Neutrinos in gravitational collapse: The Dirac formalism

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The massless Dirac equations for the neutrino are studied in the background geometry relevant to a gravitational collapse, i.e., a Friedmann dust interior matched onto a Schwarzschild exterior. The interior normal modes are obtained and continued consistently across the collapsing surface into the exterior. This yields the expected spectral shifts quite naturally in addition to the phenomenon of backward emission. The complete exterior modes are obtained by matching solutions at the Schwarzschild barrier. The asymptotic transmission coefficient is computed, and this includes in addition to the effects of gravitational collapse the effects of the exterior potential barrier.

I. INTRODUCTION

The problems of the interaction of neutrinos with strong gravitational fields have been considered in different contexts. For instance, Brill and Wheeler[1] studied the possibility of the gravitational self-binding of neutrinos (geons), whereas Unruh[2] showed the absence of neutrino superradiance in the Kerr metric. Kembhavi and Vishveshwara[3] studied the possible trapping of neutrinos in compact objects, while investigations of Dhurandhar and Vishveshwara[4,5] on the behavior of neutrinos during gravitational collapse in the geodesic approximation yielded interesting features like bursts and decays in late stages of collapse.

In this paper we set up the necessary formalism to study the massless Dirac equation for the neutrino in the collapse scenario. The collapse is modeled by a Friedmann dust interior matched to a Schwarzschild exterior, and our aim is to investigate the principal general-relativistic effects in the problem. To this end matter is assumed to be transparent to neutrinos, and nongravitational interactions are not considered. The work reported here is meant to set up the requisite formalism and to complement the investigation of Dhurandhar and Vishveshwara[4,5] in the geodesic approximation.

In the next section we give a brief summary of the classical background geometry. In Sec. III the Dirac equation is written down both in the interior and the exterior and the solutions obtained in the WKB approximation are matched across the Schwarzschild barrier. In Sec. IV the interior solutions are matched consistently across the junction to the exterior solution. The matched solutions exhibit the expected spectral shifts as also the backward emission. In the last section the above modes are employed to compute the asymptotic transmission coefficient and this includes in addition to the effects of gravitational collapse the barrier effects also.

II. THE BACKGROUND SPACETIME

In this section we quote without proofs details of the background geometry relevant to our purposes. For further discussions see Ref. 4 and references cited therein.

The line element in comoving coordinates for the interior of the collapsing body is given in geometric units ($c=G=1$) by the Friedman dust solution

$$ds^2=dT^2-S^2(T)[dR^2(1-\alpha R^2)^{-1}+R^2d\Omega^2]\,, \tag{2.1}$$

where $\alpha R_b{}^3=2M$. Here R_b is the R coordinate for a particle on the boundary and the scale factor $S(T)$ satisfies

$$\frac{dS}{dT}\equiv\dot{S}=\sqrt{\alpha}\left(\frac{1-S}{S}\right)^{1/2}. \tag{2.2}$$

Defining

$$\tau=\int_0^T\frac{dT}{S(T)}\,, \tag{2.3}$$

the above metric is conformal to

$$ds^2=d\tau^2-[dR^2(1-\alpha R^2)^{-1}+R^2d\Omega^2] \tag{2.4}$$

with conformal factor $S^2(T)$.

From (2.2) and (2.3) it follows that

$$T=\frac{1}{\sqrt{\alpha}}(\chi+\sin\chi\cos\chi)\,, \tag{2.5}$$

$$\tau=\frac{2}{\sqrt{\alpha}}\chi\,. \tag{2.6}$$

The exterior geometry is given in the usual Schwarzschild coordinates by

$$ds^2=\left(1-\frac{2M}{r}\right)dt^2-\left[dr^2\left(1-\frac{2M}{r}\right)^{-1}+r^2d\Omega^2\right], \tag{2.7}$$

where M is the geometrized mass of the collapsing object.

Spherical symmetry implies that the coordinates θ and ϕ may be chosen to be the same both in the interior and the exterior. The metric tensor and its derivatives are matched at the interface $R=R_b$. This gives

$$r=RS(T) \tag{2.8}$$

and we choose initial conditions so that the collapse ($T=0$) starts at Schwarzschild time $t=0$. We shall need to use the partial derivative of t and r with respect to T and R near the junction $R=R_b$. They are obtained straightforwardly and are given by

$$\left.\frac{\partial t}{\partial T}\right|_{R=R_b}=\left(1-\frac{2M}{r}\right)^{-1}(1-\alpha R_b{}^2)^{1/2}, \tag{2.9}$$

$$\left.\frac{\partial t}{\partial R}\right|_{R=R_b}=R_b S\mathring{S}\left(1-\frac{2M}{r}\right)(1-\alpha R_b{}^2)^{-1/2}, \tag{2.10}$$

$$\left.\frac{\partial r}{\partial T}\right|_{R=R_b}=R_b\mathring{S},\quad \left.\frac{\partial r}{\partial R}\right|_{R=R_b}=S. \tag{2.11}$$

III. THE DIRAC EQUATION IN THE COLLAPSE GEOMETRY

In the tetrad formalism, the massless Dirac equation in a general curved spacetime is given by

$$\gamma^a\nabla_a\psi=0, \tag{3.1}$$

where latin indices are tetrad labels and run over from 0 to 3. Further, γ^a are the 4×4 flat spacetime Dirac matrices satisfying the anticommutation relations

$$[\gamma^a,\gamma^b]_+=2\eta^{ab} \tag{3.2}$$

and $\nabla_a\psi$ the tetrad component of the extended covariant derivative of ψ:

$$\nabla_a\psi=e_a^\mu(\partial_\mu-\Gamma_\mu)\psi. \tag{3.3}$$

The greek indices are coordinate labels. And e_a^μ are the chosen tetrad fields while Γ_μ are the spinor affine connections given by

$$\Gamma_\mu=-\tfrac{1}{4}\gamma^a\gamma^b e_a^\nu e_{b\nu;\mu}. \tag{3.4}$$

The spin of the neutrino is always polarized antiparallel to its momentum and hence the neutrino satisfies an additional helicity condition expressed as

$$(1+i\gamma_5)\psi=0, \tag{3.5}$$

where

$$\gamma_5=\frac{\epsilon^{\mu\nu\alpha\beta}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta}{4!\sqrt{-g}},\quad \gamma_\mu=e_{a\mu}\gamma^a, \tag{3.6}$$

$\epsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor density.

