Neutrinos in perfect fluid spacetimes with local rotational symmetry

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The neutrino perturbations on the background of perfect fluid spacetimes with local rotational symmetry can be treated with the aid of the Debye-potential (two-component Hertz potential) formalism. This formalism reduces the Dirac equation to a single decoupled equation for a complex scalar ψ. The Weyl spinor pertaining to the neutrino is obtained by differentiating this scalar ψ. The class of spacetimes studied here contains a wide range of well-known examples. It falls naturally into three cases each of which has been investigated in detail retaining the general form of the functions appearing in the metric. It is observed that the method yields considerable information even at this general level of discussion. Finally, some specific examples, namely the Gödel universe, the anisotropic spatially homogeneous cosmological models, and the Taub spacetime are studied illustrating the above-mentioned scheme.

I. INTRODUCTION

The Hertz-potential formalism for electromagnetic perturbations on the background of curved spacetimes was introduced by Cohen and Kegeles.[1] Later, the authors extended this formalism to include perturbations of arbitrary spins, both integral and half-integral using spinors.[2] In particular, the neutrino and gravitational fields can be studied within this framework. They showed that the formalism may be applied to the generalized Goldberg-Sachs class of spacetimes which consists of those spacetimes which admit a shear-free congruence of null geodesics along the repeated principal null direction of the Weyl tensor.[3]

In this paper we shall consider a subclass of the above-mentioned generalized Goldberg-Sachs class of spacetimes, namely, the perfect fluid spacetimes with local rotational symmetry given by Ellis[4] and Ellis and Stewart.[5] It will be our purpose to investigate the behavior of neutrino perturbations superposed on the background of these spacetimes. Such a study has been carried out for electromagnetic perturbations by Dhurandhar, Vishveshwara, and Cohen[6] for the subclass and by Cohen, Vishveshwara, and Dhurandhar[7] for the Gödel universe. The subclass consists of a wide range of spacetimes, namely the Robertson-Walker models, Kerr, Schwarzschild, Gödel, Taub-NUT (Newman-Unti-Tamburino) anisotropic spatially homogeneous cosmological models, etc. The neutrino perturbations in the Robertson-Walker models and the Kerr spacetime have been already studied by Dhurandhar, Vishveshwara, and Cohen[8] and will not be included here.

We make use of the null-tetrad formalism of Newman and Penrose to examine the neutrino behavior, the null tetrad being given by Wainwright[9] for the subclass of spacetimes under consideration. In Sec. II we give the generic form of the metric and its specializations, the null tetrad, the required spin coefficients, and the equations governing the Hertz potential of the neutrino perturbations. In Sec. III we discuss in detail the equations which still are in the generic form. It is seen that a considerable amount of information is secured even at this general level of discussion. In Sec. IV we treat some important spacetimes with an example for each of the three cases, namely, the Gödel universe, anisotropic homogeneous cosmologies, and the Taub spacetime.

II. THE GENERIC FORM OF THE EQUATIONS

The generic form of the metric for perfect fluid spacetimes with local rotational symmetry is given by the line element

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$$ds^2 = -\frac{(dx^0)^2}{F^2} + X^2(dx^1)^2 + Y^2[(dx^2)^2 + t^2(dx^3)^2]$$
$$+\frac{y}{F^2}(2dx^0 - y\,dx^3)dx^3 - hX^2(2dx^1 - hdx^3)dx^3\ , \tag{2.1}$$

where F, X, and Y are in general functions of both x^0 and x^1, and t, y, and h are functions of x^2 only. The functions t, y, and, h satisfy conditions given in references cited above.[4,5]

An extremely important simplification is available in the present case. The perfect fluid spacetimes with local rotational symmetry fall into three distinct cases:

(*i*) $X=1,\ \ Y=Y(x^1),\ \ F=F(x^1),\ \ h=0\ ,$

(*ii*) $h=y=0\ ,$

(*iii*) $F=1,\ \ X=X(x^0),\ \ Y=Y(x^0),\ \ y=0\ .$

With these specializations it is easy to show that the spacetimes under consideration form a subclass of the generalized Goldberg-Sachs class of spacetimes.[9]

Wainwright[9] has given the following null tetrad for the metric (2.1):

$$k_a \equiv \frac{1}{\sqrt{2}}\left[\frac{1}{F}, -X, 0, Xh - \frac{y}{F}\right],\ \ n_a \equiv \frac{1}{\sqrt{2}}\left[\frac{1}{F}, X, 0, -\left[Xh + \frac{y}{F}\right]\right],\ \ m_a \equiv \frac{1}{\sqrt{2}}(0,0,Y,i+itY)\ . \tag{2.2}$$

We also mention the directional derivatives which will appear in the equation for the Hertz potential:

$$D \equiv k^a\frac{\partial}{\partial x^a},\ \ \Delta \equiv n^a\frac{\partial}{\partial x^a},\ \ \delta \equiv m^a\frac{\partial}{\partial x^a}\ . \tag{2.3}$$

The spin coefficients which are present in the equation are mentioned below in order to make the discussion self-contained. The other spin coefficients have been given in Ref. 6. We list the following five spin coefficients:

$$\begin{aligned}
\beta &= \frac{1}{2\sqrt{2}}\frac{1}{Y}\frac{t_{,2}}{t} + \frac{i}{2\sqrt{2}}\frac{1}{tY^2}(yY_{,0} + hY_{,1})\ ,\\
\gamma &= \frac{1}{2\sqrt{2}}\frac{F}{X}\left[X_{,0} + \frac{F_{,1}}{F^2}\right] + \frac{i}{4\sqrt{2}}\frac{1}{tY^2}\left[Xh_{,2} + \frac{y_{,2}}{F}\right],\\
\epsilon &= -\frac{1}{2\sqrt{2}}\frac{F}{X}\left[X_{,0} - \frac{F_{,1}}{F^2}\right] - \frac{i}{4\sqrt{2}}\frac{1}{tY^2}\left[Xh_{,2} - \frac{y_{,2}}{F}\right],\\
\mu &= \frac{1}{\sqrt{2}}\frac{1}{Y}\left[\frac{Y_{,1}}{X} - FY_{,0}\right] + \frac{i}{2\sqrt{2}}\frac{1}{tY^2}\left[Xh_{,2} + \frac{y_{,2}}{F}\right],\\
\tau &= \frac{i}{2\sqrt{2}}\frac{1}{tY}\left[h\left[\frac{F_{,1}}{F} - \frac{X_{,1}}{X}\right] + y\left[\frac{F_{,0}}{F} - \frac{X_{,0}}{X}\right]\right].
\end{aligned} \tag{2.4}$$

The commas denote partial derivatives with respect to the coordinates.

