PHYSICAL REVIEW D VOLUME 26, NUMBER 8 15 OCTOBER 1982

Dirac field theory in rotating coordinates

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(Received 15 May 1981)

The Dirac field in Minkowski spacetime is quantized in a rotating coordinate system. In contrast to the scalar case, the natural procedure of defining particles via the Killing vector of the rotating observer yields a canonical quantization scheme. This scheme is inequivalent to the usual Minkowski quantization, and in contrast to the scalar case the rotating observer sees in the inertial vacuum a (nonthermal) spectrum of particles and antiparticles.

I. INTRODUCTION

Recent investigations into possible alternative quantization schemes in Minkowski spacetime were motivated by Hawking's[1] quantum-field-theoretic result on black-hole evaporation. Earlier, Fulling[2] had shown that quantization in Rindler coordinates leads to an inequivalent quantization scheme and the accelerated observer sees the Minkowski vacuum as a thermal bath of particles with temperature proportional to acceleration. The relevance of these various quantization schemes in the context of black-hole evaporation was elucidated by Unruh,[3] who by considering a model particle detector in an accelerated state of motion showed that indeed the detector observes the above spectrum. In earlier papers[4] we have generalized the above results to the spin-half fields. Recently, Sciama, Candelas, and Deutch[5] have examined more critically the above results and stressed that the detector essentially measures the spectrum of vacuum fluctuations. The investigation of the dependence of the detector response on its state of motion necessitates the construction of quantization schemes in the relevant system of coordinates. In this connection Letaw and Pfautsch[6] have examined the scalar field in a rotating system of coordinates. Here we study the Dirac field, both massless and massive, in rotating coordinates and construct a set of complete orthonormal modes. Employing these we show that the natural definition of positive frequency yields, unlike the scalar case, a canonical quantization scheme. The quantization is inequivalent to the quantization in inertial coordinates and we exhibit the corresponding Bogoliubov transformation. The results are contrasted with the spin-zero case and interpreted physically in terms of the choice of vacuum states for the rotating and inertial observers.

II. THE DIRAC EQUATION IN ROTATING COORDINATES

In what follows latin indices are tetrad labels and run over 0 to 3 while greek indices are coordinate labels. The Dirac equation in general coordinates may be written as

$$\gamma^a \nabla_a \psi + i\mu\psi = 0 , \tag{1}$$

where γ^a are the 4×4 flat-spacetime Dirac matrices satisfying

$$[\gamma^a,\gamma^b]_+ = 2\eta^{ab} \tag{2}$$

and

$$\nabla_a \psi = e_a^\mu(\partial_\mu - \Gamma_\mu)\psi . \tag{3}$$

Here e_a^μ are the chosen tetrad fields and Γ_μ the spinor affine connections given by

$$\Gamma_\mu = -\tfrac{1}{4}\gamma^a\gamma^b e_a^\nu e_{b\nu;\mu} . \tag{4}$$

To proceed further we note that in coordinates adapted to an observer traveling in a circle with constant angular velocity Ω, the Minkowski space is given by

$$ds^2 = (1-\Omega^2 r^2)dt^2 - 2\Omega r^2 d\phi\, dt - r^2 d\phi^2 - dr^2 - dz^2 . \tag{5}$$

The metric is stationary and the coordinates (t,r,ϕ,z) are related to the cylindrical Minkowski coordinates (t',r',ϕ',z') by

$$t = t', \quad r = r' , \qquad \phi = \phi' - \Omega t', \quad z = z' . \tag{6}$$

With the nonvanishing tetrad field components chosen to be

$$e_0^t = e_1^r = e_3^z = 1 \; , \qquad e_0^\phi = -\Omega, \;\; e_2^\phi = \frac{1}{r} \tag{7}$$

the corresponding Γ_μ's are

$$\Gamma_t = \Omega\Gamma_\phi, \;\; \Gamma_\phi = -\tfrac{1}{2}\gamma^1\gamma^2, \;\; \Gamma_r = \Gamma_z = 0 \; . \tag{8}$$