To write down the Dirac equation in the interior we introduce the tetrad e_a^μ with nonvanishing components as given below:

$$\begin{aligned}&e_0^T=1,\quad e_2^\phi=(RS\sin\theta)^{-1},\\&e_1^\theta=(RS)^{-1},\quad e_3^R=R^{-1}(1-\alpha R^2)^{-1/2}.\end{aligned} \tag{3.7}$$

Employing Eqs. (3.4) and (3.7) the Γ_μ's after a straightforward calculation are given by

$$\begin{aligned}&\Gamma_T=0,\quad \Gamma_R=-\frac{\mathring{S}}{2}(1-\alpha R^2)^{-1/2}\gamma^0\gamma^3,\\&\Gamma_\theta=-\frac{R\mathring{S}}{2}\gamma^0\gamma^1+\frac{(1-\alpha R^2)^{1/2}}{2}\gamma^1\gamma^3,\\&\Gamma_\phi=-\frac{R\mathring{S}}{2}\gamma^0\gamma^2+\frac{(1-\alpha R^2)^{1/2}}{2}\sin\theta\gamma^2\gamma^3-\frac{\cos\theta}{2}\gamma^1\gamma^2.\end{aligned} \tag{3.8}$$

Choosing the γ matrices in the Bjorken-Drell[6] representation we find that

$$\gamma_5=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \tag{3.9}$$

Setting $\psi=(\eta_1,\eta_2)^T$ in Eq. (3.5) yields

$$\eta_1=\eta_2. \tag{3.10}$$

Inserting Eq. (3.8) in (3.1) the massless Dirac equation in the interior is given by

$$\left\{RS\gamma^0\left(\partial_T+\frac{3}{2}\frac{\mathring{S}}{S}\right)+R(1-\alpha R^2)^{1/2}\gamma^3\left(\partial_R+\frac{1}{R}\right)+\left[\gamma^1\left(\partial_\theta+\frac{\cot\theta}{2}\right)+\gamma^2\csc\theta\partial_\phi\right]\right\}\psi=0. \tag{3.11}$$

Equation (3.11) is separable and putting

$$\psi=\frac{1}{S^{3/2}}\exp[-i(\omega\tau-m\phi)](\eta,\eta)^T, \tag{3.12}$$

where

$$\eta=(R_1(r)S_1(\theta),R_2(r)S_2(\theta))^T, \tag{3.13}$$

yields the following set of coupled angular and radial equations for S_1, S_2 and R_1, R_2, respectively:

$$\left(\frac{d}{d\theta}+\frac{\cot\theta}{2}+m\csc\theta\right)S_2=-kS_1, \tag{3.14a}$$

$$\left(\frac{d}{d\theta}+\frac{\cot\theta}{2}-m\csc\theta\right)S_1=kS_2, \tag{3.14b}$$

$$\left(\frac{d}{dR^*}+\frac{(1-\alpha R^2)^{1/2}}{R}+i\omega\right)R_2=\frac{kR_1}{R}, \tag{3.15a}$$

$$\left(\frac{d}{dR^*}+\frac{(1-\alpha R^2)^{1/2}}{R}-i\omega\right)R_1=\frac{kR_2}{R}, \tag{3.15b}$$

where

$$\frac{dR^*}{dR}=\frac{1}{(1-\alpha R^2)^{1/2}}. \tag{3.16}$$

A. The angular functions

From equations (3.14) one can easily obtain the following decoupled equation for $S_2(\theta)$:

$$\left(\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}-\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta}\right)S_2(\theta)=-(k^2-\tfrac{1}{4})S_2(\theta), \tag{3.17}$$

where $s=\frac{1}{2}$. Regularity of $S_2(\theta)$ at 0 and π yields an eigenvalue equation for k. The eigenfunctions are formally the spin-weighted spherical harmonics[7] ${}_sS_l^m(\theta)$ of spin weight $s=\frac{1}{2}$ and order $l=k-\frac{1}{2}$ where

$$|k|=1,2,3,\ldots \text{ and } -|k|+\tfrac{1}{2}\leq m\leq |k|-\tfrac{1}{2}.$$

The explicit form of ${}_sS_l^m(\theta)$ may be given in terms of Jacobi polynomials for arbitrary real values of s.[8] $S_1(\theta)$ satisfies the same differential equation as $S_2(\pi-\theta)$. S_1 and S_2 are chosen to be real and we normalize S_1 and S_2 so that

$$\int S_1{}^2(\theta)d\Omega=\int S_2{}^2(\theta)d\Omega=\tfrac{1}{2}. \tag{3.18}$$

B. The radial solutions

To discuss the radial solutions it is more convenient to introduce the following linear combinations of R_1 and R_2:

$$F=R_1+R_2,\quad iG=R_1-R_2, \tag{3.19}$$

which satisfy the following equations:

$$\left((1-\alpha R^2)^{1/2}\frac{d}{dR}-\frac{k-(1-\alpha R^2)^{1/2}}{R}\right)F=-\omega G, \tag{3.20a}$$

$$\left((1-\alpha R^2)^{1/2}\frac{d}{dR}+\frac{k+(1-\alpha R^2)^{1/2}}{R}\right)G=\omega F. \tag{3.20b}$$

Eliminating F from Eq. (3.20) yields the following second-order differential equation for G:

$$\frac{d^2G}{dR^2}+\frac{2}{R}\left(1-\frac{\alpha R^2}{2(1-\alpha R^2)}\right)\frac{dG}{dR}+\frac{1}{1-\alpha R^2}\left(\omega^2-\frac{k^2+k+(1-\alpha R^2)^{1/2}-(1-\alpha R^2)}{R^2}\right)G=0. \tag{3.21}$$

For small values of the variable $\sqrt{\alpha}R$ Eq. (3.21) becomes

$$\left((\omega R)^2\frac{d^2}{d(\omega R)^2}+2(\omega R)\frac{d}{d(\omega R)}+[\omega^2R^2-k(k+1)]\right)G=0, \tag{3.22}$$

which is the spherical Bessel function of order k with argument ωR. The solutions can be conveniently chosen as $h_k^{(1)}(\omega R)$ and $h_k^{(2)}(\omega R)$. In this limit the corresponding F can be obtained from Eq. (3.20b). Using the recurrence relations for the spherical Bessel functions the solutions for F turn out to be $h_{k-1}^{(1)}(\omega R)$ and $h_{k-1}^{(2)}(\omega R)$, respectively. Putting

$$G=\frac{g}{R(1-\alpha R^2)^{1/4}}, \tag{3.23}$$

Eq. (3.21) can be put in a form amenable to the WKB treatment. This yields

$$\frac{d^2g}{dR^2}+\frac{1}{1-\alpha R^2}\left[3\alpha(1-\alpha R^2)^{1/4}+\frac{2-3\alpha R^2}{R^2}+\frac{5\alpha}{4}\frac{(2-3\alpha R^2)}{(1-\alpha R^2)^{1/2}}-\frac{(2-3\alpha R^2)^2}{2R^2(1-\alpha R^2)}+\frac{\omega^2-k^2+k+(1-\alpha R^2)^{1/2}-(1-\alpha R^2)}{R^2}\right]g=0. \tag{3.24}$$

For reasons which will be obvious when one considers the exterior situation we shall solve the above equation for large values of ω and k, i.e., $k\gg 1$, $\omega>\sqrt{\alpha}$, $\omega\simeq k$. In this limit Eq. (3.24) becomes

$$\frac{d^2g}{dR^2}+\frac{1}{1-\alpha R^2}\left(\omega^2-\frac{k(k+1)}{R^2}\right)g=0. \tag{3.25}$$

The solutions of Eq. (3.25) in the WKB approximation are given by

$$g(R)=\frac{1}{(|p|)^{1/2}}\left[A\exp\left(i\int_{R_b}^{R}p\,dR\right)+B\exp\left(-i\int_{R_b}^{R}p\,dR\right)\right], \tag{3.26}$$

where

$$p=q\omega(1-\alpha R^2)^{-1/2}, \tag{3.27a}$$

$$q=\left(1-\frac{k(k+1)}{\omega^2R^2}\right)^{1/2}. \tag{3.27b}$$

Using Eq. (3.26), G and hence F can be obtained using Eq. (3.20b). Equations (3.19) give R_1 and R_2 and hence the two independent solutions of the neutrino equation are written down as

$$\psi^{i(1)} = u^{i(1)} \exp\left(i \int_{R_b}^{R} p\, dR\right), \tag{3.28a}$$

$$u^{i(1)} = \frac{\exp[-i(\omega\tau - m\phi)]}{2RS^{3/2}} (a_+ s_1, a_- s_2, a_+ s_1, a_- s_2)^T, \tag{3.28b}$$

$$\psi^{i(2)} = u^{i(2)} \exp\left(-i \int_{R_b}^{R} p\, dR\right), \tag{3.28c}$$

$$u^{i(2)} = \frac{\exp[-i(\omega\tau - m\phi)]}{2RS^{3/2}} (a_-^* S_1, a_+^* S_2, a_-^* S_1, a_+^* S_2)^T, \tag{3.28d}$$

where

$$a_\pm(q) = \frac{1}{(|q\omega|)^{1/2}} \left[\frac{k + (1 - \alpha R^2)^{1/2}}{\omega R} + i(q \pm 1)\right]. \tag{3.29}$$

Note that these solutions are valid only for $B = [k(k+1)]^{1/2}/\omega < R < R_b$ and that the maximum value of B is R_b.

