraint), and the manifold $\mathscr{H}\approx SO(3,3)/[SO(3)+SO(3)]$, so $\dim\mathscr{H}=15-6=9$.

All of our results have been obtained at a point $p\in M$. Suitably smooth metric tensors and dual operators are obtained by working throughout with suitably smooth tensor fields in a neighborhood of p.

We believe that a result similar to Theorems 2 and 3 holds for other signatures. However, our attempts to modify the argument in Sec. III have so far failed, primarily because of the failure of Lemma 2 if T is null. In the Lorentzian case, one can define $\alpha\in\Lambda_p^1$ to be null if there exists a (real) two-form F and a vector T such that

$$F(T)=\alpha, \quad \#F(T)=0, \quad F\wedge F=0=F\wedge\#F. \quad (48)$$

Although this definition is correct if $\#$ is the Hodge dual of a Lorentzian metric, we have been unable to use it to actually construct a conformal metric.

ACKNOWLEDGMENTS

TD gratefully acknowledges an Indo-American fellowship, funded jointly by the NSF and the USIA in the United States and by the UGC in India.

[1]R. Penrose, "The nonlinear graviton," Gen. Relativ. Gravit. 7, 171 (1976); "Nonlinear gravitons and curved twistor theory" Gen. Relativ. Gravit. 7, 31 (1976).
[2]M. F. Atiyah, N. J. Hitchin, and I. M. Singer, "Self-duality in four-dimensional Riemannian geometry," Proc. R. Soc. London Ser. A **362**, 425 (1978).
[3]A. Ashtekar, "A note on helicity and self-duality," J. Math. Phys. **27**, 824 (1986).
[4]A. Ashtekar, *New Perspectives in Canonical Gravity* (Bibliopolis, Naples, 1988).