Employing Eqs. (7) and (8) the Dirac equation in rotating coordinates is explicitly given by

$$[\gamma^0(\partial_t - \Omega\partial_\phi) + \gamma^1\left[\partial_r + \frac{1}{2r}\right] + \frac{1}{r}\gamma^2\partial_\phi + \gamma^3\partial_z + i\mu]\psi = 0 \; . \tag{9}$$

We shall first discuss the neutrino ($\mu = 0$) case. In this case, as is well known, the spin of the neutrino is antiparallel to its momentum and hence it satisfies the additional condition

$$(1 + i\gamma_5)\psi = 0 \; . \tag{10}$$

In the standard representation we work in

$$\gamma_5 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{11}$$

and setting

$$\psi = (\eta_1, \eta_2)^T \tag{12}$$

in Eq. (10), where η_1 and η_2 are two-component spinors, yields,

$$\eta_1 = \eta_2 \; . \tag{13}$$

From Eq. (13) and (t,ϕ,z) independence of Eq. (9), it is obvious that the solutions for the neutrino case can be taken in the form

$$\psi(\omega,m,k;x) = \exp[-i(\omega - m\Omega)t] \times \exp[i(m\phi + kz)] \times (\eta(r),\eta(r))^T \; , \tag{14}$$

where $\eta(r)$ satisfies

$$\left[\sigma^1\left[\partial_r + \frac{1}{2r}\right] + \frac{im}{r}\sigma^2 + ik\sigma^3 - i\omega\right]\eta(r) = 0 \; . \tag{15}$$

Putting

$$\eta(r) = (R_1(r), \; R_2(r))^T \tag{16}$$

yields the following coupled equations for R_1 and R_2:

$$\left[\frac{d}{dr} + \frac{2m+1}{2r}\right]R_2 = i(\omega - k)R_1 \; , \tag{17a}$$

$$\left[\frac{d}{dr} - \frac{2m-1}{2r}\right]R_1 = i(\omega + k)R_2 \; . \tag{17b}$$

Eliminating R_1 from Eqs. (17) yields the following decoupled equation for R_2:

$$\left[r^2\frac{d^2}{dr^2} + r\frac{d}{dr} + Q^2r^2 - (m + \tfrac{1}{2})^2\right]R_2(r) = 0 \tag{18}$$

with

$$Q = +(\omega^2 - k^2)^{1/2} \; . \tag{19}$$

Equation (18) is a Bessel equation of order $(m + \frac{1}{2})$ in variable Qr, whose solutions are taken to be $J_{m+1/2}(Qr)$. Employing Eq. (17a) and the recurrence relations for the Bessel functions yields

$$R_1(r) = Q\frac{J_{m-1/2}(Qr)}{i(\omega - k)} \tag{20}$$

so that the normal modes are written as

$$\psi'(\omega,m,k;x) = \exp[-i(\tilde{\omega}t - m\phi - kz)](QJ_{m-1/2}, i(\omega-k)J_{m+1/2}, QJ_{m-1/2}, i(\omega-k)J_{m+1/2})^T \; , \tag{21}$$

where

$$\tilde{\omega} = \omega - m\Omega, \;\; J_{m\pm 1/2} \equiv J_{m\pm 1/2}(Qr) \; . \tag{21a}$$

As would be expected these solutions are the Minkowski modes transformed to the rotating coordinates since the Dirac field is a scalar under coordinate transformations.