C. The exterior solutions

The Dirac equation in the Schwarzchild background has been discussed in many contexts.[9] For completeness and to establish our notation we give these details and use them to obtain the complete WKB solutions in the presence of the Schwarszchild barrier.

With the nonvanishing exterior tetrad field components $e_{a'}^{\bar{\mu}}$ chosen to be

$$e_{0'}^{t} = \left(1 - \frac{2M}{r}\right)^{-1/2}, \quad e_{2'}^{\phi} = (r \sin\theta)^{-1/2}, \quad e_{1'}^{\theta} = (r)^{-1}, \quad e_{3'}^{r} = \left(1 - \frac{2M}{r}\right)^{1/2}, \tag{3.30}$$

where the primed and barred indices represent the exterior tetrad and coordinate labels, respectively, the corresponding Γ_μ's are given by

$$\Gamma_t = -\frac{M}{2r^2}\gamma^0\gamma^3, \quad \Gamma_r = 0,$$
$$\Gamma_\theta = \frac{1}{2}\left(1 - \frac{2M}{r}\right)^{1/2}\gamma^1\gamma^3, \tag{3.31}$$
$$\Gamma_\phi = -\frac{1}{2}\left[\cos\theta\gamma^1\gamma^2 - \left(1 - \frac{2M}{r}\right)^{1/2}\sin\theta\gamma^2\gamma^3\right],$$

and the Dirac equation becomes

$$\left\{\left(1 - \frac{2M}{r}\right)^{-1/2}\gamma^0\partial_t + \left(1 - \frac{2M}{r}\right)^{1/2}\left[\partial_r + \left(1 - \frac{2M}{r}\right)^{-1/2}\frac{M}{2r^2} + \frac{1}{r}\right]\gamma^3 + \frac{1}{r}\left[\gamma^1\left(\partial_\theta + \frac{\cot\theta}{2}\right) + \gamma^2\csc\theta\,\partial_\phi\right]\right\}\psi = 0. \tag{3.32}$$

For the neutrino the solutions are written in the form

$$\psi = \frac{\exp[-i(\bar{\omega}t - \bar{m}\phi)]}{r(1 - 2M/r)^{1/4}}(\eta, \eta)^T \tag{3.33}$$

with

$$\eta = (\bar{R}_1(r)S_1(\theta), \bar{R}_2(r)S_2(\theta))^T. \tag{3.34}$$

The radial equations are then

$$\left(\frac{d}{dr^*} - i\bar{\omega}\right)\bar{R}_1 = \frac{\bar{k}(1 - 2M/r)^{1/2}}{r}\bar{R}_2, \tag{3.35a}$$

$$\left(\frac{d}{dr^*} + i\bar{\omega}\right)\bar{R}_2 = \frac{\bar{k}(1 - 2M/r)^{1/2}}{r}\bar{R}_1. \tag{3.35b}$$

Here r^* is the usual tortoise coordinate defined by

$$\frac{dr^*}{dr} = \frac{r}{r - 2M}. \tag{3.36}$$

S_1 and S_2 satisfy the same differential equations.

Introducing $\bar{F}$ and $\bar{G}$ similar to F and G, we have

$$\left(\frac{d}{dr^*} - \frac{\bar{k}(1 - 2M/r)^{1/2}}{r}\right)\bar{F} = -\bar{\omega}\bar{G}, \quad \left(\frac{d}{dr^*} + \frac{\bar{k}(1 - 2M/r)^{1/2}}{r}\right)\bar{G} = \bar{\omega}\bar{F}. \tag{3.37}$$

The second-order differential equation for $\bar{F}$ and $\bar{G}$ may be found.[3] For large $\bar{k}$, $\bar{F}$ and $\bar{G}$ satisfy the same equation and we find

$$\left[\frac{d^2}{dr^{*2}} + \bar{\omega}^2 - \frac{\bar{k}^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]\bar{G} = 0. \tag{3.38}$$

The external potential

$$V_{\text{ext}}(r) = \frac{\bar{k}^2}{r^2}\left(1 - \frac{2M}{r}\right) \tag{3.39}$$

has a maximum at $r = 3m$ of value $\bar{k}^2(27M^2)^{-1}$. Hence at later stages of the collapse the exterior has a potential barrier which will have to be taken into account when writing out the exterior solutions. In the WKB regime the solutions of Eq. (3.38) are

$$\overline{G}=\frac{1}{(|\overline{p}|)^{1/2}}\left[\overline{A}\exp\left(i\int_{r_b^*}^{r^*}\overline{p}\,dr^*\right)+\overline{B}\exp\left(-i\int_{r_b^*}^{r^*}\overline{p}\,dr^*\right)\right], \qquad (3.40)$$

where

$$\overline{p}=\overline{q}\,\overline{\omega}, \qquad (3.41a)$$

$$\overline{q}=+\left[1-\frac{\overline{k}^2}{\overline{\omega}^2r^2}\left(1-\frac{2M}{r}\right)\right]^{1/2}. \qquad (3.41b)$$

The functions $\overline{R}_1$ and $\overline{R}_2$ may be obtained as before and the two exterior solutions are taken to be

$$\psi^{e(1)}(\overline{q})=u^{e(1)}(\overline{q})\exp\left(i\overline{\omega}\int_{r_b^*}^{r^*}\overline{q}\,dr^*\right), \qquad (3.42a)$$

$$u^{e(1)}(\overline{q})=\frac{\exp[-i(\overline{\omega}t-\overline{m}\phi)]}{2r(1-2M/r)^{1/4}}(\overline{a}_+S_1,\overline{a}_-S_2,\overline{a}_+S_1,\overline{a}_-S_2)^T, \qquad (3.42b)$$

$$\psi^{e(2)}(\overline{q})=u^{e(2)}(\overline{q})\exp\left(-i\overline{\omega}\int_{r_b^*}^{r^*}\overline{q}\,dr^*\right), \qquad (3.42c)$$

$$u^{e(2)}(\overline{q})=\frac{\exp[-i(\overline{\omega}t-\overline{m}\phi)]}{2r(1-2M/r)^{1/4}}(\overline{a}_-^*S_1,\overline{a}_+^*S_2,\overline{a}_-^*S_1,\overline{a}_+^*S_2)^T, \qquad (3.42d)$$

where

$$\overline{a}_\pm\equiv\overline{a}_\pm(\overline{q})=\frac{1}{(|\overline{q}\,\overline{\omega}|)^{1/2}}\left(\frac{\overline{k}(1-2M/r)^{1/2}}{\overline{\omega}r}+i(\overline{q}\pm1)\right). \qquad (3.43)$$

As is obvious the above solutions are valid only for values of $\overline{\omega}^2>V_{\text{ext}}(r^*)$ and in the range $\overline{\omega}^2<V_{\text{ext}}$ the two WKB solutions go like

$$(|\overline{q}\,\overline{\omega}|)^{-1/2}\exp\left(\pm\overline{\omega}\int|\overline{q}|\,dr^*\right). \qquad (3.44)$$

Near the turning points given by roots of the equation $\overline{\omega}^2=V_{\text{ext}}(r^*)$ the above forms are not applicable and the solutions have to be matched using, say, Airy functions. This case is relevant in later stages of the collapse when the Schwarzschild barrier uncovers and we exhibit the matching in detail for this case.