The Hertz potential is a complex scalar ψ which is governed by a single decoupled equation obtained in Ref. 2. The Weyl spinor for the neutrino is obtained by simply differentiating this scalar function ψ. We now simply state the equation for the complex scalar function ψ, namely, Eq. (4.13b) of Ref. 2,

$$[(\Delta + \mu - \gamma)(D + \bar{\epsilon}) - (\delta + \beta - \tau)(\bar{\delta} + \bar{\beta})]\psi = 0\ . \tag{2.5}$$

The Weyl spinor ϕ_A, $A=1,2$ is given in terms of ψ by the following relations:

$$\phi_1 = -(D + \bar{\epsilon})\psi,\ \ \phi_2 = -(\bar{\delta} + \bar{\beta})\psi\ . \tag{2.6}$$

For a field of spin s, $2s$ differentiations are necessary. In our case $s=\frac{1}{2}$. As is evident the governing equations of ψ assume relatively simple forms when we divide the problem into the three cases mentioned above. Writing out the derivatives and the spin coefficients we have the equations given below.

Case (i):

$$\left[\left[-F\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^1}+\frac{Y_{,1}}{Y}+i\frac{y_{,2}}{4tY^2F}-\frac{1}{2}\frac{F_{,1}}{F}\right]\left[-F\frac{\partial}{\partial x^0}-\frac{\partial}{\partial x^1}+\frac{F_{,1}}{2F}-i\frac{y_{,2}}{4tY^2F}\right]\right.$$
$$\left.-\frac{1}{Y^2}\left[\frac{iy}{t}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\left[-\frac{iy}{t}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\right]\psi=0\ . \quad (2.7)$$

Case (ii):

$$\left[\left[-F\frac{\partial}{\partial x^0}+\frac{1}{X}\frac{\partial}{\partial x^1}+\frac{Y_{,1}}{XY}-F\frac{Y_{,0}}{Y}-\frac{1}{2}F\frac{X_{,0}}{X}-\frac{1}{2}\frac{F_{,1}}{F}\right]\left[-F\frac{\partial}{\partial x^0}-\frac{1}{X}\frac{\partial}{\partial x^1}-\frac{1}{2}F\frac{X_{,0}}{X}+\frac{1}{2}\frac{F_{,1}}{F}\right]\right.$$
$$\left.-\frac{1}{Y^2}\left[\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\left[\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\right]\psi=0\ . \quad (2.8)$$

Case (iii):

$$\left[\left[-\frac{\partial}{\partial x^0}+\frac{1}{X}\frac{\partial}{\partial x^1}-\frac{Y_{,0}}{Y}-\frac{1}{2}\frac{X_{,0}}{X}+\frac{i}{4}\frac{Xh_{,2}}{tY^2}\right]\left[-\frac{\partial}{\partial x^0}-\frac{1}{X}\frac{\partial}{\partial x^1}-\frac{1}{2}\frac{X_{,0}}{X}+\frac{i}{4}\frac{Xh_{,2}}{tY^2}\right]\right.$$
$$\left.-\frac{1}{Y^2}\left[\frac{ih}{t}\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\left[\frac{-ih}{t}\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\right]\psi=0\ . \quad (2.9)$$

The equations, though relatively simpler than if they had been obtained without the specialization into three cases, are still quite difficult. However, by dividing the equations into two portions by using the technique of the separation of variables, it will be possible to make them more tractable. In the parallel case of electromagnetic perturbations[5] the equations were somewhat simpler. We intend to follow a similar approach to deal with the present situation.

III. DISCUSSION OF THE GOVERNING EQUATIONS

We have seen that the equations are quite complex. However, they naturally divide into two parts: (i) one portion containing the derivatives $\partial/\partial x^0$ and $\partial/\partial x^1$; (ii) the other part containing the derivatives $\partial/\partial x^2$ and $\partial/\partial x^3$.

The first part we call the "radial-temporal" part of the equation while the latter part we term the "angular" part of the operator. It may be noted that derivatives $\partial/\partial x^0$ enter into the angular part of the operator in case (i) and similarly $\partial/\partial x^1$ enters that of case (iii). But this is of no consequence as they are Killing vectors for the respective cases and the solutions contain the factors $e^{i\omega x^0}$ and e^{ikx^1}, respectively, where ω and k are constants. Hence the derivatives are reduced to merely multiplicative constants. We first discuss the solutions of the angular operator and then proceed to the more complex radial-temporal operator.

A. The solutions to the angular operator

We treat cases (ii) and (iii) explicitly and the solutions to case (i) can be easily obtained from that of case (iii) by a simple transformation. In case (i) if we assume

$$\psi=Z(x^1)X(x^2,x^3)e^{i\omega x^0} \quad (3.1)$$

in Eq. (2.7) we get a separated equation for X,

$$\left[\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{2t}+\frac{\omega y}{t}\right]\left[\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{2t}-\frac{wy}{t}\right]\chi+\alpha\chi=0\ , \tag{3.2}$$

where α is the separation constant.

The corresponding equation for case (iii) is obtained by setting

$$\psi=Z(x^0)\chi(x^2,x^3)e^{ikx^1} \tag{3.3}$$

in Eq. (2.9). The separated equation is thus

$$\left[\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{2t}-\frac{kh}{t}\right]\left[\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{2t}+\frac{kh}{t}\right]\chi+\alpha\chi=0\ . \tag{3.4}$$

It is easy to see that by replacing k by $-\omega$ and h by y in Eq. (3.4) it goes over to Eq. (3.2). Therefore we treat the cases (ii) and (iii) separately and with due detail.