From the conserved current for the Dirac equation (1) one defines an inner product on the space of the above solutions by

$$(\psi_1,\psi_2) \equiv \int_{t=\text{constant}} \sqrt{-g}\,\bar{\psi}_1\gamma^t\psi_2 d^3x \; , \tag{22}$$

where

$$\bar{\psi} = \psi^\dagger\gamma^0, \;\; \gamma^t = e_a^t\gamma^a \; . \tag{23}$$

In our case Eq. (22) reduces to the form

$$(\psi_1,\psi_2)=\int_{t=\text{constant}} r\psi_1^\dagger\psi_2 dr\, d\phi\, dz \ . \tag{24}$$

Writing $J_{m+1/2}(Qr)$ in terms of spherical Bessel functions $j_m(Qr)$ and employing the relation

$$\int_0^\infty j_l(kr)j_l(k'r)r^2dr=\frac{\pi}{2k^2}\delta(k-k')\ , \tag{25}$$

it can easily be shown that the modes, Eq. (21), are orthogonal and yield

$$(\psi'(\omega,m,k),\psi'(\omega',m',k'))=8\pi^2\,|\,\omega-k\,|\,\delta(\omega-\omega')\delta(k-k')\delta_{mm'}\ . \tag{26}$$

Thus

$$\psi(\omega,m,k;x)\equiv[2\pi(2\,|\,\omega-k\,|\,)^{1/2}]^{-1}\psi'(\omega,m,k;x) \tag{27}$$

are a convenient set of orthonormal modes for the neutrino field in rotating coordinates. It should be noticed that as expected the Dirac modes have positive norm for all values of $\tilde{\omega}$. In terms of these normal modes an arbitrary Dirac field may be expanded as

$$\Psi=\sum_{m=-\infty}^{\infty}\int_{\tilde{\omega}>0}d\omega\int_{-|\omega|}^{|\omega|}dk[a(\omega,m,k)\psi(\omega,m,k;x)+b^\dagger(\omega,m,k)\psi(-\omega,-m,k;x)]\ . \tag{28}$$

The above mode decomposition differs from the usual Minkowski expansion in that the rotating observer defines positive frequency via his Killing vector $\partial/\partial t$ so that modes with $\tilde{\omega}>0$ are his particles. For the inertial observer the natural choice is in terms of the Killing vector $\partial/\partial t'$. Thus the $\omega>0$ modes are the Minkowski particles. With this comment Eq. (28) may be inverted to yield

$$a(\omega,m,k)=(\psi(\omega,m,k;x),\Psi),\ \ \tilde{\omega}>0\ , \tag{29a}$$

$$b^\dagger(\omega,m,k)=(\psi(-\omega,-m,k;x),\Psi),\ \ \tilde{\omega}>0\ . \tag{29b}$$

The Lagrangian density for the Dirac field is given by

$$\mathscr{L}(x)=\sqrt{-g}\,(i\overline{\Psi}\gamma^a\nabla_a\Psi-\mu\overline{\Psi}\Psi)\ . \tag{30}$$

The momenta conjugate to the fields are

$$\Pi(x)\equiv\frac{\partial\mathscr{L}}{\partial\Psi_{,t}}=i\sqrt{-g}\,\Psi^\dagger\ ,\qquad \Pi^*(x)\equiv\frac{\partial\mathscr{L}}{\partial\overline{\Psi}_{,t}}=0\ . \tag{31}$$

Field quantization is effected by imposition of the equal-time anticommutation relations

$$[\Psi^a(z),\Psi^b(x')]_+=[\Pi^a(x),\Pi^b(x')]_+=0\ ,\qquad [\Pi^a(x),\Psi^b(x')]_+=i\delta(x-x')\delta^{ab} \tag{32}$$

and complex conjugates.

For our case $\mu=0$. Using Eqs. (29), (32), and (26) it follows that

$$[a(\omega,m,k),a^\dagger(\omega',m',k')]_+=\delta(\omega-\omega')\delta(k-k')\delta_{mm'}\ . \tag{33}$$

Similarly,

$$[b(\omega,m,k),b^\dagger(\omega',m',k')]_+=\delta(\omega-\omega')\delta(k-k')\delta_{mm'}\ . \tag{34}$$

All other anticommutators vanish.