D. The WKB solutions with the Schwarzschild barrier

Let r_k $(k=1,2)$ be the turning points, i.e., roots of the equation $\overline{\omega}^2=V_{\text{ext}}(r^*)$, i.e.,

$$r^3-b^2r-2Mb^2=0, \qquad (3.45a)$$

where

$$b\equiv\frac{\overline{k}}{\overline{\omega}}\text{ and } r_1<3M<r_2. \qquad (3.45b)$$

Expanding $V_{\text{ext}}(r^*)$ about r_k^* we have

$$V(r^*)\equiv V(r_k^*+\xi)=V(r_k^*)-\frac{2\overline{k}^2}{r_k^5}(r_k-2M)(r_k-3M)\xi. \qquad (3.46)$$

Putting

$$\beta^3=\left|\frac{2k^2}{r_k^5}(r_k-2M)(r_k-3M)\right| \qquad (3.47)$$

and defining $\overline{\xi}$ by

$$\overline{\xi}=-\beta\xi,\quad k=2$$
$$=\beta\xi,\quad k=1 \qquad (3.48)$$

Eq. (3.38) becomes, near r_k^*,

$$\frac{d^2\overline{G}}{d\overline{\xi}^2}-\overline{\xi}\,\overline{G}=0, \qquad (3.49)$$

which is the Airy equation with solutions $Ai(\overline{\xi})$ and $Bi(\overline{\xi})$. We schematically show our construction in Fig. 1.

Physically it is obvious that a wave incident on the barrier in region III gives rise to a reflected wave in III. In II both the growing and decaying solutions are possible giving rise to an outgoing solution in region I. We start for convenience with the outgoing wave in region I of the form

$$\overline{G}_{\text{I}}=\frac{A_{\text{I}}}{(|\overline{p}|)^{1/2}}\exp\left[i\left(\int_{r_2^*}^{r^*}\overline{p}\,dr^*+\pi/4\right)\right]. \qquad (3.50)$$

Near $r=r_2$

$$\overline{p}=\beta(-\overline{\xi})^{1/2},\quad \overline{\xi}<0 \qquad (3.51a)$$

so that

$$\int_{r_2^*}^{r^*}\overline{p}\,dr^*=\tfrac{2}{3}(-\overline{\xi})^{3/2}. \qquad (3.51b)$$

Hence near r_2, $\overline{G}_{\text{I}}$ goes like

$$\overline{G}_{\text{I}}\sim\frac{\overline{A}_{\text{I}}}{\sqrt{\beta}(|\overline{\xi}|)^{1/4}}\{\cos[\tfrac{2}{3}(-\overline{\xi})^{3/2}+\pi/4]+i\sin[\tfrac{2}{3}(-\overline{\xi})^{3/2}+\pi/4]\}, \qquad (3.52)$$

which, using the asymptotic form of Airy functions, is the combination

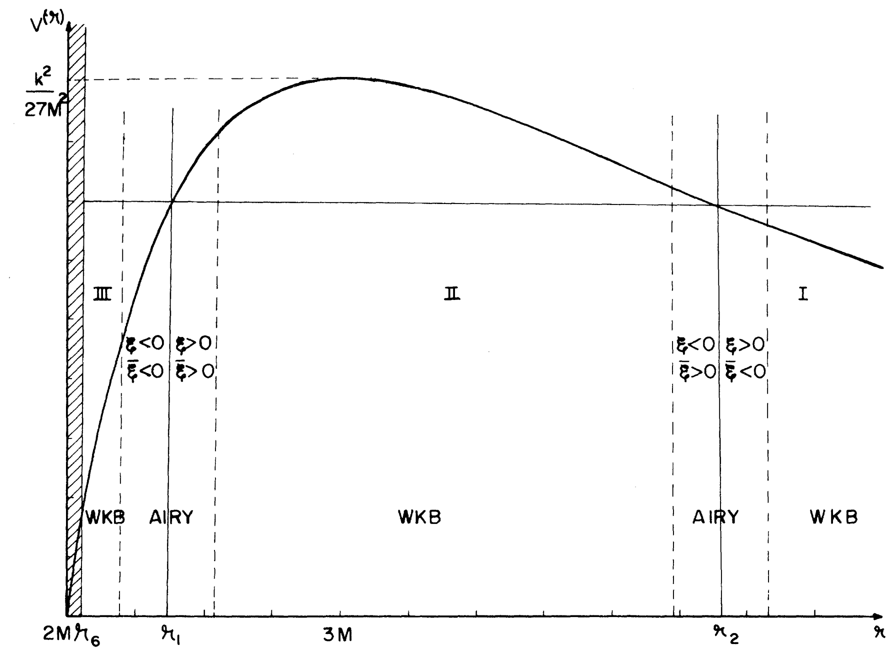

FIG. 1. The figure shows the effective potential $V_{\text{ext}}\,(r)$ and the regions of validity of the WKB approximation and the Airy approximation of the exterior wave function. $V_{\text{ext}}\,(r)$, $r\in(r_b\,,\infty)$, has a maximum of $k^2(27M^2)^{-1}$ at $r=3M$ and goes to zero as r approaches infinity. The WKB approximation is valid everywhere except in neighborhoods of r_1 and r_2, the width of the neighborhoods being given by $|\bar\xi|\sim O(10)$. In the figure the width of these regions is exaggerated.

$$\bar G_{\mathrm{I}}=\bar A_{\mathrm{I}}\left(\frac{\pi}{\beta}\right)^{1/2}[\mathrm{Bi}(\bar\xi)+i\,\mathrm{Ai}(\bar\xi)]\,. \tag{3.53}$$

This combination for $\bar\xi>0$ goes like

$$\bar G_{\mathrm{I}}\sim\frac{\bar A_{\mathrm{I}}}{\sqrt{\beta}(|\bar\xi|)^{1/4}}\left[\exp(\tfrac{2}{3}\bar\xi^{3/2})+\tfrac{1}{2}i\exp(-\tfrac{2}{3}\bar\xi^{3/2})\right]. \tag{3.54}$$

Now, near r_2,

$$|\bar p|=\beta\sqrt{\bar\xi} \tag{3.55a}$$

so that

$$\int_{r_2^*}^{r^*}|\bar p|\,dr^*=\tfrac{2}{3}(\bar\xi)^{3/2}\,. \tag{3.55b}$$

Hence $\bar G_{\mathrm{I}}$ goes into the combination

$$\bar G_{\mathrm{II}}=\frac{\bar A_{\mathrm{I}}}{(|\bar p|)^{1/2}}\left[\exp\left(\int_{r^*}^{r_2^*}|\bar p|\,dr^*\right)+\frac{i}{2}\exp\left(-\int_{r^*}^{r_2^*}|\bar p|\,dr^*\right)\right]. \tag{3.56}$$