Case (ii). The angular part of the equation for this case is obtained by the method of separation of variables,

$$\psi=Z(x^0,x^1)\chi(x^2,x^3) \tag{3.5}$$

in Eq. (2.8). The separated equation is

$$\left[\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\left[\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{1}{2}\frac{t_{,2}}{t}\right]\chi+\alpha\chi=0\ , \tag{3.6}$$

where α is the separation constant. It may be observed that this equation is simpler than the ones obtained in cases (i) and (iii). As a result it is easier to find the solutions to Eq. (3.6). From the conditions given in Refs. 3 and 4 on the function t, it can essentially assume the following functional forms:

$$(a)\ t=\sin x^2,\ \ (b)\ t=\sinh x^2,\ \ (c)\ t=x^2,\ \ (d)\ t=\text{const}\ .$$

We examine each of the above four cases in succession.

$(a)\ t=\sin x^2$

If we write

$$\chi(x^2,x^3)=\Theta(x^2)e^{imx^3} \tag{3.7}$$

in Eq. (3.6), the equation for Θ becomes

$$\left[\frac{d}{dx^2}-\frac{m}{\sin x^2}+\tfrac{1}{2}\cot x^2\right]\left[\frac{d}{dx^2}+\frac{m}{\sin x^2}+\tfrac{1}{2}\cot x^2\right]\Theta+\alpha\Theta=0\ . \tag{3.8}$$

Here we merely state the solutions; the details of the calculation and transformations have been worked out in Ref. 6. The solutions are given in terms of the Jacobi polynomials[10]:

$$\Theta(x^2)=(1-\cos x^2)^{\alpha'/2}(1+\cos x^2)^{\beta'/2}P_n^{(\alpha',\beta')}(\cos x^2)\ , \tag{3.9}$$

where $\alpha'=|m+\frac{1}{2}|$ and $\beta'=|m-\frac{1}{2}|$ and n satisfies

$$n(n+\alpha'+\beta'+1)+\tfrac{1}{2}(\alpha'+\beta')+\tfrac{1}{4}(\alpha'+\beta')^2+\tfrac{1}{4}=\alpha\ .$$

Regularity implies that n be an integer which affects the allowed values of the separation constant α.

$(b)\ t=\sinh x^2$

Setting $\chi=\Theta(x^2)e^{imx^3}$ in Eq. (3.6) gives the following equations for $\Theta(x^2)$;

$$\left[\frac{d}{dx^2}-\frac{m}{\sinh x^2}+\tfrac{1}{2}\coth x^2\right]\left[\frac{d}{dx^2}+\frac{m}{\sinh x^2}+\tfrac{1}{2}\coth x^2\right]\Theta+\alpha\Theta=0\ . \tag{3.10}$$

The solutions for Θ are given in terms of hypergeometric functions:

$$\Theta = A z^{\alpha'/2}(1-z)^{\beta'/2} F(-n, n+\alpha'+\beta'+1, \alpha'+1; z) , \tag{3.11}$$

where $z = -\sinh^2(x^2/2)$ and $\alpha' = |m+\frac{1}{2}|$, $\beta' = |m-\frac{1}{2}|$, and n is the solution of

$$n\left[n+\alpha'+\beta'+1\right] + \tfrac{1}{2}(\alpha'+\beta') + \tfrac{1}{4}(\alpha'+\beta')^2 + \tfrac{1}{4} + \alpha = 0$$

and A is a constant. The regularity conditions impose complicated constraints on the parameters.

(*c*) $t = x^2$

Again writing $\chi(x^2,x^3) = \Theta(x^2)e^{imx^3}$ we get an equation for Θ which is essentially the spherical Bessel equation, namely,

$$\frac{d^2\Theta}{d(x^2)^2} + \frac{1}{x^2}\frac{d\Theta}{dx^2} + \left[\alpha - \frac{(m+\frac{1}{2})^2}{(x^2)^2}\right]\Theta = 0 . \tag{3.12}$$

Setting $y = \sqrt{\alpha}\, x^2$ and $\Theta = \sqrt{y} f$, the equation for f is the spherical Bessel equation

$$\frac{d^2 f}{dy^2} + \frac{2}{y}\frac{df}{dy} + \left[1 - \frac{m(m+1)}{y^2}\right] f = 0 . \tag{3.13}$$

The solutions for f are $j_m(y)$ and $y_m(y)$ and hence the solutions for Θ are given by

$$\Theta = \alpha^{1/4}\sqrt{x^2} \times \begin{cases} j_m(\sqrt{\alpha}x^2), \\ y_m(\sqrt{\alpha}x^2) . \end{cases} \tag{3.14}$$

The outgoing and ingoing solutions are given by appropriate combinations of the solutions (3.14), the h_m^1 and h_m^2, respectively.

(*d*) $t = \text{const}$

Equation (3.6) simply reduces to

$$\left[\frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2} + \alpha\right]\chi = 0 . \tag{3.15}$$

The solutions are plane waves given by

$$\chi = e^{\pm ik_2x^2 \pm ik_3x^3} , \tag{3.16}$$

where the separation constant $\alpha = k_2{}^2 + k_3{}^2$.

This completes the discussion for case (ii).

Case (iii). The angular operator for this case appears in Eq. (3.4). This equation is a little more involved than the one in case (ii) due to the presence of the last term kh/t in each of the brackets. We now resort to each of the functional forms for t.

(a) $t = \sin x^2$, $h = -c\cos x^2$

The equation (3.4) with these substitutions becomes

$$\left[\frac{\partial}{\partial x^2} + \frac{i}{\sin x^2}\frac{\partial}{\partial x^3} + (ck+\tfrac{1}{2})\cot x^2\right]\left[\frac{\partial}{\partial x^2} - \frac{i}{\sin x^2}\frac{\partial}{\partial x^3} - (ck-\tfrac{1}{2})\cot x^2\right]\chi + \alpha\chi = 0 . \tag{3.17}$$

This equation is similar to Eq. (3.8) and can be similarly solved in terms of Jacobi polynomials. Since this is a more general equation we again state the solution

$$\chi = (1-\cos x^2)^{\alpha'/2}(1+\cos x^2)^{\beta'/2} P_n^{(\alpha',\beta')}(\cos x^2) e^{imx^3} ,$$

where

$$\alpha' = |ck - m - \tfrac{1}{2}| , \quad \beta' = |ck + m - \tfrac{1}{2}| ,$$

and n satisfies

$$n(n+\alpha'+\beta'+1) + \tfrac{1}{2}(\alpha'+\beta') + \tfrac{1}{4}(\alpha'+\beta')^2 - c^2k^2 + \tfrac{1}{4} = \alpha ,$$

where, again, n is an integer restricting the permitted values for the separation constant α.