The rotating observer defines his vacuum by

$$a(\omega,m,k)\,|\,0\rangle_\Omega=b(\omega,m,k)\,|\,0\rangle_\Omega=0,\ \ \tilde{\omega}>0\ . \tag{35}$$

As mentioned earlier, the rotating observer classifies as particles the modes with $\tilde{\omega}>0$. Thus his natural vacuum $|\,0\rangle_\Omega$ has no particles, i.e., $\tilde{\omega}>0$ states and all modes with $\tilde{\omega}<0$ (holes) are filled.

To relate the above creation and annihilation operators with the usual Minkowski operators recall that Eq. (21) would be precisely the Minkowski modes. However, the inertial observer calls the $\omega>0$ solutions the positive-frequency modes. Hence he has

$$\Psi=\sum_{m=-\infty}^{\infty}\int_0^\infty d\omega\int_{-\omega}^{\omega}dk\,[\hat{a}(\omega,m,k)\psi(\omega,m,k;x)+\hat{b}^\dagger(\omega mk)\psi(-\omega,-m,k;x)] \tag{36}$$

with $\hat{a}(\omega,m,k)$ and $\hat{b}(\omega,m,k)$ satisfying the canonical anticommutation relations as before. He defines his vacuum by

$$\hat{a}(\omega,m,k)\,|\,0\rangle_M=\hat{b}(\omega,m,k)\,|\,0\rangle_M =0,\quad \omega>0\;. \tag{37}$$

so that all $\omega>0$ states are empty and all "holes" $\omega<0$ are filled. From Eq. (29a)

$$a(\omega,m,k)=\int\sqrt{-g}\,d^3x\,\psi^\dagger(\omega,m,k)\Psi\;. \tag{38}$$

Expanding Ψ in terms of the Minkowski modes, Eq. (36), it can easily be shown that

$$a(\omega,m,k)=\hat{a}(\omega,m,k)\quad \omega>0\;, \tag{39a}$$

$$a(\omega,m,k)=\hat{b}^\dagger(-\omega,-m,k),\quad \omega<0\;. \tag{39b}$$

Similarly

$$b(\omega,m,k)=\hat{b}(\omega,m,k),\quad \omega>0\;, \tag{40a}$$

$$b(\omega,m,k)=\hat{a}^\dagger(-\omega,-m,k),\quad \omega<0\;. \tag{40b}$$

We thus find that the modes with $\tilde{\omega}>0$ but $\omega<0$ are related by a Bogoliubov transformation. To examine the result further let us compute the number of "rotating" particles in the Minkowski vacuum. Employing Eqs. (39) we have

$$\begin{aligned}{}_M\langle 0\,|\,N_\Omega^+(\omega,m,k)\,|\,0\rangle_M &\equiv {}_M\langle 0\,|\,a^\dagger(\omega,m,k)a(\omega,m,k)\,|\,0\rangle_M\\ &=0,\quad \omega>0\\ &=1,\quad \omega<0\;,\end{aligned} \tag{41}$$

where we have for convenience converted the continuous values ω and k to discrete ones in the usual way.

Similarly it follows that

$$\begin{aligned}{}_M\langle 0\,|\,N_\Omega^-(\omega,m,k)\,|\,0\rangle_M &\equiv {}_M\langle 0\,|\,b^\dagger(\omega,m,k)b(\omega,m,k)\,|\,0\rangle_M\\ &=0,\quad \omega>0\\ &=1,\quad \omega<0\;.\end{aligned} \tag{42}$$

The above results can be understood in a physical manner by reference to Fig. 1. From this figure in the (ω,m) plane it is obvious that in the state $|\,0\rangle_M$, all states to the right of the $\omega=0$ line are empty while all holes to the left are filled. However, to the rotating observer all states below the $\tilde{\omega}=0$ line are empty while those above are filled. Thus if the rotating observer sees Minkowski vacuum, then he find states in sector III filled (particles) and those in Sector I empty (antiparticles). The Minkowski vacuum consequently contains for the rotating observer a nonthermal spectrum of particles and antiparticles for energies, where $\tilde{\omega}$ is positive but ω is negative. However, it can be shown that the charge operator for the rotating observer