Defining

$$T=\exp\left(-\int_{r_1^*}^{r_2^*}|\bar p|\,dr^*\right), \tag{3.57}$$

Eq. (3.56) becomes

$$\bar G_{\mathrm{II}}=\frac{\bar A_{\mathrm{I}}}{(|\bar p|)^{1/2}}\left[\frac{1}{T}\exp\left(-\int_{r_1^*}^{r^*}|\bar p|\,dr^*\right)+\frac{iT}{2}\exp\left(\int_{r_1^*}^{r^*}|\bar p|\,dr^*\right)\right]. \tag{3.58}$$

Repeating a similar procedure near the point $r=r_1$ we can continue into region III and obtain the solution here as

$$\bar G_{\mathrm{III}}=\frac{\bar A_{\mathrm{I}}}{i(|\bar p|)^{1/2}}\left\{\left(\frac{1}{T}-\frac{T}{4}\right)\exp\left[i\left(\int_{r^*}^{r_1^*}\bar p\,dr^*+\pi/4\right)\right]-\left(\frac{1}{T}+\frac{T}{4}\right)\exp\left[-i\left(\int_{r^*}^{r_1^*}\bar p\,dr^*+\pi/4\right)\right]\right\}. \tag{3.59}$$

In order to facilitate the matching with the interior solution at the boundary r_b we introduce

$$T_1=\exp\left[-i\left(\int_{r_b^*}^{r_1^*}\bar p\,dr^*-\pi/4\right)\right] \tag{3.60}$$

so that we can write

$$\bar G_{\mathrm{III}}=\frac{\bar A_{\mathrm{I}}}{(|\bar p|)^{1/2}}\left[\frac{1}{T_1}\left(\frac{1}{T}-\frac{T}{4}\right)\exp\left(-i\int_{r_b^*}^{r^*}\bar p\,dr^*\right)+T_1\left(\frac{1}{T}+\frac{T}{4}\right)\exp\left(i\int_{r_b^*}^{r^*}\bar p\,dr^*\right)\right]. \tag{3.61}$$

The corresponding F's can be obtained and hence the wave function in the three regions is given by

$$\psi^e_{\rm III}=\bar{A}_{\rm I}\left[\frac{1}{T_1}\left(\frac{1}{T}-\frac{T}{4}\right)u^{e(2)}(\bar{q})\exp\left(-i\int_{r_b^*}^{r^*}\bar{p}\,dr^*\right)+T_1\left(\frac{1}{T}+\frac{T}{4}\right)u^{e(1)}(\bar{q})\exp\left(i\int_{r_b^*}^{r^*}\bar{p}\,dr^*\right)\right],$$

$$\psi^e_{\rm II}=\bar{A}_{\rm I}\left[\frac{1}{T}u^{e(2)}(-i|\bar{q}|)\exp\left(-\int_{r_1^*}^{r^*}|\bar{p}|\,dr^*\right)+\frac{iT}{2}u^{e(1)}(-i|\bar{q}|)\exp\left(\int_{r_1^*}^{r^*}|\bar{p}|\,dr^*\right)\right], \tag{3.62}$$

$$\psi^e_{\rm I}=\bar{A}_{\rm I}u^{e(1)}(\bar{q})\exp\left[i\left(\int_{r_2^*}^{r^*}\bar{p}\,dr^*+\pi/4\right)\right].$$

IV. THE JUNCTION CONDITIONS

The collapse scenario has been modeled by a Friedmann interior and a Schwarzschild exterior separated by a surface $r_b=R_bS$ across which the geometry, i.e., the metric and its derivatives is continuous. We are in fact considering *one* spacetime charted out by two different coordinate systems in two different regions. For a scalar field the boundary condition would have been the equality of values at the boundary. For the spinor field one has to be more careful since the Dirac wave function is always tied up with a choice of the Lorentz frame, in this case, a locally Lorentzian one. For convenience of setting up the Dirac equation in the interior and the exterior we have made an implicit choice of the tetrad tied up with the coordinate directions. Thus the interior wave function is tied up to the Lorentz frame along the (T,R) directions whereas the exterior ψ is with respect to the frame in the (t,r) direction (the θ and ϕ directions are the same by choice). Thus for the Dirac field the junction condition on ψ is that the appropriately rotated interior wave function equals the exterior wave function at the boundary (note that under the coordinate transformation ψ is a scalar). So,

$$\psi^e=\mathcal{S}\psi^i\,, \tag{4.1}$$

where the rotation matrix $\mathcal{S}$ depends on the Lorentz transformation as specified later. Moreover, since going from the interior to the exterior involves in addition to a tetrad transformation a coordinate transformation the derivatives would transform as

$$\nabla_{\bar\nu}\psi^e=\frac{\partial x^\mu}{\partial x^{\bar\nu}}\mathcal{S}\nabla_\mu\psi^i\,. \tag{4.2}$$

This matching of derivatives is in accordance with the theory of first-order partial differential equations in which both the value of the function and its derivatives must agree on the boundary. In the specific case of a partial differential equation in two variables the geometrical picture is that the tangent planes must match across the boundary, that is, their normals must be the same. However, our case is a little more complex. The Dirac equations in the two geometries, the interior and the exterior, are not connected merely by a coordinate transformation owing to the choice of representations of γ^μ's. Therefore, the tangent planes at the boundary are not matched in the usual way but with the help of Eq. (4.2). This equation defines a rule which uniquely specifies the tangent planes in the exterior geometry if given the tangent planes in the interior.

For the θ and ϕ coordinates Eq. (4.2) gives the anticipated results

$$k=\bar{k},\quad m=\bar{m} \tag{4.3}$$

because the θ and ϕ coordinates are chosen to be the same in both the interior and exterior geometries. For T it yields the relationship between the "energies" ω and $\bar\omega$ measured by the interior, i.e., comoving observers, and exterior, i.e., the observer at fixed coordinate r, respectively.

To obtain the matrix $\mathcal{S}$ we proceed as follows. Since e^μ_a is a coordinate vector, i.e., under a *fixed* choice of the tetrad it transforms as

$$e^{\bar\mu}_a=\frac{\partial x^{\bar\mu}}{\partial x^\nu}e^\nu_a\,, \tag{4.4}$$

we get

$$e^t_a=\left(1-\frac{2M}{r}\right)^{-1}[(1-\alpha R_b^2)^{1/2}\delta^0_a+R_b\dot{S}\delta^3_a]\,, \tag{4.5a}$$

$$e^r_a=R_b\dot{S}\delta^0_a+(1-\alpha R_b^2)^{1/2}\delta^3_a \tag{4.5b}$$

Since e^μ_a is also a Lorentz vector for a fixed value of μ it transforms like a vector under tetrad rotations

$$e^\mu_{a'}=\Lambda^a_{a'}(x)\,e^\mu_a\,, \tag{4.6}$$

where $\Lambda^a_{a'}(x)$ is a position-dependent Lorentz transformation. A straightforward computation yields

$$\Lambda^a_{a'}=\left(1-\frac{2M}{r_b}\right)^{-1/2}\begin{pmatrix}(1-\alpha R_b^2)^{1/2} & -R_b\dot{S}\\ R_b\dot{S} & (1-\alpha R_b^2)^{1/2}\end{pmatrix}. \tag{4.7}$$