(b) $t=\sinh x^2$, $h=c\cosh x^2$

Equation (3.4) reduces to

$$\left[\frac{\partial}{\partial x^2}+\frac{i}{\sinh x^2}\frac{\partial}{\partial x^3}+(\tfrac{1}{2}-ck)\coth x^2\right]\left[\frac{\partial}{\partial x^2}-\frac{i}{\sinh x^2}\frac{\partial}{\partial x^3}+(\tfrac{1}{2}+ck)\coth x^2\right]\chi+\alpha\chi=0\ . \tag{3.18}$$

As in case (ii) the equation has its solutions in terms of hypergeometric functions.[11] Setting $z=-\sinh^2(x^2/2)$ the solution is

$$\chi=z^{\alpha'/2}(1-z)^{\beta'/2}F(-n,n+\alpha'+\beta'+1,\alpha'+1;z)\ ,$$

where

$$\alpha'=|ck+m+\tfrac{1}{2}|,\quad \beta'=|ck-m+\tfrac{1}{2}|\ ,$$

and n satisfies

$$n(n+\alpha'+\beta'+1)+\tfrac{1}{2}(\alpha'+\beta')+\tfrac{1}{4}(\alpha'+\beta')^2-c^2k^2+\tfrac{1}{4}+\alpha=0\ .$$

(c) $t=x^2$, $h=A(x^2)^2+B$

Setting $\chi=\Theta(x^2)e^{imx^3}$ we have the following equation for Θ:

$$\left[\frac{d}{dx^2}-\frac{kA(x^2)^2+kB+m-\frac{1}{2}}{x^2}\right]\left[\frac{d}{dx^2}+\frac{kA(x^2)^2+kB+m+\frac{1}{2}}{x^2}\right]\Theta+\alpha\Theta=0\ . \tag{3.19}$$

The simplification for this equation leads to

$$\frac{d^2\Theta}{d(x^2)^2}+\frac{1}{x^2}\frac{d\Theta}{dx^2}+\left[\alpha+A'-\frac{1}{(x^2)^2}\left[(A'x^2+B')^2+B'+\tfrac{1}{4}\right]\right]\Theta=0\ , \tag{3.20}$$

where $A'=kA$ and $B'=kB+m$.

Writing $y=A'(x^2)^2$ we get the equation in the new independent variable y as

$$y\frac{d^2\Theta}{dy^2}+\frac{d\Theta}{dy}+\frac{1}{4}\left[\frac{\alpha}{A^1}+1-\frac{1}{y}\left[(y+B')^2+B'+\tfrac{1}{4}\right]\right]\Theta=0\ . \tag{3.21}$$

Changing both the dependent and the independent variables in the equation by the transformations $\Theta=f/\sqrt{y}$ and $y=iz$ we get the Whittaker equation for f,

$$\frac{d^2f}{dz^2}+\left[-\frac{1}{4}+\frac{i}{4z}\left(\frac{\alpha}{A'}+1-2B'\right)+\frac{1}{z^2}\left[\frac{1}{4}-\frac{(B'+\frac{1}{2})^2}{4}\right]\right]f=0\ . \tag{3.22}$$

The solutions are readily given for f in terms of confluent hypergeometric functions,[11]

$$f=e^{-z/2}z^{n+1/2}\times\begin{cases}F(\frac{1}{2}+\mu-K,1+2\mu;z),\\ U(\frac{1}{2}+\mu-K,1+2\mu;z)\ ,\end{cases} \tag{3.23}$$

where $K=(i/4)(\alpha/A'+1-2B')$ and $\mu=(B'+\frac{1}{2})/2$. Recapitulating all the transformations for the variables, we write the solution for $\Theta(x^2)$ in the final form

$$\Theta\sim\frac{e^{-(i/2)A'(x^2)^2}}{x^2}(x^2)^{2\mu+1}\times\begin{cases}F(\frac{1}{2}+\mu-K,1+2\mu;z),\\ U(\frac{1}{2}+\mu-K,1+2\mu;z)\ .\end{cases} \tag{3.24}$$

(d) $t=\text{const}=A$, $h=Bx^2+C$

This brings us to the final case and the simplest one in the above functional forms of t. As before the equation reduces to

$$\frac{d^2\Theta}{d(x^2)^2}+[\alpha+A'-(A'x^2+B')^2]\Theta=0\ , \tag{3.25}$$

where

$$A'=\frac{kB}{A}\ ,$$

$$B'=\frac{kC+m}{A}\ .$$

Setting $x=(A'x^2+B')^2$ we have the equation

$$\frac{d^2\Theta}{dx^2}+\frac{1}{2x}\frac{d\Theta}{dx}+\left[\frac{\mu}{4x}-\tfrac{1}{4}\right]\Theta=0\ , \tag{3.26}$$

where $\mu=(\alpha+A')/A'$. The final change of the dependent variable $\Theta=x^{-1/4}f$ obtains the Whittaker equation for f,

$$\frac{d^2f}{dx^2}+\left[-\tfrac{1}{4}+\frac{\mu}{4x}+\frac{3}{16x^2}\right]f=0\ . \tag{3.27}$$

The solutions for Θ are finally given by

$$\Theta(x)\sim e^{-x/2}x^{3/2}\times\begin{cases}F\left[\tfrac{3}{4}-\dfrac{\mu}{4},\tfrac{3}{2};x\right]\\[2ex] U\left[\tfrac{3}{4}-\dfrac{\mu}{4},\tfrac{3}{2};x\right]\ .\end{cases} \tag{3.28}$$

This completes the investigation of the solutions to the angular operator.

B. The solutions to the radial-temporal operator

These solutions represent the part complementary to the angular operator solutions found above. There is a close analogy between the behavior of both the operators in the electromagnetic and the neutrino cases. In case (i) the operator is purely temporal while in case (iii) it is purely radial. In case (ii), however, no such simplification exists and one needs to investigate the combined interlocked spacetime development of the Debye potential. In cases (i) and (iii) the development of the Debye potential is purely temporal and radial, respectively. But the equation for case (ii) is extremely complicated and thus one needs to impose strong constraints on the functions F, X, and Y to extract some information.