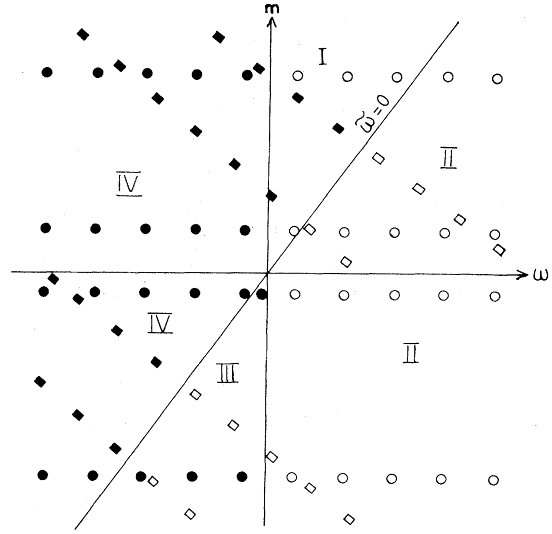

FIG. 1. The figure shows the vacuum states of the inertial and rotating observers. The open circles (squares) indicate that the Minkowski (rotating) vacuum has no "particles." The dark circles (squares) indicate the filled negative-energy states as defined by the Minkowski (rotating) observer. The observers agree on their definitions of particles and antiparticles in sectors II and IV, respectively. To the rotating observer, the Minkowski vacuum seems to contain particles (sector III) and antiparticles (sector I).

$$Q_\Omega\equiv\sum_{m=-\infty}^{\infty}\int_{\tilde{\omega}>0}d\omega\int_{-|\omega|}^{|\omega|}dk\,[a^\dagger(\omega,m,k)a(\omega,m,k)-b^\dagger(\omega,m,k)b(\omega,m,k)] \tag{43}$$

is identical to that of the Minkowski observer Q_M. Hence as would be desired he does not see any charge in the inertial vacuum.

These results must be contrasted with those obtained for the scalar field.[6] Unlike the spin-zero case we have been able to define particles as modes with $\tilde{\omega}>0$ and obtain a canonical quantization scheme with $a^\dagger$ and $b^\dagger$ being interpreted as creation operators for particles and antiparticles, respectively. This difference between the spin-half and -zero cases is due to the following reason. For the scalar field the positive-frequency solutions have a positive norm whereas the negative-frequency solutions have a negative norm, so that the sign of the norm together with the canonical commutation relation identifies the creation operators of the field. Consequently canonical quantization was not possible for the scalar case because though the frequency was classified via $\tilde{\omega}$ the norm of the modes was still determined by the sign of ω and not of $\tilde{\omega}$. It is precisely for this reason that quantization in rotating coordinates turned out to be equivalent to that in inertial coordinates. For the Dirac field on the other hand, all modes positive and negative have a positive norm so that the criterion of the norm is not available to define the creation operators. Thus in this case we define particles as modes with positive frequency, i.e., $\tilde{\omega}>0$ and this as proved above leads to a canonical quantization scheme. One point should be noted. Since the canonical anticommutation relations do not force an identification of annihilation and creation operators, they may be redefined by a shift of the zero of the energy scale. Thus if we expand the field as

$$\begin{aligned}\Psi=&\sum_{m>0}\int_{m\Omega}^{\infty}d\omega\int_{-\omega}^{\omega}dk\,[a'(\omega,m,k)\psi(\omega,m,k)+b'^{\dagger}(\omega,m,k)\psi(-\omega,-m,k)]\\&+\sum_{m<0}\int_{0}^{\infty}d\omega\int_{-\omega}^{\omega}dk\,[a'(\omega,m,k)\psi(\omega,m,k)+b'^{\dagger}(\omega,m,k)\psi(-\omega,-m,k)]\\&+\sum_{m<0}\int_{m\Omega}^{0}d\omega\int_{\omega}^{-\omega}dk\,[a'(\omega,m,k)\psi(-\omega,-m,k)+b'^{\dagger}(\omega,m,k)\psi(\omega,m,k)]\ ,\end{aligned}\tag{44}$$

the $a'(\omega,m,k)$ and $b'(\omega,m,k)$ will still satisfy the canonical anticommutation relations. However, these operators are not related to the Minkowski operators by a Bogoliubov transformation.