Setting

$$\tanh\delta \equiv R_b \dot{S}(1-\alpha R_b^2)^{-1/2} \tag{4.8}$$

it is seen that Eq. (4.7) is a boost by $(-\delta)$ in the 0–3 plane. Under this transformation the spinor ψ transforms by

$$\mathcal{S} = \exp\left(\frac{\delta}{2}\alpha_3\right), \tag{4.9}$$

which, after a straightforward calculation, can be put in the form

$$\mathcal{S} = \frac{1}{2}\left(1-\frac{2M}{r_b}\right)^{-1/4}[(\epsilon_- + \epsilon_+) + (\epsilon_- - \epsilon_+)\alpha_3], \tag{4.10}$$

where

$$\epsilon_\pm = [(1-\alpha R_b^2)^{1/2} \pm \sqrt{\alpha} R_b \tan\chi_b]^{1/2}. \tag{4.11}$$

We now continue consistently the interior solution $\psi^{i(1)}$ which represents an outwardgoing wave as seen by a comoving observer to the outside. Let us assume that $\psi^{i(1)}$ goes into A_1 times the exterior solution $\psi^{e(1)}$ with the appropriate rotation, that is,

$$\mathcal{S}\psi^{i(1)} = A_1 \psi^{e(1)}, \tag{4.12}$$

where ω and $\bar{\omega}$ would be related by using Eq. (4.2) in the form

$$\mathcal{S}\nabla_T \psi^{i(1)} = \frac{\partial x^{\bar{\nu}}}{\partial T}\nabla_{\bar{\nu}}\psi^{e(1)}. \tag{4.13}$$

Employing Eqs. (4.10), (4.12), and (4.13) we get the four relations

$$\frac{a_+\epsilon_- \exp(-i\omega\tau_b)}{\sqrt{S_b}} = \bar{a}_+ A_1 \exp(-i\bar{\omega}t_b), \tag{4.14a}$$

$$\frac{a_-\epsilon_- \exp(-i\omega\tau_b)}{\sqrt{S_b}} = \bar{a}_- A_1 \exp(-i\bar{\omega}t_b), \tag{4.14b}$$

$$\frac{1}{S_b}(\omega - \dot{S}) = \left[\left(\bar{\omega} + \frac{M}{2r_b^2}\right)(1-\alpha R_b^2)^{1/2} + \sqrt{\alpha} R_b \tan\chi_b \bar{p}\right]\left(1-\frac{2M}{r_b}\right)^{-1}, \tag{4.15a}$$

$$\frac{1}{S_b}(\omega - \dot{S}) = \left[\left(\bar{\omega} - \frac{M}{2r_b^2}\right)(1-\alpha R_b^2)^{1/2} + \sqrt{\alpha} R_b \tan\chi_b \bar{p}\right]\left(1-\frac{2M}{r_b}\right)^{-1}, \tag{4.15b}$$

where Eqs. (4.14) have been used to simplify (4.15). In the adiabatic approximation $\omega \gg \dot{S}$ and since $\bar{\omega} \gg M/2r_b^2$ Eqs. (4.15a) and (4.15b) become identical:

$$\frac{\omega}{S_b} = \bar{\omega}\left(1-\frac{2M}{r_b}\right)^{-1}[(1-\alpha R_b^2)^{1/2} + \sqrt{\alpha} R_b \tan\chi_b \bar{q}]. \tag{4.16}$$

Equation (4.16) may be inverted and after some algebra we obtain

$$\bar{\omega} = \frac{\omega}{S_b}[(1-\alpha R_b^2)^{1/2} - \sqrt{\alpha} R_b \tan\chi_b q], \tag{4.17}$$

where the proper sign has to be chosen after solving the quadratic equation for $\bar{\omega}$.

Equation (4.17) gives the energy as measured by the observer at fixed r (Schwarzschild observer) with the expected spectral shifts. The S_b factor in the denominator is the correct "cosmological" blue shift whereas the bracketed terms represent the Doppler shift between the two frames chosen by the Friedmann and Schwarzschild observers, respectively. The other well-known gravitational red-shift for the Schwarzschild field would be obtained if one remembers that t is not the proper time. If this effect were also taken into account the observed frequency will contain an additional factor of $(1-2M/r_b)^{-1/2}$. These results are in agreement with those of Ref. 4. What remains to be checked now is the consistency of Eqs. (4.14) which are two equations for one parameter A_1 which depend on the epoch of matching χ_b.

The equations are consistent if

$$\frac{\bar{a}_+}{\bar{a}_-} = \frac{a_+\epsilon_-}{a_-\epsilon_+}. \tag{4.18}$$

To check this we use the relation between ω and $\bar{\omega}$, Eq. (4.16), and after some calculation it can be shown that the task of proving Eq. (4.18) is equivalent to proving

$$\frac{1+\bar{q}}{1-\bar{q}} = \left(\frac{1+q}{1-q}\right)\frac{\epsilon_-}{\epsilon_+}. \tag{4.19}$$

To this end we first solve the quadratic for $\bar{q}$ in terms of q. The algebra is fairly long and yields

$$\bar{q} = \frac{-\sqrt{\alpha} R_b(1-q^2)(1-\alpha R_b^2)^{1/2}\tan\chi_b \pm q(1-\alpha R_b^2\sec^2\chi_b)}{1-\alpha R_b^2(1+q^2\tan^2\chi_b)}. \tag{4.20}$$

With our convention that q be non-negative we choose the positive sign in Eq. (4.20). With this choice it is straightforward to check that Eq. (4.19) holds and consequently Eqs. (4.14) are consistent. Note, however, that the relation (4.20) cannot be taken for all values of B, since for $B > B_m$, where B_m is the root of the equation

$$q(1-\alpha R_b^2\sec^2\chi_b)=\sqrt{\alpha}R_b(1-q^2)(1-\alpha R_b^2)^{1/2}\tan\chi_b\,, \tag{4.21}$$

$\bar{q}$ becomes negative. Thus the type-1 interior solution $\psi^{i(1)}$ can be matched to the type-1 exterior mode $\psi^{e(1)}$ only for values of B in the range $0 < B < B_m$. Note that B_m is a function of the epoch χ_b. We shall show subsequently that the values of B given by $B_m < B < R_b$ the type-1 interior solution can be consistently matched only with the type-2 exterior solution $\psi^{e(2)}$. This is the phenomenon of backward emission[11] wherein a wave traveling outwards with respect to the comoving observer is traveling inwards as seen by the Schwarzschild observer and which has been discussed in the geodesic approximation.[4] Equation (4.14) can now be solved for A_1. Thus

$$A_1=\frac{\epsilon_-}{\sqrt{S_b}}\exp[i(\bar{\omega}t_b-\omega\tau_b+\phi_+-\bar{\phi}_+)]\left(\frac{1+q}{1+\bar{q}}\,\frac{\bar{q}\bar{\omega}}{q\omega}\right)^{1/2}, \tag{4.22}$$

where

$$\tan\phi_+=\left(\frac{q+1}{k}\right)\omega R\,, \tag{4.23a}$$

$$\tan\bar{\phi}_+=\left(\frac{\bar{q}+1}{k}\right)\frac{\bar{\omega}r_b}{(1-2M/r_b)^{1/2}}\,. \tag{4.23b}$$

A. The backward emission

In this section we investigate whether one can match $\psi^{i(1)}$ with $\psi^{e(2)}$ for any value of B. The procedure is exactly as before and we write down the results.