Second, the separation constant α which occurs in both the angular operator and the radial-temporal operator assumes only certain fixed values being an eigenvalue of the angular operator. This must be borne in mind while finding solutions in the case of the radial-temporal equation.

Case (i). We now revert to Eq. (2.7) and set

$$\psi=Z(x^1)X(x^2,x^3)e^{-i\omega x^0}\ .$$

After carrying out the separation one obtains an equation for $Z(x^1)$:

$$\left[\frac{d}{dx^1}+\frac{Y_{,1}}{Y}+i\omega F+G\right]\left[-\frac{d}{dx^1}+i\omega F-G\right]Z+\frac{\alpha Z}{Y^2}=0\ , \tag{3.29}$$

where

$$G=-\frac{1}{2}\frac{F_{,1}}{F}+\frac{ic}{4Y^2F}\ .$$

This equation determines the development of the Debye potential in the x^1 direction. Equation (3.29) may be written out explicitly:

$$\frac{d^2Z}{d(x^1)^2}+\left[\frac{Y_{,1}}{Y}-\frac{F_{,1}}{F}+\frac{ic}{2Y^2F}\right]\frac{dZ}{dx^1}+\left[G^2+\omega^2F^2-\frac{Y_{,1}}{Y}(i\omega F-G)-i\omega F_{,1}-G_{,1}-\frac{\alpha}{Y^2}\right]Z=0\ . \tag{3.30}$$

The term $Y_{,1}/Y-F_{,1}/F$ multiplying dZ/dx^1 can be "transformed away" by making the following change in the variable x^1. We define a parameter u by

$$u=\int\frac{F}{Y}dx^1\ . \tag{3.31}$$

Then Eq. (3.30) assumes the form

$$\frac{d^2Z}{du^2}+\frac{ic}{2YF^2}\frac{dZ}{du}+\frac{1}{F^2}(JY^2-\alpha)Z=0\ , \tag{3.32}$$

where

$$J=G^2+\omega^2F^2-\frac{Y_{,1}}{Y}(i\omega F-G)-i\omega F_{,1}-G_{,1}\ .$$

The middle term in dZ/du can now be transformed away by changing the dependent variable, namely, Z. We define

$$\widetilde{Z}=Z\exp\left[\frac{ic}{4}\int\frac{du}{YF^2}\right]. \tag{3.33}$$

This immediately furnishes an equation for $\widetilde{Z}$:

$$\frac{d^2\widetilde{Z}}{du^2}+\left[J\frac{Y^2}{F^2}-\frac{\alpha}{F^2}-\tfrac{1}{4}g^2-\tfrac{1}{2}g'\right]\widetilde{Z}=0\ , \tag{3.34}$$

where

$$g=\frac{ic}{2}\frac{1}{YF^2}\ \text{and}\ g'=\frac{dg}{du}\ .$$

This equation is in the WKB[12] form and for high values of ω can be easily solved by the above-mentioned method. The equation

$$\frac{d^2\widetilde{Z}}{du^2}+p^2\widetilde{Z}=0 \tag{3.35}$$

has the solutions $(A/\sqrt{p})\exp(\pm i\int p\,du)$, where A is a constant and

$$p^2=\frac{1}{F^2}\left[JY^2+\frac{c^2}{16Y^2F^2}+\frac{ic}{4Y}\left[\frac{Y_{,1}}{Y}+2\frac{F_{,1}}{F}\right]-\alpha\right].$$

The WKB approximation is valid whenever $(1/p^2)dp/du<<1$.

We now briefly mention the method for securing solutions to case (iii).

Case (iii). The corresponding equation to (3.29) of case (i) is the following which has been obtained after setting $\psi=Z(x^0)\chi(x^2,x^3)e^{ikx^1}$:

$$\left[\frac{d}{dx^0}-\frac{ik}{X}+\frac{Y_{,0}}{Y}-H\right]\left[\frac{d}{dx^0}+\frac{ik}{X}-H\right]Z+\frac{\alpha}{Y^2}Z=0\ , \tag{3.36}$$

where

$$H=\frac{1}{2}\frac{X_{,0}}{X}-\frac{ic}{4}\frac{X}{Y^2}\ .$$

This equation expresses the time development of the function Z and hence of the Debye potential ψ. The further analysis of Eq. (3.36) is carried on as in case (i). Equation (3.36) becomes

$$\frac{d^2Z}{d(x^0)^2}+\left[\frac{Y_{,0}}{Y}-2H\right]\frac{dZ}{dx^0}+\left[\frac{\alpha}{Y^2}-\frac{ikX_{,0}}{X^2}-H_{,0}+\frac{k^2}{X^2}+H^2+i\frac{kY_{,0}}{XY}-\frac{Y_{,0}H}{Y}\right]Z=0\ . \tag{3.37}$$

To eliminate the first derivative in Eq. (3.37) the following transformation is necessary:

$$\widetilde{Z}=\left[\frac{Y}{X}\right]^{1/2}Z\exp\left[\frac{i}{4}\int\frac{X}{Y^2}dx^0\right]. \tag{3.38}$$

The equation then assumes the form

$$\frac{d^2\tilde{Z}}{d(x^0)^2}+\left[\frac{k^2}{X^2}+\frac{\alpha}{Y^2}+\frac{ik}{X}\left[\frac{X_{,0}}{X}+\frac{Y_{,0}}{Y}\right]+\frac{1}{4}\left[\frac{Y_{,0}}{Y}\right]^2-\frac{1}{2}\frac{Y_{,00}}{Y}\right]\tilde{Z}=0\,. \tag{3.39}$$

For large values of k the equation reduces to

$$\frac{d^2\tilde{Z}}{d(x^0)^2}+\left[\frac{k^2}{X^2}+\frac{\alpha}{Y^2}\right]\tilde{Z}=0\,. \tag{3.40}$$

This equation may be solved by the WKB method once the functions X and Y are given.

Case (ii). This case, when treated in the most general way, has no Killing vectors and hence no immediate simplification is possible. However, we mention the relevant equation and consider an important case in which somewhat strong restrictions are imposed. This case will be useful when we consider specific examples in the following section.