Defining the vacuum by

$$a'(\omega,m,k)\,|\,0\rangle'_{\Omega}=b'(\omega,m,k)\,|\,0\rangle'_{\Omega}=0,\quad \tilde{\omega}>0 \tag{45}$$

we find that analogous to the scalar case one obtains a quantization equivalent to the Minkowski one. However, in this construction one has sidestepped a desirable feature of the earlier construction in that particles for rotating observers have $\tilde{\omega}>0$ while absence of $\tilde{\omega}<0$ states are antiparticles.

The other important question to be answered is whether the particle spectrum in the vacuum and the excitation spectrum of the detector in the vacuum are identical. The extension of this analysis[6] to the spin-half case seems more involved and is currently under investigation.

III. THE MASSIVE DIRAC FIELD

The above results can be extended to the massive Dirac field in a straightforward manner. In this case, we do not have the chirality conditions, Eq. (10). Putting

$$\psi(\omega,m,k;x)=\exp[-i(\tilde{\omega}t-m\phi-kz)]\times(\eta_1(r),\eta_2(r))^T \tag{46}$$

in Eq. (9) yields

$$\left[\sigma^1\left[\frac{d}{dr}+\frac{1}{2r}\right]+\frac{im}{r}\sigma^2+ik\sigma^3\right]\eta_2=i(\omega-\mu)\eta_1\ , \tag{47a}$$

$$\left[\sigma^1\left[\frac{d}{dr}+\frac{1}{2r}\right]+\frac{im}{r}\sigma^2+ik\sigma^3\right]\eta_1=i(\omega+\mu)\eta_2\ . \tag{47b}$$

Eliminating η_1 from Eqs. (47) we have

$$\left[r^2\frac{d^2}{dr^2}+r\frac{d}{dr}-m^2+\sigma^3 m-\tfrac{1}{4}+(\omega^2-\mu^2-k^2)r^2\right]\eta_2=0\ . \tag{48}$$

Setting

$$\eta_1(r)=(R_-(r),R_+(r))^T\ , \tag{49}$$

we find $R_\mp$ satisfy

$$\left[r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+q^2r^2-(m\mp\tfrac{1}{2})\right]R_\mp=0 \tag{50}$$

with

$$q=+(\omega^2-\mu^2-k^2)^{1/2}\equiv(\omega_0^2-k^2)^{1/2} \tag{51}$$

so that

$$R_{\mp}(r)=J_{(m\mp 1/2}(qr)\ . \tag{52}$$

Employing Eq. (47a) η_1 can be obtained and as before it can be shown that the normalized solutions are given by

$$\psi(\omega,m,k;x)=(4\pi\,|\,\omega-\mu\,|\,^{1/2})^{-1}\exp[-i(\tilde{\omega}t-m\phi-kz)] \times((Q+ik)J_{m-1/2}(qr),\ -(Q+ik)J_{m+1/2}(qr)\ ,\ i(\omega-\mu)J_{m-1/2}(qr),\ i(\omega-\mu)J_{m+1/2}(qr))^T\ . \tag{53}$$

An arbitrary massive Dirac field is now expressed as

$$\Psi=\sum_{m=-\infty}^{\infty}\int_{\tilde{\omega}>0,\,|\omega|>\mu}d\omega\int_{-|\omega_0|}^{|\omega_0|}dk\,[a(\omega,m,k)\psi(\omega,m,k;x)+b^{\dagger}(\omega,m,k)\psi(-\omega,-m,k;x)]\ . \tag{54}$$

From now on construction is essentially the same as before and conclusions similar to the massless case are obtained in a straightforward way.

ACKNOWLEDGMENTS

I thank C. S. Shukre and C. V. Vishveshwara for many helpful discussions.

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