Matching ψ we get

$$\frac{a_+\epsilon_-\exp(-i\omega\tau_b)}{\sqrt{S_b}}=\bar{a}_-^*A_b\exp(-i\bar{\omega}t_b)\,, \tag{4.24a}$$

$$\frac{a_-\epsilon_+\exp(-i\omega\tau_b)}{\sqrt{S_b}}=\bar{a}_+^*A_b\exp(-i\bar{\omega}t_b)\,, \tag{4.24b}$$

with

$$\frac{\omega}{S_b}=\bar{\omega}\left(1-\frac{2M}{r_b}\right)^{-1}[(1-\alpha R_b^2)^{1/2}-\sqrt{\alpha}R_b\tan\chi_b\bar{q}]\,. \tag{4.25}$$

Consistency of Eqs. (4.24) is equivalent to showing

$$\frac{1-\bar{q}}{1+\bar{q}}=\frac{1+q}{1-q}\,\frac{\epsilon_-}{\epsilon_+}\,, \tag{4.26}$$

where

$$\bar{q}=\frac{\sqrt{\alpha}R_b(1-q^2)(1-\alpha R_b^2)^{1/2}\tan\chi_b\pm q(1-\alpha R_b^2\sec^2\chi_b)}{1-\alpha R_b^2(1+q^2\tan^2\chi_b)}\,. \tag{4.27}$$

With the positive sign Eq. (4.26) does not hold whereas for the negative sign it does. However, in this case $\bar{q}$ is positive only for $B_m < B < R_b$ since the numerator is exactly the negative of Eq. (4.20). Thus $\psi^{i(1)}$ can be consistently matched to $\psi^{e(2)}$ only in the range $B_m < B < R_b$ and in this case A_b is given by

$$A_b=\frac{\epsilon_-}{\sqrt{S_b}}\exp[i(\bar{\omega}t_b-\omega\tau_b+\phi_+-\bar{\phi}_-)]\left(\frac{1+q}{1-\bar{q}}\,\frac{\bar{q}\bar{\omega}}{q\omega}\right)^{1/2} \tag{4.28}$$

with

$$\tan\bar{\phi}_-=\left(\frac{\bar{q}-1}{k}\right)\frac{\bar{\omega}r}{(1-2M/r_b)^{1/2}}\,. \tag{4.29}$$

For completeness we mention that $\psi^{i(2)}$ can be similarly matched to $\psi^{e(2)}$ for all values of B between 0 and R_b whereas $\psi^{i(2)}$ cannot be matched at all to $\psi^{e(1)}$ for any possible value of B. These results physically mean that a wave traveling inward with respect to the comoving observer cannot be forward moving with respect to the Schwarzschild observer but always inward moving. In this case the coefficient A_2 is given by

$$A_2=\frac{\epsilon_-}{\sqrt{S_b}}\exp[i(\bar{\omega}t_b-\omega\tau_b+\bar{\phi}_--\phi_-)]\left(\frac{1-q}{1-\bar{q}}\,\frac{\bar{q}\bar{\omega}}{q\omega}\right)^{1/2}, \tag{4.30}$$

where

$$\bar{\omega}=\frac{\omega}{S_b}[(1-\alpha R_b^2)^{1/2}+\sqrt{\alpha}R_b\tan\chi_b q] \tag{4.31}$$

and

$$\tan\phi_-=\left(\frac{q-1}{k}\right)\omega R\,. \tag{4.32}$$

The above effects can be looked upon as a reflection of the difference in the state of motion of the two types of observers or as due to the dragging effect of the collapsing matter on the neutrinos. One may also remark that the situation concerning the matching of the ingoing and outgoing waves at the boundary is reversed if the object were expanding instead of collapsing.

V. THE ASYMPTOTIC FLUX

We now employ the above matched modes to compute the asymptotic transmission coefficient in some relevant physical situations. To this end recall that the Dirac equation defines a conserved current J^μ:

$$\nabla_\mu J^\mu = 0\,, \tag{5.1}$$

where

$$J^\mu = \bar\psi \gamma^\mu \psi \tag{5.2a}$$

with

$$\gamma^\mu = e^\mu_a \gamma^a \text{ and } \bar\psi = \psi^\dagger \gamma^0\,. \tag{5.2b}$$

From Eq. (5.1) the time rate of the number of particles flowing past the $R=$ constant surface is given by

$$\left.\frac{\partial N}{\partial T}\right|_{R=\text{constant}} = \int_{R=\text{constant}} \sqrt{-g}\, J^R\, d\theta\, d\phi, \tag{5.3}$$

where

$$N = \int \sqrt{-g}\, J^T\, dR\, d\theta\, d\phi\,. \tag{5.4}$$

The corresponding rate for the Schwarzschild observer is given by

$$\left.\frac{\partial N}{\partial t}\right|_{r=\text{constant}} = \int_{r=\text{constant}} \sqrt{-g}\, J^r d\theta\, d\phi\,. \tag{5.5}$$

We define the transmission coefficient K as the ratio of the two fluxes:

$$K = \frac{(\partial N/\partial t)_{r_0}}{(\partial N/\partial t)_{R_0}} = \frac{\int_{r=r_0}\sqrt{-g}\, J^r\, d\theta\, d\phi}{\int_{R=R_0}\sqrt{-g}\, J^R d\theta\, d\phi}\,. \tag{5.6}$$

Before calculating K let us discuss the possible cases that may arise. These cases depend on three factors: (1) the radius of the collapsing object when the neutrino crosses the boundary, (2) whether the neutrino is forward or backward emitted, and (3) whether the frequency is above or below the potential barrier. We discuss in detail one case and the quote the results for the others. Consider the epoch wherein the collapse has proceeded past $r=3M$. The Schwarzschild barrier is now uncovered and those modes with the values of $\bar\omega^2 > k^2(27M^2)^{-1}$ will only be marginally affected. These correspond to values of $b < 3\sqrt{3}M$. Of the modes with $b > 3\sqrt{3}M$ those that are backward emitted will fall into the eventually formed black hole whereas those that are forward emitted will hit the barrier and may tunnel through it. The probability of penetration turns out to be so negligible that these modes are almost totally reflected.[12] The classical analog of such confinement has been discussed within the framework of geodesic formalism.[4]

Let us now calculate K for modes with $B < B_m$ and $b < 3\sqrt{3}M$. For these modes, neglecting the small reflected wave, the wave function in the entire spacetime is given by $\psi^{i(1)}$ in the interior and $A_1\psi^{e(1)}$ in the exterior. With these values of the wave function a straightforward computation yields

$$\int_{R=R_0} \sqrt{-g}\, J^R d\theta\, d\phi = (S_0\omega)^{-1}\,, \tag{5.7a}$$

$$\int_{r=r_0} \sqrt{-g}\, J^r\, d\theta\, d\phi = |A_1|^2(\bar\omega)^{-1}\,. \tag{5.7b}$$

Substituting for A_1 we finally obtain

$$K \equiv K_f(\bar q) = \frac{S_0}{S_b}\epsilon_-^{\,2}\, \frac{1+q}{1+\bar q}\, \frac{|\bar q|}{q}\,, \tag{5.8}$$

where K_f denotes transmission without the barrier. For modes with $B<B_m$ and $b > 3\sqrt{3}M$, the corresponding wave functions are given by

$$\begin{aligned} \psi^e_1 &= A_1 u^{e(1)} \exp\left[i\left(\int_{r_2^*}^{r^*} \bar p\, dr^* + \pi/4\right)\right]\,, \\ \psi^i &= T_1\left(\frac{1}{T} + \frac{T}{4}\right)\psi^{i(1)}\,. \end{aligned} \tag{5.9}$$

Employing these K comes out to be

$$K = K_f(\bar q)\left(\frac{1}{T} + \frac{T}{4}\right)^{-2} \tag{5.10}$$

which for small values of $T \ll 1$ becomes

$$K = K_f(\bar q) T^2\,. \tag{5.11}$$

The remaining cases are summarized in Table I.