The radial-temporal equation is given below as

$$\left[-F\frac{\partial}{\partial x^0}+\frac{1}{X}\frac{\partial}{\partial x^1}+\frac{Y_{,1}}{XY}-F\frac{Y_{,0}}{Y}-\frac{1}{2}F\frac{X_{,0}}{X}-\frac{1}{2}\frac{F_{,1}}{F}\right]\left[-F\frac{\partial}{\partial x^0}-\frac{1}{X}\frac{\partial}{\partial x^1}-\frac{1}{2}F\frac{X_{,0}}{X}+\frac{1}{2}\frac{F_{,1}}{F}\right]Z+\frac{\alpha}{Y^2}Z=0 \tag{3.41}$$

We now consider an important simplification which will be required when we discuss the anisotropic, homogeneous cosmologies in the next section. We assume that X and Y are functions of x^0 only and $F=1$. This reduces the equation to a tractable form,

$$\left[-\frac{\partial}{\partial x^0}+\frac{1}{X}\frac{\partial}{\partial x^1}-\frac{Y_{,0}}{Y}-\frac{1}{2}\frac{X_{,0}}{X}\right]\left[-\frac{\partial}{\partial x^0}-\frac{1}{X}\frac{\partial}{\partial x^1}-\frac{1}{2}\frac{X_{,0}}{X}\right]Z+\frac{\alpha}{Y^2}Z=0\,. \tag{3.42}$$

Clearly $\partial/\partial x^1$ is a Killing vector and we may set

$$Z=e^{ikx^1}f(x^0)\,. \tag{3.43}$$

The equation for f then becomes

$$\frac{d^2f}{d(x^0)^2}+\left[\frac{X_{,0}}{X}+\frac{Y_{,0}}{Y}\right]\frac{df}{dx^0}+\left[-\frac{1}{4}\left[\frac{X_{,0}}{X}\right]^2+\frac{1}{2}\frac{X_{,00}}{X}-\frac{ikX_{,0}}{X^2}+\frac{k^2}{X^2}+\frac{ikY_{,0}}{Y}+\frac{1}{2}\frac{X_{,0}Y_{,0}}{XY}+\frac{\alpha}{Y^2}\right]f=0\,. \tag{3.44}$$

The first derivative term can be made to vanish by defining a suitable independent variable. Define

$$\tilde{f}=f\sqrt{XY}\,.$$

Then the equation for $\tilde{f}$ becomes

$$\frac{d^2\tilde{f}}{d(x^0)^2}+\left[G+\frac{\alpha}{Y^2}\right]\tilde{f}=0\,, \tag{3.45}$$

where

$$G=\frac{X_{,00}}{X}+\frac{1}{2}\frac{Y_{,00}}{Y}+\frac{X_{,0}Y_{,0}}{XY}-\frac{1}{2}\left[\frac{X_{,0}}{X}\right]^2-\frac{1}{4}\left[\frac{Y_{,0}}{Y}\right]^2-\frac{ik}{X}\left[\frac{X_{,0}}{X}-\frac{Y_{,0}}{Y}\right]+\frac{k^2}{X^2}\,.$$

In the next section we will consider certain important specific spacetimes which can be treated by the methods described above.

IV. SPECIFIC EXAMPLES OF SPACETIMES

The examples which we investigate, one for each case, will illustrate the foregoing material. We have chosen some typical examples the first of which, namely, the Gödel universe, we examine from first principles. In the case of the other two, namely, the anisotropic spatially homogeneous cosmologies and the Taub spacetime, we use the above results. The results obtained will then be compared with the electromagnetic ones. Some comparisons and contrasts will be highlighted.

A. The Gödel universe

The Gödel universe[13] is described by the metric

$$ds^2=-(dx^0)^2+(dx^1)^2+(dx^2)^2-2e^{\alpha x^2}dx^0dx^3-\tfrac{1}{2}e^{2\alpha x^2}(dx^3)^2\ , \tag{4.1}$$

where α is a parameter relating to the "rotation" of the universe. This is a dust (pressure=0) solution of Einstein's equations.

Comparison of (4.1) with the generic form of the metric for perfect fluid spacetimes with local rotational symmetry immediately furnishes the relevant functions:

$$F=1,\ \ X=1,\ \ Y=1,\ \ y=e^{\alpha x^2},\ \ t=\frac{1}{\sqrt{2}}e^{\alpha x^2},\ \ h=0\ .$$

The solution falls under case (i) in the classification scheme. Using Eq. (2.7) immediately gives the differential equation for the Debye potential ψ:

$$\begin{aligned}\Bigg[&\left[-\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^1}+\frac{i}{2\sqrt{2}}\alpha\right]\left[-\frac{\partial}{\partial x^0}-\frac{\partial}{\partial x^1}-\frac{i\alpha}{2\sqrt{2}}\right]\\ &-\left[i\sqrt{2}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}+i\sqrt{2}e^{-\alpha x^2}\frac{\partial}{\partial x^3}+\tfrac{1}{2}\alpha\right]\left[-i\sqrt{2}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}-i\sqrt{2}e^{-\alpha x^2}\frac{\partial}{\partial x^3}+\tfrac{1}{2}\alpha\right]\Bigg]\psi=0\ .\end{aligned} \tag{4.2}$$

This partial differential equation completely describes the behavior of the potential ψ. We observe that $\partial/\partial x_0$, $\partial/\partial x^1$, and $\partial/\partial x^3$ are Killing vectors, and hence we may set

$$\psi=e^{-ik_0x^0+ik_1x^1+ik_3x^3}Z(x^2)\ . \tag{4.3}$$

With this substitution and after rearranging terms we have an equation for $Z(x^2)$:

$$\frac{d^2Z}{d(x^2)^2}+\alpha\frac{dZ}{dx^2}+\left[\frac{\alpha^2}{4}-2(k_0-k_3e^{-\alpha x^2})^2-\sqrt{2}k_3e^{-\alpha x^2}-\left[k_1+\frac{\alpha}{2\sqrt{2}}\right]^2+k_0{}^2\right]Z=0\ . \tag{4.4}$$

The terms in the square brackets can be rearranged in ascending powers of $e^{-\alpha x^2}$ which will eventually aid in the solution:

$$\frac{d^2Z}{d(x^2)^2}+\alpha\frac{dZ}{dx^2}+[A+Be^{-\alpha x^2}+Ce^{-2\alpha x^2}]Z=0\ , \tag{4.5}$$

where

$$A=-\left[k_0{}^2+k_1{}^2+\frac{k_1\alpha}{\sqrt{2}}-\frac{\alpha^2}{8}\right],\ \ B=4k_0k_3-\sqrt{2}\alpha k_3,\ \ C=-2k_3{}^2\ .$$

The equation (4.5) can be reduced to the following form by the substitution $u=e^{-\alpha x^2}$:

$$\frac{d^2Z}{du^2}+\frac{1}{\alpha^2}\left[\frac{A}{u^2}+\frac{B}{u}+C\right]Z=0\ . \tag{4.6}$$

Finally by the transformation $v=2\sqrt{2}k_3u/\alpha$ the equation may be brought into the Whittaker form,

$$\frac{d^2Z}{dv^2}+\left[-\frac{1}{4}+\left[\frac{2\sqrt{2}k_0-\alpha}{2\alpha}\right]\frac{1}{v}-\frac{1}{\alpha^2v^2}\left[k_0{}^2+k_1{}^2+\frac{k_1\alpha}{\sqrt{2}}-\frac{\alpha^2}{8}\right]\right]Z=0\,. \tag{4.7}$$

The solutions for Z are the confluent hypergeometric functions with the parameters K and μ:

$$Z=e^{-v/2}v^{\mu+1/2}\times\begin{cases}F(\frac{1}{2}+\mu-K,1+2\mu;v),\\ U(\frac{1}{2}+\mu-K,1+2\mu;v)\,,\end{cases} \tag{4.8}$$

where

$$K=\frac{2\sqrt{2}k_0-\alpha}{2\alpha}$$

and

$$\mu^2=\frac{k_0{}^2+k_1{}^2+k_1\alpha/\sqrt{2}}{\alpha^2}+\frac{1}{8}\,.$$

Although the expression (4.8) gives the exact solutions it seems difficult to extract useful information from it. To examine the problem from the physical point of view, it is necessary to go back to Eq. (4.6). It is evident that the solutions are oscillatory when $A+Bu+Cu^2>0$ and damped when $A+Bu+Cu^2<0$. The coordinate u assumes values between 0 and ∞ as x^2 varies from ∞ to $-\infty$. One observes that Eq. (4.6) is slightly different from the one for the electromagnetic case.[6] This is due to the difference in the spins of the neutrino and the photon. However, in the geometric optics limit (high-frequency limit) the equations agree. The results coincide with the behavior of the null geodesics. This is seen as follows. Since $\partial/\partial x^0$, $\partial/\partial x^1$, and $\partial/\partial x^3$ are Killing vectors, k_0, k_1, and k_3 are constant along the null geodesic where k^α, $\alpha=0,1,2,3$ is the tangent vector to the null geodesic. This fact immediately implies that for motion of the zero-mass particle to be possible, we must have

$$(k^2)^2=(k^0)^2-(k^1)^2+2\frac{k_0k_3}{u}+\frac{1}{2}\frac{(k^3)^2}{u^2}\geq 0\,. \tag{4.9}$$

In the high-frequency limit we have

$$A=-(k_0{}^2+k_1{}^2),\quad B=4k_0k_3\,,$$

and

$$C=-2k_3{}^2\,,$$

and the inequality $A+Bu+Cu^2\geq 0$ reduces to (4.9) on raising the indices with the help of the metric tensor. To examine in detail the behavior of the wave function it is necessary to study the quadratic

$$Q(u)=A+Bu+Cu^2\,. \tag{4.10}$$

We note that for $u\simeq 0$ and u large $Q(u)<0$, implying that the solutions are damped. The solutions are oscillatory only if the maximum value of $Q(u)$ becomes positive. This can be checked by solving the quadratic $Q(u)=0$. The roots are given by

$$u_\pm=\frac{1}{\sqrt{2}k_3}\left[\sqrt{2}k_0\pm(k_0{}^2-k_1{}^2)^{1/2}\right]\,. \tag{4.11}$$

The solutions are oscillatory when the roots are real, i.e., when $k_0{}^2>k_1{}^2$ and when $u_-<u<u_+$. For $u<u_-$ and $u>u_+$ the solutions are damped. This behavior of the solutions agrees satisfactorily with that of the null geodesics and the electromagnetic perturbations.

B. Anisotropic spatially homogeneous cosmological models for a dust source

These models[14] are described by the following line element:

$$ds^2=-dt^2+X^2dx^2+Y^2dy^2+Z^2dz^2\,, \tag{4.12}$$

where X, Y, and Z are given functions of t. The models are spatially homogeneous but anisotropic as the name suggests and belong to a Bianchi type-I universe. The general form (4.12) is not locally rotationally symmetric but can be made so by setting two of the functions X, Y, and Z equal. To be specific we set $Y=Z$. The functions X and Y are given by

$$X=t[\tfrac{9}{2}Mt(t+\Sigma)]^{-1/3}$$

and

$$Y=[\tfrac{9}{2}Mt(t+\Sigma)]^{2/3}\,,$$

where Σ is the anisotropy parameter. Since $h=y=0$ the model comes under case (ii) of the classification.

We set $\psi=Z(x^0)e^{i\vec{k}\cdot\vec{r}}$ in Eq. (2.8), where $\vec{k}\equiv(k_1,k_2,k_3)$ and $\vec{r}=(x^1,x^2,x^3)$ and the dot product is Euclidean. The equation becomes

$$\frac{d^2Z}{dt^2}+\left[\frac{\dot X}{X}+\frac{\dot Y}{Y}\right]\frac{dZ}{dt}+\left[-\frac{1}{4}\left[\frac{\dot X}{X}\right]^2+\frac{1}{2}\frac{\ddot X}{X}-ik\frac{\dot X}{X^2}+\frac{k^2}{X^2}+\frac{ik\dot Y}{XY}+\frac{1}{2}\frac{\dot X\dot Y}{XY}+\frac{k_2{}^2k_3{}^2}{Y^2}\right]Z=0\ , \qquad (4.13)$$

where the dot represents differentiations with respect to the cosmological time t. The term in dZ/dt can be made to vanish by defining a suitable dependent variable $\bar Z$:

$$\bar Z=\sqrt{XY}Z\ . \qquad (4.14)$$

The equation thus assumes the form

$$\frac{d^2Z}{dt^2}+\left[G+\frac{k_2{}^2+k_3{}^2}{Y^2}\right]Z=0\ , \qquad (4.15)$$

where G is the function given in Eq. (3.45).