It should be mentioned that in all the backward-emission cases our results are given with the understanding that in the actual case one would be considering the wave packet formed from our

TABLE I. The truth table showing the asymptotic transmission K, in the various cases that arise depending on the radius of the collapsing object when the neutrino crosses the boundary ($r \lessgtr 3M$), on whether it is forward ($B < B_m$) or backward emitted ($B > B_m$), and on whether the frequency is above ($b < 3\sqrt{3}M$) or below ($b > 3\sqrt{3}M$) the barrier. Here 1 represents a true statement and 0 a false statement, e.g., in the first row all the three statements are true and in this case the transmission is given by $K_f(\bar q)$.

$r < 3M$	$B < B_m$	$b < 3\sqrt{3}M$	K
1	1	1	$K_f(\bar q)$
1	1	0	$K_f(\bar q)\left(\frac{1}{T}+\frac{T}{4}\right)^{-2}$
1	0	1	0
1	0	0	0
0	1	1	$K_f(\bar q)$
0	1	0	$K_f(\bar q)$
0	0	1	0
0	0	0	$K_f(-\bar q)\left(\frac{4-T^2}{4+T^2}\right)^2$

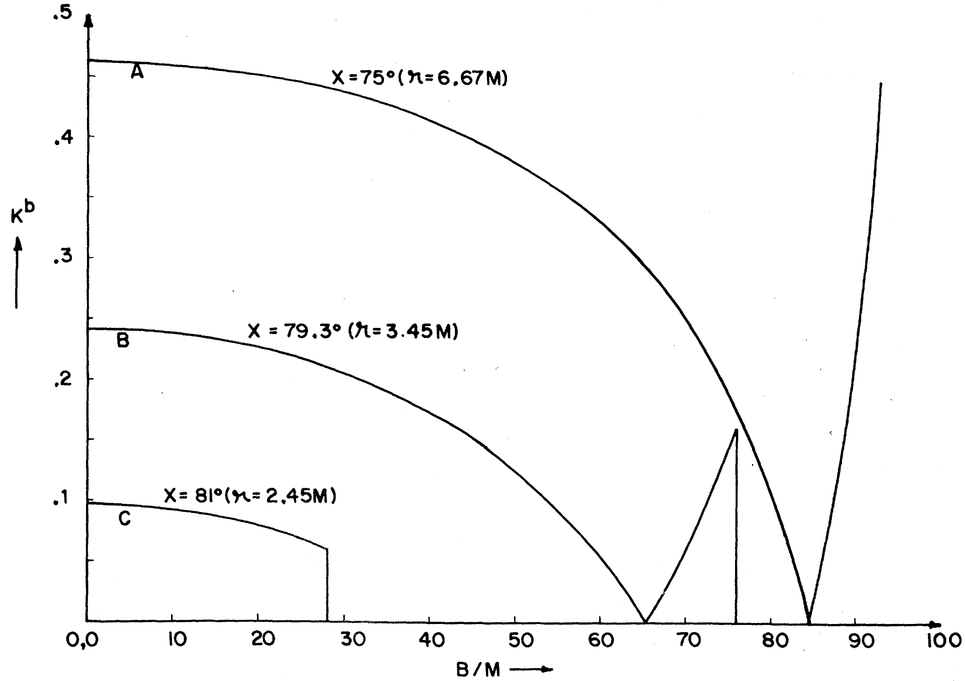

FIG. 2. The plot of asymptotic transmission K^b against the interior impact parameter B for various epochs χ_b is shown. The curves fall into three different types labeled A, B, C depending on the epoch of matching, i.e., on whether $r_b > 3\sqrt{3}M\ (1-\alpha R_b^2)$, $3M < r_b < 3\sqrt{3}M\,(1-\alpha R_b^2)$, and $2M < r_b < 3M$, respectively. The initial portion of the curve represents the forward emission whereas the second branch in cases A and B represents the backward-emitted waves reflected from the barrier. The vertical portion in the graphs B and C indicates the limiting value of B beyond which the modes are confined (i.e., not observed at infinity). The apparent discontinuity in the derivative of K^b is a consequence of the WKB approximation since the factor T varies rapidly near $b = 3\sqrt{3}M$.

type-1 and -2 solutions. It is this which yields the zero asymptotic flux for the backward-emitted cases except the last one which we shall discuss in slightly more detail. Consider modes which are backward emitted at an epoch when $r \gg 3M$. Since the backward-emitted modes will not catch up with the infalling surface[13] those modes with $b > 3\sqrt{3}M$ will be reflected by the barrier present in the exposed Schwarzschild exterior. This would seem like the usual scattering problem in the Schwarzschild background and one can obtain the complete solution in terms of the WKB solutions of Sec. III. We write these down to make obvious the quoted value of K for this case

$$\psi_{\rm I}^e = A_b\left\{\left(\frac{1}{T}+\frac{T}{4}\right)u^{e(2)}\exp\left[-i\left(\int_{r_2^*}^{r^*}\bar{p}\,dr^* + \pi/4\right)\right] - \left(\frac{1}{T}-\frac{T}{4}\right)u^{e(1)}\exp\left[i\left(\int_{r_2^*}^{r^*}\bar{p}\,dr^* + \pi/4\right)\right]\right\},$$

$$\psi_{\rm II}^e = A_b\left[\frac{T}{2}u^{e(2)}(-i\,|\bar{p}|)\exp\left(\int_{r^*}^{r_2^*}|\bar{p}|\,dr^*\right) - \frac{i}{T}u^{e(1)}(-i\,|\bar{p}|)\exp\left(-\int_{r^*}^{r_2^*}|\bar{p}|\,dr^*\right)\right], \tag{5.12}$$

$$\psi_{\rm III}^e = \frac{A_b}{T_1}\psi^{e(2)},$$

$$\psi^i = \left(\frac{1}{T}+\frac{T}{4}\right)\psi^{i(1)}T_2, \tag{5.13}$$

where

$$T_2 = \exp\left(-i\int_{r_2^*}^{r_b^*}\bar{p}\,dr^*\right). \tag{5.14}$$

This yields

$$K = \left(\frac{1}{T}-\frac{T}{4}\right)^2\left(\frac{1}{T}+\frac{T}{4}\right)^{-2}K_f(-\bar{q}), \tag{5.15}$$

which for $T \ll 1$ becomes

$$K \simeq (1-T^2)K_f(-\bar{q}). \tag{5.16}$$

The value of K if the inner surface R_0 is chosen to be R_b is denoted by K^b. It should be noted that K is related to K^b by a simple scale factor

$$K=\left(\frac{S_0}{S_b}\right)K^b. \tag{5.17}$$

In Fig. 2 we give a plot of transmission K^b as a function of the source impact parameter B. This summarizes the features which we have discussed in the previous sections. Further investigations would involve constructing wave packets to study the interesting features like flux profiles discussed in Ref. 5 and the necessary formalism for these purposes has been set up in this work.

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[12]It should be recalled that our analysis is for high values of k and ω which may be more relevant in astrophysical contexts. In this limit the transmission coefficient varies from very low values close to zero to minimum values close to unity within a very short energy range. The problem for small values of k and ω can be done from the exact equation given in the text by numerical integration.

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