The most important simplification is obtained in the high-frequency limit. In this approximation (4.15) becomes

$$\frac{d^2Z}{dt^2}+\left[\frac{k_1{}^2}{X^2}+\frac{k_2{}^2+k_3{}^2}{Y^2}\right]Z=0\ . \qquad (4.16)$$

From this we observe that the solutions are always oscillatory for large values of the constants k_1, k_2, and k_3. For small values of t,

$$X\sim t^{2/3}\Sigma^{-1/3}$$

and

$$Y\sim(t\Sigma)^{2/3}\ ,$$

and anisotropy plays an important role in the perturbations. The oscillations of $Z(t)$ are rapid. However for large t, $X\sim t^{1/3}$ and $Y\sim t^{4/3}$ and the anisotropy parameter Σ becomes less effective. The anisotropy dies out and has little effect on the neutrino perturbations. Since both X and Y increase as t increases the oscillations of the Debye potential gradually slow down. The term in Y decays quicker and therefore in the later stages, the parameters k_2 and k_3 are less significant. As t increases further the oscillations due to the constants k_2 and k_3 become arbitrarily slow; the behavior is similar to the electromagnetic perturbations.

The case of Kantowski-Sachs spacetimes, which we examined in the electromagnetic case, can be studied in much the same way as the foregoing. We shall not discuss these cases here.

C. The Taub spacetime

The Taub spacetime[15] is an interesting one as it exhibits rather peculiar properties. It comes under case (iii) of the classification scheme when one suitably transforms the time coordinate. The geometry is described by the line element

$$ds^2=-(dx^0)^2+(2n)^2U(d\chi+\cos\theta d\phi)^2+(t^2+n^2)(d\theta^2+\sin^2\theta d\psi^2)\ , \qquad (4.17)$$

where x^0 is the newly defined time coordinate by the equation

$$x^0=\int\frac{dt}{\sqrt{U}} \qquad (4.18)$$

and $U(t)=-1+2(mt+n^2)/(t^2+n^2)$ where m and n are positive constants. The spacetime has singularities at $t=t_\pm=m\pm(m^2+n^2)^{1/2}$. Across the $t=t_\pm$ surfaces the spacetime may be extended to the NUT space.

Comparing (4.17) with the generic form of the metric, we find that $F=1$, $X=2n\sqrt{U}$, $t=\sin\theta$, $h=-c\cos\theta$ and $Y=(t^2+n^2)^{1/2}$. We write the perturbation as

$$\psi=Z(x^0)e^{ik\chi}\Theta(\theta,\phi) \qquad (4.19)$$

and substitute it in Eq. (2.9). This substitution yields equations for both Z and Θ. The equation has been already discussed in the previous section case (iii)(a) $t=\sin x^2$, $h=-c\cos x^2$. The solutions for Θ are given in terms of the Jacobi polynomials. The constant c has value unity.

The equation for the temporal development $Z(x^0)$ is the following:

$$\left[\frac{d}{dx^0}-\frac{ik}{X}+\frac{Y_{,0}}{Y}-H\right]\left[\frac{d}{dx^0}+\frac{ik}{X}-H\right]Z+\frac{\alpha}{Y^2}Z=0\ , \qquad (4.20)$$

where

$$H=\frac{1}{2}\frac{X_{,0}}{X}-\frac{i}{4}\frac{X}{Y^2}\ .$$

The transformation

$$\tilde Z=\frac{(t^2+n^2)^{1/2}\exp\left[\frac{i}{2}\tan^{-1}\frac{t}{n}\right]}{[4n^2(t_+-t)(t-t_-)]^{1/4}}Z \qquad (4.21)$$

transforms the equation to a form which can be tackled by the WKB approximation. The equation assumes the form

$$\frac{d^2\widetilde{Z}}{d(x^0)^2}+\left[\frac{k^2}{X^2}+\frac{\alpha}{Y^2}+\frac{ik}{X}\left[\frac{X_{,0}}{X}+\frac{Y_{,0}}{Y}\right]+\frac{1}{4}\left[\frac{Y_{,0}}{Y}\right]^2-\frac{1}{2}\frac{Y_{,00}}{Y}\right]\widetilde{Z}=0\ . \quad (4.22)$$

It is interesting to examine the equation for large values of k. With this approximation only the terms in k^2 and α remain in the equation. Writing the functional forms of both X and Y, we have the equation

$$\frac{d^2\widetilde{Z}}{d(x^0)^2}+\left[\frac{k^2}{4n^2U}+\frac{\alpha}{t^2+n^2}\right]\widetilde{Z}=0\ . \quad (4.23)$$

As $t\rightarrow t_\pm$, $U\rightarrow 0$, the oscillations become extremely rapid and $\widetilde{Z}\sim e^{ik/2n\sqrt{U}}$. In this limit we find that

$$\frac{1}{U}\sim\pm\frac{t_\pm{}^2+n^2}{2(m^2+n^2)^{1/2}}\frac{1}{(t_\pm-t)}\ .$$

Therefore the oscillations speed up as the square root of the time lapse from the singularities. The term in α has no such drastic effect.

This completes the discussion of the specific examples which we have worked out to illustrate the method. Several such examples may be examined on the above lines.

V. CONCLUSION

We find that the neutrino behavior in the perfect fluid spacetimes with local rotational symmetry lends itself to investigation by the two-component Hertz-potential or the Debye-potential formalism. The entire information of the neutrino perturbation is contained in a single complex scalar which obeys a decoupled equation. The problem can be divided into three cases each of which is fully studied in a general manner. It is seen that even this general approach yields considerable information about the perturbations without having to resort to any specific functional forms. It is observed that in the high-frequency limit the equations show remarkable resemblance to the ones pertaining to electromagnetic perturbations. This is in fact consistent as both the perturbations, the neutrino and the electromagnetic, behave in a way similar to the corresponding null geodesics, since both the neutrino and the photon are zero-mass particles. Finally some specific examples are studied, one for each of the three special cases mentioned in the text. These specific examples help to get insight into the actual machinery of the methods applied. Furthermore the present calculations can perhaps be used for the study of astrophysical phenomena involving neutrinos in physically interesting spacetimes such as the anisotropic spatially homogeneous cosmologies.

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